Biased Domination Games

Tharit Sereekiatdilok

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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บทคัดย่อ

เกมโดมิเนชัน (domination game) บนกราฟ G เป็นเกมที่ประกอบไปด้วยผู้เล่น 2 คน ได้แก่ *โดมิเนเตอร์ (Dominator)* และ*สตอลเลอร์ (Staller)* โดยแต่ละตาผู้เล่นจะผลัดกันเลือกจุดหนึ่งจุดบนกราฟ G หลังจากนั้นจุดที่ถูกเลือกจะ*โดมิเนท (dominate)* ย่านใกล้เคียงปิด (closed neighborhood) ของจุดนั้น ในแต่ละตานั้นผู้เล่นจะต้องเลือกจุดที่โดมิเนทจุดเพิ่มขึ้นอย่างน้อยหนึ่งจุดเสมอ เกมจะจบลงเมื่อทุกจุดบนกราฟถูกโดมิเนท โดมิเนเตอร์จะพยายามทำให้เกมจบด้วยการทำให้จำนวนครั้งในการเลือกจุดทั้งหมดน้อยที่สุด ในทางกลับกันสตอลเลอร์จะพยายามยืดเกมให้ยาวนานที่สุดเท่าที่เป็นไปได้ เราจะเรียกเกมโดมิเนชันว่า *เกมที่ 1* เมื่อโดมิเนเตอร์เป็นผู้เริ่มเกมและ *เกมที่ 2* เมื่อสตอลเลอร์เป็นผู้เริ่มเกม ถ้าผู้เล่นทั้งคู่เล่นเกมโดยใช้วิธีที่ดีที่สุดจนเกมจบแล้วจำนวนครั้งในการเลือกจุดทั้งหมดในเกมจะถูกเรียกว่า *เลขเกมโดมิเนชัน (game domination number)*

ในการศึกษาครั้งนี้ เราขอเสนอ*เกมไบแอสโดมิเนชัน (biased domination game)* ที่อยู่ในรูปแบบที่ถูกขยายขึ้นจากเกมโดมิเนชัน โดยแต่ละตาผู้เล่นสามารถเลือกจุดได้มากกว่าหนึ่งจุด ในทำนองเดียวกัน ถ้าผู้เล่นทั้งคู่เล่นเกมโดยใช้วิธีที่ดีที่สุดจนเกมจบแล้วจำนวนครั้งในการเลือกจุดทั้งหมดในเกมจะถูกเรียกว่า *เลขไบแอสเกมโดมิเนชัน (biased game domination number)* เราศึกษาความสัมพันธ์ของเลขไบแอสเกมโดมิเนชันของเกมไบแอสโดมิเนชันต่าง ๆ นอกจากนี้เรายังคำนวณเลขไบแแอสเกมโดมิเนชันบนบางกราฟ เช่น กราฟวัฏจักร (cycle) และกราฟกำลังของกราฟวัฏจักร (power of cycle) ยิ่งไปกว่านั้นเราศึกษาวิธีการการเลือกจุดแบบพิเศษที่เรียกว่า *การเลือกแบบมินิมอล (minimal move)* และ*การเลือกแบบแมกซิมอล (maximal move)* ซึ่งเราหาสมบัติบางประการของเลขไบแอสเกมโดมิเนชันบนกราฟที่สามารถเลือกจุดเหล่านี้ได้ ท้ายที่สุดเราคำนวณค่าเลขไบแอสเกมโดมิเนชันบนกราฟกำลังของกราฟวัฏจักร (power of cycle) และหาการเล่นที่เหมาะสมที่สุดโดยใช้การเลือกจุดแบบพิเศษ

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ABSTRACT

A Domination game on a graph G is a game of two players, called *Dominator* and *Staller*, on a graph. The players take turns to perform a move by choosing a vertex in the graph. Vertices in the closed neighborhood of a chosen vertex are said to be *dominated*. A move u is legal if it creates at least one new dominated vertex. The game is ended when all vertices in the graph are dominated. Dominator tries to end the game as soon as possible, while Staller tries to prolong the game. In the domination game, if Dominator starts the game, this game is said to be *Game* 1. Otherwise, it is said to be *Game* 2. If both players play optimally in a domination game on a graph G, the number of moves when the game is ended is called the *game domination numbers*.

In this research, we introduce an extended version of a domination game on a graph, called a *biased domination game*, in which Dominator and Staller play more than one move in each turn. Similarly, we define the *biased game domination number* as the number of moves in an ended biased domination game which both players play with optimal strategies. We study relations of biased game domination numbers between various games. In addition, we study two special types of moves, called *minimal moves* and *maximal moves*. Some properties of the biased game domination numbers on a graph where the special moves are always available is studied. Lastly, the biased game domination numbers of powers of a cycle are explicitly computed, together with optimal strategies using a special move.

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Tharit Sereekiatdilok

Contents

Chapter 1

Introduction

Domination games were introduced in [1] as a game of two players, called Dominator and Staller, on a graph. The two players take turns to choose a vertex (pick a move) in a graph. A move is legal if its closed neighborhood is not contained in the closed neighborhood of vertices which have been chosen before. That is, for a sequence of previuosly picked moves $u_1, u_2, u_3, \ldots, u_{n-1}$, the player can pick a move u_n if and only if $N[u_n] \nsubseteq \bigcup_{i=1}^{n-1} N[u_i]$. Vertices in a closed neighborhood of chosen vertices are then called *dominated*. Note that every vertex v in $N[u]$ is dominated by choosing a vertex u. The game is ended when all vertices in a graph are dominated. In the domination game, if Dominator starts the game, this game is said to be a *Game* 1. Otherwise, it is said to be a *Game* 2. If both players play optimally in a domination game on a graph G, the number of moves when the game is ended is called a *domination number*, denoted by $\gamma_g(G)$ and $\gamma_g'(G)$ in a Game 1 and Game 2, respectively.

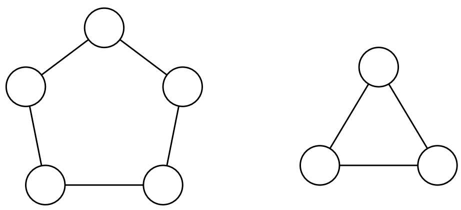
Many aspects of domination games have been studied. Game domination numbers on various graphs, such as trees [2], forests [3], paths and cycles [4], powers of cycles [5], disjoint union of paths and cycles [6], have been computed. Possible values of domination numbers of unions of graphs are studied [7]. Bound on domination numbers have been studied, see [8, 9, 10, 11, 12, 13] for examples.

Some variations of the game have also been studied. The total domination game has been introduced in [14], in which a move u will dominate its open neighborhood $N(u)$ instead of its closed neighborhood $N[u]$. So a move is legal if and only if its open neighborhood is not contained in the open neighborhood of all vertices chosen before. Similarly, bound on total domination numbers have been studied in [15, 16, 17, 18], and total domination numbers themselves were computed for some families of graphs, such as cycles and paths [19] and a family of cyclic bipartite graphs [20]. Recently, some other variations based on the definition of legal moves have been proposed in [21].

Since players pick only one move per turn in the standard domination game, we study the term of biased positional games for domination games that each player takes a different number of moves. In this thesis, we introduce a variation of a domination game called a *biased domination game* or simply called a *biased game* where Dominator and Staller can pick more than one move for each turn. Similar to the definition of the game domination number, we define the *biased game domination number*. For $D, S \subseteq \mathbb{N}$, if in each turn Dominator and Staller can pick $m \in D$ and $n \in S$ and play m and n legal moves, respectively, then the game is called (D, S)*-biased domination game* or (D, S)*-game*. The biased domination numbers are denoted by $\gamma_{(D,S)}(G)$ for Game 1 and $\gamma'_{(D,S)}(G)$ for Game 2. We called (D, S) as the *biased ordered pair*. In the case of $(D, S) = (\{\delta\}, \{\sigma\})$, the game is called (δ, σ)*-biased domination game* or (δ, σ)*-game*. The biased domination numbers are denoted by $\gamma_{(\delta,\sigma)}(G)$ for Game 1 and $\gamma'_{(\delta,\sigma)}(G)$ for Game 2. Note that in a (D, S)-game, a player does not need to play the same number of moves in every turn. For example, if $D = \{3, 5\}$, Dominator can play 3 moves in the first turn and play 5 moves in the second turn.

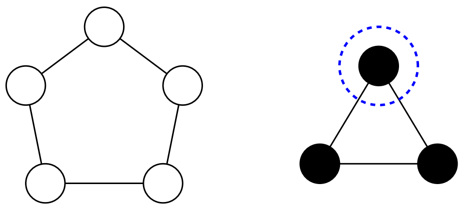
Remark 1.1. The $(1, 1)$-game is the original domination game.

Example 1.2. Consider $G = C_5 \sqcup C_3$. We will show that $\gamma_{(\{1,2\},\{1\})}(G) = 3$.



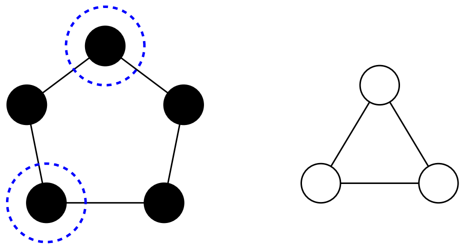
We first notice that G has 8 vertices. Since each move can dominate at most 3 vertices, at least 3 moves are required to end the game. Hence, the biased game domination number is at least 3.

Next, we can show that Dominator can force the game to end in 3 moves. So we can conclude that $\gamma_{(\{1,2\},\{1\})}(G) = 3$. In the first turn, Dominator starts the game by picking 1 or 2 moves. If Dominator decides to pick 1 move in C_3, as shown in the picture below, Staller then picks any move on C_5 and Dominator finishes the game in 3 moves.



Note that if Dominator were to pick only one move on a graph C_5 as the first turn, Staller can pick a move on C_5 that dominates exactly one new vertex and forces the game to end in at least 4 moves. Hence this is not a move in Dominator's optimal strategy.

Similarly, Dominator can also decide to pick 2 move in his first turn. If two moves are picked as in the following picture, Dominator can force the game to end in 3 moves as well. (In the next turn, any move of Staller must ends the game.) This is also another optimal strategy for Dominator.



The main goal of this thesis is to study the relation between biased domination games having different biased ordered pairs. In Chapter 2, we review some definitions and results related to domination games. We define two special moves in Chapter 3 and show some properties of the biased game domination numbers when such special moves are available. In Chapter 4, examples of graphs with special moves are presented. We also explicitly compute the biased game domination number of the power of a cycle, and give optimal strategies for both players using the special moves.

Chapter 2

Preliminaries

In this chapter, we collect the definitions and theorems which will be used later throughout the thesis.

2.1 Domination Games

In 2010, B. Brešar, S. Klavžar and D.F. Rall introduced the domination game. Moreover, they consider Game 1 in a domination game such that Dominator (resp. Staller) is allowed, but not obligated, to skip exactly one move in the game. That is, there is at most one turn such that Dominator (resp. Staller) may decide to pass. After the game is ended, the number of moves in a game where both players are playing optimally, is denoted by $\gamma_g^{dp}(G)$ (resp. $\gamma_g^{sp}(G)$). We call this game the *Dominator-pass game* (resp. *Staller-pass* game).

Lemma 2.1 ([1])**.** For any graph G,

$$\gamma_g^{sp}(G) \leq \gamma_g(G) + 1, \gamma_g'(G) \leq \gamma_g^{sp}(G) + 1 \text{ and } \gamma_g^{dp}(G) \geq \gamma_g(G) - 1.$$

This Lemma is applied to prove a relation of game domination numbers between Game 1 and Game 2.

Theorem 2.2 ([1])**.** *For any graph G,*

$$\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma(G) + 2.$$

Definition 2.3 ([1])**.** Let k and l be positive integers. We say the pair (k, l) is *realizable* if there is a graph G such that $\gamma_g(G) = k$ and $\gamma_g'(G) = l$.

In 2013, W.B. Kinnersley. D.B. West and R. Zamani defined the partially dominated graph for simplifying the proof of a domination game.

Definition 2.4 ([8])**.** Let G be a graph and S be a subset of $V(G)$. A *partially dominated graph* $G|S$ is a graph G in which all vertices in S are already dominated. Consider a partially dominated graph $G|S$, a vertex v is called *saturated* if all of its closed neighborhood is dominated. If we delete all saturated vertex and all edges which both of their endpoints are dominated in a partially dominated graph $G|S$, we called this graph a *residual graph* $\lfloor G|S \rfloor$.

Remark 2.5. A domination game on a partially dominated graph $G|S$ is considered as same as a residual graph $\lfloor G|S \rfloor$.

Theorem 2.6 (Continuation Principle, [8])**.** *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_g(G|A) \geq \gamma_g(G|B)$ and $\gamma'_g(G|A) \geq \gamma'_g(G|B)$.*

Theorem 2.7 ([8])**.** *For any graph G, we have $|\gamma'_g(G) - \gamma_g(G)| \leq 1$.*

We say that a graph G is a *no-minus* graph when $\gamma_g(G|S) \leq \gamma'_g(G|S)$ for every subset S of $V(G)$.

Theorem 2.8 ([8])**.** *Forests are no-minus graphs.*

In 2015, P. Dorbec, G. Košmrlj and G. Renault found bounds of game domination numbers on unions of graphs.

Theorem 2.9 ([7])**.** *Let $G_1|S_1$ and $G_1|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ realizes (k, k) and $G_2|S_2$ realizes (l, m) where $m \in \{l, l+1\}$, then the disjoint union $(G_1 \sqcup G_2)|(S_1 \sqcup S_2)$ realizes $(k + l, k + m)$.*

Theorem 2.10 ([7])**.** *Let $G_1|S_1$ and $G_1|S_2$ be partially dominated no-minus graphs that realizes $(k, k+1)$ and $(l, l+1)$, respectively. Then*

$$k + l \leq \gamma_g((G_1 \sqcup G_2)|(S_1 \sqcup S_2)) \leq k + l + 1$$
$$k + l + 1 \leq \gamma'_g((G_1 \sqcup G_2)|(S_1 \sqcup S_2)) \leq k + l + 2.$$

2.2 Domination Games on Some Graphs

In 2017, G. Košmrlj studied the game domination number on paths and cycles by considering some partially dominated graphs of paths and cycles and classifying some families of a graph.

Definition 2.11 ([4])**.** We classify a graph G that realizes a pair (k, l) as follows:

1. a graph G is a *plus* graph if $l = k + 1$;
2. a graph G is a *equal* graph if $l = k$;
3. a graph G is a *minus* graph if $l = k - 1$.

To compute game domination numbers on paths and cycles, Košmrlj defined some special partially dominated graph of paths.

Definition 2.12 ([4])**.** A partially dominated graph P''_n is a graph P_{n+2} that both leaves are already dominated. A partially dominated graph P'_n is a graph P_{n+1} that exactly one leaf is already dominated.

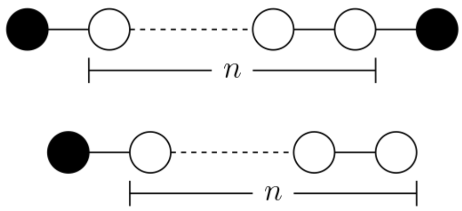


Figure 2.1 Partially dominated paths P''_n (top) and P'_n (bottom).

Theorem 2.13 ([4])**.** *For any nonnegative integer n, we have*

$$\gamma_g(P''_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1, & n \equiv 3 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil, & \textit{otherwise}, \end{cases}$$

$$\gamma'_g(P''_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

In addition, letting $i_r = (i \mod 4)$ and $j_r = (j \mod 4)$ for any $i + j = n$. Then

$$\gamma_g(P''_i \sqcup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j) + 1, & (i_r, j_r) \in \{2,3\} \times \{2,3\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j), & \textit{otherwise}, \end{cases}$$

$$\gamma'_g(P''_i \sqcup P''_j) = \begin{cases} \gamma_g(P''_i) + \gamma_g(P''_j), & (i_r, j_r) \in \{0,1\} \times \{0,1\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 2, & (i_r, j_r) \in \{(2,3), (3,2), (3,3)\}, \\ \gamma_g(P''_i) + \gamma_g(P''_j) + 1, & \textit{otherwise}. \end{cases}$$

Theorem 2.14 ([4]). *For any nonnegative integer n, we have*

$$\gamma_g(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1, & n \equiv 3 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise}, \end{cases}$$

$$\gamma'_g(C_n) = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil - 1, & n \equiv 2 \mod 4, \\ \left\lceil \frac{n-1}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

Theorem 2.15 ([4]). *For any nonnegative integer m and n, we have*

$$\gamma_g(P''_m \sqcup P''_n) = \gamma_g(P''_m \sqcup P'_n) = \gamma_g(P'_m \sqcup P'_n)$$

and

$$\gamma'_g(P''_m \sqcup P''_n) = \gamma'_g(P''_m \sqcup P'_n) = \gamma'_g(P'_m \sqcup P'_n).$$

Moreover, we have $\gamma_g(P''_m) = \gamma_g(P'_m)$ and $\gamma'_g(P''_m) = \gamma'_g(P'_m)$.

Theorem 2.16 ([4]). *For any nonnegative integer n, we have*

$$\gamma_g(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1, & n \equiv 3 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise}, \end{cases}$$

$$\gamma'_g(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

In 2020, N. Chantarachada and C. Worawannotai computed the formula of game domination numbers on powers of cycles and found optimal strategies of both players.

Definition 2.17. For positive integers p and n, the *p-th power C_n^p of an n-cycle* C_n has the following vertex set and edge set,

$$V(C_n^p) = \{0, 1, 2, \dots, n-1\},$$
$$E(C_n^p) = \{\{i, i \pm 1\}, \{i, i \pm 2\}, \dots, \{i, i \pm p\} : i \in V(C_n^p)\}$$

where the operations $+$ and $-$ are considered under modulo n.

Theorem 2.18 ([5]). *Let $G = C_n^p$. If $n = (2p+2)q + r$ where $q, r \in \mathbb{N} \cup \{0\}$ and $0 \leq r < 2p + 2$, then*

1. $\gamma_g(G) = 2q + [r \neq 0]$,
2. $\gamma'_g(G) = 2q + [r = 2p + 1]$

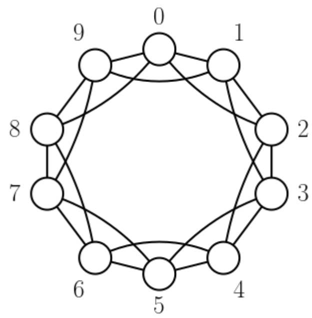


Figure 2.2 The graph C_{10}^2

where $[x] = 1$ *if the statement* x *is true and* $[x] = 0$ *if* x *is false.*
Moreover, an optimal strategy for Staller is when (s)he always makes a move that dominates exactly one new vertex, while an optimal strategy for Dominator is when (s)he always makes a move that dominates as many new vertices as possible without creating a new dominated component (except the move that starts the game).

In 2015, C. Bujtás established the upper bounds of game domination numbers on *isolate-free* forests, i.e., forests without isolated vertices by considering the conjecture in [8].

Conjecture 2.19 ([8])**.** *If* G *is an isolate-free forest of order* n*, then*

$$\gamma_g(G) \leq \frac{3n}{5} \text{ and } \gamma_g'(G) \leq \frac{3n+2}{5}.$$

Theorem 2.20 ([3])**.** *If* G *is an isolate-free forest of order* n *in which no two leaves have distance* 4*, then*

$$\gamma_g(G) \leq \frac{3n}{5} \text{ and } \gamma_g'(G) \leq \frac{3n+1}{5}.$$

Theorem 2.21 ([3])**.** *If* G *is an isolate-free forest of order* n*, then*

$$\gamma_g(G) \leq \frac{5n}{8} \text{ and } \gamma_g'(G) \leq \frac{5n+2}{8}.$$

Chapter 3

Biased Domination Games

In this chapter, we show some relations between biased game domination numbers having different biased ordered pairs, under the condition that some certain moves are always available. Those moves will be called *minimal* and *maximal* moves.

3.1 Special Moves

In [4] and [5], we observed that some special moves are used in optimal strategies in the domination games on paths and powers of cycles. So we define a modification of these special moves in biased domination games.

Definition 3.1. For any biased domination games on $G|C$, we define as following.

1. Moves u and u' are the *same* move with respect to C if their newly dominated vertices set are the same set, i.e.,

$$N[u] \setminus C = N[u'] \setminus C.$$

2. If a move u dominates only one new vertex v, then it is a *minimal move* with respect to C. In this case, a vertex v is *minimally dominated* by u and denoted by $m_C[u]$. It is easy to see that minimal moves u and u' are same if and only if $m_C[u] = m_C[u']$.
3. A move u is a *maximal move* with respect to C if there is a newly dominated vertex v such that for any move dominating v, its set of newly dominated vertices must be contained in the set of newly dominated vertices of a move u. That is, a maximal move u with respect to C has a vertex $v \in N[u] \setminus C$ such that for all u' which $v \in N[u'] \setminus C$, we have

$$N[u'] \setminus C \subseteq N[u] \setminus C. \tag{3.1}$$

If we define $N_{G|C}[u] = N_G[u] \setminus C$, a move u is a *maximal move* in $G|C$ if there is a newly dominated vertex v such that for any other move u' dominating v in $G|C$, we have

$$N_{G|C}[u'] \subseteq N_{G|C}[u].$$

In other words, there is $v \in N_{G|C}[u]$ such that

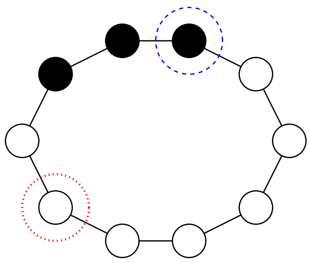
$$\text{if } v \in N_{G|C}[u'], \text{ then } N_{G|C}[u'] \subseteq N_{G|C}[u].$$

In this case, we say that vertex v is *maximally dominated* by a maximal move u, and a set of all maximally dominated vertices by a maximal move u is denoted by $M_C[u]$.

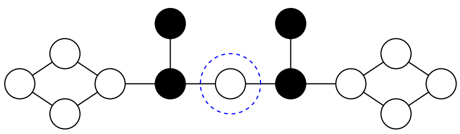
Remark 3.2. If a maximal move u maximally dominates a vertex v, there is no valid move dominating v after picking a move u.

Example 3.3. The following are examples of minimal and maximal moves in partially dominated graphs. Minimal moves and maximal moves are indicated by blue dashed lines and red dotted lines, respectively.

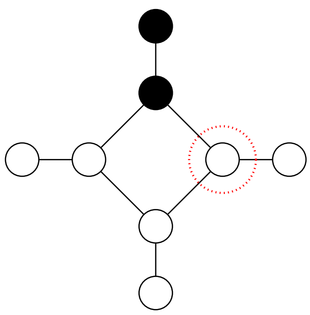
1. The following partially dominated graph has both minimal move and maximal move.



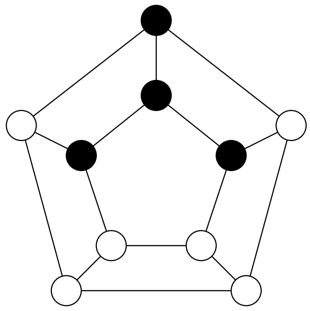
2. The following partially dominated graph has a minimal move but no maximal move.



3. The following partially dominated graph has a maximal move but no minimal move.



4. The following partially dominated graph has neither maximal move nor minimal move.



In this thesis, the condition "the player can always make a minimal (resp. maximal) move" means the player can make such move at every of his turn after the game has already started. This means we do not apply the condition to the first move of the game.

Proposition 3.4. *In a biased domination game on a graph G, if both players can always make a minimal move, then G is connected or G is an edgeless graph (a union of isolated vertices).*

Proof. Assume that G is not connected and has at least one edge. Then there is a component C that is not an isolated vertex. Thus for all vertices in this component, their closed neighborhood has size at least 2, that is, if a vertex u is in this component, then $|N[u]| \geq 2$. Consider the game that the first move happened at another component. So no players can make a minimal move at the component C. □

Proposition 3.5. *Let G be a connected graph and D be a set of vertices in G. In a biased domination game on a residual graph $\lfloor G|D \rfloor$, a vertex u is a minimal move if and only if u is a dominated leaf or all vertices $v \in N(u)$ are already dominated.*

Proof. Assume that u is a a minimal move and there is a vertex $v \in N(u)$ not dominated on $\lfloor G|D \rfloor$. Note that no two dominated vertices are adjacent in $\lfloor G|D \rfloor$. Since v is not dominated, u is already dominated. Since u dominates exactly one new vertex, u minimally dominates a vertex v. So u must be a leaf on $\lfloor G|D \rfloor$. Hence u is a dominated leaf.

Conversely, it is easy to see that if a vertex u is a dominated leaf or all vertices $v \in N(u)$ are already dominated in $\lfloor G|D \rfloor$, then u is a minimal move. □

Proposition 3.6. *If a player can always make a minimal (resp. maximal) move on a graph G, then a player can always make a minimal (resp. maximal) move on a graph H where $V(H) = V(G) \sqcup \{u_1, u_2, \ldots, u_n\}$ and $E(H) = E(G) \sqcup \{u_i v \mid i \in [n], v \in V(G)\}$.*

Proof. Suppose that a player can always make a minimal move on a graph G and the game on a graph H has been started. If there is a move u_i picked in a game, then all vertices in $V(G)$ are already dominated. So an undominated vertex v must be in the set $\{u_1, u_2, \ldots, u_n\}$. Since $N_H(v) = V(G)$ and all vertices in $V(G)$ are already dominated, a move v is a minimal move. If there is no move u_i picked in a game, then all picked moves are in $V(G)$. This implies that $u_1, u_2, \ldots, u_n$ are already dominated. Thus there is no newly dominated vertex from the set $\{u_1, \ldots, u_n\}$. Since the player can always make a minimal move v in a graph G and $N_H[v] = N_G[v] \sqcup \{u_1, u_2, \ldots, u_n\}$, a move v is also a minimal on a graph H.

Similarly, this proof can be applied for a maximal move. If there is a move u_i picked in a game, any vertex in G is a maximal move in H. If there is no move u_i picked in a game, a maximal move in G is also a maximal move in H. Hence, a maximal move in H always exists. □

Lemma 3.7. Let u and u' be maximal moves with respect to C. The following are equivalent.

1. Moves u and u' are the same moves with respect to C.
2. $M_C[u] = M_C[u']$.
3. $M_C[u] \cap M_C[u'] \neq \emptyset$.

Proof. Let u and u' be maximal moves with respect to C.

First, assume that the moves u and u' are the same moves, i.e., $N[u] \setminus C = N[u'] \setminus C$. Let $v \in M_C[u]$. Since u is a maximal move, for any move w dominating v,

$$N[w] \setminus C \subseteq N[u] \setminus C = N[u'] \setminus C.$$

Then $v \in M_C[u']$. So $M_C[u] \subseteq M_C[u']$. Similarly, $M_C[u] \supseteq M_C[u']$. Hence $M_C[u] = M_C[u']$.

Next, it is clear that $M_C[u] = M_C[u']$ implies $M_C[u] \cap M_C[u'] \neq \emptyset$ as u and u' are maximal moves.

Finally, suppose that $M_C[u] \cap M_C[u'] \neq \emptyset$. We then let $v \in M_C[u] \cap M_C[u']$. Then v is maximally dominated by u and u'. By the definition of maximal moves u and u', then we have

$$\begin{aligned} N[u] \setminus C &\subseteq N[u'] \setminus C, \\ N[u] \setminus C &\supseteq N[u'] \setminus C. \end{aligned}$$

So $N[u] \setminus C = N[u'] \setminus C$. This means that moves u and u' are the same moves with respect to C. □

Remark 3.8. We can say that maximal moves u and u' are the same moves with respect to C if and only if $M_C[u] = M_C[u']$. Moreover, for any distinct maximal moves u and u', $M_C[u] \cap M_C[u'] = \emptyset$.

Lemma 3.9. For any distinct maximal moves u and u' maximally dominating a vertex v and v', respectively, there is no move such that dominating v and v'.

Proof. Consider the game with the dominated vertex set C and maximal moves u and u' maximally dominating a vertex v and v', respectively. Assume that there is a move w which is not u nor u' and dominating v and v'. Then $v \in N[w] \setminus C$. Since u maximally dominates v, we have $N[w] \setminus C \subseteq N[u] \setminus C$. But w also dominates v', so $v' \in N[w] \setminus C$. Hence $v' \in N[u] \setminus C$. Since u' maximally dominates v', $N[u] \setminus C \subseteq N[u'] \setminus C$. Similarly, we have $N[u'] \setminus C \subseteq N[u] \setminus C$. This implies that maximal moves u and u' are the same moves, a contradiction. □

Proposition 3.10. *In a biased domination game on a graph G, if a move u is a maximal move that maximally dominates a vertex v, then all vertices in $N(v) \setminus N[u]$ are already dominated.*

Proof. Let C is the set of previous moves and $w \in N(v) \setminus N[u]$. Since u maximally dominates a vertex v,

$$\{w\} \setminus N[C] \subseteq N[w] \setminus N[C] \subseteq N[u] \setminus N[C].$$

Since $w \notin N[u]$,

$$\begin{aligned} \{w\} \setminus N[C] &= (\{w\} \setminus N[u]) \setminus N[C] \\ &\subseteq (N[w] \setminus N[u]) \setminus N[C] \\ &\subseteq (N[u] \setminus N[u]) \setminus N[C] = \emptyset. \end{aligned}$$

Then $\{w\} \subseteq N[C]$, i.e., $w \in N[C]$. This means that w is already dominated. □

Definition 3.11 ([1])**.** Let G and H be graphs. A *corona* $\tilde{G}$ with base G is the graph obtained from adding a leaf as a neighbor to each vertex of G. We say H is a *generalized corona* with base G if H is constructed from G by adding at least one leaf as neighbor to each vertex of G.

Proposition 3.12. *Let G be a graph. If a graph H is a generalized corona with base G, then players can always make a maximal move on H.*

Proof. Consider a game on a graph H that has been started but has not ended. Let $V = V(H) \setminus V(G)$, i.e., the set of leaves that are added to G to construct H. Note that if all vertices in V are dominated, then the game has already ended. Since a game on a graph H is not ended, there is an undominated leaf $v \in V$. Then its unique neighbor u is a maximal move on a graph H since u newly dominates v and the only moves that can dominate v are u and v. □

3.2 Biased Domination Games

From Theorem 2.6, we can extend it to biased domination games, using similar proof which is based on the *imagination strategy* [1]. The following result follows from the observation in [1].

Lemma 3.13. Let G be a graph and $D, S \subseteq \mathbb{N}$. Consider a Dominator-start (resp. Staller-start) (D, S)-biased game.

1. If Dominator has a strategy to make the game end within k moves when Staller plays optimally, then $\gamma_{(D,S)}(G) \leq k$ (resp. $\gamma'_{(D,S)}(G) \leq k$).
2. If Staller has a strategy to make the game end in at least k moves when Dominator plays optimally, then $\gamma_{(D,S)}(G) \geq k$ (resp. $\gamma'_{(D,S)}(G) \geq k$).

Proof. The result follows from the definition of biased game domination numbers. □

Theorem 3.14. *For any graph G,*

$$\gamma(G) \leq \gamma_{(D,S)}(G).$$

Proof. Since $\gamma(G)$ is the number of the smallest dominating set, we know that $\gamma(G)$ is the smallest number of moves that can end the biased domination game on a graph G. (If the number of moves is less than $\gamma(G)$, the game still does not end.) So $\gamma(G) \leq \gamma_{(D,S)}(G)$. □

Theorem 3.15 (Continuation Principle of the Biased Domination Game)**.** *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$ and $\gamma'_{(\delta,\sigma)}(G|A) \geq \gamma'_{(\delta,\sigma)}(G|B)$.*

Proof. Assume that $A \subseteq B \subseteq V(G)$. We will show that $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$ and $\gamma'_{(\delta,\sigma)}(G|A) \geq \gamma'_{(\delta,\sigma)}(G|B)$.

To show $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$, we let the real game be a biased domination game on $G|A$ where Dominator plays optimally, and let the imagined game be a biased domination game on $G|B$ imagined and optimally played by Staller. The number of moves in the real game and the imagined game when the games ended are denoted by R and I, respectively. Then $\gamma_{(\delta,\sigma)}(G|A) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G|B)$. Thus it is enough to show that $R \geq I$.

In each turn, when Dominator plays moves in the real game, Staller copies such moves to the imagined game. Staller then responds optimally in the imagined game and copies the moves back to the real game. Note that every Staller's move in the imagined game is always legal in the real game, but a Dominator's move in the real game is not necessary legal in the imagined game as $A \subseteq B$.

If all Dominator's moves in the real game are legal in the imagined game, then both games may be ended at the same time or there is some undominated vertices left in the real game. So $R \geq I$.

If there exists a move of Dominator in the real game that cannot be copied to the imagined game. This means that all vertices in the closed neighborhood of such move are already dominated in the imagined game. Staller then imagined that Dominator picks a random legal move in the imagined game and continues the game. The game continues until the imagined game ends or there is another move of Dominator in the real game which is not legal in the imagined game. In the later case, Staller then imagines another random legal move for Dominator in the imagined game.

We notice that at every turn in the game, the dominated vertices in the real game are also dominated in the imagined game. This means the real game cannot ends before the imagined game. Hence $R \geq I$. Thus, $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$.

The proof above always works whether it is Dominator or Staller who plays the first move. Thus the same proof can be directly applied to Game 2. □

Using the same proof, we also obtain the continuation principle for (D, S)-biased domination games.

Theorem 3.16. *The continuation principle also holds for a version of (D, S)-biased domination game.*

To prove the theorems in this section, we define a version of Dominator-pass games and Staller-pass games for biased domination games as follows.

Definition 3.17. In a (δ, σ)-game on a graph G, if Staller is allowed to pass some moves in each turn (except the first move of each turn) in total of at most n moves per game, then we define such game as an *n-Staller-pass-(δ, σ)-game* or *$sp(n)$-(δ, σ)-game*. The number of moves in an $sp(n)$-(δ, σ)-game when both players play optimally are denoted by $\gamma_{sp(n),(\delta,\sigma)}(G)$ in game 1 and $\gamma'_{sp(n),(\delta,\sigma)}(G)$ in game 2. Similar notation, $dp(n)$, is used for *n-Dominator-pass* games.

Remark 3.18. We note that a *turn* in a (δ, σ)-game is comprised of δ *moves* for Dominator and σ *moves* for Staller. For pass games, since the order of moves in a turn by the same player does not matter, we can assume that the player plays a certain number of consecutive moves and then skip the rest. In order to prevent an empty turn, we forbid skipping the first move as shown in Definition 3.17.

Note that Lemma 3.13 also holds for Staller-pass biased game and Dominator-pass biased game.

Theorem 3.19. *For any graph G and $i \geq 0$,*

$$\gamma_{sp(i),(\delta,\sigma)}(G) \leq \gamma_{sp(i+1),(\delta,\sigma)}(G).$$

Proof. We can consider $\gamma_{sp(i),(\delta,\sigma)}(G)$ as the number of moves that Dominator plays optimally and Staller skips at most i moves on an $sp(i+1)$-(δ, σ)-biased domination game. By Lemma 3.13, $\gamma_{sp(i),(\delta,\sigma)}(G) \leq \gamma_{sp(i+1),(\delta,\sigma)}(G)$. □

Theorem 3.20. *For any graph G and $i \geq 0$,*

$$\gamma_{dp(i+1),(\delta,\sigma)}(G) \leq \gamma_{dp(i),(\delta,\sigma)}(G).$$

Proof. We consider the biased game domination number $\gamma_{dp(i),(\delta,\sigma)}(G)$ as the number of moves that Staller plays optimally and Dominator skips at most i moves on an $dp(i+1)$-(δ, σ)-biased domination game. By Lemma 3.13, $\gamma_{dp(i+1),(\delta,\sigma)}(G) \leq \gamma_{dp(i),(\delta,\sigma)}(G)$. □

3.2.1 Biased Games and Minimal Moves

Theorem 3.21. *For any graph G, if Staller can always make a minimal move, then*

$$\gamma_{sp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Proof. First,we will show that $\gamma_{sp(1),(\delta,\sigma)}(G) \leq \gamma_{(\delta,\sigma)}(G)$ by using the imagination strategy in [1], we consider a situation where Staller is playing Game 1 of an $sp(1)$-(δ,σ)-biased domination game (Real Game: RG) with an optimal strategy while Dominator imagines and plays Game 1 of a (δ,σ)-biased domination game (Imagined Game: IG) optimally. Let the real game and the imagined game end in R and I moves, respectively. By Lemma 3.13, $R \geq \gamma_{sp(1),(\delta,\sigma)}(G)$ and $I \leq \gamma_{(\delta,\sigma)}(G)$. So it is enough to prove that $R \leq I$. In the real game, Staller has to play σ moves at each turn, except possibly at one turn where either $\sigma - 1$ moves (a pass of the last move) or σ moves is allowed. Whenever Staller plays σ moves at a turn, Dominator copies each move of Staller to the imagined game, responds optimally in the imagined game, and copies the moves back to the real game. If Staller does not skip a move until the game ends, the sequences of moves are formed as the following, where d_i^j and s_i^j denote the j-th move in the i-th turn of Dominator and Staller, respectively.

$$\text{RG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, d_2^1 \ldots, d_2^\delta, s_2^1, \ldots, s_2^\sigma, \ldots$$
$$\text{IG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, d_2^1 \ldots, d_2^\delta, s_2^1, \ldots, s_2^\sigma, \ldots$$

This means that both games are played with the same sequence of moves. Thus $R = I$.

If Staller decides to play only $\sigma - 1$ moves at turn k, Dominator copies each move of Staller up to such move to the imagined game. Then Dominator imagined that Staller makes the σ-th move s_k^* that dominates exactly one new vertex v_k. This is a minimal move, which is always available by the assumption. Now the sequences of moves are formed as the following.

$$\text{RG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^{\sigma-1}, \times$$
$$\text{IG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^{\sigma-1}, s_k^*$$

Dominator then responses optimally in the imagined game and copies the moves back to the real game.

The game continues with Staller playing exactly σ moves at each turn. Note that all moves by Dominator in the imagined game are always legal in the real game. On the other hand, a move by Staller in the real game may not be legal in the imagined game.

If there is an illegal move at turn $m > k$ in the imagined game, says s_m^θ for some $1 \leq \theta \leq \sigma$. First, we consider when the move is s_m^1. We have the following sequences of moves.

$$\begin{aligned} \text{RG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, \times, \ldots, d_m^\delta, s_m^1 \\ \text{IG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^*, \ldots, d_m^\delta \end{aligned}$$

Since s_m^1 is not legal in the imagined game, we have

$$\emptyset \neq N[s_m^1] \setminus N[C] \subseteq N[s_k^*] \setminus N[C] = \{v_k\}$$

where C is the set of all vertices played before s_m^1 in the real game, i.e.,

$$C = \left\{ \bigcup_{j=1}^{m-1} \left(\bigcup_{i=1}^{\delta} \{d_j^i\} \cup \bigcup_{i=1}^{\sigma} \{s_j^i\} \right) \cup \bigcup_{i=1}^{\delta} \{d_m^i\} \right\} \setminus \{s_k^\sigma\}.$$

Hence $N[s_m^1] \setminus N[C] = \{v_k\}$. That is s_m^1 dominates only one new vertex v_k. This means both games now have the same set of dominated vertices.

When Staller picks the rest of the moves in the turn, Dominator skips s_m^1 and copies these moves to the imagined game, and also imagines that Staller picks another minimal move s_m^* (newly dominating a vertex v_m) as the last move. Thus the sequences of moves in the both games are as follows.

$$\begin{aligned} \text{RG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, \times, \ldots, d_m^\delta, s_m^1, s_m^2, s_m^3, \ldots, s_m^{\sigma-1}, s_m^\sigma \\ \text{IG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^*, \ldots, d_m^\delta, s_m^2, s_m^3, s_m^4, \ldots, s_m^\sigma, s_m^* \end{aligned}$$

Note that $s_m^2, s_m^3, s_m^4, \ldots, s_m^\sigma$ are all legal in the imagined game since

$$N[s_m^1] \setminus N[C] = N[s_k^*] \setminus N[C] = \{v_k\}.$$

The same computation is applied when it is the move s_m^θ for $\theta > 1$ which is not legal in the imagined game. The vertex v_k is the only new vertex dominated by s_m^θ, and the set of all dominated vertices in both games are now the same. Dominator then imagines a minimal move s_m^*. So we get the following sequences of moves.

$$\begin{aligned} \text{RG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, \times, \ldots, d_m^\delta, \ldots, s_m^{\theta-1}, s_m^\theta, s_m^{\theta+1}, \ldots, s_m^{\sigma-1}, s_m^\sigma \\ \text{IG: } & d_1^1, \ldots, d_1^\delta, s_1^1, \ldots, s_1^\sigma, \ldots, s_k^1, \ldots, s_k^*, \ldots, d_m^\delta, \ldots, s_m^{\theta-1}, s_m^{\theta+1}, s_m^{\theta+2}, \ldots, s_m^\sigma, s_m^* \end{aligned}$$

We can always repeat the same procedure if there is a move in the real game that is not legal in the imagined game. At the end, both games end at the same move or the imagined game ends before the real game. In the first case, we have $R = I - 1$. In the second case, since v_m is the only vertex left undominated by not playing s_m^*, it is this unique vertex left undominated in the real game when the imagined game has ended. So $R = I$.

From both cases, we have $R \leq I$. Since $\gamma_{sp(1),(\delta,\sigma)}(G) \leq R$ and $I \leq \gamma_{(\delta,\sigma)}(G)$, we then have $\gamma_{sp(1),(\delta,\sigma)}(G) \leq \gamma_{(\delta,\sigma)}(G)$. By Theorem 3.19, $\gamma_{sp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G)$. □

Theorem 3.22. *For any graph G and $i \geq 0$, if Staller can always make a minimal move, then*

$$\gamma_{sp(i+1),(\delta,\sigma)}(G) = \gamma_{sp(i),(\delta,\sigma)}(G).$$

Proof. Let the real game be the $sp(i+1)$-(δ,σ)-game with an optimal strategy of Staller and the imagined game be the $sp(i)$-(δ,σ)-game imagined by Dominator and played with his optimal strategy. Dominator copies all the moves of Staller from the real game to the imagined game, up to the i-th time Staller skipped the move. At the $(i+1)$-th skip, Dominator imagines a random minimal move as in the proof of Theorem 3.21. The same analysis can then be directly applied. □

From Theorem 3.21 and Theorem 3.22, we immediately get the following corollary.

Corollary 3.23. *For any graph G and $i \geq 0$, if Staller can always make a minimal move, then*

$$\gamma_{sp(i),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Using Corollary 3.23, we can now compare the biased game domination numbers with different σ.

Theorem 3.24. *For any graph G, if Staller can always make a minimal move, then*

$$\gamma_{(\delta,j)}(G) \leq \gamma_{(\delta,\sigma)}(G)$$

for all $1 \leq j \leq \sigma$.

Proof. Let $j \leq \sigma$. We can consider a (δ, j)-biased game as a situation in a Staller-pass (δ,σ)-game where Staller passes $\sigma - j$ moves in every of his turn until the game ends. Let k be a number a lot larger than the possible total number of all passed moves by Staller in this game. Then $\gamma_{(\delta,j)}(G)$ is a number of moves in the $sp(k)$-(δ,σ)-game where Dominator plays optimally and Staller passes $\sigma - j$ moves for each turn until the game ends. This implies that $\gamma_{(\delta,j)}(G) \leq \gamma_{sp(k),(\delta,\sigma)}(G)$. By Corollary 3.23, we have $\gamma_{(\delta,j)}(G) \leq \gamma_{(\delta,\sigma)}(G)$. □

Theorem 3.25. *Let G be a graph and $D, S, S' \subseteq \mathbb{N}$. If Staller can always make a minimal move and* $\max(S) \leq \max(S')$*, then*

$$\gamma_{(D,S)}(G) \leq \gamma_{(D,S')}(G).$$

Proof. We can prove this theorem in the same manner as Theorem 3.24. However, we present another proof of this theorem by using some properties of minimal moves and the imagination strategy.

While Dominator is playing a (D, S)-game (real game) where Staller plays optimally, Dominator imagines a (D, S')-game (imagined game) where Staller always picks $\max(S')$ moves until the game ends. Dominator copies Staller's moves in the real game to the imagined game, imagines some additional Staller's moves, responds with optimal moves and copies them to the real game. For each turn of the real game, if Staller picks the number of moves less than $\max(S')$ moves, then Dominator imagines Staller picks extra minimal moves until the number of moves is $\max(S')$. Note that a Dominator's move in the imagined game is always legal in the real game but a Staller's move in the real game may be illegal in the imagined game. If there is an illegal move, then Dominator imagines Staller picks a minimal move instead of copying.

Let R and I be numbers of moves when real game and imagined game end, respectively. Since Staller plays optimally in the real game, then $R \geq \gamma_{(D,S)}(G)$. Similarly, $I \leq \gamma_{(D,S')}(G)$ since Dominator plays optimally in the imagined game. So we will show that $R \leq I$.

Let p be the total numbers of extra minimal moves in the imagined game when Staller picks less than $\max(S')$ moves in all turns of the real game, and let q be the number of minimal moves imagined instead of illegal copying to the imagined game. We have 3 cases for consideration.

1. If $p = 0$, then Staller always picks $\max(S')$ moves in the real game. This means that the imagined game is played exactly the same real game. So $R = I$.
2. If $p > 0$ and $q = 0$, then there has been $I - p$ moves in the real game when the imagined game ends. Since the extra moves in the imagined game are minimal moves, there are at most p undominated vertices in the real game. When the imagined game ends, the real game will end within at most p additional moves. This implies that $R \leq (I - p) + p = I$.
3. If $p > 0$ and $q > 0$, then there has been $I - p$ moves in the real game when the imagined game ends. There are $I - p - q$ moves that are in both games, p extra minimal moves in the imagined game, and q minimal moves in the

imagined game which are imagined instead of q moves in the real game. From only the $I-p-q$ moves, there are at most $p+q$ undominated vertices in the real game. The q moves dominate at least q addtional vertices in the real game, hence we are left with at most p vertices in the real game. So when the imagined game ends, the real game will end within at most p additional moves. This implies that $R \leq (I-p)+p = I$.

Hence $R \leq I$. Therefore $\gamma_{(D,S)}(G) \leq \gamma_{(D,S')}(G)$. □

Corollary 3.26. *Let G be a graph and $D, S, S' \subseteq \mathbb{N}$. If Staller can always make a minimal move and* $\max(S) = \max(S')$, *then*

$$\gamma_{(D,S)}(G) = \gamma_{(D,S')}(G).$$

3.2.2 Biased Games and Maximal Moves

Similar to the previous section, we want to compare biased game domination numbers on biased games with different δ. We first consider Dominator-pass games.

Theorem 3.27. *For any graph G, if Dominator can always make a maximal move (except possibly at the first move of the game), then*

$$\gamma_{dp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Proof. Let the real game (RG) be a $dp(1)$-(δ,σ)-game with an optimal strategy of Dominator and the (δ,σ)-game be a game imagined by Staller (IG) and played with an optimal strategy of Staller. The number of moves in the real game and the imagined game when the games end are denoted by R and I, respectively. Then $\gamma_{dp(1),(\delta,\sigma)}(G) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G)$. We claim that $R \geq I$.

If Dominator decides not to pass a move in the real game, then Staller copies all of Dominator's moves from the real game to the imagined game, responds optimally and copies the moves back to the real game. Since the two games are identical, we have $R = I$.

If Dominator decides to pass a move at turn k in the real game, Staller imagines that Dominator picks a maximal move d_k^* in the imagined game. So it dominates a vertex w_k such that all legal moves dominating w_k in the real game are illegal in the imagined game, see Equation (3.1). The sequences of moves are formed as the following:

$$\text{RG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, d_k^1, \ldots, d_k^{\delta-1}, \times$$
$$\text{IG: } d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, d_k^1, \ldots, d_k^{\delta-1}, d_k^*$$

The game continues by Staller playing optimally in the imagined game then copying the moves back to the real game. Note that all moves by Staller in the imagined game are always legal in the real game. However, a move by Dominator in the real game may not be legal in the imagined game.

If there is no illegal move until one of the games ends, we know that the vertex w_k still remains undominated in the real game. (All legal moves dominating w_k in the real game are illegal in the imagined game.) So the imagined game ends, while the real game has at least one vertex w_k remaining. Thus the real game needs at least one extra move, the vertex w_k itself, to finish the game. Including the move d_k^* imagined by Staller, we have $R \geq I$.

Whenever there is an illegal copying from the real game to the imagined game, Staller imagines that Dominator picks a new maximal move instead of such illegal move. Assume that in the last illegal copying, Staller imagined a maximal move which dominates w such that all legal moves moves dominating w in the real game are illegal in the imagined game.

We know when the imagined game ends, the real game must have at least one undominated vertex w. Thus the real game needs at least an extra move, the vertex w itself, to finish the game. Excluding the skip, we have $R \geq I$.

Hence $R \geq I$ in every case. Since $\gamma_{dp(1),(\delta,\sigma)}(G) \geq R$ and $I \geq \gamma_{(\delta,\sigma)}(G)$, we have $\gamma_{dp(1),(\delta,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$. By Theorem 3.20, $\gamma_{dp(1),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G)$. □

Theorem 3.28. *For any graph G and $i \geq 0$, if Dominator can always make a maximal move (except the first move of a game), then*

$$\gamma_{dp(i+1),(\delta,\sigma)}(G) = \gamma_{dp(i),(\delta,\sigma)}(G).$$

Proof. Let the $dp(i+1)$-(δ,σ)-game be the real game with an optimal strategy of Dominator and the $dp(i)$-(δ,σ)-game be the imagined game by Staller with an optimal strategy. Staller copies all the moves of Dominator from the real game to the imagined game, up to the i-th time Dominator skipped the move. The rest of the proof is similar to Theorem 3.28. □

Corollary 3.29. *For any graph G and $i \geq 0$, if Dominator can always make a maximal move (except the first move of a game), then*

$$\gamma_{dp(i),(\delta,\sigma)}(G) = \gamma_{(\delta,\sigma)}(G).$$

Theorem 3.30. *For any graph G, if Dominator can always make a maximal move (except the first move of a game), then*

$$\gamma_{(j,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$$

for all $1 \leq j \leq \delta$.

Proof. Let $j \leq \delta$. We can consider a (j, σ)-biased game as a situation in a Dominator-pass game where Dominator passes $\delta - j$ moves at every turn until the game ends. Let k be a number a lot larger than the possible total number of all passed moves by Dominator in this game. Then $\gamma_{(j,\sigma)}(G)$ is a number of moves in the $dp(k)$-(δ, σ)-game where Staller plays optimally and Dominator passes $\sigma - j$ moves for each turn until the game ends. This implies that $\gamma_{(j,\sigma)}(G) \geq \gamma_{dp(k),(\delta,\sigma)}(G)$. By Corollary 3.29, we have $\gamma_{(j,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$. □

Theorem 3.31. *Let G be a graph and $D, D', S \subseteq \mathbb{N}$. If Dominator can always make a maximal move and* $\max(D) \leq \max(D')$*, then*

$$\gamma_{(D,S)}(G) \geq \gamma_{(D',S)}(G).$$

Proof. While Staller is playing a (D, S)-game (real game) where Dominator plays optimally, Staller imagines a (D', S)-game (imagined game) where Dominator always picks $\max(D')$ moves for each turn until the game ends. Staller copies Dominator's moves in the real game to the imagined game, imagines some additional Dominator's moves, responds with optimal moves and copies them to the real game. If for each turn of the real game, Dominator picks the number of moves less than $\max(D')$, then Staller imagines Dominator picks maximal moves until the number of moves is $\max(D')$. Note that a Staller's move in the imagined game is always legal in the real game but a Dominator's move in the real game may be illegal in the imagined game. If there is an illegal move for copying, then Staller imagines Dominator picks a maximal move instead of copying.

Let R and I be numbers of moves when real game and imagined game end, respectively. Then $R \leq \gamma_{(D,S)}(G)$ and $I \geq \gamma_{(D',S)}(G)$. So we will show that $R \geq I$.

Let p be the total number of extra maximal moves in the imagined game when Dominator picks the number of moves less than $\max(D')$ over all turns of the real game and let q be the number of maximal moves imagined instead of illegal copying to the imagined game. We consider 3 cases as following.

1. If $p = 0$, then Dominator always picks $\max(D')$ moves in the real game. This means that the imagined game is played as same as the real game. So $R = I$.

2. If $p > 0$ and $q = 0$, then there has been $I - p$ moves in the real game when the imagined game ends. Since there are p distinct maximal moves played in the imagined game, then there are at least p undominated vertices (which are maximally dominated by those maximal moves in the imagined game) in the real game. By Lemma 3.7, these vertices are distinct, so we need at least p moves for dominating these undominated vertices in the real game. This means at least p moves are required for ending the real game after the imagined game ends. This implies that $R \geq (I - p) + p = I$.
3. If $p > 0$ and $q > 0$, then there has been $I - p$ moves in the real game when the imagined game ends. There are $I - p - q$ moves that are in both games, p extra maximal moves in the imagined game, and q maximal moves in the imagined game which are imagined instead of q moves in the real game. From only the $I - p - q$ moves, there are at least $p + q$ distinct undominated vertices (which are maximally dominated by those maximal moves in the imagined game) in the real game since there are $p + q$ distinct maximal moves played in the imagined game and by Lemma 3.7. From the $p + q$ distinct undominated vertices in the real game, the q moves dominate at most q additional vertices in the real game, and hence we are left with at least p undominated vertices in the real game. So when the imagined game ends, at least p moves are required for ending the real game. This implies that there are at least p undominated vertices remaining in the real game. This implies that $R \geq (I - p) + p = I$.

Hence $R \geq I$. Therefore $\gamma_{(D,S)}(G) \geq \gamma_{(D',S)}(G)$. $\square$

Corollary 3.32. *Let G be a graph and $D, D', S, S' \subseteq \mathbb{N}$. If Dominator can pick a maximal move for all turns (except the first move in Game 1), Staller can pick a minimal move for all turns (except the first move in Game 2),* $\max(D) \leq \max(D')$ *and* $\max(S) \leq \max(S')$*, then*

$$\gamma_{(D',S)}(G) \leq \gamma_{(D,S)}(G) \leq \gamma_{(D,S')}(G) \text{ and } \gamma_{(D',S)}(G) \leq \gamma_{(D',S')}(G) \leq \gamma_{(D,S')}(G).$$

From sections 3.2 and 3.3, we have results as follows.

Corollary 3.33. *Let G be a graph and $D, D', S, S' \subseteq \mathbb{N}$. If Dominator can pick a maximal move for all turns (except the first move in Game 1), Staller can pick a minimal move for all turns (except the first move in Game 2),* $\max(D) = \max(D')$ *and* $\max(S) = \max(S')$*, then*

$$\gamma_{(D',S)}(G) = \gamma_{(D,S)}(G) = \gamma_{(D',S')}(G) = \gamma_{(D,S')}(G) = \gamma_{(\max(D),\max(S))}(G).$$

Remark 3.34. All theorems in this thesis hold for Game 2 of a biased domination game.

Chapter 4

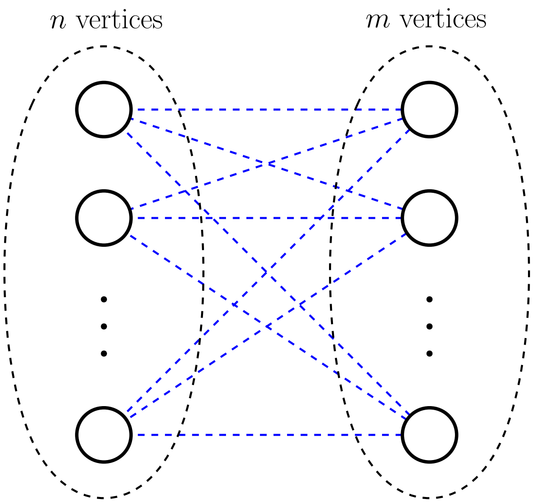
Examples of Special Moves

In this chapter, we give examples of graphs that a special move is always available and explicitly compute the biased game domination numbers on the powers of a cycle.

4.1 Examples of Minimal Moves

4.1.1 Edgeless Graph and Complete Bipartite Graph

It is obvious that the players can always make a minimal move on an edgeless graph.



If we have n isolated vertices that there is always a minimal move on the graph and we add the m vertices and construct edges as in Proposition 3.6, then we obtain a complete bipartite graph $K_{m,n}$. Moreover, the players can always make a minimal move on a complete bipartite graph $K_{m,n}$.

4.1.2 Powers of a Cycle and Powers of a Path

For positive integers p and n, the *p-th power* C_n^p *of a cycle* C_n has the following vertex set and edge set,

$$V(C_n^p) = \{0, 1, 2, \dots, n-1\},$$
$$E(C_n^p) = \{\{i, i \pm 1\}, \{i, i \pm 2\}, \dots, \{i, i \pm p\} : i \in V(C_n^p)\}$$

where the operations $+$ and $-$ are considered under modulo n.

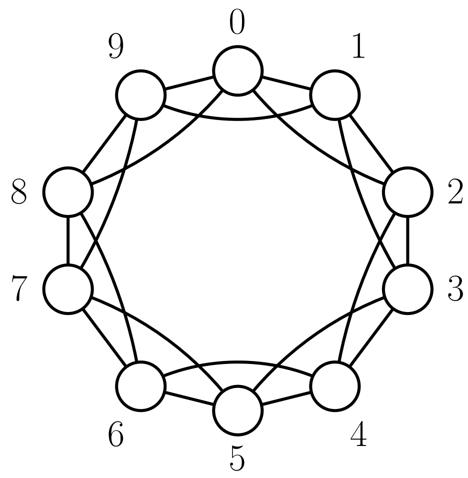


Figure 4.1 A power of a cycle C_{10}^2

We consider a moment during a game on a graph C_n^p that is already started but still not ended. WLOG, we can assume that there are dominated vertices $0, 1, 2, \dots, 2p$ and undominated vertex $2p+1$. Consider the move $p+1$. The move $p+1$ dominates vertices $1, 2, 3, \dots, 2p+1$ but vertices $1, 2, 3, \dots, 2p$ are already dominated. Then the move $p+1$ newly dominated only a vertex $2p+1$. So the move $p+1$ is a minimal move. Hence the players can always make a minimal move on a power of a cycle.

Similarly, we can apply this proof that a player can always make a minimal move on cycles, paths and powers of a path.

4.2 Examples of Maximal Moves

4.2.1 Edgeless Graph and Complete Bipartite Graph

It is obvious that the players can always make a maximal move (the move that maximally dominates itself) on an edgeless graph.

If we have n isolated vertices that there is always a maximal move on the graph and we add the m vertices and construct edges as in Proposition 3.6, then we obtain a complete bipartite graph $K_{m,n}$. Moreover, the players can always make a maximal move on a complete bipartite graph $K_{m,n}$.

4.2.2 Powers of a Cycle and Powers of a Path

We consider the started but not ended game on a graph C_n^p. WLOG, there are dominated vertices $0, 1, 2, \ldots, 2p$ and undominated vertex $2p+1$. We will show that the move $3p+1$ is a maximal move that maximally dominates the vertex $2p+1$. Notice that there are only vertices $p+1, p+2, \ldots, 3p+1$ that can newly dominate the vertex $2p+1$ and for $i = p+1, p+2, \ldots, 3p+1$, the move i newly dominates $2p+1, \ldots, i+p$. Then the set of newly dominated vertices by the move $i \neq 3p+1$ is a proper subset of the set of newly dominated vertices by the move $3p+1$. So the move $3p+1$ is a maximal move maximally that dominates the vertex $2p+1$. Hence the players can always make a maximal move on a power of a cycle.

Similarly, we can apply this proof that a player can always make a maximal move on cycles, paths and powers of a path.

4.2.3 Trees and Forests

We consider the started but not ended game on a tree. We consider a domination game on a partially dominated tree. Conside two cases as follows.

1. If there is an undominated leaf on a tree, its unique neighbor is a maximal move maximally dominating this leaf since the other move that dominates the leaf itself.
2. If there is no undominated leaf, we repeatedly delete all dominated leaves until there exists an undominated leaf whose unique neighbor is a maximal move as described in Case 1. Notice that a tree that all leaves are deleted is also a tree.

Hence there always is a maximal move on a tree. Similarly, we can apply this proof to show that a player can always make a maximal moves on forests.

4.2.4 Sunlet Graphs

The n-sunlet graph is a corona with a cycle C_n as the base. By Proposition 3.12, the players can always make a maximal move on the n-sunlet graph.

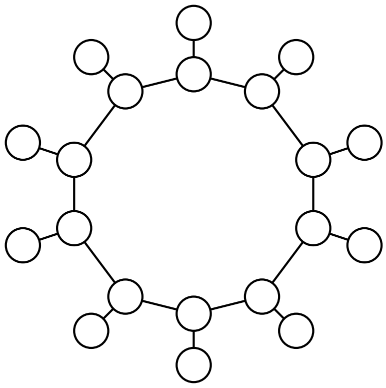


Figure 4.2 A 10-sunlet obtained by a cycle C_{10}

4.3 Biased Domination Numbers on Powers of Cycles

We will explicitly compute the biased game domination number of powers of cycles and give optimal strategies of both players on powers of cycles. The results are based on [5].

Theorem 4.1. *Let $p, n, \delta, \sigma \in \mathbb{N}$ and $G = C_n^p$. If $n = ((2p+1)\delta + \sigma)q + r$ where $q, r \in \mathbb{N} \cup \{0\}$ and $0 \leq r < (2p+1)\delta + \sigma$, then*

$$\gamma_{(\delta,\sigma)}(G) = (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta)$$

where $[x] = 1$ if the statement x is true and $[x] = 0$ if x is false.
Moreover, an optimal strategy for Staller is when (s)he always makes a move that dominates exactly one new vertex, while an optimal strategy for Dominator is when (s)he always makes a move that dominates as many new vertices as possible without creating a new dominated component (except the move that starts the game).

Proof. For the lower bound of $\gamma_{(\delta,\sigma)}(G)$, we consider the situation when Staller always picks a move that dominates exactly one new vertex. The lower bound of $\gamma_{(\delta,\sigma)}(G)$ will be obtained from Lemma 3.13. We know that on a graph C_n^p, a single move dominates at most $2p+1$ new vertices. Since a move by Staller always dominates exactly one new vertex, Dominator picks δ moves and Staller picks σ moves per round, we see that both players can dominate at most $(2p+1)\delta+\sigma$ new vertices per round. Thus the first $(\delta+\sigma)q$ moves (the first q rounds) dominate at most $((2p+1)\delta+\sigma)q$ vertices.

If $r = 0$, then $n = ((2p+1)\delta+\sigma)q$. Thus it requires at least $(\delta+\sigma)q$ moves to end the game.

If $0 < r \leq (2p+1)\delta$, there are at least r undominated vertices after q rounds. We need at least $\left\lceil \frac{r}{2p+1} \right\rceil$ additional moves. By Lemma 3.13, we have

$$\gamma_{(\delta,\sigma)}(G) \geq (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil$$

If $r > (2p+1)\delta$, then the first $(\delta+\sigma)q+\delta$ moves (the first q rounds and the $(q+1)$-th turn of Dominator) dominate at most $((2p+1)\delta+\sigma)q+(2p+1)\delta$ vertices. Thus there exist at least $r-(2p+1)\delta$ undominated vertices at the beginning of the $(q+1)$-th turn of Staller. Since Staller plays with the proposed strategy, it requires at least $r-(2p+1)\delta$ additional moves. So we need at least $(\delta+\sigma)q+\delta+r-(2p+1)\delta$ moves in total. By Lemma 3.13, we have

$$\gamma_{(\delta,\sigma)}(G) \geq (\delta+\sigma)q + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).$$

From three cases, we get

$$\gamma_{(\delta,\sigma)}(G) \geq (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).$$

For the upper bound, we consider the situation when Dominator always dominates as many new vertices as possible without creating a new dominated component (except the first move). Let M be the number of moves when the game is ended and let $k = \left\lfloor \frac{M}{\delta+\sigma} \right\rfloor$. We consider the following sequence of moves from the begining to the end of the game.

$$d_1^1, d_1^2, \ldots, d_1^\delta, s_1^1, s_1^2, \ldots, s_1^\sigma, \ldots, d_k^1, \ldots, s_k^\sigma(, d_{k+1}^1, \ldots, s_{k+1}^{\sigma-1}) \tag{4.1}$$

where d_i^j and s_i^j are the j-th move at the i-th round of Dominator and Staller, respectively. The moves in parenthesis indicate that the game may end at either the move s_k^σ (full k rounds), or $d_{k+1}^1, d_{k+1}^2, \ldots, d_{k+1}^\delta, s_{k+1}^1, \ldots, s_{k+1}^{\sigma-2}, s_{k+1}^{\sigma-1}$ (in the $(k+1)$-th round). Let $f(v)$ denote the number of new dominated vertices when a player choose a vertex v. By the current Dominator's strategy, it is clear that

$$f(d_1^1) = f(d_1^2) = \cdots = f(d_1^\delta) = 2p+1.$$

Now we consider three cases depending on the last move.

Case 1: The last move is s_k^σ. Consider new dominated vertices from start to end (the move d_1^1 to the move s_k^σ in (4.1)). Since the game ends in full k rounds, Dominator and Staller play exactly δk and σk moves, respectively, during the game. If Staller picks α moves that create new dominated components, then Staller can force Dominator to dominate less than $2p+1$ new vertices for $\beta \leq \alpha$

times. Thus Staller dominates $2p + 1$ new vertices for α moves and dominates at least one new vertex for the other $\sigma k - \alpha$ moves. Hence the number of vertices dominated by Staller is

$$\sum_{j=1}^{k} \sum_{i=1}^{\sigma} f(s_j^i) \geq \alpha(2p+1) + (\sigma k - \alpha). \tag{4.2}$$

On the other hand, Dominator dominates at least one new vertex for β moves and dominates $2p + 1$ vertices for the other $\delta k - \beta$ moves. Hence the number of vertices that newly dominated by Dominator is

$$\sum_{j=1}^{k} \sum_{i=1}^{\delta} f(d_j^i) \geq \beta + (\delta k - \beta)(2p+1). \tag{4.3}$$

From (4.2) and (4.3), we have

$$\begin{aligned} n &= \sum_{j=1}^{k} \left(\sum_{i=1}^{\delta} f(d_j^i) + \sum_{i=1}^{\sigma} f(s_j^i) \right) \\ &\geq \beta + (\delta k - \beta)(2p+1) + \alpha(2p+1) + (\sigma k - \alpha) \\ &= \delta k(2p+1) - \beta(2p) + \sigma k + \alpha(2p) \\ &= \delta k(2p+1) + \sigma k + (\alpha - \beta)(2p) \\ &\geq ((2p+1)\delta + \sigma)k. \end{aligned}$$

This implies that $k \leq \left\lfloor \dfrac{n}{(2p+1)\delta + \sigma} \right\rfloor$ as k is an integer. Thus

$$\begin{aligned} M &= (\delta + \sigma)k \\ &\leq (\delta + \sigma) \left\lfloor \frac{n}{(2p+1)\delta + \sigma} \right\rfloor \\ &\leq (\delta + \sigma) \left\lfloor \frac{n}{(2p+1)\delta + \sigma} \right\rfloor + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil \\ &\quad + [r > (2p+1)\delta](\delta + r - (2p+1)\delta) \\ &= (\delta + \sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil \\ &\quad + [r > (2p+1)\delta](\delta + r - (2p+1)\delta) \end{aligned}$$

by the assumption $n = (\delta + \sigma)q + r$ where $q = \left\lfloor \dfrac{n}{(2p+1) + \sigma} \right\rfloor$.

Case 2: The last move is d_{k+1}^{θ} where $1 \leq \theta \leq \delta$. Consider new dominated vertices from start to end (the move d_1^1 to the move d_{k+1}^{θ} in (4.1)). This means Dominator makes a total number of $\delta k + \theta$ moves while Staller makes a total number of σk moves. If Staller picks α moves that create new dominated components, then Staller can force Dominator to dominate less than $2p+1$ new vertices for $\beta \leq \alpha+1$ times. Thus Staller dominates $2p+1$ new vertices for α moves and dominates at least one new vertex for the other $\sigma k - \alpha$ moves. Hence the number of vertices that newly dominated by Staller is

$$\sum_{j=1}^{k}\sum_{i=1}^{\sigma} f(s_j^i) \geq \alpha(2p+1) + (\sigma k - \alpha). \tag{4.4}$$

Dominator dominates at least one new vertex for β moves and dominates $2p+1$ vertices for the other $\delta k + \theta - \beta$ moves. Hence the number of vertices that newly dominated by Dominator is

$$\sum_{j=1}^{k}\sum_{i=1}^{\delta} f(d_j^i) + \sum_{i=1}^{\theta} f(d_{k+1}^i) \geq \beta + (\delta k + \theta - \beta)(2p+1). \tag{4.5}$$

From (4.4) and (4.5), we have

$$\begin{aligned}
n &= \sum_{j=1}^{k}\left(\sum_{i=1}^{\delta} f(d_j^i) + \sum_{i=1}^{\sigma} f(s_j^i)\right) + \sum_{i=1}^{\theta} f(d_{k+1}^i) \\
&\geq \beta + (\delta k + \theta - \beta)(2p+1) + \alpha(2p+1) + (\sigma k - \alpha) \\
&= \delta k(2p+1) + \theta(2p+1) - \beta(2p) + \sigma k + \alpha(2p) \\
&= \delta k(2p+1) + \theta(2p+1) + \sigma k + (\alpha - \beta)(2p) \\
&\geq ((2p+1)\delta + \sigma)k + \theta(2p+1) - 2p \\
&= ((2p+1)\delta + \sigma)k + (\theta - 1)(2p+1) + 1.
\end{aligned}$$

This implies that

$$k \leq \left\lfloor \frac{n - ((\theta-1)(2p+1)+1)}{(2p+1)\delta + \sigma} \right\rfloor$$

since k is an integer. We then have

$$\begin{aligned}
M &= (\delta + \sigma)k + \theta \\
&\leq (\delta + \sigma)\left\lfloor \frac{n - ((\theta-1)(2p+1)+1)}{(2p+1)\delta + \sigma} \right\rfloor + \theta.
\end{aligned}$$

From the assumption, we have $n = ((2p+1)\delta+\sigma)q + r$ where $q = \left\lfloor \frac{n}{(2p+1)\delta+\sigma} \right\rfloor$. Also note that $(\theta-1)(2p+1)+1 \leq (2p+1)\delta + \sigma$ as $\theta \leq \delta$. Thus

$$\left\lfloor \frac{n - ((\theta-1)(2p+1)+1)}{(2p+1)\delta + \sigma} \right\rfloor = \begin{cases} q-1 & \text{if } r < (\theta-1)(2p+1)+1 \\ q & \text{if } r \geq (\theta-1)(2p+1)+1. \end{cases}$$

Hence,

$$M \leq \begin{cases} (\delta+\sigma)(q-1) + \theta & \text{if } r < (\theta-1)(2p+1)+1 \\ (\delta+\sigma)q + \theta & \text{if } r \geq (\theta-1)(2p+1)+1. \end{cases}$$

When $r < (\theta-1)(2p+1)+1$, we see that

$$\begin{aligned}
M &\le (\delta+\sigma)(q-1)+\theta \\
&= (\delta+\sigma)q-\delta-\sigma+\theta \\
&\le (\delta+\sigma)q-\delta-\sigma+\delta \\
&\le (\delta+\sigma)q \\
&\le (\delta+\sigma)q+[r \le (2p+1)\delta]\left\lceil \frac{r}{2p+1} \right\rceil \\
&\quad +[r > (2p+1)\delta](\delta+r-(2p+1)\delta).
\end{aligned}$$

The last inequality holds since $\left\lceil \frac{r}{2p+1} \right\rceil$ and $\delta+r-(2p+1)\delta$ are nonnegative. When $r \ge (\theta-1)(2p+1)+1$, we have $\frac{r}{2p+1} \ge \theta-1+\frac{1}{2p+1}$ and

$$\left\lceil \frac{r}{2p+1} \right\rceil \ge \theta-1+\left\lceil \frac{1}{2p+1} \right\rceil = \theta-1+1=\theta.$$

Hence

$$\begin{aligned}
M &\le (\delta+\sigma)q+\theta \\
&\le (\delta+\sigma)q+\left\lceil \frac{r}{2p+1} \right\rceil \\
&\le (\delta+\sigma)q+[r \le (2p+1)\delta]\left\lceil \frac{r}{2p+1} \right\rceil \\
&\quad +[r > (2p+1)\delta](\delta+r-(2p+1)\delta).
\end{aligned}$$

The last inequality holds since for $r > (2p+1)\delta$, we have

$$\left\lceil \frac{r}{2p+1} \right\rceil = \delta+\left\lceil \frac{r-(2p+1)\delta}{2p+1} \right\rceil \le \delta+r-(2p+1)\delta.$$

From two cases, we have

$$M \le (\delta+\sigma)q+[r \le (2p+1)\delta]\left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta+(r-(2p+1)\delta).$$

Case 3: The last move is s_{k+1}^{θ} where $1 \le \theta \le \sigma-1$. Consider new dominated vertices from start to end (the move d_1^1 to the move s_{k+1}^{θ} in (4.1)). If Staller picks α moves that create new dominated components, then Staller can force Dominator to dominate less than $2p+1$ new vertices for $\beta \le \alpha$ times. Thus Staller dominates $2p+1$ new vertices for α moves and dominates at least one new vertex for $\sigma k+\theta-\alpha$ moves. Hence the number of vertices that newly dominated by Staller is

$$\sum_{j=1}^{k}\sum_{i=1}^{\sigma} f(s_j^i)+\sum_{i=1}^{\theta} f(s_{k+1}^i) \ge \alpha(2p+1)+(\sigma k+\theta-\alpha). \tag{4.6}$$

Dominator dominates at least one new vertex for β moves and dominates $2p+1$ vertices for the other $\delta(k+1)-\beta$ moves. Hence the number of vertices that newly dominated by Dominator is

$$\sum_{j=1}^{k+1}\sum_{i=1}^{\delta} f(d_j^i) \geq \beta + (\delta(k+1)-\beta)(2p+1). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\begin{aligned}
n &= \sum_{j=1}^{k+1}\sum_{i=1}^{\delta} f(d_j^i) + \sum_{j=1}^{k}\sum_{i=1}^{\sigma} f(s_j^i) + \sum_{i=1}^{\theta} f(d_{k+1}^i) \\
&\geq \beta + (\delta(k+1)-\beta)(2p+1) + \alpha(2p+1) + (\sigma k + \theta - \alpha) \\
&= \delta(k+1)(2p+1) - \beta(2p) + \alpha(2p) + \sigma k + \theta \\
&= ((2p+1)\delta + \sigma)k + \delta(2p+1) + \theta + (\alpha - \beta)(2p) \\
&\geq ((2p+1)\delta + \sigma)k + (2p+1)\delta + \theta.
\end{aligned}$$

This implies that

$$k \leq \left\lfloor \frac{n - ((2p+1)\delta + \theta)}{(2p+1)\delta + \sigma} \right\rfloor$$

since k is an integer. We then have

$$\begin{aligned}
M &= (\delta + \sigma)k + \delta + \theta \\
&\leq (\delta + \sigma)\left\lfloor \frac{n - ((2p+1)\delta + \theta)}{(2p+1)\delta + \sigma} \right\rfloor + \delta + \theta
\end{aligned}$$

By the assumption, we have $n = ((2p+1)\delta + \sigma) + r$ where $q = \left\lfloor \frac{n}{(2p+1)\delta + \sigma} \right\rfloor$. Also note that $(2p+1)\delta + \theta \leq (2p+1)\delta + \sigma$ as $\theta < \sigma$. Thus

$$\left\lfloor \frac{n - ((2p+1)\delta + \theta)}{(2p+1)\delta + \sigma} \right\rfloor = \begin{cases} q-1 & \text{if } r < (2p+1)\delta + \theta \\ q & \text{if } r \geq (2p+1)\delta + \theta. \end{cases}$$

Hence,

$$M \leq \begin{cases} (\delta + \sigma)(q-1) + \delta + \theta & \text{if } r < (2p+1)\delta + \theta \\ (\delta + \sigma)q + \delta + \theta & \text{if } r \geq (2p+1)\delta + \theta. \end{cases}$$

When $r < (2p+1)\delta + \theta$, we see that

$$\begin{aligned}
M &\leq (\delta + \sigma)(q-1) + \delta + \theta \\
&= (\delta + \sigma)q - \delta - \sigma + \delta + \theta \\
&\leq (\delta + \sigma)q - \delta - \sigma + \delta + \sigma \\
&= (\delta + \sigma)q \\
&\leq (\delta + \sigma)q + [r \leq (2p+1)\delta]\left\lceil \frac{r}{2p+1} \right\rceil \\
&\quad + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).
\end{aligned}$$

The last inequality holds since $\left\lceil \frac{r}{2p+1} \right\rceil$ and $\delta + r - (2p+1)\delta$ are nonnegative. When $r \geq (2p+1)\delta + \theta$, we have $r > (2p+1)\delta$ and $\theta \leq r - (2p+1)\delta$ as $1 \leq \theta < \sigma$. Then

$$\begin{aligned}
M &\leq (\delta+\sigma)q + \delta + \theta \\
&\leq (\delta+\sigma)q + \delta + r - (2p+1)\delta \\
&= (\delta+\sigma)q + [r > (2p+1)\delta](\delta + r - (2p+1)\delta) \\
&\leq (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil \\
&\quad + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).
\end{aligned}$$

From two cases, we have

$$M \leq (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).$$

By Lemma 3.13, we have $\gamma_{(\delta,\sigma)}(G) \leq M$ and then

$$\gamma_{(\delta,\sigma)}(G) \leq (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta).$$

Since the lower bound and the upper bound match, we can conclude that

$$\gamma_{(\delta,\sigma)}(G) = (\delta+\sigma)q + [r \leq (2p+1)\delta] \left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta)$$

as desired. □

Remark 4.2. The optimal strategies for Dominator and Staller in Theorem 4.1 are indeed to always play a maximal move and a minimal move, respectively. However, the property of the maximal move is not directly used in the proof. So it is interesting to see whether we can interpret the proof in a new angle by using the property of the maximal move.

Corollary 4.3. *For $n \in \mathbb{N}$,*

$$\gamma_{(\delta,\sigma)}(C^p_{n+\delta(2p+1)+1}) = \gamma'_{(\delta,\sigma)}(C^p_{n+2p+1}) + \delta.$$

Moreover, optimal strategies are

1. *Staller always make a move that dominates exactly one new vertex (except the move that starts the game),*
2. *Dominator always make a move that dominates as many new vertices as possible without creating a new dominated component (except the move that starts the game).*

Proof. Let $A = \{d_1^1, d_1^2, \ldots, d_1^\delta\}$ be a set of the first δ Dominator's moves and s_1^1 be the first Staller's move of Game 1 on a graph $C^p_{n+\delta(2p+1)+1}$ using the optimal strategies obtained from Theorem 4.1. On the other hand, we let u be any first move of Game 2 on a graph C^p_{n+2p+1}. We see that both games are now equivalent. Hence

$$\gamma_{(\delta,\sigma)}(C^p_{n+\delta(2p+1)+1}) = \delta + \gamma'_{(\delta,\sigma)}(C^p_{n+2p+1}).$$

It is also clear that the same set of optimal strategies can be applied to Game 2 on C^p_{n+2p+1}. □

Corollary 4.4. *For $n \in \mathbb{N}$,*

$$\gamma'_{(\delta,\sigma)}(C^p_{n+2p+\sigma}) = \gamma_{(\delta,\sigma)}(C^p_n) + \sigma.$$

Proof. The proof is similar to Corollary 4.3 by comparing Game 2 on a graph $C^p_{n+2p+\sigma}$ at the move right after the first turn of Staller following the first move of Dominator (with optimal strategies of both players) and Game 1 on a graph C^p_n at the move right after any first move of Dominator. □

Corollary 4.5. *Let $p, n \in \mathbb{N}$, $D, S \subseteq \mathbb{N}$, $\max D = \delta$, $\max S = \sigma$ and $G = C^p_n$. If $n = ((2p+1)\delta + \sigma)q + r$ where $q, r \in \mathbb{N} \cup \{0\}$ and $0 \leq r < (2p+1)\delta + \sigma$, then*

$$\gamma_{(D,S)}(G) = (\delta+\sigma)q + [r \leq (2p+1)\delta]\left\lceil \frac{r}{2p+1} \right\rceil + [r > (2p+1)\delta](\delta + r - (2p+1)\delta)$$

where $[x] = 1$ if the statement x is true and $[x] = 0$ if x is false.
Moreover, an optimal strategy for Staller is when he always makes σ moves that dominates exactly one new vertex for each turn, while an optimal strategy for Dominator is when he always makes δ moves that dominates as many new vertices as possible without creating a new dominated component (except the move that starts the game) for each turn.

Chapter 5

Conclusions

In this thesis, we have introduced a biased version of domination games. Under the condition that a minimal move (resp. a maximal move) is always available, we can compare the biased game domination number of two biased games with different number of moves for Staller (resp. Dominator) in each turn. The property that a graph always has a minimal move or a maximal move available is still rather strong. So it is interesting to know what other collections of graphs have this property, or what other conditions give the same results as in Theorem 3.24 and Theorem 3.30. We found the explicit formula of biased game domination numbers of powers of a cycle.

Moreover, in the case of powers of a cycle, we found very simple optimal strategies for both players using the special moves. It is interesting to see whether such special moves can be also used in other graphs, or whether there are other optimal strategies for the game on powers of a cycle.

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