Almost Ideals and Fuzzy Almost Ideals of Ordered Semirings

Pokpong Srimora

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
2023

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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ชื่อวิทยานิพนธ์ เกือบไอดีล และเกือบไอดีลวิภัชนัยของกึ่งริงอันดับ
ผู้เขียน นายปกป้อง ศรีโมรา
สาขาวิชา คณิตศาสตร์
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บทคัดย่อ

กึ่งริงอันดับคือระบบ $(S, +, \cdot, \leq)$ ซึ่งประกอบด้วยเซตไม่ว่าง S โดยที่ $(S, +, \cdot)$ เป็นกึ่งริง $(S, \leq)$ เป็นเซตอันดับบางส่วน และสำหรับทุก $a, b, c \in S$ ถ้า $a \leq b$ แล้ว $a+c \leq b+c,\ c+a \leq c+b$ และ $ac \leq bc,\ ca \leq cb$

ในงานวิจัยนี้ได้มีการนิยาม เกือบกึ่งริงย่อยอันดับ เกือบไอดีลอันดับ เกือบควอซี-ไอดีลอันดับ เกือบไบ-ไอดีลอันดับ และ เกือบไอดีลภายในอันดับของกึ่งริงอันดับ ยิ่งไปกว่านั้น ได้มีการนิยาม เกือบกึ่งริงย่อยอันดับวิภัชนัย เกือบไอดีลอันดับวิภัชนัย เกือบควอซี-ไอดีลอันดับวิภัชนัย เกือบไบ-ไอดีลอันดับวิภัชนัย และ เกือบไอดีลภายในอันดับวิภัชนัยของกึ่งริงอันดับ รวมทั้งศึกษาสมบัติและความสัมพันธ์ต่างๆ ของเกือบไอดีลอันดับและเกือบไอดีลอันดับวิภัชนัยชนิดต่างๆ ของกึ่งริงอันดับ

สุดท้ายนี้ได้ให้นิยามของ ไตร-ควอซีไอดีล และ ไตร-ควอซีไอดีลวิภัชนัยของกึ่งริงอันดับและศึกษาสมบัติ และความสัมพันธ์ของไตร-ควอซีไอดีล และไตร-ควอซีไอดีลวิภัชนัยของกึ่งริงอันดับ รวมไปถึงการให้นิยามของเกือบไตร-ควอซีไอดีลอันดับและเกือบไตร-ควอซีอันดับวิภัชนัยของกึ่งริงอันดับและศึกษาสมบัติและความสัมพันธ์ต่างๆ ของเกือบไตร-ควอซีไอดีลอันดับ และเกือบไตร-ควอซีไอดีลอันดับวิภัชนัยของกึ่งริงอันดับ

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Academic Year 2022

Abstract

An ordered semiring is a system $(S, +, \cdot, \leq)$ consisting of a nonempty set S such that $(S, +, \cdot)$ is a semiring, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S$, if $a \leq b$, then $a + c \leq b + c$, $c + a \leq c + b$ and $ac \leq bc$, $ca \leq cb$.

In this research, we introduce the concepts of almost ordered subsemirings, almost ordered ideals, almost ordered quasi-ideals, almost ordered bi-ideals, almost ordered interior-ideals of ordered semirings and investigate their properties. Moreover, we define fuzzy almost ordered subsemirings, fuzzy almost ordered ideals, fuzzy almost ordered quasi-ideals, fuzzy almost ordered bi-ideals and fuzzy almost ordered interior-ideals of ordered semirings and provide some relationships.

Finally, we define tri-quasi ideals and fuzzy tri-quasi ideals of ordered semirings and investigate some properties and relationships between them. In addition, we introduce the notion of almost ordered tri-quasi ideals and fuzzy almost ordered tri-quasi ideals of ordered semirings and give some relationship between them.

Acknowledgments

I would like to express my deepest and sincere gratitude to my advisor, Assoc. Prof. Dr. Ronnason Chinram and my co-advisor Asst. Prof. Dr. Winita Yonthanthum, for suggestions, guidance, and criticism which are very useful for this research and let me carry out my success.

I would like to thank the examining committee Dr. Pattarawan Singavananda and Asst. Prof. Dr. Montakarn Petapirak for their constructive comments and suggestions.

I also would like to thank all of my teacher who educated me in all previous lectures.

I extend my thanks to seniors, my classmates, and my friends for their helps and suggestions.

Finally, I am very grateful to Faculty of Science Research Fund, Prince of Songkla University for a financial support throughout acadmic years.

Pokpong Srimora

Contents

Chapter 1

Introduction

Ideal theory play an important role in semigroups and many types of algebraic structures. The notion of bi-ideals in semigroups was first introduced by Good and Hughes [1] in 1952. Steinfeld [2] introduced the concepts of quasi-ideals in semigroups in 1956. The definition of interior ideals of semigroups has been introduced by Lajos in [3]. In 1958, Iséki [4] studied quasi-ideals of semirings without zero. The concept of bi-ideals in associative rings was introduced in 1970 [5]. Rao [8] studied tri-quasi-ideals in Γ-semirings. This ideal generalizes ideals, left ideals, right ideals, bi-ideals, quasi-ideals and interior ideals.

Fuzzy set theory was developed by Zadeh [15] as an extension of the classical notion of set. In 1971, Rosenfeld [16] introduced the fuzzification of algebraic structures as well as fuzzy subgroups. The concept of fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals of semigroups were studied in [9, 10]. In 1995, Hong, Jun and Meng [11] considered the fuzzifications of interior ideals of semigroups.

Almost ideals of semigroups were first introduced by Grosek and Satko [13] in 1980. The concept of almost ideals was used to study almost ideals in many algebraic structures. The definition of almost subsemigroups and fuzzy almost subsemigroups of semigroups was introduced in [14]. The notion of almost subsemiring and fuzzy almost subsemiring of semirings were defined in [17]. In 1981, Bogdanovic [18] introduced the notion of almost bi-ideals in semigroups. Wattanatripop, Chinram and Changphas defined almost quasi-ideals and fuzzy almost bi-ideals of semigroups in [19, 20], respectively. Murugadas, Kalpana and Vetrivel [21] defined fuzzy almost quasi-ideals in semigroups. Recently, almost bi-ideals, almost quasi-ideals, fuzzy almost bi-ideals and fuzzy almost quasi-ideals in ordered semigroups were defined in [22]. Kaopusek, Kaewnoi and Chinram [23] introduced the concepts of almost interior-ideals of semigroups. Furthermore, W. Krailoet, A. Simuen, R. Chinram and P. Petchkaew [24] defined the notions of fuzzy almost interior ideals in semigroups.

An ordered semiring is an interesting generalization of semirings. In 2011, Gan and Jiang [6] defined ordered ideals in ordered semirings. The notion of ordered quasi-ideals and ordered bi-ideals of ordered semirings were defined in [7]. The notions of fuzzy ideals and fuzzy interior-ideals of ordered semirings was introduced in [12]. In this thesis, we define almost ordered subsemirings, almost ordered quasi-ideals, almost ordered bi-ideals, almost ordered interior-ideals, fuzzy almost ordered subsemirings, fuzzy almost ordered quasi-ideals, fuzzy almost ordered bi-ideals and fuzzy almost ordered interior-ideals in ordered semirings and study the relationship between them. Moreover, we define tri-quasi ideals and fuzzy tri-quasi ideals of ordered semirings and investigate some properties and relationships between them. In addition, we introduce the notion of almost ordered tri-quasi ideals and fuzzy almost ordered tri-quasi ideals of ordered semirings and give some relationship between them.

Chapter 2

Preliminaries

In this chapter, we collect the definitions and theorems which will be used later in the study of this thesis.

2.1 Ordered semirings

Definition 2.1.1. A *semiring* $(R, +, \cdot)$ is a nonempty set R together with two binary operations $+$ and $\cdot$ satisfying the following axioms :

1. $(R, +)$ is a semigroup,
2. $(R, \cdot)$ is a semigroup, and
3. the distributive laws hold in R.

Definition 2.1.2. An *ordered semiring* is a system $(S, +, \cdot, \leq)$ consisting of a nonempty set S such that

1. $(S, +, \cdot)$ is a semiring,
2. $(S, \leq)$ is a partially ordered set, and
3. for all $a, b, c \in S$, if $a \leq b$, then $a + c \leq b + c$, $c + a \leq c + b$ and $ac \leq bc$, $ca \leq cb$.

Throughout this thesis, let S be an ordered semiring.

Definition 2.1.3. Let A and B be nonempty subsets of S. Define

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\},$$
$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

For $x \in S$ and $\emptyset \neq A \subseteq S$, let $Ax = A\{x\}$ and $xA = \{x\}A$. For $n \in \mathbb{N}$, let $A^n = \underbrace{A \cdots A}_{n \text{ copies}}$.

Theorem 2.1.4. Let A, B, C and D be nonempty subsets of S. Then the following statements hold.

(1) $A + (B + C) = (A + B) + C$ and $A(BC) = (AB)C$.

(2) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$ and $AC \subseteq BD$.

Definition 2.1.5. Let A be a nonempty subset of S. Define

$$(A\,] = \{\, x \in S \mid x \leq a \text{ for some } a \in A \,\}.$$

Theorem 2.1.6. Let A, B, C and D be nonempty subsets of S. Then the following statements hold.

(1) $A \subseteq (A\,]$ and $\big((A\,]\big] = (A\,]$.

(2) If $A \subseteq B$, then $(A\,] \subseteq (B\,]$.

(3) $(A\,] + (B\,] \subseteq (A + B\,]$ and $(A\,](B\,] \subseteq (AB\,]$.

(4) If $A \subseteq B$ and $C \subseteq D$, then $(A + C\,] \subseteq (B + D\,]$ and $(AC\,] \subseteq (BD\,]$.

Definition 2.1.7. Let A be a nonempty subset of S.

1. A is called a *subsemiring* of S if $A + A \subseteq A$ and $A^2 \subseteq A$.
2. A is called a *left ideal* (resp. *right ideal*) of S if $A + A \subseteq A$, $SA \subseteq A$ (resp. $AS \subseteq A$) and $(A\,] = A$.
3. A is called an *ideal* of S if A is both a left ideal and a right ideal of S.
4. A is called a *quasi-ideal* of S if $A + A \subseteq A$, $SA \cap AS \subseteq A$ and $(A\,] = A$.
5. A is called a *bi-ideal* of S if A is a subsemiring of S, $ASA \subseteq A$ and $(A\,] = A$.
6. A is called an *interior-ideal* of S if A is a subsemiring of S, $SAS \subseteq A$ and $(A\,] = A$.

Definition 2.1.8. An element $a \in S$ is said to be *idempotent* if $a = a^2$. S is called an *idempotent ordered semiring* if every element of S is idempotent.

2.2 Fuzzy ideals of ordered semirings

Definition 2.2.1. A *fuzzy subset* of a set X is a function $f : X \to [0,1]$. For each $x \in X$, the value of $f(x)$ is called the *degree of membership* of x.

Definition 2.2.2. Let f and g be fuzzy subsets of a set X, we say that $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Definition 2.2.3. Let $\{f_i\}_{i\in I}$ be a collection of fuzzy subsets of a set X. Define fuzzy subsets $\bigcap\limits_{i\in I} f_i$ and $\bigcup\limits_{i\in I} f_i$ of X by

$$\Big(\bigcap_{i\in I} f_i\Big)(x) = \inf_{i\in I}\{f_i(x)\} \text{ and } \Big(\bigcup_{i\in I} f_i\Big)(x) = \sup_{i\in I}\{f_i(x)\} \text{ for all } x \in X.$$

Definition 2.2.4. Let X be a nonempty set. The *characteristic function* of a subset A of X is a fuzzy subset C_A of X defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2.5. Let X be a nonempty set. For $s \in X$ and $\alpha \in (0,1]$, a *fuzzy point* s_α of a set X is a fuzzy subset of X defined by

$$s_\alpha(x) = \begin{cases} \alpha & \text{if } x = s, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2.6. Let f be a fuzzy subset of X. The *support* of f is defined by

$$supp(f) = \{x \in X \mid f(x) \neq 0\}.$$

Definition 2.2.7. Let f be a fuzzy subset of X and $t \in [0,1]$. The set

$$f_t = \{x \in X \mid f(x) \geq t\}$$

is called the *level subset* of f.

Definition 2.2.8. Let f and g be two fuzzy subsets of S and $x \in S$. The *multiplication* of f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab} \min\{f(a), g(b)\} & \text{if } \ x = ab \ \text{ for some } \ a, b \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and the *addition* of f and g, denoted by $f + g$, is defined by

$$(f + g)(x) = \begin{cases} \sup\limits_{x=a+b} \min\{f(a), g(b)\} & \text{if } \ x = a + b \ \text{ for some } \ a, b \in S, \\ 0 & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$, let $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ copies}}$.

Theorem 2.2.9. Let f, g, h and k be fuzzy subsets of S. Then the following properties hold.

(1) $(f \circ g) \circ h = f \circ (g \circ h)$ and $(f + g) + h = f + (g + h)$.

(2) If $f \subseteq g$ and $h \subseteq k$, then $f + h \subseteq g + k$ and $f \circ h \subseteq g \circ k$.

Definition 2.2.10. Let f be a fuzzy subset of S. Define the fuzzy subset $(f\,]$ of S by

$$(f\,](x) = \sup_{x \leq y} f(y) \ \text{ for all } \ x \in S.$$

Theorem 2.2.11. Let f, g, h and k be fuzzy subsets of S. Then the following statements hold.

(1) $f \subseteq (f\,]$.

(2) If $f \subseteq g$, then $(f\,] \subseteq (g\,]$.

(3) If $f \subseteq g$ and $h \subseteq k$, then $(f + h\,] \subseteq (g + k\,]$ and $(f \circ h\,] \subseteq (g \circ k\,]$.

Definition 2.2.12. Let f be a fuzzy subset of S.

1. f is called a *fuzzy subsemiring* of S if $f+f \subseteq f$ and $f \circ f \subseteq f$.
2. f is called a *fuzzy left ideal* (resp. *fuzzy right ideal*) of S if $f+f \subseteq f$, $C_S \circ f \subseteq f$ (resp. $f \circ C_S \subseteq f$) and $(f\,] = f$.
3. f is called a *fuzzy ideal* of S if f is both a fuzzy left ideal and a fuzzy right ideal of S.
4. f is called a *fuzzy quasi-ideal* of S if $f+f \subseteq f$, $(C_S \circ f) \cap (f \circ C_S) \subseteq f$ and $(f\,] = f$.
5. f is called a *fuzzy bi-ideal* of S if f is a fuzzy subsemiring of S, $f \circ C_S \circ f \subseteq f$ and $(f\,] = f$.
6. f is called a *fuzzy interior-ideal* of S if f is a fuzzy subsemiring of S, $C_S \circ f \circ C_S \subseteq f$ and $(f\,] = f$.

Theorem 2.2.13. Let A and B be two nonempty subsets of S. Then the following statements hold.

(1) $A \subseteq B$ if and only if $C_A \subseteq C_B$.

(2) $C_A + C_B = C_{A+B}$ and $C_A \circ C_B = C_{AB}$.

(3) $(C_A\,] = C_{(A\,]}$.

Theorem 2.2.14. Let f and g be fuzzy subsets of S. Then the following statements hold.

(1) $f \neq 0$ if and only if $supp(f) \neq \emptyset$.

(2) If $f \subseteq g$, then $supp(f) \subseteq supp(g)$.

(3) $supp(f \cup g) = supp(f) \cup supp(g)$ and $supp(f \cap g) = supp(f) \cap supp(g)$.

(4) If $f \neq 0$ and $g \neq 0$, then

$$supp(f+g) = supp(f) + supp(g) \text{ and } supp(f \circ g) = supp(f)supp(g).$$

(5) If $f \neq 0$, then $supp(f\,] = \big(supp(f)\big]$.

Proof. (1) For any $x \in S$, we have $f(x) \neq 0$ if and only if $x \in supp(f)$. This implies that $f \neq 0$ if and only if $supp(f) \neq \emptyset$.

(2) Assume that $f \subseteq g$. Let $x \in supp(f)$. Then $g(x) \geq f(x) \neq 0$. It follows that $x \in supp(g)$. Hence, $supp(f) \subseteq supp(g)$.

(3) Let $x \in S$. Then

$$\begin{aligned} x \in supp(f \cup g) &\Longleftrightarrow (f \cup g)(x) \neq 0 \\ &\Longleftrightarrow f(x) \neq 0 \ \text{ or } \ g(x) \neq 0 \\ &\Longleftrightarrow x \in supp(f) \cup supp(g). \end{aligned}$$

Then $supp(f \cup g) = supp(f) \cup supp(g)$. Similarly, we can show that

$$supp(f \cap g) = supp(f) \cap supp(g).$$

(4) Assume that $f \neq 0$ and $g \neq 0$. Let $x \in S$. Then

$$\begin{aligned} x \in supp(f + g) &\Longleftrightarrow (f + g)(x) \neq 0 \\ &\Longleftrightarrow x = a + b \ \text{ for some } \ a \in supp(f), b \in supp(g) \\ &\Longleftrightarrow x \in supp(f) + supp(g). \end{aligned}$$

Then $supp(f + g) = supp(f) + supp(g)$. Similarly, we can show that

$$supp(f \circ g) = supp(f)supp(g).$$

(5) Assume that $f \neq 0$. Let $x \in S$. Then

$$\begin{aligned} x \in supp(f\,] &\Longleftrightarrow (f\,](x) \neq 0 \\ &\Longleftrightarrow x \leq y \ \text{ for some } \ y \in supp(f) \\ &\Longleftrightarrow x \in \big(supp(f)\big]. \end{aligned}$$

Hence, $supp(f\,] = \big(supp(f)\big]$. □

Theorem 2.2.15. Let $f_1, f_2, ..., f_n$ be nonzero fuzzy subsets of S and g be a fuzzy subset of S. Then

(1) $(f_1 \circ f_2 \circ \cdots \circ f_n] \cap g \neq 0$ if and only if $\big(supp(f_1)supp(f_2)\cdots supp(f_n)\big] \cap supp(g) \neq \emptyset$.

(2) $(f_1 + f_2 + \cdots + f_n] \cap g \neq 0$ if and only if $\big(supp(f_1) + supp(f_2) + \cdots + supp(f_n)\big] \cap supp(g) \neq \emptyset$.

Proof. We only prove (1). The proof of (2) is similar to (1).

(1) By Theorem 2.2.14, we have

$$\begin{aligned}(f_1 \circ f_2 \circ \cdots \circ f_n] \cap g \neq 0 &\iff supp\big((f_1 \circ f_2 \circ \cdots \circ f_n] \cap g\big) \neq \emptyset \\ &\iff supp(f_1 \circ f_2 \circ \cdots \circ f_n] \cap supp(g) \neq \emptyset \\ &\iff \big(supp(f_1 \circ f_2 \circ \cdots \circ f_n)\big] \cap supp(g) \neq \emptyset \\ &\iff \big(supp(f_1)supp(f_2)\cdots supp(f_n)\big] \cap supp(g) \neq \emptyset.\end{aligned}$$

This completes the proof. □

Theorem 2.2.16. Let f be a nonzero fuzzy subset of S. Then the following statements hold.

(1) If f is a fuzzy subsemiring of S, then $supp(f)$ is a subsemiring of S.

(2) If f is a fuzzy ideal of S, then $supp(f)$ is an ideal of S.

(3) If f is a fuzzy quasi-ideal of S, then $supp(f)$ is a quasi-ideal of S.

(4) If f is a fuzzy bi-ideal of S, then $supp(f)$ is a bi-ideal of S.

(5) If f is a fuzzy interior-ideal of S, then $supp(f)$ is an interior-ideal of S.

Proof. We only prove (2) and (3). The proof of (1), (4) and (5) are similar to (2).

(2) Assume that f is a fuzzy ideal of S. Since $f + f \subseteq f$, by Theorem 2.2.14, $supp(f) + supp(f) = supp(f + f) \subseteq supp(f)$. Since $C_S \circ f \subseteq f$, we get $S\big(supp(f)\big) = supp(C_S)supp(f) = supp(C_S \circ f) \subseteq supp(f)$. Similarly, we can show that $\big(supp(f)\big)S \subseteq supp(f)$. Since $(f] = f$, by Theorem 2.2.14, we get $\big(supp(f)\big] = supp(f] = supp(f)$. Hence, $supp(f)$ is an ideal of S.

(3) Assume that f is a fuzzy quasi-ideal of S. Since $f+f \subseteq f$, by Theorem 2.2.14, we have $supp(f)+supp(f) \subseteq supp(f)$. Since $(f \circ C_S) \cap (C_S \circ f) \subseteq f$, by Theorem 2.2.14, we have

$$\begin{aligned}\big(supp(f)\big)S \cap S\big(supp(f)\big) &= \big(supp(f)supp(C_S)\big) \cap \big(supp(C_S)supp(f)\big)\\ &= supp(f \circ C_S) \cap supp(C_S \circ f)\\ &= supp\big((f \circ C_S) \cap (C_S \circ f)\big)\\ &\subseteq supp(f).\end{aligned}$$

Since $(f\,] = f$, by Theorem 2.2.14, we get $\big(supp(f)\big] = supp(f\,] = supp(f)$. Hence, $supp(f)$ is a quasi-ideal of S. □

Chapter 3

Almost ordered subsemirings and their fuzzifications

In this chapter, we introduce the concepts of almost ordered subsemirings and fuzzy almost ordered subsemirings in ordered semirings. Some relationships between almost ordered subsemirings and fuzzy almost ordered subsemirings in ordered semirings are provided.

3.1 Almost ordered subsemirings of ordered semirings

Definition 3.1.1. Let A be a nonempty subset of S. A is called an *almost ordered subsemiring* of S if $(A+A]\cap A \neq \emptyset$ and $(A^2\,]\cap A \neq \emptyset$.

Theorem 3.1.2. Every subsemiring of S is an almost ordered subsemiring of S.

Proof. Let A be a subsemiring of S. Then $A+A\subseteq A$ and $A^2\subseteq A$. This implies that $A+A=(A+A)\cap A\subseteq (A+A]\cap A$ and $A^2=A^2\cap A\subseteq (A^2\,]\cap A$. Thus $(A+A]\cap A\neq\emptyset$ and $(A^2\,]\cap A\neq\emptyset$. Hence, A is an almost ordered subsemiring of S. □

The converse of Theorem 3.1.2 is not generally true as shown in the following example.

Example 3.1.3. Let $S_1=\{a,b,c,d\}$. Define binary operations $+$ and $\cdot$ on S_1 given by the following tables

$+$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	b	c	d
d	d	b	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	b

.

Define a relation $\leq$ on S_1 by

$$\leq=\{\,(a,a),(b,b),(c,c),(d,d),(b,d)\,\}.$$

Then $(S_1, +, \cdot, \leq)$ is an ordered semiring. Let $A = \{a, d\}$. We have $a \leq a + a$, $a \leq a^2$ and $a \in A$, so $a \in (A + A] \cap A$ and $a \in (A^2] \cap A$, that is $(A + A] \cap A \neq \emptyset$ and $(A^2] \cap A \neq \emptyset$. Thus A is an almost ordered subsemiring but not a subsemiring of S_1 because $A^2 = \{a, b\} \nsubseteq A$.

Proposition 3.1.4. Let a be any element of S. Then $A = \{a, a + a, a^2\}$ is an almost ordered subsemiring of S.

Proof. Since $a + a \in (A + A) \cap A \subseteq (A + A] \cap A$, we have $(A + A] \cap A \neq \emptyset$. Since $a^2 \in A^2 \cap A \subseteq (A^2] \cap A$, we get $(A^2] \cap A \neq \emptyset$. Hence, A is an almost ordered subsemiring of S. □

Theorem 3.1.5. Let A and B be any two nonempty subsets of S. If $A \subseteq B$ and A is an almost ordered subsemiring of S, then B is also an almost ordered subsemiring of S.

Proof. Assume that A is an almost subsemiring of S such that $A \subseteq B$. Then $(A+A] \cap A \neq \emptyset$ and $(A^2] \cap A \neq \emptyset$. Since $A \subseteq B$, we have $(A+A] \cap A \subseteq (B+B] \cap B$ and $(A^2] \cap A \subseteq (B^2] \cap B$. This implies that $(B + B] \cap B \neq \emptyset$ and $(B^2] \cap B \neq \emptyset$. Hence, B is an almost ordered subsemiring of S. □

Corollary 3.1.6. The union of almost subsemirings of S is also an almost ordered subsemiring of S.

3.2 Fuzzy almost ordered subsemirings of ordered semirings

Definition 3.2.1. Let f be a fuzzy subset of S. We call f a *fuzzy almost ordered subsemiring* of S if $(f + f] \cap f \neq 0$ and $(f \circ f] \cap f \neq 0$.

Theorem 3.2.2. Let f be a fuzzy subset of S. Then f is a fuzzy almost ordered subsemiring of S if and only if $supp(f)$ is an almost ordered subsemiring of S.

Proof. By Theorem 2.2.15, we have

$$(f \circ f] \cap f \neq 0 \text{ if and only if } \big(supp(f)^2\big] \cap supp(f) \neq \emptyset \text{ and}$$
$$(f + f] \cap f \neq 0 \text{ if and only if } \big(supp(f) + supp(f)\big] \cap supp(f) \neq \emptyset.$$

This completes the proof. □

Corollary 3.2.3. Let A be a nonempty subset of S. Then C_A is a fuzzy almost ordered subsemiring of S if and only if A is an almost ordered subsemiring of S.

Proof. Since $A = supp(C_A)$, the results follow from Theorem 3.2.2. □

Theorem 3.2.4. Every nonzero fuzzy subsemiring of S is a fuzzy almost ordered subsemiring of S.

Proof. Let f be a nonzero fuzzy subsemiring of S. By Theorem 2.2.16 (1), $supp(f)$ is a subsemiring of S. By Theorem 3.1.2, $supp(f)$ is an almost ordered subsemiring of S and by Theorem 3.2.2, f is a fuzzy almost ordered subsemiring of S. □

The converse of Theorem 3.2.4 does not hold in general. We consider the following example.

Example 3.2.5. From Example 3.1.3, we have $A = \{a, d\}$ is an almost ordered subsemiring of S_1 but is not a subsemiring of S_1. Define a fuzzy subset f of S_1 by

$$f(x) = \begin{cases} 0.5 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We have $supp(f) = A$. By Theorem 3.2.2, f is a fuzzy almost ordered subsemiring of S_1. Since $supp(f)$ is not a subsemiring of S_1, by Theorem 2.2.14 (1), f is not a fuzzy subsemiring of S_1.

Theorem 3.2.6. Let f be a fuzzy almost ordered subsemiring of S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is also a fuzzy almost ordered subsemiring of S.

Proof. By Theorem 3.2.2, $supp(f)$ is an almost ordered subsemiring of S. Since $f \subseteq g$, it follows that $supp(f) \subseteq supp(g)$. By Theorem 3.1.5, $supp(g)$ is an almost ordered subsemiring of S. Again by Theorem 3.2.2, g is a fuzzy almost ordered subsemiring of S. □

Corollary 3.2.7. The union of fuzzy almost ordered subsemirings of S is also a fuzzy almost ordered subsemiring of S.

Chapter 4

Almost ordered ideals and their fuzzifications

In this chapter, we define almost ordered ideals and fuzzy almost ordered ideals in ordered semirings and we give the relationship between them.

4.1 Almost ordered ideals of ordered semirings

Definition 4.1.1. A nonempty subset I of S is called an *almost ordered ideal* of S if $(I+I] \cap I \neq \emptyset$, $(sI] \cap I \neq \emptyset$ and $(Is] \cap I \neq \emptyset$ for all $s \in S$.

Theorem 4.1.2. Every ideal of S is an almost ordered ideal of S.

Proof. Let I be an ideal of S. Since $I + I \subseteq (I + I]$ and $I + I \subseteq I$, we have $(I + I] \cap I \neq \emptyset$. Let $s \in S$. We have $sI \subseteq (sI]$ and $sI \subseteq SI \subseteq I$, so $(sI] \cap I \neq \emptyset$. Similarly, we have $(Is] \cap I \neq \emptyset$. Hence, I is an almost ordered ideal of S. □

The converse of Theorem 4.1.2 does not hold in general. We consider the following example

Example 4.1.3. Let $S_2 = \{a, b, c, d, e\}$. Define the binary operations $+$ and $\cdot$ on S_2 show in the table

$+$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

$\cdot$	a	b	c	d	e
a	a	a	a	d	d
b	a	b	b	d	e
c	a	c	c	d	e
d	d	d	d	a	a
e	d	d	d	a	a

.

Define a relation $\leq$ on S_2 by

$$\leq = \{ (a, a), (b, b), (c, c), (d, d), (e, e), (b, c) \}.$$

Then $(S_2, +, \cdot, \leq)$ is an ordered semiring. Let $I = \{c, d, e\}$. Since $c \leq c + c$ and $c \in I$, we get $c \in (I + I\,] \cap I$. Thus $(I + I\,] \cap I \neq \emptyset$. We see that

$$\begin{aligned}
&(Ia\,] \cap I = \{a, d\} \cap \{c, d, e\} \neq \emptyset,\ (aI\,] \cap I = \{a, d\} \cap \{c, d, e\} \neq \emptyset,\\
&(Ib\,] \cap I = \{b, c, d\} \cap \{c, d, e\} \neq \emptyset,\ (bI\,] \cap I = \{b, d, e\} \cap \{c, d, e\} \neq \emptyset,\\
&(Ic\,] \cap I = \{b, c, d\} \cap \{c, d, e\} \neq \emptyset,\ (cI\,] \cap I = \{b, c, d, e\} \cap \{c, d, e\} \neq \emptyset,\\
&(Id\,] \cap I = \{a, d\} \cap \{c, d, e\} \neq \emptyset,\ (dI\,] \cap I = \{a, d\} \cap \{c, d, e\} \neq \emptyset,\\
&(Ie\,] \cap I = \{a, e\} \cap \{c, d, e\} \neq \emptyset,\ (eI\,] \cap I = \{a, d\} \cap \{c, d, e\} \neq \emptyset.
\end{aligned}$$

Then I is an almost ordered ideal of S_2. We have $a = ca \in IS_2$ but $a \notin I$, so $IS \nsubseteq I$. Therefore, I is not an ideal of S_2.

Theorem 4.1.4. Every almost ordered ideal of an ordered semiring S is an almost ordered subsemiring of S.

Proof. Let I be an almost ordered ideal of S. Then $(sI\,] \cap I \neq \emptyset$ for all $s \in S$. Let $a \in I$. Then $\emptyset \neq (aI\,] \cap I \subseteq (I^2\,] \cap I$. This implies that $(I^2\,] \cap I \neq \emptyset$. Hence, I is an almost ordered subsemiring of S. □

Theorem 4.1.5. Let A be an almost ordered ideal of S. If B is a subset of S containing A, then B is also an almost ordered ideal of S.

Proof. Let B be a subset of S containing A. Then $\emptyset \neq (A+A\,] \cap A \subseteq (B+B\,] \cap B$, $\emptyset \neq (sA\,] \cap A \subseteq (sB\,] \cap B$ and $\emptyset \neq (As\,] \cap A \subseteq (Bs\,] \cap B$ for all $s \in S$. Hence, B is an almost ordered ideal of S. □

The following corollary is a direct consequence of Theorem 4.1.5.

Corollary 4.1.6. The union of almost ordered ideals of an ordered semiring S is also an almost ordered ideal of S.

4.2 Fuzzy almost ordered ideals of ordered semirings

Definition 4.2.1. A fuzzy subset f of S is called a *fuzzy almost ordered ideal* of S if $(f + f\,] \cap f \neq 0$, $(s_\alpha \circ f\,] \cap f \neq 0$ and $(f \circ s_\alpha\,] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.

Theorem 4.2.2. Let f be a fuzzy subset of S. Then f is a fuzzy almost ordered ideal of S if and only if $supp(f)$ is an almost ordered ideal of S.

Proof. By Theorem 2.2.15 we have

$$\begin{aligned}
(f+f\,]\cap f\neq 0 &\text{ if and only if } \big(supp(f)+supp(f)\,\big]\cap supp(f)\neq\emptyset,\\
(s_\alpha\circ f\,]\cap f\neq 0 &\text{ if and only if } \Big(s\big(supp(f)\big)\Big]\cap supp(f)\neq\emptyset, \text{and}\\
(f\circ s_\alpha\,]\cap f\neq 0 &\text{ if and only if } \Big(\big(supp(f)\big)s\Big]\cap supp(f)\neq\emptyset.
\end{aligned}$$

This completes the proof. □

Corollary 4.2.3. Let A be a nonempty subset of S. Then C_A is a fuzzy almost ordered ideal of S if and only if A is an almost ordered ideal of S.

Theorem 4.2.4. Let f be a nonzero fuzzy ideal of S. Then f is a fuzzy almost ordered ideal of S.

Proof. Since $f\neq 0$, by Theorem 2.2.16 (2), $supp(f)$ is an ideal of S. By Theorem 4.1.2, $supp(f)$ is an almost ordered ideal of S. By Theorem 4.2.2, f is a fuzzy almost ordered ideal of S. □

The converse of Theorem 4.2.4 is not generally true as shown in the following example.

Example 4.2.5. From Example 4.1.3, we have $I=\{c,d,e\}$ is an almost ordered ideal of S_2 but is not an ideal of S_2. Define a fuzzy subset f of S_2 by

$$f(x)=\begin{cases}0.5 & \text{if } x\in I,\\ 0 & \text{otherwise.}\end{cases}$$

We have $supp(f)=I$. By Theorem 4.2.2, f is a fuzzy almost ordered ideal of S_2. Since $supp(f)$ is not an ideal of S_2, by Theorem 2.2.16 (2), f is not a fuzzy ideal of S_2.

Theorem 4.2.6. Every fuzzy almost ordered ideal of S is a fuzzy almost ordered subsemiring of S.

Proof. Let f be a fuzzy almost ordered ideal of S. By Theorem 4.2.2, $supp(f)$ is an almost ordered ideal of S. By Theorem 4.1.4, $supp(f)$ is an almost ordered subsemiring of S, so f is a fuzzy almost ordered subsemiring of S by Theorem 3.2.2. □

Theorem 4.2.7. Let f and g be fuzzy subsets of S such that $f \subseteq g$. If f is a fuzzy almost ordered ideal of S, then g is also a fuzzy almost ordered ideal of S.

Proof. Assume that f is a fuzzy almost ordered ideal of S such that $f \subseteq g$. Then $supp(f)$ is an almost ordered ideal of S. Since $f \subseteq g$, we have $supp(f) \subseteq supp(g)$. By Theorem 4.1.5, $supp(g)$ is an almost ordered ideal of S. Hence, g is a fuzzy almost ordered ideal of S. □

Corollary 4.2.8. The union of fuzzy almost ordered ideals of S is also a fuzzy almost ordered ideal of S.

Definition 4.2.9. A fuzzy almost ordered ideal f of S is called *minimal* if for all fuzzy almost ordered ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 4.2.10. Let f be a fuzzy subset of S. Then f is a minimal fuzzy almost ordered ideal of S if and only if $supp(f)$ is a minimal almost ordered ideal of S.

Proof. Suppose that f is a minimal fuzzy almost ordered ideal of S. Then $supp(f)$ is an almost ordered ideal of S. Let I be an almost ordered ideal of S such that $I \subseteq supp(f)$. Define a fuzzy subset g of S by

$$g(x) = \begin{cases} \dfrac{f(x)}{2} & \text{if } \ x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Since $I \subseteq supp(f)$, we have $f(x) \neq 0$ for all $x \in I$, so $g(x) \neq 0$ for all $x \in I$. This implies that $supp(g) = I$. Then g is a fuzzy almost ordered ideal of S and $g \subseteq f$. Since f is minimal, we have $I = supp(g) = supp(f)$. So $supp(f)$ is minimal.

Conversely, suppose that $supp(f)$ is a minimal almost ordered ideal of S. Then f is a fuzzy almost ordered ideal of S. Let g be a fuzzy almost ordered ideal of S with $g \subseteq f$, so $supp(g) \subseteq supp(f)$. We have $supp(g)$ is an almost ordered ideal of S. Since $supp(f)$ is minimal, $supp(g) = supp(f)$. Hence, f is minimal. □

Corollary 4.2.11. Let $\emptyset \neq A \subseteq S$. Then C_A is a minimal fuzzy almost ordered ideal of S if and only if A is a minimal almost ordered ideal of S.

Chapter 5

Almost ordered quasi-ideals and their fuzzifications

In this chapter, we define almost ordered quasi-ideals and fuzzy almost ordered quasi-ideals in ordered semirings. Moreover, we give some relationships between almost ordered quasi-ideals and their fuzzification.

5.1 Almost ordered quasi-ideals of ordered semirings

Definition 5.1.1. A nonempty subset Q of S is called an *almost ordered quasi-ideal* of S if $(Q+Q] \cap Q \neq \emptyset$ and $(sQ] \cap (Qs] \cap Q \neq \emptyset$. for all $s \in S$.

Theorem 5.1.2. Let Q be a quasi-ideal of S such that $sQ \cap Qs \neq \emptyset$ for all $s \in S$. Then Q is an almost ordered quasi-ideal of S.

Proof. Since $Q + Q \subseteq (Q + Q]$ and $Q + Q \subseteq Q$, we have $(Q + Q] \cap Q \neq \emptyset$. Let $s \in S$. We have $sQ \cap Qs \subseteq SQ \cap QS \subseteq Q$. Thus

$$\emptyset \neq sQ \cap Qs = sQ \cap Qs \cap Q \subseteq (sQ] \cap (Qs] \cap Q.$$

Hence, Q is an almost ordered quasi-ideal of S. □

The converse of Theorem 5.1.2 is not generally true as shown in the following example

Example 5.1.3. Let $S_3 = \{0, a, b, c\}$. Define binary operations $+$ and $\cdot$ on S_3 as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

.

Define a relation $\leq$ on S_3 by

$$\leq = \{ (a,a), (b,b), (c,c), (0,0), (0,a), (0,b), (0,c), (a,b), (a,c), (b,c) \}.$$

Then $(S_3, +, \cdot, \leq)$ is an ordered semiring. Let $Q = \{0, a, c\}$. Then $(Q + Q\,] = S_3$, so $(Q + Q\,] \cap Q = S_3 \cap Q = Q \neq \emptyset$. We have $0 \leq 0s$ and $0 \leq s0$ for all $s \in S_3$, so $0 \in (sQ\,] \cap (Qs\,] \cap Q$ for all $s \in S_3$. Thus Q is an almost ordered quasi-ideal of S_3. We see that $b = bc = cb \in S_3Q \cap QS_3$ but $b \notin Q$. Thus $S_3Q \cap QS_3 \not\subseteq Q$. Hence, Q is not a quasi-ideal of S_3.

Theorem 5.1.4. Every almost ordered quasi-ideal of S is an almost ordered ideal of S.

Proof. Let Q be an almost ordered quasi-ideal of S. Let $s \in S$. Then

$$\emptyset \neq (sQ\,] \cap (Qs\,] \cap Q \subseteq (sQ\,] \cap Q \text{ and } \emptyset \neq (sQ\,] \cap (Qs\,] \cap Q \subseteq (Qs\,] \cap Q.$$

Therefore, Q is an almost ordered ideal of S. □

The following result is a direct consequence of Theorem 5.1.4 and Theorem 4.1.4.

Corollary 5.1.5. Every almost ordered quasi-ideal of S is an almost ordered subsemiring of S.

The converse of Theorem 5.1.4 is not generally true as shown in the following example.

Example 5.1.6. Consider the ordered semiring S_2 in Example 4.1.3. We have $I = \{c, d, e\}$ is an almost ordered ideal of S_2. We see that $(Ie\,] \cap (eI\,] \cap I = \emptyset$. Then I is not an almost ordered quasi-ideal of S_2.

Theorem 5.1.7. Let A and B be any two nonempty subsets of S. If $A \subseteq B$ and A is an almost ordered quasi-ideal of S, then B is also an almost ordered quasi-ideal of S.

Proof. Assume that A is an almost ordered quasi-ideal of S with $A \subseteq B$. Then $\emptyset \neq (A + A\,] \cap A \subseteq (B + B\,] \cap B$ and $\emptyset \neq (sA\,] \cap (As\,] \cap A \subseteq (sB\,] \cap (Bs\,] \cap B$ for all $s \in S$. Hence, B is an almost ordered quasi-ideal of S. □

The following corollary is a direct consequence of Theorem 5.1.7.

Corollary 5.1.8. The union of almost ordered quasi-ideals of S is also an almost ordered quasi-ideal of S.

5.2 Fuzzy almost ordered quasi-ideals of ordered semirings

Definition 5.2.1. A fuzzy subset f of S is called a *fuzzy almost ordered quasi-ideal* of S if $(f+f\,]\cap f \neq 0$ and $(s_\alpha \circ f\,]\cap(f\circ s_\alpha\,]\cap f\neq 0$ for all $s\in S$, and $\alpha\in(0,1]$.

Theorem 5.2.2. Let f be a fuzzy subset of an ordered semiring S. Then f is a fuzzy almost ordered quasi-ideal of S if and only if $supp(f)$ is an almost ordered quasi-ideal of S.

Proof. By Theorem 2.2.15, we have

$$(f+f\,]\cap f\neq 0 \text{ if and only if } \big(supp(f)+supp(f)\,\big]\cap supp(f)\neq\emptyset.$$

Let $s\in S$ and $\alpha\in(0,1]$. By Theorem 2.2.14, we get

$$\begin{aligned}(s_\alpha\circ f\,]\cap(f\circ s_\alpha\,]\cap f\neq 0 &\iff supp\big((s_\alpha\circ f\,]\cap(f\circ s_\alpha\,]\cap f\big)\neq\emptyset\\ &\iff supp(s_\alpha\circ f\,]\cap supp(f\circ s_\alpha\,]\cap supp(f)\neq\emptyset\\ &\iff \Big(s\big(supp(f)\big)\Big]\cap\Big(\big(supp(f)\big)s\Big]\cap supp(f)\neq\emptyset,\end{aligned}$$

which completes the proof of the theorem. □

Corollary 5.2.3. Let A be a nonempty subset of S. Then C_A is a fuzzy almost ordered quasi-ideal of S if and only if A is an almost ordered quasi-ideal of S.

Theorem 5.2.4. Let $f\neq 0$ be a fuzzy quasi-ideal of S such that for all $s\in S$, $s\big(supp(f)\big)\cap\big(supp(f)\big)s\neq\emptyset$. Then f is a fuzzy almost ordered quasi-ideal of S.

Proof. Since $f\neq 0$, by Theorem 2.2.16, $supp(f)$ is a quasi-ideal of S. Since $s\big(supp(f)\big)\cap\big(supp(f)\big)s\neq\emptyset$, by Theorem 5.1.2, $supp(f)$ is an almost ordered quasi-ideal of S. □

Corollary 5.2.5. If S has a zero element 0, then every nonzero fuzzy quasi-ideal of S is a fuzzy almost ordered quasi-ideal of S.

Proof. Assume that S has a zero element 0. Let $f\neq 0$ be a fuzzy quasi-ideal of S. Then $supp(f)$ is a quasi-ideal of S. Let $a\in supp(f)$. Since $0=0a=a0$, we have $0\in S\big(supp(f)\big)\cap\big(supp(f)\big)S\subseteq supp(f)$. Since $0=s0=0s$ for all $s\in S$, we get $0\in s\big(supp(f)\big)\cap\big(supp(f)\big)s$ for all $s\in S$. Hence, f is a fuzzy almost ordered quasi-ideal of S by Theorem 5.2.4. □

The converse of Theorem 5.2.4 is not always true as shown in the following example.

Example 5.2.6. From Example 5.1.3, we have $Q = \{0, a, c\}$ is an almost quasi-ideal of S_3 but is not a quasi-ideal of S_3 . Define fuzzy subset f of S_3 by

$$f(x) = \begin{cases} 0.5 & \text{if} \;\; x \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

We have $supp(f) = Q$. By Theorem 5.2.2, f is a fuzzy almost ordered quasi-ideal of S_3. Since $supp(f)$ is not a quasi-ideal of S_3, by Theorem 2.2.16 (3), f is not a fuzzy quasi-ideal of S_3.

Theorem 5.2.7. Every fuzzy almost ordered quasi-ideal of S is a fuzzy almost ordered ideal of S.

Proof. Let f be a fuzzy almost ordered quasi-ideal of S. Then $supp(f)$ is an almost ordered quasi-ideal of S. By Theorem 5.1.4, $supp(f)$ is an almost ordered ideal of S. Therefore, f is a fuzzy almost ordered ideal of S. □

From Theorem 5.2.7 and Theorem 4.2.6, we have the following corollary.

Corollary 5.2.8. Every fuzzy almost ordered quasi-ideal of S is a fuzzy almost ordered subsemiring of S.

The converse of Theorem 5.2.7 is not always true as shown in the following example.

Example 5.2.9. Frorm Example 5.1.6, we have $I = \{c, d, e\}$ is an almost ordered ideal of S_2 but it is not an almost ordered quasi-ideal of S_2. By Corollary 4.2.3 C_I is a fuzzy almost ordered ideal of S_2. Since I is not an almost ordered quasi-ideal of S_2, by Corollary 5.2.3, C_I is not a fuzzy almost ordered quasi-ideal of S_2.

Theorem 5.2.10. Let f and g be fuzzy subsets of S such that $f \subseteq g$. If f is a fuzzy almost ordered quasi-ideal of S, then g is also a fuzzy almost ordered quasi-ideal of S.

Proof. Assume that f is a fuzzy almost orderd quasi-ideal of S. Then $supp(f)$ is an almost ordered quasi-ideal of S with $supp(f) \subseteq supp(g)$, so $supp(g)$ is an almost ordered quasi-ideal of S. Hence, g is a fuzzy almost ordered quasi-ideal of S. □

Corollary 5.2.11. The union of fuzzy almost ordered quasi-ideals of S is also a fuzzy almost ordered quasi-ideal of S.

Definition 5.2.12. A fuzzy almost ordered quasi-ideal f of S is called *minimal* if for all fuzzy almost ordered quasi-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 5.2.13. Let f be a fuzzy subset of S. Then f is a minimal fuzzy almost ordered quasi-ideal of S if and only if $supp(f)$ is a minimal almost ordered quasi-ideal of S.

Proof. The proof is similar to Theorem 4.2.10. □

Corollary 5.2.14. Let A be a nonempty subset of S. Then C_A is a minimal fuzzy almost ordered quasi-ideal of S if and only if A is a minimal almost ordered quasi-ideal of S.

Chapter 6

Almost ordered bi-ideals and their fuzzifications

In this chapter, we define almost ordered bi-ideals and fuzzy almost ordered bi-ideals in ordered semirings. Some relationships between almost ordered bi-ideals and fuzzy almost ordered bi-ideals of ordered semirings are provided.

6.1 Almost ordered bi-ideals of ordered semirings

Definition 6.1.1. A nonempty subset B of S is called an *almost ordered bi-ideal* of S if B is an almost ordered subsemiring of S and $(BsB\,] \cap B \neq \emptyset$ for all $s \in S$.

Theorem 6.1.2. Every bi-ideal of S is an almost ordered bi-ideal of S.

Proof. Let B be a bi-ideal of S. Since B is a subsemiring of S, by Theorem 3.1.2, B is an almost ordered subsemiring of S. Let $s \in S$. Then

$$(BsB\,] \subseteq (BSB\,] \subseteq (B\,] = B.$$

It follows that $\emptyset \neq (BsB\,] = (BsB\,] \cap B$. Hence, B is an almost ordered bi-ideal of S. □

The converse of Theorem 6.1.2 is not generally true as shown in the following example.

Example 6.1.3. Let $S_4 = \{a, b, c, d\}$. Define binary operations $+$ and $\cdot$ on S_4 given by the following tables:

$+$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

.

Define a relation $\leq$ on S_4 by

$$\leq = \{\,(a, a), (b, b), (c, c), (d, d), (c, a)\,\}.$$

Then $(S_4, +, \cdot, \leq)$ is an ordered semiring. Let $B = \{a, b\}$. Then $a \in (B + B\,] \cap B$ and $a \in (B^2\,] \cap B$, so B is an almost ordered subsemiring of S_4. For all $x \in S_4$, we have

$$(BxB\,] \cap B = \{a, b, c\} \cap \{a, b\} \neq \emptyset.$$

Hence, B is an almost ordered bi-ideal of S_4. We see that $(B\,] = \{a, b, c\} \neq B$. Then B is not a bi-ideal of S_4.

Theorem 6.1.4. Every almost ordered ideal of S is an almost ordered bi-ideal of S.

Proof. Let I be an almost ordered ideal of S. Let $s \in S$ and $a \in I$. Then $\emptyset \neq (asI\,] \cap I \subseteq (IsI\,] \cap I$. Hence, I is an almost ordered bi-ideal of S. □

The following corollary is a direct consequence of Theorem 5.1.4 and Theorem 6.1.4.

Corollary 6.1.5. Every almost ordered quasi-ideal of S is an almost ordered bi-ideal of S.

The converse of Theorem 6.1.4 is not generally true as shown in the following example.

Example 6.1.6. Consider the ordered semiring S_4 in Example 6.1.3, we have $B = \{a, b\}$ is an almost ordered bi-ideal of S_4. We see that

$$(dB\,] \cap B = \big(\{d\}\big] \cap \{a, b\} = \{d\} \cap \{a, b\} = \emptyset.$$

Hence, B is not an almost ordered ideal of S_4.

Theorem 6.1.7. Let A be an almost ordered bi-ideal of S. If B is a subset of S such that $A \subseteq B$, then B is also an almost ordered bi-ideal of S.

Proof. Let B be a subset of S such that $A \subseteq B$. Since A is an almost ordered subsemiring of S, it follows from Theorem 3.1.5 that B is also an almost ordered subsemiring of S. For all $s \in S$, we have $\emptyset \neq (AsA\,] \cap A \subseteq (BsB\,] \cap B$. Therefore, B is an almost ordered bi-ideal of S. □

Corollary 6.1.8. The union of almost ordered bi-ideals of S is also an almost ordered bi-ideal of S.

6.2 Fuzzy almost ordered bi-ideals of ordered semirings

Definition 6.2.1. A fuzzy subset f of S is called a *fuzzy almost ordered bi-ideal* of S if f is a fuzzy almost ordered subsemiring of S and $(f \circ s_\alpha \circ f\,] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.

Theorem 6.2.2. Let f be a fuzzy subset of S. Then f is a fuzzy almost ordered bi-ideal of S if and only if $supp(f)$ is an almost ordered bi-ideal of S.

Proof. By Theorem 3.2.2, we have f is a fuzzy almost ordered subsemiring of S if and only if $supp(f)$ is an almost ordered subsemiring of S. Let $s \in S$ and $\alpha \in (0, 1]$. By Theorem 2.2.15, we have

$$(f \circ s_\alpha \circ f\,] \cap f \neq 0 \text{ if and only if } \Big(\big(supp(f)\big)s\big(supp(f)\big)\Big] \cap supp(f) \neq \emptyset.$$

This completes the proof. □

Corollary 6.2.3. Let A be a nonempty subset of S. Then C_A is a fuzzy almost ordered bi-ideal of S if and only if A is an almost ordered bi-ideal of S.

Theorem 6.2.4. If f is a nonzero fuzzy bi-ideal of S, then f is a fuzzy almost ordered bi-ideal of S.

Proof. Assume that f is a nonzero fuzzy bi-ideal of S. Then $supp(f)$ is a fuzzy bi-ideal of S, so $supp(f)$ is an almost ordered bi-ideal of S. Hence, f is a fuzzy almost ordered bi-ideal of S. □

The converse of Theorem 6.2.4 is not generally true as shown in the following example.

Example 6.2.5. From Example 6.1.3, we have $B = \{a, b\}$ is an almost bi-ideal but it is not a bi-ideal of S_4. By Corollary 6.2.3, C_B is a fuzzy almost ordered bi-ideal of S_4. Since B is not a bi-ideal of S_4, by Theorem 2.2.14 (4), C_B is not a fuzzy bi-ideal of S_4.

Theorem 6.2.6. Every fuzzy almost ordered ideal of S is a fuzzy almost ordered bi-ideal of S.

Proof. Let f be a fuzzy almost ordered ideal of S. Then $supp(f)$ is an almost ordered ideal of S, by Theorem 6.1.4, $supp(f)$ is an almost ordered bi-ideal of S. Hence, f is a fuzzy almost ordered bi-ideal of S. □

The following corollary is a direct consequence of Theorem 5.2.7 and Theorem 6.2.6.

Corollary 6.2.7. Every fuzzy almost ordered quasi-ideal of S is a fuzzy almost ordered bi-ideal of S.

The converse of Theorem 6.2.6 is not generally true as shown in the following example.

Example 6.2.8. Consider the ordered semiring S_4 in Example 6.1.6, we have B is an almost ordered bi-ideal of S_4 but it is not an almost ordered ideal of S_4. Then C_B is a fuzzy almost ordered bi-ideal of S_4 but it is not a fuzzy almost ordered ideal of S_4.

Theorem 6.2.9. Let f and g be fuzzy subsets of S such that $f \subseteq g$. If f is a fuzzy almost ordered bi-ideal of S, then g is also a fuzzy almost ordered bi-ideal of S.

Proof. Suppose that f is a fuzzy almost ordered bi-ideal of S with $f \subseteq g$. Then $supp(f)$ is an almost ordered bi-ideal of S such that $supp(f) \subseteq supp(g)$. It follows that $supp(g)$ is an almost ordered bi-ideal of S. Then g is a fuzzy almost ordered bi-ideal of S. □

Corollary 6.2.10. The union of fuzzy almost ordered bi-ideals of S is also a fuzzy almost ordered bi-ideal of S.

Definition 6.2.11. A fuzzy almost ordered bi-ideal f of S is called *minimal* if for all fuzzy almost ordered bi-ideal g of S with $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 6.2.12. Let f be a fuzzy subset of S. Then f is a minimal fuzzy almost ordered bi-ideal of S if and only if $supp(f)$ is a minimal almost ordered bi-ideal of S.

Proof. The proof is similar to Theorem 4.2.10. □

Corollary 6.2.13. Let A be a nonempty subset of S. Then C_A is a minimal fuzzy almost ordered bi-ideal of S if and only if A is a minimal almost ordered bi-ideal of S.

Chapter 7

Almost ordered interior-ideals and their fuzzifications

In this chapter, we define almost ordered interior-ideals and fuzzy almost ordered interior-ideals in ordered semirings and we give some relationships between them.

7.1 Almost ordered interior-ideals of ordered semirings

Definition 7.1.1. A nonempty subset I of S is called an *almost ordered interior-ideal* of S if I is an almost ordered subsemiring of S and $(sIt] \cap I \neq \emptyset$ for all $s, t \in S$.

Theorem 7.1.2. Every interior-ideal of S is an almost ordered interior-ideal of S.

Proof. Let I be an interior-ideal of S. Since I is a subsemiring of S, by Theorem 3.1.2, I is an almost ordered subsemiring of S. Let $s, t \in S$. Then

$$(sIt] \subseteq (SIS] \subseteq (I] = I.$$

It follows that $\emptyset \neq (sIt] = (sIt] \cap I$. Therefore, I is an almost ordered interior-ideal of S. □

The converse of Theorem 7.1.2 is not always true as shown in the following example.

Example 7.1.3. Let $S_5 = \{a, b, c, d, e\}$. Define binary operations $+$ and $\cdot$ on S_5 as shown in the following table:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d
e	e	d	d	d	e

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	b	b	b
d	a	a	b	b	b
e	a	a	b	b	b

.

Define a relation $\leq$ on S_5 by

$$\leq = \{ (a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,d),(c,d),(e,d) \}.$$

Then $(S_5, +, \cdot, \leq)$ is an ordered semiring. Let $I = \{a, d\}$. We have $a \leq a + a$, $a \leq a^2$ and $a \in I$. Then $a \in (I + I\,] \cap I$ and $a \in (I^2\,] \cap I$, so I is an almost ordered subsemiring of S_5. Since $a = sat$ for all $s, t \in S_5$ and $a \in I$, it follows that $a \in (sIt\,] \cap I$ for all $s, t \in S_5$. Then I is an almost ordered interior-ideal of S_5 but is not an interior-ideal of S_5 because $b = dd \in I^2$ but $b \notin I$, that is, $I^2 \nsubseteq I$.

Theorem 7.1.4. Let A be an almost ordered interior-ideal of S. If B is a subset of S containing A, then B is also an almost ordered interior-ideal of S.

Proof. Let B be a subset of S containing A. By Theorem 3.1.5, B is an almost ordered subsemiring of S. We also have $\emptyset \neq (sAt\,] \cap A \subseteq (sBt\,] \cap B$ for all $s, t \in S$. Hence, B is an almost ordered interior-ideal of S. □

Corollary 7.1.5. The union of almost ordered interior-ideals of S is also an almost ordered interior-ideal of S.

7.2 Fuzzy almost ordered interior-ideals of ordered semirings

Definition 7.2.1. A fuzzy subset f of S is called a *fuzzy almost ordered interior-ideal* of S if f is a fuzzy almost ordered subsemiring of S and $(s_\alpha \circ f \circ t_\beta\,] \cap f \neq 0$ for all $s, t \in S$ and $\alpha, \beta \in (0, 1]$.

Theorem 7.2.2. Let f be a fuzzy subset of S. Then f is a fuzzy almost ordered interior-ideal of S if and only if $supp(f)$ is an almost ordered interior-ideal of S.

Proof. By Theorem 3.2.2, f is a fuzzy almost ordered subsemiring of S if and only if $supp(f)$ is an almost ordered subsemiring of S. Let $s, t \in S$ and $\alpha, \beta \in (0, 1]$. By Theorem 2.2.15, we have

$$(s_\alpha \circ f \circ t_\beta\,] \cap f \neq 0 \text{ if and only if } \Big(s\big(supp(f)\big)t\Big] \cap supp(f) \neq \emptyset,$$

which completes the proof of the theorem. □

Corollary 7.2.3. Let A be a nonempty subset of S. Then C_A is a fuzzy almost ordered interior-ideal of S if and only if A is an almost ordered interior-ideal of S.

Theorem 7.2.4. If f is a nonzero fuzzy interior-ideal of S, then f is a fuzzy almost ordered interior-ideal of S.

Proof. Assume that f is a nonzero fuzzy interior-ideal of S. By Theorem 2.2.16 (5), $supp(f)$ is an interior-ideal of S, so $supp(f)$ is an almost ordered interior-ideal of S. Hence, f is a fuzzy almost ordered interior-ideal of S. □

The converse of Theorem 7.2.4 is not generally true as shown in the following example.

Example 7.2.5. From Example 7.1.3, we have $I = \{a, d\}$ is an almost ordered interior-ideal of S_5 but it is not an interior-ideal of S_5. Therefore, C_I is a fuzzy almost ordered inteior-ideal of S_5 but it is not a fuzzy inteior-ideal of S_5.

Theorem 7.2.6. Let f be a fuzzy almost ordered interior-ideal of S and g be a fuzzy subsets of S such that $f \subseteq g$. Then g is also a fuzzy almost ordered interior-ideal of S.

Proof. Since f is a fuzzy almost ordered interior-ideal of S, we get $supp(f)$ is an almost ordered interior-ideal of S. Since $supp(f) \subseteq supp(g)$, we have $supp(g)$ is an almost ordered interior-ideal of S. Hence, g is a fuzzy almost ordered interior-ideal of S. □

Corollary 7.2.7. The union of fuzzy almost ordered interior-ideals of S is also a fuzzy almost ordered interior-ideal of S.

Definition 7.2.8. A fuzzy almost ordered interior-ideal f of S is called *minimal* if for all fuzzy almost ordered interior-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 7.2.9. Let f be a fuzzy subset of S. Then f is a minimal fuzzy almost ordered interior-ideal of S if and only if $supp(f)$ is a minimal almost ordered interior-ideal of S.

Proof. The proof is similar to Theorem 4.2.10. □

Corollary 7.2.10. Let A be a nonempty subset of S. Then C_A is a minimal fuzzy almost ordered interior-ideal of S if and only if A is a minimal almost ordered interior-ideal of S.

Chapter 8

Tri-quasi ideals of ordered semirings

In this chapter, we introduce the notion of tri-quasi ideals and fuzzy tri-quasi ideals of ordered semirings. Moreover, we define almost ordered tri-quasi ideals and fuzzy almost ordered tri-quasi ideals of ordered semirings and we give some relationships between them.

8.1 Tri-quasi ideals of ordered semirings

Definition 8.1.1. A nonempty subset Q of S is said to be a *tri-quasi ideal* of S if Q is a subsemiring of S, $Q^2SQ^2 \subseteq Q$ and $(Q\,] = Q$.

Example 8.1.2. Consider the ordered semiring S_1 in Example 3.1.3, and let $Q = \{a, b\}$. Then we have

$$Q + Q = \{a, b\} + \{a, b\} = \{a, b\} \subseteq Q \text{ and } Q^2 = \{a, b\}^2 = \{a, b\} \subseteq Q.$$

Then Q is a subsemiring of S_1. We see that

$$Q^2S_1Q^2 = \{a, b\}^2S_1\{a, b\}^2 = \{a, b\} \subseteq Q \text{ and } (Q\,] = \big(\{a, b\}\big] = \{a, b\} = Q.$$

Hence, Q is a tri-quasi ideal of S_1.

The following theorem shows some relationships between tri-quasi-ideals and other ideals of ordered semirings.

Theorem 8.1.3. Let S be an ordered semiring. Then

(1) Every left (right) ideal of S is a tri-quasi ideal of S.

(2) Every ideal of S is a tri-quasi ideal of S.

(3) Every quasi-ideal of S is a tri-quasi ideal of S.

(4) Every bi-ideal of S is a tri-quasi ideal of S.

(5) Every interior-ideal of S is a tri-quasi ideal of S.

Proof. We will prove (4) and (5). Part $(1), (2)$ and (3) follows directly from (4).

(4) Let B be a bi-ideal of S. Then $B^2SB^2 = B(BSB)B \subseteq BSB \subseteq B$. Hence, B is a tri-quasi ideal of S.

(5) Let I be an interior-ideal of S. Then

$$I^2SI^2 = (IIS)I^2 \subseteq (SIS)I^2 \subseteq I^3 \subseteq SIS \subseteq I.$$

Therefore, I is a tri-quasi ideal of S. □

Definition 8.1.4. For nonempty subsets A of S, define the subset ΣA of S by

$$\Sigma A = \Big\{ \sum_{i=1}^{n} a_i \mid a_i \in A \text{ and } n \in \mathbb{N} \Big\}.$$

Theorem 8.1.5. Let A and B be nonempty subsets of S. Then the following properties hold.

(1) $A \subseteq \Sigma A$ and $\Sigma A = A$ if and only if $A + A \subseteq A$.

(2) $\Sigma A + \Sigma A \subseteq \Sigma A$.

(3) $\Sigma(\Sigma A) = \Sigma A$.

(4) $(\Sigma A)B \subseteq \Sigma AB$ and $B(\Sigma A) \subseteq \Sigma BA$.

(5) $(\Sigma A)(\Sigma B) \subseteq \Sigma AB$.

(6) $\Sigma(A] \subseteq (\Sigma A]$.

Theorem 8.1.6. Let S be an ordered semiring. Then

(1) The intersection of a right ideal and a left ideal of S is a tri-quasi ideal of S.

(2) The intersection of an interior ideal and a bi-ideal of S is a tri-quasi ideal of S.

(3) If B is a bi-ideal of S and A is a nonempty subset of S, then $(\Sigma AB]$ and $(\Sigma BA]$ are tri-quasi ideals of S.

Proof. (1) Let R be a right ideal and L be a left ideal of S. Since $RL \subseteq R \cap L$, we have $R \cap L \neq \emptyset$. Since R and L are subsemirings of S, we get $R \cap L$ is a subsemiring of S. Since

$$(R \cap L)^2S(R \cap L)^2 \subseteq R(R \cap L)S(R \cap L)^2 \subseteq RS \subseteq R \text{ and}$$
$$(R \cap L)^2S(R \cap L)^2 \subseteq (R \cap L)^2S(R \cap L)L \subseteq SL \subseteq L,$$

it follows that $(R\cap L)^2S(R\cap L)^2 \subseteq R\cap L$. We also have $(R\cap L\,] \subseteq (R\,]\cap(L\,] = R\cap L$. Hence, $R \cap L$ is a tri-quasi ideal of S.

(2) Let B be a bi-ideal of S and I be an interior ideal of S. Since $BIB \subseteq B \cap I$, we have $B \cap I \neq \emptyset$. Since B and I are subsemirings of S, we get $B \cap I$ is a subsemiring of S. Since

$$(B \cap I)^2S(B \cap I)^2 \subseteq BBSBB \subseteq BBB \subseteq BSB \subseteq B \text{ and}$$
$$(B \cap I)^2S(B \cap I)^2 \subseteq IISII \subseteq SISIS \subseteq SIS \subseteq I,$$

it follows that $(B\cap I)^2S(B\cap I)^2 \subseteq B\cap I$. We also have $(B\cap I\,] \subseteq (B\,]\cap(I\,] = B\cap I$. Hence, $B \cap I$ is a tri-quasi ideal of S.

(3) Let A be a nonempty subset of S and B be a bi-ideal of S. Then $(\Sigma AB\,] + (\Sigma AB\,] \subseteq (\Sigma AB + \Sigma AB\,] \subseteq (\Sigma AB\,]$ and we have

$$(\Sigma AB\,](\Sigma AB\,] \subseteq \big((\Sigma AB)(\Sigma AB)\big] \subseteq (\Sigma ABAB\,] \subseteq (\Sigma ABSB\,] \subseteq (\Sigma AB\,].$$

Thus $(\Sigma AB\,]$ is a subsemiring of S. We have

$$\begin{aligned}(\Sigma AB\,]^2S(\Sigma AB\,]^2 &\subseteq \big((\Sigma AB)^2S(\Sigma AB)^2\big]\\ &\subseteq \big((\Sigma ABAB)S(\Sigma ABAB)\big]\\ &\subseteq \big((\Sigma ABABS)(\Sigma ABAB)\big]\\ &\subseteq (\Sigma ABABSABAB\,]\\ &\subseteq (\Sigma ABSB\,]\\ &\subseteq (\Sigma AB\,].\end{aligned}$$

Since $\big((\Sigma AB\,]\big] = (\Sigma AB\,]$, we get $(\Sigma AB\,]$ is a tri-quasi ideal of S. Similarly, we can show that $(\Sigma BA\,]$ is a tri-quasi ideal of S. □

Theorem 8.1.7. Let A be a nonempty subset of S. Then $(\Sigma A^2SA^2\,]$ is a tri-quasi ideal of S.

Proof. Since

$$(\Sigma A^2SA^2\,] + (\Sigma A^2SA^2\,] \subseteq (\Sigma A^2SA^2 + \Sigma A^2SA^2\,] \subseteq (\Sigma A^2SA^2\,] \text{ and}$$
$$(\Sigma A^2SA^2\,]^2 \subseteq \big((\Sigma A^2SA^2)^2\big] \subseteq (\Sigma A^2SA^2A^2SA^2\,] \subseteq (\Sigma A^2SA^2\,],$$

we have $(\Sigma A^2SA^2\,]$ is a subsemiring of S. We also have

$$\begin{aligned}(\Sigma A^2SA^2\,]^2S(\Sigma A^2SA^2\,]^2 &\subseteq \big((\Sigma A^2SA^2)^2S(\Sigma A^2SA^2)^2\big]\\ &\subseteq \big((\Sigma A^2SA^2A^2SA^2)S(\Sigma A^2SA^2A^2SA^2)\big]\\ &\subseteq \big((\Sigma A^2SA^2A^2SA^2S)(\Sigma A^2SA^2A^2SA^2)\big]\\ &\subseteq \big((\Sigma A^2SA^2A^2SA^2SA^2SA^2A^2SA^2)\big]\\ &\subseteq (\Sigma A^2SA^2\,].\end{aligned}$$

Since $\big((\Sigma A^2SA^2\,]\big] = (\Sigma A^2SA^2\,]$, we get $(\Sigma A^2SA^2\,]$ is a tri-quasi ideal of S. □

Theorem 8.1.8. Let $\{Q_i\}_{i\in I}$ be a collection of tri-quasi ideals of S. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is a tri-quasi ideal of S.

Proof. Assume that $\bigcap_{i\in I} Q_i \neq \emptyset$. Then $\bigcap_{i\in I} Q_i$ is a subsemiring of S. We have

$$\Big(\bigcap_{i\in I} Q_i\Big)^2 S\Big(\bigcap_{i\in I} Q_i\Big)^2 \subseteq Q_i^2SQ_i^2 \subseteq Q_i \text{ for all } i\in I \text{ and}$$
$$\Big(\bigcap_{i\in I} Q_i\Big] \subseteq \bigcap_{i\in I}(Q_i\,] = \bigcap_{i\in I} Q_i.$$

It follows that $\Big(\bigcap_{i\in I} Q_i\Big)^2 S\Big(\bigcap_{i\in I} Q_i\Big)^2 \subseteq \bigcap_{i\in I} Q_i$. Hence, $\bigcap_{i\in I} Q_i$ is a tri-quasi ideal of S. □

Theorem 8.1.9. Let Q be a nonempty subset of S and $Q = (Q^2]$. Then the following statements are equivalent :

(1) Q is a tri-quasi ideal of S.

(2) There exist a right ideal R and a left ideal L of S such that $RL \subseteq Q \subseteq R\cap L$.

(3) Q is a left ideal of some right ideal of S.

(4) Q is a right ideal of some left ideal of S.

Proof. (1) ⇒ (2) : Assume that Q is a tri-quasi ideal of S. Let $R = (\Sigma QS\,]$ and $L = (\Sigma SQ\,]$. It is easy to see that R and L are a right ideal and a left ideal of S, respectively. Then we have

$$RL \subseteq (\Sigma QSSQ\,] = \big(\Sigma(Q^2\,]SS(Q^2\,]\big] \subseteq \big(\Sigma(Q^2SQ^2\,]\big] \subseteq (\Sigma Q^2SQ^2\,] \subseteq (\Sigma Q\,] = Q \text{ and}$$

$$Q = (Q^2\,] \subseteq (QS\,] \cap (SQ\,] \subseteq (\Sigma QS\,] \cap (\Sigma SQ\,] = R \cap L.$$

$(2) \Rightarrow (3):$ Assume that there exist a right ideal R and a left ideal L of S such that $RL \subseteq Q \subseteq R \cap L$. We have $(\Sigma QS\,]$ is a right ideal of S and $Q = (Q^2\,] \subseteq (\Sigma QS\,]$. We see that

$$(\Sigma QS\,]Q \subseteq \big(\Sigma(R \cap L)S\big](R \cap L) \subseteq (\Sigma RS\,]L \subseteq (R]L = RL \subseteq Q.$$

Since $(Q\,] = \big((Q^2\,]\big] = (Q^2\,] = Q$, we obtain that Q is a left ideal of a right ideal $(\Sigma QS\,]$ of S.
$(3) \Rightarrow (4):$ Assume that Q is a left ideal of some right ideal R of S. Then $RQ \subseteq Q$ and $RS \subseteq R$. Thus

$$Q = (Q^2\,] \subseteq (\Sigma SQ\,] \text{ and } Q(\Sigma SQ\,] \subseteq R(\Sigma SQ\,] \subseteq (\Sigma RSQ\,] \subseteq (\Sigma RQ\,] \subseteq (Q\,] = Q.$$

Therefore, Q is a right ideal of a left ideal $(\Sigma SQ\,]$ of S.
$(4) \Rightarrow (1):$ Assume that Q is a right ideal of some left ideal L of S. Then $QL \subseteq Q$ and $SL \subseteq L$. Thus $Q^2SQ^2 \subseteq Q^2SQL \subseteq QSL \subseteq QL \subseteq Q$. Hence, Q is a tri-quasi ideal of S. □

Definition 8.1.10. An ordered semiring with multiplicative identity is called a *division ordered semiring* if for each non-zero element has multiplicative inverse.

Theorem 8.1.11. If S is a division ordered semiring, then the only tri-quasi ideals of S are $\{0\}$ and S.

Proof. Assume that S is a division ordered semiring. Let Q be a tri-quasi ideal of S such that $Q \neq \{0\}$. Let $a \in Q$ with $a \neq 0$. Then there exists $b \in S$ such that $ab = 1 = ba$. Let $x \in S$. Then $x = xba \in SQ$ and $x = abx \in QS$. This implies that $QS = S = SQ$. Thus

$$S = SQ = QSQ = Q(SQ)Q = Q(QS)QQ \subseteq Q.$$

Therefore, $S = Q$. □

Definition 8.1.12. An element $a \in S$ is said to be *regular* if $a \leq axa$ for some $x \in S$, and S is called a *regular ordered semiring* if every element of S is regular.

Theorem 8.1.13. If S has a multiplicative identity, then S is regular if and only if $Q \cap I \cap L \subseteq (QIL]$ for any tri-quasi ideal Q, ideal I and left ideal L of S.

Proof. Assume that S is regular. Let Q, I and L be a tri-quasi ideal, an ideal and a left ideal of S, respectively. Let $a \in Q \cap I \cap L$. Since S is regular, $a \in (aSa]$, so $a \in (aSa] \subseteq \big((aSa]S(aSa]\big] \subseteq (aSaSaSa] \subseteq (QIL]$. Hence, $Q \cap I \cap L \subseteq (QIL]$.

Conversely, suppose that $Q \cap I \cap L \subseteq (QIL]$ for any tri-quasi ideal Q, ideal I and left ideal L of S. Let $a \in S$. Then

$$a \in (aS] \cap S \cap (Sa] \subseteq \big((aS]S(Sa]\big] \subseteq \big((aSSSa]\big] \subseteq (aSa].$$

Hence, S is regular. □

Theorem 8.1.14. If S is regular and commutative, then the following statements hold:

(1) $Q = (Q^2SQ^2]$ for all tri-quasi ideal Q of S.

(2) Every tri-quasi ideal of S is an ideal of S.

Proof. (1) Let Q be a tri-quasi ideal of S. Then $(Q^2SQ^2] \subseteq (Q] = Q$. Let $a \in Q$. Since S is regular, we have $a \leq axa$ for some $x \in S$. Then

$$a \leq axa \leq (axa)x(axa) = aa(xxx)aa \in Q^2SQ^2.$$

So $a \in (Q^2SQ^2]$. This shows that $Q \subseteq (Q^2SQ^2]$. It follows that $Q = (Q^2SQ^2]$.

(2) Let Q be a tri-quasi ideal of S. By (1), $Q = (Q^2SQ^2]$. Thus

$$QS = (Q^2SQ^2]S \subseteq (Q^2SSQ^2] \subseteq (Q^2SQ^2] \subseteq (Q] = Q.$$

Since S is commutative, $SQ = QS \subseteq Q$. Therefore, Q is an ideal of S. □

Theorem 8.1.15. If S has a multiplicative identity and S is commutative, then S is regular if and only if $Q = (Q^2SQ^2]$ for every tri-quasi ideal Q of S.

Proof. Assume that S is regular. Let Q be a tri-quasi ideal of S. By Theorem 8.1.14(1), $Q = (Q^2SQ^2]$.

Conversely, suppose that $Q = (Q^2SQ^2]$ for all tri-quasi ideal Q of S. Let

$a \in S$. By Theorem 8.1.3(1), we get $(aS\,] \cap (Sa\,]$ is a tri-quasi ideal of S. Thus

$$\begin{aligned} a \in (aS\,] \cap (Sa\,] &= \Big(\big((aS\,] \cap (Sa\,]\big)^2 S\big((aS\,] \cap (Sa\,]\big)^2\Big] \\ &\subseteq \big((aS\,]^2 S (Sa\,]^2\big] \\ &\subseteq \Big(\big((aS)^2 S (Sa)^2\big]\Big] \\ &\subseteq (aSa\,]. \end{aligned}$$

Hence, S is regular. □

Theorem 8.1.16. Let T be a nonempty subset of S. If S is idempotent, then T is a tri-quasi ideal of S if and only if there exist a right ideal R and a left ideal L of S such that $T = (\Sigma RL\,]$.

Proof. Suppose that T is a tri-quasi ideal of S. Let $R = (\Sigma TS\,]$, $L = (\Sigma ST\,]$. Then

$$(\Sigma RL\,] = \big(\Sigma(\Sigma TS\,](\Sigma ST\,]\big] \subseteq \big(\Sigma(\Sigma TSST\,]\big] \subseteq \big((\Sigma TSST\,]\big] \subseteq (\Sigma T^2 S T^2\,] \subseteq T$$

and $T \subseteq TTTT \subseteq TSST \subseteq (\Sigma TS\,](\Sigma ST\,] \subseteq \big(\Sigma(\Sigma TS\,](\Sigma ST\,]\big] = (\Sigma RL\,]$. Hence, $T = (\Sigma RL\,]$. Conversely, assume that $T = (\Sigma RL\,]$ where R is a right ideal and L is a left ideal of S. Since R is a bi-ideal of S, by Theorem 8.1.3(3), we get T is a tri-quasi ideal of S. □

Definition 8.1.17. S is called a *tri-quasi-simple ordered semiring* if S has no tri-quasi ideals other than S itself.

Theorem 8.1.18. If S is a left and a right simple, then S is a tri-quasi simple ordered semiring.

Proof. Let Q be a tri-quasi ideal of S. Then $(\Sigma SQ\,]$ is a left ideal of S and $(\Sigma QS\,]$ is a right ideal of S. So $S = (\Sigma SQ\,]$ and $S = (\Sigma QS\,]$. Thus

$$S = (\Sigma SQ\,] = \big(\Sigma(\Sigma QS\,]Q\,\big] \subseteq \big(\Sigma(\Sigma QSQ]\big] \subseteq (\Sigma QSQ] \subseteq (\Sigma Q^2 SQ] \subseteq (\Sigma Q^2 S Q^2] \subseteq Q.$$

Hence, S is tri-quasi simple. □

Theorem 8.1.19. Let S be an ordered semiring. Then the following statements are equivalent:

(1) S is tri-quasi simple.

(2) $Q(a) = S$ for all $a \in S$, where $Q(a)$ is the smallest tri-quasi ideal of S generated by a.

(3) $(a^2Sa^2\,] = S$ for all $a \in S$.

Proof. (1) $\Rightarrow$ (2): Assume that S is tri-quasi simple. Then $S = Q(a)$ for all $a \in S$.
(2) $\Rightarrow$ (1): Assume that $S = Q(a)$ for all $a \in S$. Let A be a tri-quasi ideal of S and $a \in A$. Thus $S = Q(a) \subseteq A$, so $S = A$. Hence, S is tri-quasi simple.
(1) $\Rightarrow$ (3): Suppose that S is tri-quasi simple. Let $a \in S$. By Theorem 8.1.7, $(a^2Sa^2\,]$ is a tri-quasi ideal of S. Then $(a^2Sa^2\,] = S$.
(3) $\Rightarrow$ (1): Suppose that $(a^2Sa^2\,] = S$ for all $a \in S$. Let Q be a tri-quasi ideal of S and $q \in Q$. Thus $S = (q^2Sq^2\,] \subseteq (Q^2SQ^2\,] \subseteq (Q\,] = Q$. Hence, S is a tri-quasi simple ordered semiring. □

Definition 8.1.20. A tri-quasi ideal Q of S is said to be a *minimal tri-quasi ideal* of S if Q does not contain any other tri-quasi ideal of S.

Theorem 8.1.21. Let Q be a tri-quasi ideal of S. If Q is a tri-quasi simple subsemiring of S, then Q is a minimal tri-quasi ideal of S.

Proof. Assume that Q is a tri-quasi simple subsemiring of S. Let C be a tri-quasi ideal of S with $C \subseteq Q$. Then C is a subsemiring of Q, $C^2QC^2 \subseteq C^2SC^2 \subseteq C$ and $(C\,] = C$. Thus C is a tri-quasi ideal of Q. Since Q is tri-quasi simple, we get $C = Q$. Hence Q is a minimal tri-quasi ideal of S. □

Theorem 8.1.22. Let R and L be nonempty subsets of S. If R is a minimal right ideal of S and L is a minimal left ideal of S, then $(\Sigma RL\,]$ is a minimal tri-quasi ideal of S.

Proof. Let R be a minimal right ideal of S and L be a minimal left ideal of S. Since R is a bi-ideal of S, by Theorem 8.1.3 (3), we get $(\Sigma RL\,]$ is a tri-quasi ideal of S. Let A be a tri-quasi ideal of S with $A \subseteq (\Sigma RL\,]$. Let $a \in A$. Then

$$(Sa^2\,] \subseteq \big(S(\Sigma RL\,]^2\,\big] \subseteq \big(S(\Sigma RLRL\,]\big] \subseteq \big((\Sigma SRLRL\,]\big] \subseteq \big((\Sigma L\,]\big] \subseteq \big((L\,]\big] \subseteq L \quad \text{and}$$
$$(a^2S\,] \subseteq \big((\Sigma RL\,]^2S\big] \subseteq \big((\Sigma RLRL\,]S\,\big] \subseteq \big((\Sigma RLRLS\,]\big] \subseteq \big((\Sigma R\,]\big] \subseteq \big((R\,]\big] \subseteq R.$$

By the minimality of L and R, we get $L = (Sa^2\,]$ and $R = (a^2S\,]$. Thus

$$(\Sigma RL\,] = \big(\Sigma(a^2S\,](Sa^2\,]\big] \subseteq \big(\Sigma(a^2Sa^2\,]\big] \subseteq \big(\Sigma(A^2SA^2\,]\big] \subseteq \big(\Sigma(A\,]\big] \subseteq \big((\Sigma A\,]\big] \subseteq A.$$

Hence, $(\Sigma RL\,] = A$. Therefore, $(\Sigma RL\,]$ is a minimal tri-quasi ideal of S. □

For any $a \in S$, let $L(a)$ be the smallest left ideal of S generated by a and $R(a)$ be the smallest right ideal of S generated by a.

Theorem 8.1.23. Let Q be a tri-quasi ideal of S. If S is regular and commutative, then the following statements are equivalent:

(1) Q is a minimal tri-quasi ideal of S.

(2) $R(x) = R(y)$ for all $x, y \in Q$.

(3) $L(x) = L(y)$ for all $x, y \in Q$.

Proof. (1) $\Rightarrow$ (2): Assume that Q is a minimal tri-quasi ideal of S. Let $x, y \in Q$. By Theorem 8.1.3(1) and Theorem 8.1.8, $R(x) \cap Q$ is a tri-quasi ideal of S. Since $R(x) \cap Q \subseteq Q$, by the minimality of Q, we get $R(x) \cap Q = Q$, so $Q \subseteq R(x)$. Then $y \in R(x)$. Thus $R(y) \subseteq R(x)$. Similarly, we can show that $R(x) \subseteq R(y)$. Hence, $R(x) = R(y)$.
(2) $\Rightarrow$ (3): Let $x, y \in Q$. Since S is commutative, $L(x) = R(x) = R(y) = L(y)$.
(3) $\Rightarrow$ (1): Suppose that $L(x) = L(y)$ for all $x, y \in Q$. Let A be a tri-quasi ideal of S with $A \subseteq Q$. Let $x \in Q$ and $a \in A$. By Theorem 8.1.14(2), A is a left ideal of S. Thus $x \in L(x) = L(a) \subseteq A$. Hence, Q is minimal. □

8.2 Fuzzy tri-quasi ideals of ordered semirings

Definition 8.2.1. A fuzzy subset f of an ordered semiring S is called a *fuzzy tri-quasi-ideal* of S if f is a fuzzy subsemiring of S, $f^2 \circ C_S \circ f^2 \subseteq f$ and $(f\,] = f$.

Example 8.2.2. Consider the ordered semiring S_3 in Example 5.1.3. Define a fuzzy subset f of S_3 by

$$f(0) = 1, f(a) = 0.8, f(b) = 0.6 \text{ and } f(c) = 0.3.$$

Then f is a fuzzy tri-quasi ideal of S_3.

Theorem 8.2.3. Let f be a fuzzy subset of S. Then f is a fuzzy tri-quasi ideal of S if and only if f satisfies the following conditions.

(1) $f(x+y) \geq \min\{f(x), f(y)\}$ and $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

(2) $f(abscd) \geq \min\{f(a), f(b), f(c), f(d)\}$ for all $a, b, s, c, d \in S$.

(3) For all $x, y \in S$, if $x \leq y$, then $f(x) \geq f(y)$.

Proof. Assume that f is a fuzzy tri-quasi ideal of S. For the first condition, let $x, y \in S$. We have $f(x+y) \geq (f+f)(x+y) \geq \min\{f(x), f(y)\}$. Similarly, we get $f(xy) \geq (f \circ f)(xy) \geq \min\{f(x), g(y)\}$. To prove the second condition, let $a, b, s, c, d \in S$. Then $f(abscd) \geq (f^2 \circ C_S \circ f^2)(abscd) \geq \min\{f(a), f(b), f(c), f(d)\}$. Finally, let $x, y \in S$ such that $x \leq y$. Then $f(x) = (f\,](x) = \sup_{x \leq y} f(y) \geq f(y)$.

Conversely, suppose that the three conditions hold. Let $x \in S$. If $x \notin S+S$, then $(f+f)(x) = 0 \leq f(x)$. Suppose that $x \in S+S$. Then

$$(f+f)(x) = \sup_{x=a+b} \min\{f(a), f(b)\} \leq \sup_{x=a+b} f(a+b) = f(x).$$

This shows that $f + f \subseteq f$. Similarly, we can show that $f \circ f \subseteq f$. Then f is a fuzzy subsemiring of S. Next, we will show that $f^2 \circ C_S \circ f^2 \subseteq f$. Let $x \in S$. If $x \notin S^5$, then $(f^2 \circ C_S \circ f^2)(x) = 0 \leq f(x)$. Assume that $x \in S^5$. Then

$$\begin{aligned}
(f^2 \circ C_S \circ f^2)(x) &= \sup_{x=abscd} \min\{f(a), f(b), C_S(s), f(c), f(d)\} \\
&= \sup_{x=abscd} \min\{f(a), f(b), f(c), f(d)\} \\
&\leq \sup_{x=abscd} f(abscd) \\
&= f(x).
\end{aligned}$$

Thus $f^2 \circ C_S \circ f^2 \subseteq f$. Finally we will show that $(f\,] = f$. Let $x \in S$. Then $f(x) \geq f(y)$ for all $y \in S$ with $x \leq y$. We have $(f\,](x) = \sup_{x \leq y} f(y) \leq f(x)$, so $(f\,] = f$. Hence, f is a fuzzy tri-quasi ideal of S. $\square$

Theorem 8.2.4. The following statements hold.

(1) Every fuzzy left (right) ideal of S is a fuzzy tri-quasi ideal of S.

(2) Every fuzzy ideal of S is a fuzzy tri-quasi ideal of S.

(3) Every fuzzy quasi-ideal of S is a fuzzy tri-quasi ideal of S.

(4) Every fuzzy bi-ideal of S is a fuzzy tri-quasi ideal of S.

(5) Every fuzzy interior ideal of S is a fuzzy tri-quasi ideal of S.

Proof. We will prove (4) and (5). Part $(1), (2)$ and (3) follows directly from (4).

(4) Let f be a fuzzy bi-ideal of S. Then $f^2 \circ C_S \circ f^2 \subseteq f \circ C_S \circ f \subseteq f$. Hence, f is a fuzzy tri-quasi ideal of S.

(5) Let f be a fuzzy interior ideal of S. Then $f^2 \circ C_S \circ f^2 \subseteq C_S \circ f \circ C_S \subseteq f$. Hence, f is a fuzzy tri-quasi ideal of S. □

Theorem 8.2.5. Let Q be a nonempty subset of S. Then Q is a tri-quasi ideal of S if and only if C_Q is a fuzzy tri-quasi ideal of S.

Proof. Assume that Q is a tri-quasi ideal of S. Then $C_Q + C_Q = C_{Q+Q} \subseteq C_Q$, $C_Q \circ C_Q = C_{Q^2} \subseteq C_Q$, $C_Q^2 \circ C_S \circ C_Q^2 = C_{Q^2SQ^2} \subseteq C_Q$ and $(C_Q] = C_{(Q]} = C_Q$. Hence, C_Q is a fuzzy tri-quasi ideal of S.

Conversely, assume that C_Q is a fuzzy tri-quasi ideal of S. Since $C_{Q+Q} = C_Q + C_Q \subseteq C_Q$, we have $Q + Q \subseteq Q$. Similarly, we can show that $Q^2 \subseteq Q$. Since $C_{Q^2SQ^2} = C_Q^2 \circ C_S \circ C_Q^2 \subseteq C_Q$, we have $Q^2SQ^2 \subseteq Q$. Since $C_{(Q]} = (C_Q] = C_Q$, it follows that $(Q] = Q$. Therefore, Q is a tri-quasi ideal of S. □

Theorem 8.2.6. Let f be a fuzzy subset of S. Then f is a fuzzy tri-quasi ideal of S if and only if for any $t \in [0,1]$, if $f_t \neq \emptyset$, then f_t is a tri-quasi ideal of S.

Proof. Assume that f is a fuzzy tri-quasi ideal of S and $t \in [0,1]$ such that $f_t \neq \emptyset$. Let $x, y \in f_t$. Then we have

$$f(x+y) \geq \min\{f(x), f(y)\} \geq \min\{t, t\} = t \text{ and}$$
$$f(xy) \geq \min\{f(x), f(y)\} \geq \min\{t, t\} = t,$$

so $x+y, xy \in f_t$. Thus f_t is a subsemiring of S. Let $x \in {f_t}^2 S {f_t}^2$. Then $x = abscd$, where $a, b, c, d \in f_t$ and $s \in S$. Thus $f(abscd) \geq \min\{f(a), f(b), f(c), f(d)\} \geq t$. Thus $x \in f_t$. This shows that ${f_t}^2 S {f_t}^2 \subseteq f_t$. Let $x \in S$ such that $x \leq y$ for some $y \in f_t$. Thus $f(x) \geq f(y) \geq t$, so $x \in f_t$. Hence, f_t is a tri-quasi ideal of S.

Conversely, assume that f_t is a tri-quasi ideal of S for all $t \in [0,1]$ with $f_t \neq \emptyset$. Let $x, y \in S$ and $t = \min\{f(x), f(y)\}$. Then $t \in [0,1]$ and $x, y \in f_t$. By assumption, f_t is a tri-quasi ideal of S. Then $x + y \in f_t$ and $xy \in f_t$. Thus

$$f(x+y) \geq t = \min\{f(x), f(y)\} \quad \text{and} \quad f(xy) \geq t = \min\{f(x), f(y)\}.$$

Let $a, b, s, c, d \in S$ and $t = \min\{f(a), f(b), f(c), f(d)\}$. Then $a, b, d, c \in f_t$, so $abscd \in f_t$. It follows that $f(abscd) \geq t = \min\{f(a), f(b), f(c), f(d)\}$. Let $x \in S$ such that $x \leq y$. Let $t = f(y)$. Then $y \in f_t$, so $x \in (f_t] = f_t$. Thus $f(x) \geq f(y)$. Therefore, f is a fuzzy tri-quasi ideal of S. □

Theorem 8.2.7. Let $\{f_i\}_{i \in I}$ be a collection of fuzzy tri-quasi ideals of S. Then $\bigcap_{i \in I} f_i$ is a fuzzy tri-quasi ideal of S.

Proof. We have

$$\Big(\bigcap_{i \in I} f_i\Big) + \Big(\bigcap_{i \in I} f_i\Big) \subseteq f_i + f_i \subseteq f_i \text{ and } \Big(\bigcap_{i \in I} f_i\Big)^2 \subseteq f_i^2 \text{ for all } i \in I.$$

It follows that $\Big(\bigcap_{i \in I} f_i\Big) + \Big(\bigcap_{i \in I} f_i\Big) \subseteq \bigcap_{i \in I} f_i$ and $\Big(\bigcap_{i \in I} f_i\Big)^2 \subseteq \bigcap_{i \in I} f_i$. Then $\bigcap_{i \in I} f_i$ is a fuzzy subsemiring of S. We see that

$$\Big(\bigcap_{i \in I} f_i\Big)^2 \circ C_S \circ \Big(\bigcap_{i \in I} f_i\Big)^2 \subseteq f_i^2 \circ C_S \circ f_i^2 \subseteq f_i \text{ for all } i \in I.$$

Then $\Big(\bigcap_{i \in I} f_i\Big)^2 \circ C_S \circ \Big(\bigcap_{i \in I} f_i\Big)^2 \subseteq \bigcap_{i \in I} f_i$. We have $\Big(\bigcap_{i \in I} f_i\Big] \subseteq (f_i] = f_i$ for all $i \in I$, so $\Big(\bigcap_{i \in I} f_i\Big] \subseteq \bigcap_{i \in I} f_i$. Hence, $\bigcap_{i \in I} f_i$ is a fuzzy tri-quasi ideal of S. □

Theorem 8.2.8. If S is regular and commutative, then every fuzzy tri-quasi ideal of S is a fuzzy ideal of S.

Proof. Let f be a fuzzy tri-quasi ideal of S and $a, b \in S$. Then $a \leq axa$ for some $x \in S$ and $b \leq byb$ for some $y \in S$. Thus

$$ba \leq baxa \leq b(axa)x(axa) = aabxxxaa \text{ and } ba \leq byba \leq (byb)y(byb)a = bbyyyabb.$$

Then we have

$$f(ba) \geq f(aabxxxaa) \geq \min\{f(a), f(a), f(a), f(a)\} = f(a) \text{ and}$$
$$f(ba) \geq f(bbyyyabb) \geq \min\{f(b), f(b), f(b), f(b)\} = f(b).$$

Hence, f is a fuzzy ideal of S. □

Definition 8.2.9. S is called *fuzzy tri-quasi simple* if every fuzzy tri-quasi ideal of S is a constant function.

Theorem 8.2.10. S is tri-quasi simple if and only if S is fuzzy tri-quasi simple.

Proof. Assume that S is a tri-quasi simple ordered semiring. Let f be a fuzzy tri-quasi ideal of S and $x, y \in S$. By Theorem 8.1.19, $x \leq y^2 s_1 y^2$ and $y \leq x^2 s_2 x^2$ for some $s_1, s_2 \in S$. Then

$$\begin{aligned} f(x) &\geq f(y^2 s_1 y^2) \\ &\geq \min\{f(y), f(y), f(y), f(y)\} \\ &= f(y) \geq f(x^2 s_2 x^2) \\ &\geq \min\{f(x), f(x), f(x), f(x)\} \\ &= f(x). \end{aligned}$$

Thus f is a constant function. Hence, S is fuzzy tri-quasi simple.

Conversely, suppose that S is fuzzy tri-quasi simple. Let Q be a tri-quasi ideal of S. By Theorem 8.2.5, C_Q is a fuzzy tri-quasi ideal of S. Let $x \in S$ and $q \in Q$. Since C_Q is a constant function, we get $C_Q(x) = C_Q(q) = 1$, so $x \in Q$. Thus $S = Q$. Hence, S is tri-quasi simple. □

Theorem 8.2.11. If S is a fuzzy tri-quasi simple ordered semiring, then S is a simple ordered semiring.

Proof. Assume that S is a fuzzy tri-quasi simple ordered semiring. By Theorem 8.2.10, S is tri-quasi simple. Let I be an ideal of S. By Theorem 8.1.3(2), I is a tri-quasi ideal of S. Thus $I = S$. Hence, S is simple. □

Definition 8.2.12. A fuzzy tri-quasi ideal f is called *minimal* if for each nonzero fuzzy tri-quasi ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 8.2.13. If f is a nonzero fuzzy tri-quasi ideal of S, then $supp(f)$ is a tri-quasi ideal of S.

Proof. Assume that f is a nonzero fuzzy tri-quasi ideal of S. By Theorem 2.2.16, $supp(f)$ is a subsemiring of S. Since $f^2 \circ C_S \circ f^2 \subseteq f$, we get

$$\big(supp(f)\big)^2 S\big(supp(f)\big)^2 = supp(f^2 \circ C_S \circ f^2) \subseteq supp(f).$$

Since $(f] = f$, we have $\big(supp(f)\big] = supp(f\,] = supp(f)$. Hence, $supp(f)$ is a tri-quasi ideal of S. $\square$

The converse of Theorem 8.2.13 is not generally true as shown in the following example.

Example 8.2.14. Let $X = \{a, b\}$. Then $\big(P(X), \cup, \cap, \subseteq\big)$ is an ordered semiring. Define a fuzzy subset f of $P(X)$ by

$$f(\emptyset) = 1, f\big(\{a\}\big) = 0.8, f\big(\{b\}\big) = 0.3 \text{ and } f(X) = 0.6.$$

Then $supp(f) = P(X)$ is a tri-quasi ideal of $P(X)$. We see that

$$f\big(X \cap X \cap \{b\} \cap X \cap X\big) = f\big(\{b\}\big) = 0.3 < 0.6 = \min\big\{f(X), f(X), f(X), f(X)\big\}.$$

Hence, f is not a fuzzy tri-quasi ideal of S.

Theorem 8.2.15. Let Q be a tri-quasi ideal of S. Then Q is a minimal tri-quasi ideal of S if and only if C_Q is a minimal fuzzy tri-quasi ideal of S.

Proof. Assume that Q is a minimal tri-quasi ideal of S. By Theorem 8.2.5, C_Q is a fuzzy tri-quasi ideal of S. Let g be a nonzero fuzzy tri-quasi ideal of S such that $g \subseteq C_Q$. Then $supp(g) \subseteq supp(C_Q) = Q$. By Theorem 8.2.13, $supp(g)$ is a tri-quasi ideal of S. By the minimality of Q, we have $supp(g) = Q = supp(C_Q)$. Hence, C_Q is a minimal fuzzy tri-quasi ideal of S.

Conversely, suppose that C_Q is a minimal fuzzy tri-quasi ideal of S. Let A be a tri-quasi ideal of S such that $A \subseteq Q$. Then C_A is a fuzzy tri-quasi ideal of S such that $C_A \subseteq C_Q$. Since C_Q is minimal, we get $supp(C_A) = supp(C_Q)$. Thus $A = supp(C_A) = supp(C_Q) = Q$. Hence, Q is minimal. $\square$

8.3 Almost ordered tri-quasi ideals of ordered semirings

Definition 8.3.1. A nonempty subset Q of S is called an *almost ordered tri-quasi ideal* of S if Q is an almost ordered subsemiring of S and $(Q^2sQ^2] \cap Q \neq \emptyset$ for all $s \in S$.

Theorem 8.3.2. Every tri-quasi ideal of S is an almost ordered tri-quasi ideal of S.

Proof. Let Q be a tri-quasi ideal of S. Since Q is a subsemiring of S, by Theorem 3.1.2, Q is an almost ordered subsemiring of S. Let $s \in S$. We have

$$(Q^2sQ^2\,] \subseteq (Q^2SQ^2\,] \subseteq (Q\,] = Q.$$

Then $\emptyset \neq (Q^2sQ^2\,] = (Q^2sQ^2\,] \cap Q$. Hence, Q is an almost ordered tri-quasi ideal of S. $\square$

The converse of Theorem 8.3.2 does not hold in general. We consider the following example.

Example 8.3.3. Let $S_6 = \{a, b, c, d, e\}$. Define the binary operations $+$ and $\cdot$ on S shown in the table

$+$	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

$\cdot$	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

.

Define a relation $\leq$ on S_6 by

$$\leq = \{\,(a, a), (b, b), (c, c), (d, d), (e, e)\,\}.$$

Then $(S_6, +, \cdot, \leq)$ is an ordered semiring. Let $Q = \{a, b\}$. We have $a \leq a + a$, $a \leq a^2$ and $a \in Q$. Then $a \in (Q + Q\,] \cap Q$ and $a \in (Q^2\,] \cap Q$, so Q is an almost ordered subsemiring of S_6. We see that $(Q^2sQ^2\,] \cap Q = S_6 \cap Q = Q \neq \emptyset$ for all $s \in S_6$. Then Q is an almost ordered tri-quasi ideal of S_6. We see that $e = b^2ab^2 \in Q^2S_6Q^2$ but $e \notin Q$, so $Q^2S_6Q^2 \nsubseteq Q$. Therefore, Q is not a tri-quasi ideal of S_6.

Theorem 8.3.4. Let S be an ordered semiring. Then

(1) Every almost ordered ideal of S is an almost ordered tri-quasi ideal of S.

(2) Every almost ordered quasi-ideal of S is an almost ordered tri-quasi ideal of S.

(3) Every almost ordered bi-ideal of S is an almost ordered tri-quasi ideal of S.

(4) Every almost ordered interior-ideal of S is an almost ordered tri-quasi ideal of S.

Proof. We will prove (3) and (4). Part (1) and (2) follows directly from (3).

(3) Let B be an almost ordered bi-ideal of S and $b \in B$. Let $s \in S$. Then $\emptyset \neq (BbsbB] \cap B \subseteq (B^2sB^2] \cap B$. Hence, B is an almost ordered tri-quasi ideal of S.

(4) Let I be an almost ordered interior-ideal of S and $a \in I$. Let $s \in S$ Then $\emptyset \neq (aIsa^2] \cap I \subseteq (I^2sI^2] \cap I$. Therefore, I is an almost ordered tri-quasi ideal of S. □

The converse of Theorem 8.3.4 does not hold in general. We consider the following example.

Example 8.3.5. Consider the ordered semiring S_6 in Example 8.3.3, we have $Q = \{a, b\}$ is an almost ordered tri-quasi ideal of S_6. We see that $(QcQ] \cap Q = \emptyset$. Then Q is not an almost ordered bi-ideal of S_6. We also have $(cQa] \cap Q = \emptyset$. Thus Q is not an almost ordered interior-ideal of S_6.

Theorem 8.3.6. Let A and B be any two nonempty subsets of S. If $A \subseteq B$ and A is an almost ordered tri-quasi-ideal of S, then B is also an almost ordered tri-quasi-ideal of S.

Proof. Suppose that A is an almost ordered tri-quasi ideal of S with $A \subseteq B$. By Theorem 3.1.5, B is an almost ordered subsemiring of S. Let $s \in S$. We have $(A^2sA^2] \cap A \subseteq (B^2sB^2] \cap B$. This implies that $(B^2sB^2] \cap B \neq \emptyset$. Therefore, B is an almost ordered tri-quasi ideal of S. □

Corollary 8.3.7. The union of almost ordered tri-quasi ideals of S is also an almost ordered tri-quasi ideal of S.

8.4 Fuzzy almost ordered tri-quasi ideals of ordered semirings

Definition 8.4.1. A fuzzy subset f of S is called a *fuzzy almost ordered tri-quasi ideal* of S if f is a fuzzy almost ordered subsemiring of S and $(f^2 \circ s_\alpha \circ f^2] \cap f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.

Theorem 8.4.2. Let f be a fuzzy subset of S. Then f is a fuzzy almost ordered tri-quasi ideal of S if and only if $supp(f)$ is an almost ordered tri-quasi ideal of S.

Proof. By Theorem 3.2.2, we have f is fuzzy almost ordered subsemiring of S if and only if $supp(f)$ is an almost ordered subsemiring of S. Let $s \in S$ and $\alpha \in (0, 1]$. By Theorem 2.2.15, we have

$$(f^2 \circ s_\alpha \circ f^2] \cap f \neq 0 \text{ if and only if } \Big(\big(supp(f)\big)^2 s \big(supp(f)\big)^2\Big] \cap supp(f) \neq \emptyset.$$

This completes the proof. □

Corollary 8.4.3. Let $\emptyset \neq A \subseteq S$. Then C_A is a fuzzy almost ordered tri-quasi ideal of S if and only if A is an almost ordered tri-quasi ideal of S.

Theorem 8.4.4. Every nonzero fuzzy tri-quasi ideal of S is a fuzzy almost ordered tri-quasi ideal of S.

Proof. Let f be a nonzero fuzzy tri-quasi ideal of S. Then $supp(f)$ is a tri-quasi ideal of S, so $supp(f)$ is an almost ordered tri-quasi ideal of S. Hence, f is a fuzzy almost ordered tri-quasi ideal of S. □

The converse of Theorem 8.4.4 is not generally true as shown in the following example.

Example 8.4.5. From Example 8.3.3, we have $Q = \{a, b\}$ is an almost ordered tri-quasi ideal of S_6 but it is not a tri-quasi ideal of S_6. Then C_Q is a fuzzy almost ordered tri-quasi ideal of S_6 but it is not a fuzzy tri-quasi ideal of S_6.

Theorem 8.4.6. Let S be an ordered semiring. Then

(1) Every fuzzy almost ordered ideal of S is a fuzzy almost ordered tri-quasi ideal of S.

(2) Every fuzzy almost ordered quasi-ideal of S is a fuzzy almost ordered tri-quasi ideal of S.

(3) Every fuzzy almost ordered bi-ideal of S is a fuzzy almost ordered tri-quasi ideal of S.

(4) Every fuzzy almost ordered interior-ideal of S is a fuzzy almost ordered tri-quasi ideal of S.

Proof. We will prove (3) and (4). Part (1) and (2) follows directly from (3).

(3) Let f be a fuzzy almost ordered bi-ideal of S. Then $supp(f)$ is an almost ordered bi-ideal of S. By Theorem 8.3.4 (3), $supp(f)$ is an almost ordered tri-quasi ideal of S. Thus f is a fuzzy almost ordered tri-quasi ideal of S.

(4) Let f be a fuzzy almost ordered interior-ideal of S. Then $supp(f)$ is an almost ordered interior-ideal of S. By Theorem 8.3.4 (4), we have $supp(f)$ is an almost ordered tri-quasi ideal of S. It follows that f is a fuzzy almost ordered tri-quasi ideal of S. □

The converse of Theorem 8.4.6 does not hold in general. We consider the following example.

Example 8.4.7. From Example 8.3.5, we have $Q = \{a, b\}$ is an almost tri-quasi ideal of S_6 but it is not an almost ordered bi-ideal (resp. interior-ideal) of S_6. Hence, C_Q is a fuzzy almost ordered tri-quasi ideal of S_6 but it is not a fuzzy almost ordered bi-ideal (resp. interior-ideal) of S_6.

Theorem 8.4.8. Let f and g be fuzzy subsets of S such that $f \subseteq g$. If f is a fuzzy almost ordered tri-quasi ideal of S, then g is also a fuzzy almost ordered tri-quasi ideal of S.

Proof. Suppose that f is a fuzzy almost ordered tri-quasi ideal of S with $f \subseteq g$. Then $supp(f)$ is an almost ordered tri-quasi ideal of S with $supp(f) \subseteq supp(g)$, so $supp(g)$ is an almost ordered tri-quasi ideal of S. Hence, g is a fuzzy almost ordered tri-quasi ideal of S. □

Corollary 8.4.9. The union of fuzzy almost ordered tri-quasi ideals of S is also a fuzzy almost ordered tri-quasi ideal of S.

Definition 8.4.10. A fuzzy almost ordered tri-quasi ideal f of an ordered semiring S is called *minimal* if for all fuzzy almost ordered tri-quasi ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 8.4.11. Let f be a fuzzy subset of S. Then f is a minimal fuzzy almost ordered tri-quasi ideal of S if and only if $supp(f)$ is a minimal almost ordered tri-quasi ideal of S.

Proof. The proof is similar to Theorem 4.2.10. □

Corollary 8.4.12. Let A be a nonempty subset of S. Then C_A is a minimal fuzzy almost ordered tri-quasi ideal of S if and only if A is a minimal almost ordered tri-quasi ideal of S.

Chapter 9

Conclusion

An ordered semiring is an interesting algebraic structure. It is a semiring under partial order, two most simple algebraic structures on the same set. Many authors investigated interesting results in this algebraic system. Also, the concept of fuzzy subsets is interesting to play with it.

In Chapter 3, we define almost ordered subsemirings and fuzzy almost ordered subsemirings in ordered semirings. The union of two almost ordered subsemirings [fuzzy almost ordered subsemirings] is also an almost ordered subsemiring [fuzzy almost ordered subsemiring]. Moreover, we investigate some relationships between almost ordered subsemirings and fuzzy almost ordered subsemirings of ordered semirings.

In Chapter 4, we define almost ordered ideals and fuzzy almost ordered ideals in ordered semirings. Every almost ordered ideal [fuzzy almost ordered ideal] is an almost ordered subsemiring [fuzzy almost ordered subsemiring] but the converse is not true in general. The union of two almost ordered ideals [fuzzy almost ordered ideals] is also an almost ordered ideal [fuzzy almost ordered ideal]. Moreover, some relationships between almost ordered ideals and fuzzy almost ordered ideals of ordered semirings are provided.

In Chapter 5, we define almost ordered quasi-ideals and fuzzy almost ordered quasi-ideals in ordered semirings. Every almost ordered quasi-ideal [fuzzy almost ordered quasi-ideal] is an almost ordered ideal [fuzzy almost ordered ideal] but the converse is not true in general. The union of two almost ordered quasi-ideals [fuzzy almost ordered quasi-ideals] is also an almost ordered quasi-ideal [fuzzy almost ordered quasi-ideal]. We also give some relationships between almost ordered quasi-ideals and fuzzy almost ordered quasi-ideals of ordered semirings.

In Chapter 6, we introduce the notion of almost ordered bi-ideals and fuzzy almost ordered bi-ideals of ordered semirings. Every almost ordered quasi-ideal [fuzzy almost ordered quasi-ideal] is an almost ordered bi-ideal [fuzzy almost ordered bi-ideal] but the converse is not true in general. The union of two almost ordered bi-ideals [fuzzy almost ordered bi-ideals] is also an almost ordered bi-ideal [fuzzy almost ordered bi-ideal]. We also give some relationships between almost ordered bi-ideals and fuzzy almost ordered bi-ideals of ordered semirings.

In Chapter 7, we define almost ordered interior-ideals and fuzzy almost ordered interior-ideals of ordered semirings. The union of two almost ordered interior-ideals [fuzzy almost ordered interior-ideals] is also an almost ordered interior-ideal [fuzzy almost ordered interior-ideal]. Moreover, we investigate some relationships between almost ordered interior-ideals and fuzzy almost ordered interior-ideals of ordered semirings.

In Chapter 8, we introduce the notion of tri-quasi ideals of ordered semirings. This generalizes the concepts of ideals, quasi-ideals, bi-ideals and interior-ideals. We also give the characterization of regular ordered semirings by the properties of their tri-quasi ideals. Moreover, we define fuzzy tri-quasi ideals of ordered semirings and give some relationships between tri-quasi ideals and fuzzy tri-quasi ideals of ordered semirings. In addition, we define almost ordered tri-quasi ideals and fuzzy almost ordered tri-quasi ideals of ordered semirings. The union of two almost ordered tri-quasi ideals [fuzzy almost ordered tri-quasi ideals] is also an almost tri-quasi ideal [fuzzy almost ordered tri-quasi ideal]. Finally, we give some relationships between almost ordered tri-quasi-ideals and fuzzy almost ordered tri-quasi ideals of ordered semirings.

Bibliography

[1] R. A. Good and D. R. Hughes, Associated for a semigroup, Bulletin of the American Mathematical Society, 58 (1952), 624-625.

[2] O. Steinfeld, Ueber die Quasiideale von Halbgruppen, Publicationes Mathematicae Debrecen. 4 (1956), 262–275.

[3] S. Lajos, (m, k, n)-ideals in semigroups, Notes on Semigroups II, Department of Mathematics, Karl Marx University of Economics, Budapest, 1 (1976), 12-19.

[4] K. Iséki, Quasiideals in semirings without zero, Proceedings of the Japan Academy. 34 (2), (1958), 79-81.

[5] S. Lajos, F.A. Szasz. On the bi-ideals in associative rings, Proceedings of the Japan Academy. 46 (6) (1970), 505-507.

[6] A. P. Gan and Y. L. Jiang, On ordered ideals in ordered semirings, Journal of Mathematical Research and Exposition. 31 (6), (2011), 989-996.

[7] P. Palakawong na Ayutthaya and B. Pibaljommee, Characterizations of regular ordered semirings by ordered quasi-ideals, International Journal of Mathematics and Mathematical Sciences, (2016), Article ID 4272451, 8 pages.

[8] M. M. K. Rao, Tri-quasi-ideals of Γ-semirings, Discussiones Mathematicae - General Algebra and Applications, 41 (2021), 33–44.

[9] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy sets and Systems. 5 (1981), 203-215.

[10] N. Kuroki, Fuzzy semiprime quasi-ideals in semigroups, Information Sciences. 75 (3), (1993), 201–211.

[11] S. M. Hong, Y. B. Jun and J. Meng, Fuzzy interior ideals in semigroups, Indian Journal of Pure and Applied Mathematics 26 (9), (1995), 859-863.

[12] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy Information and Engineering. 6 (1), (2014), 101–114.

[13] O. Grosek and L. Satko, A new notion in the theory of semigroups, Semigroup Forum. 20 (1980), 233-240.

[14] A. Simuen, A. Iampan and R. Chinram, A novel of ideals and fuzzy ideals of Gamma-semigroups, Journal of Mathematics, (2021), Article ID 6638299, 14 pages.

[15] L. A. Zadeh, Fuzzy sets, Information and Control. 8 (1965), 338–353.

[16] A. Rosenfeld, Fuzzy groups, Journal of Mathematical Analysis and Applications. 35 (1971), 512-517.

[17] R. Rittichuai, A. Iampan, R. Chinram and P. Singavananda, Almost subsemirings and fuzzifications, International Journal of Mathematics and Computer Science. 17 (4), (2022), 1491-1497.

[18] S. Bogdanovic, Semigroups in which some bi-ideal is a group, Review of Research Faculty of Science-University of Novi Sad. 11 (1981), 261-266.

[19] K. Wattanatripop, R. Chinram and T. Changphas, Quasi-A-ideals and fuzzy A-ideals in semigroups, Journal of Discrete Mathematical Sciences and Cryptography. 21 (5), (2018), 1131-1138.

[20] K. Wattanatripop, R. Chinram and T. Changphas, Fuzzy almost bi-ideals in semigroups, International Journal of Mathematics and Computer Science 13 (1), (2018), 51-58.

[21] P. Murugadas, K. Kalpana and V. Vetrivel, Fuzzy almost quasi-ideals in semigroups, Malaya Journal of Matematik. S (1), (2019), 310-313.

[22] S. Suebsung, R. Chinram, W. Yonthanthum, K. Hila, and A. Iampan, On almost bi-ideals and almost quasi-ideals of ordered semigroups and their fuzzifications, Icic Express Letters. 16 (2), (2022), 127-135.

[23] N. Kaopusek, T. Kaewnoi and R. Chinram, On almost interior ideals and weakly almost interior ideals of semigroups, Journal of Discrete Mathematical Sciences and Cryptography, DOI: 10.1080/09720529.2019.1696917, (2020), 1-6.

[24] W. Krailoet, A. Simuen, R. Chinram and P. Petchkaew, A note on fuzzy almost interior ideals in semigroups, International Journal of Mathematics and Computer Science. 16 (2), (2021), 803-808.

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