Magnifying Elements in the Generalized Semigroups of Transformations Preserving an Equivalence Relation

Thananya Kaewnoi

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ชื่อวิทยานิพนธ์ สมาชิกขยายในกึ่งกรุปการแปลงวางนัยทั่วไปคงสภาพความสัมพันธ์สมมูล

ผู้เขียน นางสาวธนัญญา แก้วน้อย

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บทคัดย่อ

สมาชิก a ของกึ่งกรุป S ถูกเรียกว่าสมาชิกขยายซ้าย (ขวา) ถ้ามีสับเซตแท้ M ของ S ที่ทำให้ $aM = S$ $(Ma = S)$ กำหนดให้ $T(X)$ และ $P(X)$ แทนกึ่งกรุปการแปลงเต็มและกึ่งกรุปการแปลงบางส่วนบนเซต X ตามลำดับ สำหรับความสัมพันธ์สมมูล E และผลแบ่งกั้น $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ บนเซต X กำหนด

$$T_E(X) = \{\alpha \in T(X) \mid \text{ถ้า } (x,y) \in E \text{ แล้ว } (x\alpha, y\alpha) \in E\},$$

$$P_E(X) = \{\alpha \in P(X) \mid \text{ถ้า } (x,y) \in E \text{ แล้ว } (x\alpha, y\alpha) \in E\},$$

$$T(X,\mathcal{P}) = \{\alpha \in T(X) \mid X_i\alpha \subseteq X_i \text{ สำหรับทุก } i \in \Lambda\}$$

และ

$$P(X,\mathcal{P}) = \{\alpha \in P(X) \mid X_i\alpha \subseteq X_i \text{ สำหรับทุก } i \in \Lambda\}$$

ได้ว่า $T_E(X)$, $P_E(X)$, $T(X,\mathcal{P})$ และ $P(X,\mathcal{P})$ เป็นกึ่งกรุปภายใต้การประกอบของฟังก์ชัน

วัตถุประสงค์หลักของวิทยานิพนธ์นี้ ให้สมบัติของสมาชิกขยายในกึ่งกรุป $T_E(X)$, $P_E(X)$, $T_E(X,\mathcal{P}) = T_E(X) \cap T(X,\mathcal{P})$ และ $P_E(X,\mathcal{P}) = P_E(X) \cap P(X,\mathcal{P})$ นอกจากนี้ จะให้เงื่อนไขจำเป็นและเพียงพอในการเป็นสมาชิกขยายซ้ายและสมาชิกขยายขวาสำหรับสมาชิกในกึ่งกรุปดังกล่าวข้างต้น

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ABSTRACT

An element a of a semigroup S is called a left (right) magnifying element if there exists a proper subset M of S such that $aM = S$ $(Ma = S)$. Let $T(X)$ and $P(X)$ denote the semigroup of the full and partial transformations on a nonempty set X, respectively. For an equivalence relation E and a partition $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ on the set X, let

$$T_E(X) = \{\alpha \in T(X) \mid (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\},$$

$$P_E(X) = \{\alpha \in P(X) \mid (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\},$$

$$T(X, \mathcal{P}) = \{\alpha \in T(X) \mid X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\},$$

and

$$P(X, \mathcal{P}) = \{\alpha \in P(X) \mid X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\}$$

Then $T_E(X)$, $P_E(X)$, $T(X, \mathcal{P})$ and $P(X, \mathcal{P})$ are semigroups under the composition of functions, as well.

The main purpose of this thesis is to provide the properties of magnifying elements in the semigroups $T_E(X)$, $P_E(X)$, $T_E(X, \mathcal{P}) = T_E(X) \cap T(X, \mathcal{P})$ and $P_E(X, \mathcal{P}) = P_E(X) \cap P(X, \mathcal{P})$. Futhermore, the necessary and sufficient conditions for elements in these semigroups to be a left or right magnifying element are established.

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Thananya Kaewnoi

Contents

Chapter 1

Introduction and Preliminaries

A *semigroup* is a system $(S, *)$ consisting of a nonempty set S together with the binary associative operation $*$, i.e., $x*y$ belongs to S and $(x*y)*z = x*(y*z)$ for all elements x, y, z in S. For example, $(\mathbb{N}, +)$, $(\mathbb{N}, \times)$, $(\mathbb{R}, +)$ and $(\mathbb{R}, \times)$ are semigroups. Nevertheless, $(\mathbb{N}, -)$ is not a semigroup since $2 - 3 = -1 \notin \mathbb{N}$.

For convenience, we write S instead of $(S, *)$ and let xy stand for $x * y$ for any elements x, y in S. For a semigroup S, a subset T of a semigroup S is called a *subsemigroup* of S if T is a semigroup under the operation of S. The intersection of any set of subsemigroups of S is either an emptyset or a subsemigroup of S.

An element a of a semigroup S is called a *left (right) magnifying element* if $aM = S$ $(Ma = S)$ for some proper subset M of S. If such a proper subset M related to a left (right) magnifying element a is a subsemigroup of S, then a is called a strong left (right) magnifying element. The notion of magnifying elements in the semigroup was originally proposed by Ljapin [9] in 1963. In 1969, Tolo [12] showed that a regular semigroup S containing a left identity element is factorizable, i.e., it can be written as the set product of proper subsemigroups A and B of S if a left magnifying element exists in S. In 1992, the neccesary and sufficient conditions of the existence of magnifying elements in any semigroups were established by Catino and Migliorini [1]; moreover, they improved the results of Tolo [12] by showing the existence of a left magnifying element in a regular semigroup with a left or right identity.

Let $T(X)$ denote the set of all functions on a nonempty set X, i.e.,

$$T(X) = \{\alpha \mid \alpha \text{ is a function from } X \text{ to itself}\}.$$

The set $T(X)$ is a semigroup under the composition of functions which is called the *full transformation semigroup on* X. In 1996, the necessary and sufficient conditions for elements in any subsemigroup of $T(X)$ with identity to be left or right magnifying elements were established by K.D. Magill, Jr. [11]. Furthermore, he applied the result in some specific transformation semigroups, e.g., the semigroup of all linear transformations of a vector space, the semigroup of all continuous selfmaps of the topological space. Later, Gutan [5] constructed the semigroup S containing both left and strong left magnifying elements. This result answers the question that was queried by K.D. Magill, Jr. in [11]. A year later, Gutan [6] showed that every semigroup containing magnifying elements is factorizable. Besides, Gutan established the definitions of good, very good, and bad magnifying elements by using the notion of magnifying elements in a semigroup which has been introduced by Ljapin [9]. Gutan provided the method for obtaining semigroups having good left magnifying elements such that none of those is a very good magnifying element in [7]; moreover, Gutan and Kisielewicz constructed a semigroup having both good and bad magnifying elements in [8]. Recently, the study of magnifying elements in transformation semigroup have been developing by many authors. The necessary and sufficient conditions for elements in the generalizations of $T(X)$ to be magnifying elements are determined, e.g., Chinram, Petchkaew and Buapradist [4] worked on $T(X)$ which is determined by a partition of the set X. Chinram and Buapradist focused on the set of all functions in $T(X)$ such that the range of restricted function to Y, where Y is a fixed nonempty subset of X, and the set of all functions in $T(X)$ whose range is a nonempty subset Y of X in [3] and [2], respectively.

Let $P(X)$ denote the set of all functions from all subsets of X to X, i.e.,

$$P(X) = \{\alpha : A \longrightarrow X \mid A \subseteq X \text{ and } \alpha \text{ is a function}\}.$$

The semigroup $P(X)$ is a generalization of $T(X)$ which is called the *partial transformation semigroup on* X. Recently, Luangchaisri, Changphas and Phanlert [10] extended the Magill's results [11] to $P(X)$.

Let E be an equivalence relation and $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition of a nonempty set X. Next, we introduce a generalization of $P(X)$ which is defined as follows:

The *full transformation semigroup on* X *preserving a partition* $\mathcal{P}$, denoted by $T(X, \mathcal{P})$, is defined as

$$T(X, \mathcal{P}) = \{\alpha \in T(X) \mid X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\}.$$

Note that if $\mathcal{P} = \{X\}$, then $T(X, \mathcal{P}) = T(X)$ and if $\mathcal{P} = \{\{x\} \mid x \in X\}$, then $T(X, \mathcal{P})$ is a singleton set containing the identity map on X.

In 2018, Chinram, Petchkaew, and Buapradist [4] investigated magnifying elements in $T(X,\mathcal{P})$.

Theorem 1.0.1. [4] *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition of a set X. A semigroup $T(X,\mathcal{P})$ has left and right magnifying elements if and only if X_i is infinite for some $i \in \Lambda$.*

Theorem 1.0.2. [4] *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition of a set X and X_i is infinite for some $i \in \Lambda$. A function α is a left magnifying element of $T(X,\mathcal{P})$ if and only if α is one-to-one but not onto.*

Theorem 1.0.3. [4] *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition of a set X and X_i is infinite for some $i \in \Lambda$. A function α is a right magnifying element of $T(X,\mathcal{P})$ if and only if α is onto but not one-to-one.*

The *partial transformations semigroup on X preserving a partition $\mathcal{P}$*, denoted by $P(X,\mathcal{P})$, is defined as

$$P(X,\mathcal{P}) = \{\alpha \in P(X) \mid X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\}.$$

Note that if $\mathcal{P} = \{X\}$, then $P(X,\mathcal{P}) = P(X)$ and if $\mathcal{P} = \{\{x\} \mid x \in X\}$, then $T(X,\mathcal{P})$ is the set of all restrictions of the identity function on a set X to a subset A of X.

The *full transformation semigroup on X preserving an equivalence relation E*, denoted by $T_E(X)$, is defined as

$$T_E(X) = \{\alpha \in T(X) \mid (x,y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

The *partial transformation semigroup on X preserving an equivalence relation E*, denoted by $P_E(X)$, is defined as

$$P_E(X) = \{\alpha \in P(X) \mid (x,y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}.$$

Clearly, $T_E(X)$ and $P_E(X)$ contain the identity map on X are subsemigroups of $T(X)$ and $P(X)$, respectively. Furthermore, if the equivalence relation E is trivial, i.e., $E = X \times X$ or E is the identity relation, then $T_E(X) = T(X)$ and $P_E(X) = P(X)$.

Many authors have extensively studied the transformation semigroups that preserve an equivalence relation in many aspects, e.g., regularity, Green's equivalences, natural partial orders. However, no one has yet studied the magnifying elements in this semigroup which is determined by a partition of a nonempty set X. Consequently, we will study magnifying elements in $T(X)$ and $P(X)$ that preserves both an equivalence relation and a partition on a nonempty set X.

We denote the set of the *full and partial transformations on X preserving both an equivalence relation E and a partition $\mathcal{P}$* by $T_E(X,\mathcal{P})$ and $P_E(X,\mathcal{P})$, respectively. Note that

$$T_E(X,\mathcal{P}) = T_E(X) \cap T(X,\mathcal{P})$$

and

$$P_E(X,\mathcal{P}) = P_E(X) \cap P(X,\mathcal{P}).$$

Then $T_E(X,\mathcal{P})$ and $P_E(X,\mathcal{P})$ are semigroups under the composition of functions since the identity map id_X on X belongs to $T_E(X)\cap T(X,\mathcal{P})$ and $P_E(X)\cap P(X,\mathcal{P})$.

Note that if the equivalence relation E is trivial, i.e., $E = X \times X$ or E is an identity relation, then $T_E(X,\mathcal{P}) = T(X,\mathcal{P})$ and $P_E(X,\mathcal{P}) = P(X,\mathcal{P})$. If $\mathcal{P} = \{X\}$, then $T_E(X,\mathcal{P}) = T_E(X)$ and $P_E(X,\mathcal{P}) = P_E(X)$. Moreover, $T_E(X,\mathcal{P}) = T(X)$ and $P_E(X,\mathcal{P}) = P(X)$ if the equivalence relation E is trivial and the partition $\mathcal{P} = \{X\}$. If all elements in $\mathcal{P}$ are singleton sets, then $T(X,\mathcal{P})$ is a singleton set containing the identity map on X and $P_E(X,\mathcal{P})$ is the set of all restrictions of the identity function on a set X to a subset A of X.

The main purpose of this thesis is to provide the properties of magnifying elements in the following transformation semigroups:

1. $T_E(X)$,
2. $P_E(X)$,
3. $T_E(X,\mathcal{P})$,
4. $P_E(X,\mathcal{P})$.

Futhermore, we justify the existence of magnifying elements in these semigroups. In particular, the necessary and sufficient conditions for elements to be a left or right magnifying element are established.

Throughout the rest of this chapter, we denote a partition on a nonempty set X by $\mathcal{P} = \{X_i \mid i \in \Lambda\}$, which is a collection of nonempty subsets of X satisfying $X = \bigcup_{i\in\Lambda} X_i$ with $X_i \cap X_j = \emptyset$ for all $i, j \in \Lambda$ such that $i \neq j$. Let $X = \{x_j \mid j \in \Lambda'\}$ and E be an equivalence relation on a nonempty set X. The equivalence class of $x \in X$ determined by E is denoted by $[x]_E = \{y \in X \mid (x, y) \in E\}$. Let $X/E = \{[x]_E \mid x \in X\}$ and $(X_i, x_j) = X_i \cap [x_j]_E$ for $i \in \Lambda$ and $j \in \Lambda'$. For any functions α, β and $x \in X$, the notations $x\alpha$ and $x\alpha\beta$ are used instead of $\alpha(x)$ and $(\beta \circ \alpha)(x)$, respectively. The image of α is denoted by ran α.

Chapter 2

Magnifying elements in $T_E(X)$ and $P_E(X)$

In this chapter, we provide the necessary and sufficient conditions for elements in the full and partial transformation semigroups preserving an equivalence relation to be a left or right magnifying element.

2.1 Magnifying elements in $T_E(X)$

Recall that $T_E(X) = \{\alpha \in T(X) \mid (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}$, where E is an equivalence relation on X, is a semigroup under the composition of functions. A function $\alpha \in T_E(X)$ is called a left (right) magnifying element if there exists a proper subset M of $T_E(X)$ such that $\alpha M = T_E(X)$ $(M\alpha = T_E(X))$.

2.1.1 Left magnifying elements in $T_E(X)$

In this subsection, we provide the necessary and sufficient conditions for elements in $T_E(X)$ to be a left magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 2.1.1. If α is a left magnifying element in $T_E(X)$, then α is one-to-one.

Proof. Assume that α is a left magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $\alpha M = T_E(X)$. Since the identity map id_X on X belongs to $T_E(X)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. This implies that α is one-to-one. □

However, the converse of Lemma 2.1.1 is not true since there exists no proper subset M of $T_E(X)$ such that $id_X M = T_E(X)$.

Lemma 2.1.2. Let α be a left magnifying element in $T_E(X)$. For any $x, y \in X$, $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.

Proof. The necessity is obvious. Conversely, let α be a left magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $\alpha M = T_E(X)$. Since $id_X \in T_E(X)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. It follows that $x = xid_X = x\alpha\beta$ and $y = yid_X = y\alpha\beta$. Therefore, $(x, y) = (x\alpha\beta, y\alpha\beta) \in E$ since $\beta \in T_E(X)$. □

Lemma 2.1.3. If α is a left magnifying element in $T_E(X)$, then $\alpha M = \alpha T_E(X)$ for some proper subset M of $T_E(X)$.

Proof. Assume that α is a left magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $\alpha M = T_E(X)$. Clearly, $\alpha M \subseteq \alpha T_E(X)$ and $\alpha T_E(X) \subseteq T_E(X) = \alpha M$. This shows that $\alpha M = \alpha T_E(X)$. □

Lemma 2.1.4. If $\alpha \in T_E(X)$ is bijective, then α is not a left magnifying element.

Proof. Assume that $\alpha \in T_E(X)$ is bijective. So α^{-1} is also bijective. Suppose to the contrary that α is a left magnifying element. By definition, there exists a proper subset M of $T_E(X)$ such that $\alpha M = T_E(X)$. By Lemma 2.1.3, $\alpha M = \alpha T_E(X)$. Hence $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha T_E(X) = T_E(X)$, which is a contradiction. Therefore, α is not a left magnifying element. □

The next corollary follows by Lemma 2.1.1, Lemma 2.1.2 and Lemma 2.1.4.

Corollary 2.1.5. *If α is a left magnifying element in $T_E(X)$, then α is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Lemma 2.1.6. If $\alpha \in T_E(X)$ is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$, then α is a left magnifying element.

Proof. Assume that α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. For each $x \in \text{ran}\,\alpha$, there exists $y_x \in X$ such that $y_x\alpha = x$.
Case 1: $|[x]_E| = 1$ for all $x \in X$.
Let $x_0 \in X$ and $M = \{\beta \in T_E(X) \mid x\beta = x_0 \text{ for all } x \notin \text{ran}\,\alpha\}$. Clearly, M is a proper subset of $T_E(X)$ since the constant map $\beta' \in T_E(X)$ (for all $x \in X$, $x\beta' = x'$

where $x' \in X$ such that $x' \neq x_0$) does not belong to M. To show that $\alpha M = T_E(X)$, let $\gamma \in T_E(X)$ and define $\beta \in T_E(X)$ for all $x \in X$ by

$$x\beta = \begin{cases} y_x\gamma & \text{if } x \in \operatorname{ran}\alpha, \\ x_0 & \text{if } x \notin \operatorname{ran}\alpha. \end{cases}$$

Clearly, $\beta \in M$. Since α is one-to-one and $y_{x\alpha}\alpha = x\alpha$, $y_{x\alpha} = x$ and hence for each $x \in X$, $x\alpha\beta = y_{x\alpha}\gamma = x\gamma$. This shows that $\alpha\beta = \gamma$, which implies $\alpha M = T_E(X)$. Therefore, α is a left magnifying element in $T_E(X)$.

Case 2: $|[x]_E| > 1$ for some $x \in X$.

For each $x \in X$, choose $a_x \in [x]_E$ (if $(x, y) \in E$, we must choose $a_x = a_y$). Let $I = \{a_x \mid x \in X\}$. Clearly, $I \neq X$ since $|[x]_E| > 1$ for some $x \in X$. Let $M = \{\beta \in T_E(X) \mid x\beta \in I \text{ for all } x \notin \operatorname{ran}\alpha\}$. Clearly, M is a proper subset of $T_E(X)$. To show that $\alpha M = T_E(X)$, let $\gamma \in T_E(X)$ and define $\beta \in T_E(X)$ for all $x \in X$ by

$$x\beta = \begin{cases} y_x\gamma & \text{if } x \in \operatorname{ran}\alpha, \\ a_{x'\gamma} & \text{if } x \notin \operatorname{ran}\alpha \text{ and } \exists x' \in X \text{ such that } (x, x'\alpha) \in E, \\ a_x & \text{otherwise.} \end{cases}$$

To show that $\beta \in M$, let $(a, b) \in E$. Then $a, b \in [x]_E$ for some $x \in X$.

Case I: $a, b \in \operatorname{ran}\alpha$.

Then there exist $y_a, y_b \in X$ such that $y_a\alpha = a$ and $y_b\alpha = b$. By assumption, we have $(y_a, y_b) \in E$. Therefore, $(a\beta, b\beta) = (y_a\gamma, y_b\gamma) \in E$ since $\gamma \in T_E(X)$.

Case II: $a, b \notin \operatorname{ran}\alpha$.

Case i: $\operatorname{ran}\alpha \cap [x]_E \neq \emptyset$.

Then we can choose $c \in \operatorname{ran}\alpha \cap [x]_E$ and hence there exists $y_c \in X$ such that $y_c\alpha = c$. Since $(a, b) \in [x]_E$, we have $(a, y_c\alpha), (b, y_c\alpha) \in E$. Thus $(a\beta, b\beta) = (a_{y_c\gamma}, a_{y_c\gamma}) \in E$.

Case ii: $\operatorname{ran}\alpha \cap [x]_E = \emptyset$.

Then there exists no $c \in X$ such that $(a, c\alpha), (b, c\alpha) \in E$. Therefore, $(a\beta, b\beta) = (a_x, a_x) \in E$.

Next, we may assume that $a \in \operatorname{ran}\alpha$ and $b \notin \operatorname{ran}\alpha$.

Case III: $a \in \operatorname{ran}\alpha$, $b \notin \operatorname{ran}\alpha$.

Consider $b \notin \operatorname{ran}\alpha$, there exists $y_a \in X$ such that $(b, y_a\alpha) = (b, a) \in E$.

Therefore, $(a\beta, b\beta) = (y_a\gamma, a_{y_a\gamma}) \in E$ because $a_{y_a\gamma} \in [y_a\gamma]_E$.

Since α is one-to-one and $y_{x\alpha}\alpha = x\alpha$, $y_{x\alpha} = x$ and hence for each $x \in X$, we have $x\alpha\beta = y_{x\alpha}\gamma = x\gamma$. This shows that $\alpha\beta = \gamma$, which implies $\alpha M = T_E(X)$. Therefore, α is a left magnifying element in $T_E(X)$. □

The following examples illustrate the ideas of the proof given in Lemma 2.1.6.

Example 2.1.7. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } x = y.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1\}, \{2\}, \{3\}, \{4\}, \ldots\}$. Let $\alpha \in T_E(X)$ be defined by $x\alpha = 2x$ for all $x \in X$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

It is obvious that α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Lemma 2.1.6, the function α is a left magnifying element. Let $M = \{\beta \in T_E(X) \mid (2x+1)\beta = 2 \text{ for all } x \in \mathbb{N}\}$ and γ be any function in $T_E(X)$. Then there exists $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate the ideas by considering the element γ of $T_E(X)$, which is defined by $x\gamma = 4x$ for all $x \in X$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in T_E(X)$ for all $x \in X$ by

$$x\beta = \begin{cases} 2x & \text{if } x \text{ is even,} \\ 2 & \text{if } x \text{ is odd.} \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix}.$$

Clearly, $\beta \in M$ and we have

$$\begin{aligned} \alpha\beta &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

Example 2.1.8. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } x \equiv y \mod 2.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}\}$. Let $\alpha \in T_E(X)$ be defined by $x\alpha = x + 2$ for all $x \in \mathbb{N}$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

It is obvious that α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Lemma 2.1.6, the function α is a left magnifying element. Let $M = \{\beta \in T_E(X) \mid 1\beta, 2\beta \in \{1, 2\}\}$ and γ be any function in $T_E(X)$. Then there exists $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate the ideas by considering the element γ of $T_E(X)$, which is defined by $x\gamma = x + 1$ for all $x \in X$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in T_E(X)$ by $1\beta = 2, 2\beta = 1$ and $x\beta = x - 1$ for all positive integers $x > 2$. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\begin{aligned} \alpha\beta &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

Theorem 2.1.9. *A function $\alpha \in T_E(X)$ is a left magnifying element if and only if α is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Proof. It follows by Corollary 2.1.5 and Lemma 2.1.6. □

If $E = X \times X$, then the following corollary holds.

Corollary 2.1.10. *A function $\alpha \in T(X)$ is a left magnifying element if and only if α is one-to-one but not onto.*

2.1.2 Right magnifying elements in $T_E(X)$

In this subsection, we provide the necessary and sufficient conditions for elements in $T_E(X)$ to be a right magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 2.1.11. If α is a right magnifying element in $T_E(X)$, then α is onto.

Proof. Assume that α is a right magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $M\alpha = T_E(X)$. Since the identity map id_X on X belongs to $T_E(X)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. This implies that α is onto. □

Lemma 2.1.12. Let α be a right magnifying element in $T_E(X)$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.

Proof. Assume that α is a right magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $M\alpha = T_E(X)$. Since $id_X \in T_E(X)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. Let $x, y \in X$ be such that $(x, y) \in E$. It follows that $x\beta\alpha = xid_X = x$ and $y\beta\alpha = yid_X = y$. Choose $a = x\beta$ and $b = y\beta$. Clearly, $(a, b) = (x\beta, y\beta) \in E$ since $\beta \in T_E(X)$. Therefore, the proof is completed. □

Lemma 2.1.13. If α is a right magnifying element in $T_E(X)$, then $M\alpha = T_E(X)\alpha$ for some proper subset M of $T_E(X)$.

Proof. Assume that α is a right magnifying element in $T_E(X)$. By definition, there exists a proper subset M of $T_E(X)$ such that $M\alpha = T_E(X)$. Clearly, $M\alpha \subseteq T_E(X)\alpha$ and $T_E(X)\alpha \subseteq T_E(X) = M\alpha$. This shows that $M\alpha = T_E(X)\alpha$. □

Lemma 2.1.14. If $\alpha \in T_E(X)$ is bijective, then α is not a right magnifying element.

Proof. Assume that $\alpha \in T_E(X)$ is bijective. So α^{-1} is also bijective. Suppose to the contrary that α is a right magnifying element. By definition, there exists a proper subset M of $T_E(X)$ such that $M\alpha = T_E(X)$. By Lemma 2.1.13, $M\alpha = T_E(X)\alpha$. Hence $M = M\alpha\alpha^{-1} = T_E(X)\alpha\alpha^{-1} = T_E(X)$, which is a contradiction. Therefore, α is not a right magnifying element. □

The next corollary follows by Lemma 2.1.11, Lemma 2.1.12 and Lemma 2.1.14.

Corollary 2.1.15. *If $\alpha \in T_E(X)$ is a right magnifying element, then α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Lemma 2.1.16. If $\alpha \in T_E(X)$ is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in T_E(X)$ is onto but not one-to-one and for any $(x, y) \in E$, there is $(a, b) \in E$ with $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in T_E(X) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $T_E(X)$ since the identity map id_X on X does not belong to M. Let γ be any function in $T_E(X)$. Since α is onto, for each $x \in X$ there exists $y_x \in X$ such that $y_x\alpha = x\gamma$ (if $x_1\gamma = x_2\gamma$, we must choose $y_{x_1} = y_{x_2}$ and if $(a\gamma, b\gamma) \in E$, we must choose $(y_a, y_b) \in E$). Define $\beta \in T(X)$ by $x\beta = y_x$ for all $x \in X$. To show that $\beta \in T_E(X)$, let $a, b \in X$ be such that $(a, b) \in E$. Since $\gamma \in T_E(X)$, $(a\gamma, b\gamma) \in E$. By assumption, there exists $(y_a, y_b) \in E$ such that $y_a\alpha = a\gamma$ and $y_b\alpha = b\gamma$. Hence $(a\beta, b\beta) = (y_a, y_b) \in E$. Since α is not one-to-one, there are distinct elements $x, y \in X$ such that $x\alpha = y\alpha$. Then at least one of x and y does not belong to ran β and hence β is not onto. So $\beta \in M$. Futhermore, for all $x \in X$, we see that $x\beta\alpha = y_x\alpha = x\gamma$. This shows that $\beta\alpha = \gamma$, which implies $M\alpha = T_E(X)$. Therefore, α is a right magnifying element. □

The following examples illustrate the ideas of the proof given in Lemma 2.1.16.

Example 2.1.17. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{2} \rfloor = \lfloor \frac{y}{2} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1\}, \{2, 3\}, \{4, 5\}, \ldots\}$. Let $\alpha \in T_E(X)$ be defined by $1\alpha = 1, 2\alpha = 2, 3\alpha = 3$ and $x\alpha = x - 2$ for all positive integers $x > 3$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

It is obvious that α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 2.1.16, the function α is a right magnifying element. Let $M = \{\beta \in T_E(X) \mid \beta \text{ is not onto}\}$ and $\gamma \in T_E(X)$ be any function. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element γ of $T_E(X)$, which is defined by $x\gamma = \lfloor \frac{x+2}{2} \rfloor$ for all $x \in X$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in T_E(X)$ by $1\beta = 1$ and for all $x \in X$, $(2x)\beta = (2x+1)\beta = x+3$. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & \cdots \end{pmatrix} = \gamma.$$

Example 2.1.18. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } x \equiv y \mod 2.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{2, 4, 6, \ldots\}, \{1, 3, 5, \ldots\}\}$. Let $\alpha \in T_E(X)$ be defined by $1\alpha = 1, 2\alpha = 2$ and $x\alpha = x - 2$ for all positive integers $x > 2$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

It is obvious that α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 2.1.16, the function α is a right magnifying element. Let $M = \{\beta \in T_E(X) \mid \beta \text{ is not onto}\}$ and $\gamma \in T_E(X)$ be any function. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element γ of $T_E(X)$, which is defined by $x\gamma = x + 2$ for all $x \in X$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in T_E(X)$ by $x\beta = x + 4$ for all $x \in X$. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} = \gamma.$$

Theorem 2.1.19. *A function $\alpha \in T_E(X)$ is a right magnifying element if and only if α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Proof. This follows by Corollary 2.1.15 and Lemma 2.1.16. □

If $E = X \times X$, then the following corollary holds.

Corollary 2.1.20. *A function $\alpha \in T(X)$ is a right magnifying element if and only if α is onto but not one-to-one.*

2.2 Magnifying elements in $P_E(X)$

Recall that $P_E(X) = \{\alpha \in P(X) \mid (x, y) \in E$ implies $(x\alpha, y\alpha) \in E\}$, where E is an equivalence relation on X, is a semigroup under the composition of functions. A function $\alpha \in P_E(X)$ is called a left (right) magnifying element if there exists a proper subset M of $P_E(X)$ such that $\alpha M = P_E(X)$ $(M\alpha = P_E(X))$.

2.2.1 Left magnifying elements in $P_E(X)$

In this subsection, we provide the necessary and sufficient conditions for elements in $P_E(X)$ to be a left magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 2.2.1. If α is a left magnifying element in $P_E(X)$, then α is one-to-one and dom $\alpha = X$.

Proof. Assume that α is a left magnifying element in $P_E(X)$. By definition, there is a proper subset M of $P_E(X)$ such that $\alpha M = P_E(X)$. Since the identity map id_X on X belongs to $P_E(X)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. This implies that α is one-to-one and dom $\alpha = X$. □

However, the converse of Lemma 2.2.1 is not true in general since there is no proper subset M of $P_E(X)$ such that $id_X M = P_E(X)$.

Lemma 2.2.2. Let α be a left magnifying element in $P_E(X)$. For any $x, y \in X$, $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.

Proof. The necessity is obvious. Conversely, assume that α is a left magnifying element in $P_E(X)$. By definition, there exists a proper subset M of $P_E(X)$ such that

$\alpha M = P_E(X)$. Since $id_X \in P_E(X)$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. It follows that $x = xid_X = x\alpha\beta$ and $y = yid_X = y\alpha\beta$. Thus $(x, y) = (x\alpha\beta, y\alpha\beta) \in E$ since $\beta \in P_E(X)$. $\square$

Lemma 2.2.3. If α is a left magnifying element in $P_E(X)$, then $\alpha M = \alpha P_E(X)$ for some proper subset M of $P_E(X)$.

Proof. Assume that α is a left magnifying element in $P_E(X)$. By definition, there exists a proper subset M of $P_E(X)$ such that $\alpha M = P_E(X)$. Clearly, $\alpha M \subseteq \alpha P_E(X)$ and $\alpha P_E(X) \subseteq P_E(X) = \alpha M$. This shows that $\alpha M = \alpha P_E(X)$. $\square$

Lemma 2.2.4. If $\alpha \in P_E(X)$ is bijective on X, then α is not a left magnifying element.

Proof. Assume that $\alpha \in P_E(X)$ is bijective on X. So α^{-1} is also bijective on X. Suppose to the contrary that α is a left magnifying element. By definition, there is a proper subset M of $P_E(X)$ such that $\alpha M = P_E(X)$. By Lemma 2.2.3, we have $\alpha M = \alpha P_E(X)$. Then $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha P_E(X) = P_E(X)$, which is a contradiction. Therefore, α is not a left magnifying element. $\square$

The next corollary follows by Lemma 2.2.1, Lemma 2.2.2 and Lemma 2.2.4.

Corollary 2.2.5. *If α is a left magnifying element in $P_E(X)$, then α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$, implies $(x, y) \in E$.*

Lemma 2.2.6. If $\alpha \in P_E(X)$ is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$, then α is a left magnifying element in $P_E(X)$.

Proof. Assume that $\alpha \in P_E(X)$ is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. Let $M = \{\beta \in P_E(X) \mid \text{dom } \beta \subseteq \text{ran } \alpha\}$. Since α is not onto, ran $\alpha \neq X$ and hence dom $\beta \neq X$ for all $\beta \in M$. So M is a proper subset of $P_E(X)$ since id_X does not belong to M. To show that $\alpha M = P_E(X)$, let $\gamma \in P_E(X)$. For $x \in (\text{dom}\gamma)\alpha$, there exists $y_x \in \text{dom } \gamma$ such that $y_x\alpha = x$. Define $\beta \in P(X)$ by $x\beta = y_x\gamma$ if $x \in (\text{dom}\gamma)\alpha$. Clearly, dom $\beta \subseteq$ ran α. To claim that $\beta \in P_E(X)$, let $a, b \in (\text{dom}\gamma)\alpha$. Then there exist $y_a, y_b \in \text{dom } \gamma$ such that $y_a\alpha = a$ and $y_b\alpha = b$. By assumption, $(y_a, y_b) \in E$. Then $(a\beta, b\beta) = (y_a\gamma, y_b\gamma) \in E$ since $\gamma \in P_E(X)$. So $\beta \in M$. Since α is one-to-one and $y_{x\alpha}\alpha = x\alpha$, $y_{x\alpha} = x$ and hence for $x \in \text{dom } \gamma$, $x\alpha\beta = y_{x\alpha}\gamma = x\gamma$. This shows that $\alpha\beta = \gamma$, which implies $M\alpha = T_E(X)$. Therefore, α is a left magnifying element. $\square$

The following examples illustrate the ideas of the proof given in Lemma 2.2.6.

Example 2.2.7. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{2} \rfloor \equiv \lfloor \frac{y}{2} \rfloor \mod 2.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{2, 3, 6, \ldots\}, \{1, 4, 5, \ldots\}\}$. Let $\alpha \in P_E(X)$ be defined by $1\alpha = 2$, $x\alpha = x + 2$ for all positive integers $x > 1$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 6 & 7 & 10 & 11 & 14 & 15 & \cdots \end{pmatrix}.$$

It is obvious that α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Lemma 2.2.6, the function α is a left magnifying element. Let $M = \{\beta \in P_E(X) \mid \text{dom } \beta \subseteq \mathbb{N} \setminus \{1, 3\}\}$ and let $\gamma \in P_E(X)$ be any function. Then there exists $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate the ideas by considering the element γ of $P_E(X)$, which is defined by $x\gamma = x - 2$ for all positive integers $x > 3$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ - & - & 4 & 5 & 8 & 9 & \cdots & - & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in P_E(X)$ by $x\beta = x - 4$ for all possitive integers $x > 5$. For convenience, we write β as

$$\beta = \begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ - & - & 2 & 3 & 6 & 7 & \cdots & - & - & - & 4 & 5 & 8 & 9 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\alpha\beta = \begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ 4 & 5 & 8 & 9 & 12 & 13 & \cdots & 2 & 6 & 7 & 10 & 11 & 14 & 15 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ - & - & 2 & 3 & 6 & 7 & \cdots & - & - & - & 4 & 5 & 8 & 9 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 10 & 11 & \cdots & 1 & 4 & 5 & 8 & 9 & 12 & 13 & \cdots \\ - & - & 4 & 5 & 8 & 9 & \cdots & - & 2 & 3 & 6 & 7 & 10 & 11 & \cdots \end{pmatrix} = \gamma.$$

Example 2.2.8. Let $X = \mathbb{Z} \times \mathbb{Z}$. Define a relation E on X by

$$((a, b), (c, d)) \in E \text{ if and only if } a = c.$$

It is clear that E is an equivalence relation on X. Let $\alpha \in P_E(X)$ be defined by $(a, b)\alpha = (2a, 2b)$ for all $a, b \in \mathbb{Z}$. Then α is one-to-one but not onto and for any $(a, b), (c, d) \in X$, if $((a, b)\alpha, (c, d)\alpha) \in E$, then $((a, b), (c, d)) \in E$. By Lemma 2.2.6, the function α is a left magnifying element. Let $M = \{\beta \in P_E(X) \mid \text{dom } \beta \subseteq \text{ran } \alpha\}$ and $\gamma \in P_E(X)$ be any function. Then there exists $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate the ideas by considering the element γ of $P_E(X)$, which is defined by $(a, b)\gamma = (a+1, b+2)$ for all $a, b \in \mathbb{Z}$. To get the required result, define a function $\beta \in P_E(X)$ by $(2k, 2l)\beta = (k + 1, l + 2)$ for all $k, l \in \mathbb{Z}$. So $\beta \in M$ and we have $(a, b)\alpha\beta = ((a, b)\alpha)\beta = (2a, 2b)\beta = (a + 1, b + 2) = (a, b)\gamma$ for all $a, b \in \mathbb{Z}$, which shows that $\alpha\beta = \gamma$.

Theorem 2.2.9. *A function $\alpha \in P_E(X)$ is a left magnifying element if and only if α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Proof. It follows from Corollary 2.2.5 and Lemma 2.2.6. □

If $E = X \times X$, then the following corollary holds.

Corollary 2.2.10. *A function $\alpha \in P(X)$ is a left magnifying element if and only if α is one-to-one but not onto on X.*

2.2.2 Right magnifying elements in $P_E(X)$

In this subsection, we provide the necessary and sufficient conditions for elements in $P_E(X)$ to be a right magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 2.2.11. If α is a right magnifying element in $P_E(X)$, then α is onto.

Proof. Assume that α is a right magnifying element in $P_E(X)$. By definition, there exists a proper subset M of $P_E(X)$ with $M\alpha = P_E(X)$. Since the identity map id_X on X belongs to $P_E(X)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. This implies that α is onto. □

Lemma 2.2.12. Let α be a right magnifying element in $P_E(X)$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.

Proof. Assume that α is a right magnifying element in $P_E(X)$. By definition, there exists a proper subset M of $P_E(X)$ such that $M\alpha = P_E(X)$. Since $id_X \in P_E(X)$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. Let $x, y \in X$ be such that $(x, y) \in E$. It follows that $x\beta\alpha = xid_X = x$ and $y\beta\alpha = yid_X = y$. Choose $a = x\beta$ and $b = y\beta$. Clearly, $(a, b) = (x\beta, y\beta) \in E$ since $\beta \in P_E(X)$. Therefore, the proof is completed. □

Lemma 2.2.13. If α is a right magnifying element, then $M\alpha = P_E(X)\alpha$ for some proper subset M of $P_E(X)$.

Proof. Assume that α is a right magnifying element in $P_E(X)$. By definition, there exists a proper subset M of $P_E(X)$ such that $M\alpha = P_E(X)$. Clearly, $M\alpha \subseteq P_E(X)\alpha$ and $P_E(X)\alpha \subseteq P_E(X) = M\alpha$. Therfore, $M\alpha = P_E(X)\alpha$. □

Lemma 2.2.14. If $\alpha \in P_E(X)$ is bijective on X, then α is not a right magnifying element.

Proof. Assume that $\alpha \in P_E(X)$ is bijective on X. So α^{-1} is al so bijective on X. Suppose to the contrary that α is a right magnifying element. By definition, there is a proper subset M of $P_E(X)$ such that $M\alpha = P_E(X)$. By Lemma 2.2.13, we have $M\alpha = P_E(X)\alpha$. Then $M = M\alpha\alpha^{-1} = P_E(X)\alpha\alpha^{-1} = P_E(X)$, which is a contradiction. Therefore, α is not a right magnifying element. □

The next corollary follows by Lemma 2.2.11, Lemma 2.2.12 and Lemma 2.2.14.

Corollary 2.2.15. *If $\alpha \in P_E(X)$ is a right magnifying element and dom $\alpha = X$, then α is onto but not one-to-one and for any $(x, y) \in E$ there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Lemma 2.2.16. If $\alpha \in P_E(X)$ is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$ there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in P_E(X)$ is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $P_E(X)$ since the identity map id_X on X does not belong to M. Let γ be any function in $P_E(X)$. Since α is onto, for each $x \in$ dom γ, there exists $y_x \in$ dom α such that $y_x\alpha = x\gamma$ (if $x_1\gamma = x_2\gamma$, we must choose $y_{x_1} = y_{x_2}$ and if $(a\gamma, b\gamma) \in E$, we must choose $(y_a, y_b) \in E$). Define $\beta \in P(X)$ by $x\beta = y_x$ for all $x \in$ dom γ. To show that

$\beta \in P_E(X)$, let $a, b \in X$ be such that $(a, b) \in E$. Since $\gamma \in P_E(X)$, $(a\gamma, b\gamma) \in E$. By assumption, we can choose $(y_a, y_b) \in E$ such that $y_a\alpha = a\gamma$ and $y_b\alpha = b\gamma$. Then $(a\beta, b\beta) = (y_a, y_b) \in E$. Since α is not one-to-one, there are distinct elements $x, y \in X$ such that $x\alpha = y\alpha$. Thus at least one of x and y does not belong to ran β. So $\beta \in M$. For all $x \in$ dom γ, we see that $x\beta\alpha = y_x\alpha = x\gamma$. This shows that $\beta\alpha = \gamma$, which implies that $M\alpha = P_E(X)$. Therefore, α is a right magnifying element. □

The next example illustrates the ideas of the proof given in Lemma 2.2.16.

Example 2.2.17. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \ldots\}$. Let $\alpha \in P_E(X)$ be defined by $x\alpha = x$ for all positive integers $x \leq 5$ and $x\alpha = x - 3$ for all positive integers $x > 5$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

It is obvious that α is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 2.2.16, the function α is a right magnifying element. Let $M = \{\beta \in P_E(X) \mid \beta \text{ is not onto}\}$ and γ be any function in $P_E(X)$. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element γ of $P_E(X)$, which is defined by $1\gamma = 1$, $2\gamma = 2$ $x\gamma = x - 3$ for positive integer $x > 5$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in P_E(X)$ by $x\beta = x$ for all $x \in$ dom γ. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 6 & 7 & 8 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\begin{aligned} \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 6 & 7 & 8 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & - & - & - & 3 & 4 & 5 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

Lemma 2.2.18. If $\alpha \in P_E(X)$ is onto, dom $\alpha \neq X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in P_E(X)$ is onto, dom $\alpha \neq X$ and for any $(x, y) \in E$, there is $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $P_E(X)$ since the identity map id_X on X does not belong to M. Let γ be any function in $P_E(X)$. Since α is onto, for each $x \in$ dom γ, there exists $y_x \in$ dom α such that $y_x\alpha = x\gamma$ (if $x_1\gamma = x_2\gamma$, we must choose $y_{x_1} = y_{x_2}$ and if $(a\gamma, b\gamma) \in E$, we must choose $(y_a, y_b) \in E$). Define a function $\beta \in P(X)$ by $x\beta = y_x$ for all $x \in$ dom γ. To show that $\beta \in P_E(X)$, let $a, b \in X$ be such that $(a, b) \in E$. Since $\gamma \in P_E(X)$, we have $(a\gamma, b\gamma) \in E$. By assumption, there exists $(y_a, y_b) \in E$ such that $y_a\alpha = a\gamma$ and $y_b\alpha = b\gamma$. Then $(a\beta, b\beta) = (y_a, y_b) \in E$. Since ran $\beta \subseteq$ dom $\alpha \neq X$, β is not onto. So $\beta \in M$. For all $x \in$ dom γ, we see that $x\beta\alpha = y_x\alpha = x\gamma$. This shows that $\beta\alpha = \gamma$, which implies $M\alpha = P_E(X)$. Therefore, α is a right magnifying element. □

The next example illustrates the ideas of the proof given in Lemma 2.2.18.

Example 2.2.19. Let $X = \mathbb{N}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivelence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \ldots\}$. Let $\alpha \in P_E(X)$ be defined by $3\alpha = 1, 4\alpha = 2$ and $x\alpha = x - 3$ for all positive integers $x > 5$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 1 & 2 & - & 3 & 4 & 5 & \cdots \end{pmatrix}.$$

It is obvious that α is onto, dom $\alpha \neq X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 2.2.18, the function α is a right magnifying element. Let $M = \{\beta \in P_E(X) \mid \beta \text{ is not onto}\}$ and γ be any function in $P_E(X)$. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element γ of $P_E(X)$, which is defined by $x\gamma = \lfloor \frac{x+3}{3} \rfloor$ for all positive integers $x > 2$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 2 & 2 & 2 & 3 & 3 & 3 & \cdots \end{pmatrix}.$$

To get the required result, define a function $\beta \in P_E(X)$ by $x\beta = \lfloor \frac{x+9}{3} \rfloor$ if $x = 3, 4, 5$ and $x\beta = \lfloor \frac{x+12}{3} \rfloor$ for all positive integers $x \geq 6$. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 4 & 4 & 4 & 6 & 6 & 6 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\begin{aligned} \beta\alpha &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 4 & 4 & 4 & 6 & 6 & 6 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 1 & 2 & - & 3 & 4 & 5 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ - & - & 2 & 2 & 2 & 3 & 3 & 3 & \cdots \end{pmatrix} = \gamma. \end{aligned}$$

Theorem 2.2.20. *A function $\alpha \in P_E(X)$ is a right magnifying element if and only if α is onto, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha, y = b\alpha$ and either*

1. *dom $\alpha \neq X$ or*
2. *dom $\alpha = X$ and α is not one-to-one.*

Proof. It follows by Corollary 2.2.15, Lemma 2.2.16 and Lemma 2.2.18. □

If $E = X \times X$, then the following corollary holds.

Corollary 2.2.21. *A function $\alpha \in P(X)$ is a right magnifying element if and only if α is onto and either*

1. *dom $\alpha \neq X$ or*
2. *dom $\alpha = X$ and α is not one-to-one.*

Chapter 3

Magnifying elements in $T_E(X, \mathcal{P})$ and $P_E(X, \mathcal{P})$

In this chapter, we provide the necessary and sufficient conditions for elements in the full and partial transformation semigroups preserving an equivalence relation and a partition to be a left or right magnifying element.

3.1 Magnifying elements in $T_E(X, \mathcal{P})$

Recall that, for an equivalence relation E on a nonempty set X, $[x]_E = \{y \in X \mid (x, y) \in E\}$ denotes the equivalence class of an element $x \in X$ determined by E, and we set $X/E = \{[x]_E \mid x \in X\}$. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition of $X = \{x_j \mid j \in \Lambda'\}$. Put $(X_i, x_j) = X_i \cap [x_j]_E$ for $i \in \Lambda$ and $j \in \Lambda'$. The semigroup $T_E(X, \mathcal{P}) = T_E(X) \cap T(X, \mathcal{P})$, i.e.,

$$T_E(X, \mathcal{P}) = \{\alpha \in T_E(X) \mid X_i\alpha \subseteq X_i \text{ for all } i \in \Lambda\},$$

which is a semigroup under the composition of functions. A function $\alpha \in T_E(X, \mathcal{P})$ is called a left (right) magnifying element if there exists a proper subset M of $T_E(X, \mathcal{P})$ such that $\alpha M = T_E(X, \mathcal{P})$ $(M\alpha = T_E(X, \mathcal{P}))$.

3.1.1 Left magnifying elements in $T_E(X, \mathcal{P})$

In this subsection, we provide the necessary and sufficient conditions for elements in $T_E(X, \mathcal{P})$ to be a left magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 3.1.1. If α is a left magnifying element in $T_E(X, \mathcal{P})$, then α is one-to-one.

Proof. Assume that α is a left magnifying element in $T_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $T_E(X,\mathcal{P})$ such that $\alpha M = T_E(X,\mathcal{P})$. Since the identity map id_X on X belongs to $T_E(X,\mathcal{P})$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. This implies that α is one-to-one. □

However, the converse of Lemma 3.1.1 is not true since there exists no proper subset M of $T_E(X,\mathcal{P})$ such that $id_X M = T_E(X,\mathcal{P})$.

Lemma 3.1.2. Let α be a left magnifying element in $T_E(X,\mathcal{P})$. For any $x, y \in X$, $(x,y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.

Proof. The necessity is obvious. Conversely, assume that $\alpha \in T_E(X,\mathcal{P})$ is a left magnifying element. By definition, there exists a proper subset M of $T_E(X,\mathcal{P})$ such that $\alpha M = T_E(X,\mathcal{P})$. Since $id_X \in T_E(X,\mathcal{P})$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. It follows that $x = xid_X = x\alpha\beta$ and $y = yid_X = y\alpha\beta$. Therefore, $(x,y) = (x\alpha\beta, y\alpha\beta) \in E$ since $\beta \in T_E(X,\mathcal{P})$. □

Lemma 3.1.3. If α is a left magnifying element in $T_E(X,\mathcal{P})$, then $\alpha M = \alpha T_E(X,\mathcal{P})$ for some proper subset M of $T_E(X,\mathcal{P})$.

Proof. Assume that α is a left magnifying element in $T_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $T_E(X,\mathcal{P})$ such that $\alpha M = T_E(X,\mathcal{P})$. It is clear that $\alpha M \subseteq \alpha T_E(X,\mathcal{P})$ and $\alpha T_E(X,\mathcal{P}) \subseteq T_E(X,\mathcal{P}) = \alpha M$. This shows that $\alpha M = \alpha T_E(X,\mathcal{P})$. □

Lemma 3.1.4. If $\alpha \in T_E(X,\mathcal{P})$ is bijective, then α is not a left magnifying element.

Proof. Let $\alpha \in T_E(X,\mathcal{P})$ be bijective. So α^{-1} is also bijective. Suppose that α is a left magnifying element. By definition, there is a proper subset M of $T_E(X,\mathcal{P})$ such that $\alpha M = T_E(X,\mathcal{P})$. By Lemma 3.1.3, we have $\alpha M = \alpha T_E(X,\mathcal{P})$. Then $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha T_E(X,\mathcal{P}) = T_E(X,\mathcal{P})$, which is a contradiction. Therefore, α is not a left magnifying element. □

The next corollary follows by Lemma 3.1.1, Lemma 3.1.2 and Lemma 3.1.4.

Corollary 3.1.5. *If α is a left magnifying element in $T_E(X,\mathcal{P})$, then α is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x,y) \in E$.*

Lemma 3.1.6. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If $X_i \in \mathcal{P}$ is finite for all $i \in \Lambda$, then there exists no left magnifying element in $T_E(X,\mathcal{P})$.

Proof. Suppose to contrary that there is a left magnifying element α in $T_E(X, \mathcal{P})$. By assumption and Lemma 3.1.1, we have $\alpha|_{X_i}$ is bijective for all $i \in \Lambda$. Since $X\alpha = \big(\bigcup_{i\in\Lambda} X_i\big)\alpha = \bigcup_{i\in\Lambda} X_i\alpha = \bigcup_{i\in\Lambda} X_i = X$, α is onto which is a contradiction. □

It is noticeable in Lemma 3.1.6 that if a left magnifying element exists in $T_E(X, \mathcal{P})$, then X_i is infinite for some $i \in \Lambda$. However, the converse of this statement is not true in general. It is illustrated by the following counterexample.

Example 3.1.7. Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N}\cup\{0\}\}$ where $X_0 = \{0, -1, -2, \ldots\}$ and $X_i = \{2i-1, 2i\}$ for all $i \in \mathbb{N}$, that is, $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{5, 6\}, \ldots$ Define a relation E on X by $E = \bigcup_{j=1}^{\infty}(A_j \times A_j)$ where $A_1 = \{0, \pm1, \pm2\}$ and $A_j = \{\pm(2j-1), \pm2j\}$ for all positive integers $j \geq 2$, that is, $A_2 = \{\pm3, \pm4\}$, $A_3 = \{\pm5, \pm6\}$, $A_4 = \{\pm7, \pm8\}, \ldots$ Clearly, (X_i, x) is finite for all $i \in \mathbb{N} \cup \{0\}$ and $x \in X$. Moreover, $A_i = (X_0, x_i)\dot{\cup}X_i$ where $[x_i]_E = A_i$ for all $i \in \mathbb{N}$. Note that $(X_0, x_i) \subseteq A_i$ where $[x_i]_E = A_i$ and $X_i \subseteq A_i$ for all $i \in \mathbb{N}$. Let α be one-to-one in $T_E(X, \mathcal{P})$. Then $X_i\alpha = X_i \subseteq A_i$ for all $i \in \mathbb{N}$. This forces that $(X_0, x_i)\alpha = (X_0, x_i) \subseteq A_i$ where $[x_i]_E = A_i$ for all $i \in \mathbb{N}$. Then α is onto. This implies that there exists no left magnifying element in $T_E(X, \mathcal{P})$.

Corollary 3.1.8. *If X is a finite set, then $T_E(X, \mathcal{P})$ has no left magnifying element.*

Lemma 3.1.9. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a nonempty set X such that for any $x \in X$, there is exactly one $X_i \in \mathcal{P}$ with $[x]_E \subseteq X_i$. If $\alpha \in T_E(X, \mathcal{P})$ is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$, then α is a left magnifying element of $T_E(X, \mathcal{P})$.

Proof. Assume that $\alpha \in T_E(X, \mathcal{P})$ is one-to-one but not onto and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. For each $x \in \operatorname{ran} \alpha$, there exists $y_x \in X$ such that $y_x\alpha = x$.

Case 1: $|[x]_E| = 1$ for all $x \in X$.

Let $M = \{\beta \in T_E(X, \mathcal{P}) \mid x\beta = x \text{ for all } x \notin \operatorname{ran} \alpha\}$. Clearly, M is a proper subset of $T_E(X, \mathcal{P})$. For any function γ in $T_E(X, \mathcal{P})$, define a function $\beta \in T_E(X, \mathcal{P})$ for all $x \in X$ by

$$x\beta = \begin{cases} y_x\gamma & \text{if } x \in \operatorname{ran} \alpha, \\ x & \text{if } x \notin \operatorname{ran} \alpha. \end{cases}$$

Clearly, $\beta \in M$. For any $x \in X$, $x\alpha\beta = y_{x\alpha}\gamma$. Since $y_{x\alpha}\alpha = x\alpha$ and α is one-to-one, $y_{x\alpha} = x$. Therefore, $x\alpha\beta = x\gamma$. This shows that $\alpha\beta = \gamma$ which implies that

$\alpha M = T_E(X, \mathcal{P})$.

Case 2: $|[x]_E| > 1$ for some $x \in X$.

Let $X = \{x_j \mid j \in \Lambda'\}$. For any $i \in \Lambda$ and $j \in \Lambda'$ with $(X_i, x_j) \cap \text{ran } \alpha \neq \emptyset$, we can choose $x_{ij} \in (X_i, x_j) \cap \text{ran } \alpha$. Then there exists $y_{x_{ij}} \in X_i$ such that $y_{x_{ij}}\alpha = x_{ij}$. If $(X_i, x_j) \cap \text{ran } \alpha = \emptyset$, then we choose $x_i \in X_i \cap \text{ran } \alpha$.

Let $M = \{\beta \in T_E(X, \mathcal{P}) \mid \beta \text{ is not one-to-one}\}$. Clearly, M is a proper subset of $T_E(X, \mathcal{P})$ since the identity map id_X on X does not belong to M. For any function γ in $T_E(X, \mathcal{P})$, define a function $\beta \in T(X)$ for all $x \in X$ by

$$x\beta = \begin{cases} y_x\gamma & \text{if } x \in \text{ran } \alpha, \\ y_{x_{ij}}\gamma & \text{if } x \notin \text{ran } \alpha \text{ and } x \in (X_i, x_j) \text{ such that } (X_i, x_j) \cap \text{ran } \alpha \neq \emptyset, \\ y_{x_i}\gamma & \text{if } x \notin \text{ran } \alpha \text{ and } x \in (X_i, x_j) \text{ such that } (X_i, x_j) \cap \text{ran } \alpha = \emptyset. \end{cases}$$

To show that $\beta \in T_E(X, \mathcal{P})$, let $(a, b) \in E$. Then $a, b \in [x_j]_E$ for some $x_j \in X$.

Case I: $a, b \in \text{ran } \alpha$.

Then there exist $y_a, y_b \in X$ such that $y_a\alpha = a$ and $y_b\alpha = b$. By assumption, we have $(y_a, y_b) \in E$. Hence $(a\beta, b\beta) = (y_a\gamma, y_b\gamma) \in E$.

Case II: $a, b \notin \text{ ran } \alpha$.

Since there is exactly one $X_i \in \mathcal{P}$ such that $[x]_E \subseteq X_i$ for all $x \in X$, a and b must belong to X_i for some $i \in \Lambda$. So $a, b \in (X_i, x_j)$.

Case i: $(X_i, x_j) \cap \text{ran } \alpha \neq \emptyset$. Then $(a\beta, b\beta) = (y_{x_{ij}}\gamma, y_{x_{ij}}\gamma) \in E$.

Case ii: $(X_i, x_j) \cap \text{ran } \alpha = \emptyset$. Then $(a\beta, b\beta) = (y_{x_i}\gamma, y_{x_i}\gamma) \in E$.

Next, we may assume that $a \in \text{ ran } \alpha$ and $b \notin \text{ ran } \alpha$.

Case III: $a \in \text{ ran } \alpha$ and $b \notin \text{ ran } \alpha$.

Then $(X_i, x_j) \cap$ ran $\alpha \neq \emptyset$ since $a \in (X_i, x_j) \cap$ ran α and hence choose $x_{ij} = a$. Therefore, $(a\beta, b\beta) = (y_a\gamma, y_a\gamma) \in E$.

Hence, β preserves the equivalence relation E. Moreover, it is easy to see that β preserves the partition $\mathcal{P}$ on X, as well. Since $[x]_E > 1$ for some $x \in X$ and α is one-to-one but not onto, there exists $x_0 \notin \text{ ran } \alpha$. Then $x_0\beta \in (X \setminus \{x_0\})\beta$ and hence β is not one-to-one. Therefore, $\beta \in M$. For any $x \in X$, $x\alpha\beta = y_{x\alpha}\gamma$. Since $y_x\alpha = x\alpha$ and α is one-to-one, $y_{x\alpha} = x$. Therefore, $x\alpha\beta = x\gamma$. This shows that $\alpha\beta = \gamma$ which implies that $\alpha M = T_E(X, \mathcal{P})$. □

The following examples illustrate the ideas of the proof given in Lemma 3.1.9.

Example 3.1.10. Let $X = \mathbb{Q}$ and $\mathcal{P}$ be a partition on X where $\mathcal{P} = \{\mathbb{Q}^+, \mathbb{Q}^- \cup \{0\}\}$. Let $\mathcal{A} = \mathbb{Q}^- \setminus \{-2n+1 \mid n \in \mathbb{N}\}$ and denote $A_n = \left\{\dfrac{x}{n} \mid x \in \mathbb{N} \text{ and } \gcd(x, n) = 1\right\}$ where $n \in \mathbb{N}$. Define a relation E on X by

$$E = \bigcup_{i=1}^{\infty}(A_{2i} \times A_{2i}) \cup \bigcup_{i=1}^{\infty}\Big(A_{2i-1} \cup \{-2i+1\} \times A_{2i-1} \cup \{-2i+1\}\Big) \cup (\mathcal{B} \times \mathcal{B})$$

where $\mathcal{B} = \mathcal{A} \cup \{0\}$. Thus E is an equivalence relation on X and $\mathbb{Q}^+ \in \mathcal{P}$ is infinite. Clearly, $\mathcal{B}$ and A_n are infinite for all $n \in \mathbb{N}$. Then there exist bijections $\varphi_0 : \mathcal{B} \longrightarrow \mathcal{B}$, $\varphi_2 : A_2 \longrightarrow A_1$, $\varphi_{2n} : A_{2n} \longrightarrow A_{2(n-1)}$ for all $n \geq 2$ and $\varphi_{2n-1} : A_{2n-1} \longrightarrow A_{2(n+1)-1}$ for all $n \in \mathbb{N}$. Define a function $\alpha \in T_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\varphi_i & \text{if } x \in A_i, i \in \mathbb{N}, \\ x - 2 & \text{if } x \in \{-2n+1 \mid n \in \mathbb{N}\}, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $-1 \notin$ ran α. Therefore, α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$ but α is not a left magnifying element of $T_E(X, \mathcal{P})$ since there exists no function $\beta \in T_E(X, \mathcal{P})$ such that $\alpha\beta = id_X$.

Example 3.1.11. Let $X = \mathbb{Z}$ and $\mathcal{P}$ be a partition on X such that $\mathcal{P} = \{X_1, X_2\}$ where $X_1 = \{0, -1, -2, -3, \ldots\}$ and $X_2 = \{1, 2, 3, \ldots\}$. Define a relation E on X by $E = \bigcup_{i=0}^{\infty}(A_i \times A_i)$ where $A_0 = X_1, A_1 = \{1\}, A_2 = \{2, 3\}, A_3 = \{4, 5, 6\}, \ldots$
It is obvious that $X_1 \in \mathcal{P}$ is infinite, E is an equivalence relation on X and $X/E = \{\{0, -1, -2, \ldots\}, \{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \ldots\}$. Define a function α on X by

$$x\alpha = \begin{cases} x & \text{if } x \in A_0, \\ x + i & \text{if } x \in A_i, i > 0. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \ldots & -3 & -2 & -1 & 0 & 2 & 4 & 5 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & \cdots \end{pmatrix}.$$

It is noticeable that α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Lemma 3.1.9, the function α is a left magnifying element. Let $M = \{\beta \in T_E(X, \mathcal{P}) \mid \beta \text{ is not one-to-one }\}$ and let γ be an element of $T_E(X, \mathcal{P})$. Then there exists an element $\beta \in M$ such that $\alpha\beta = \gamma$.

We will illustrate the ideas by considering the element γ of $T_E(X, \mathcal{P})$, which is defined by

$$x\gamma = \begin{cases} x & \text{if } x \in A_i, i \leq 2, \\ x + i & \text{if } x \in A_i, i > 2. \end{cases}$$

For convenience, we write γ as

$$\gamma = \begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & \cdots \end{pmatrix}.$$

Note that for all $j \geq 1$, $A_j \cap \text{ran}\,\alpha \neq A_j$ and hence there is an element $x_j \in A_j$ and $x_j \notin \text{ran}\,\alpha$. To get the desired result, for any $x_j \notin \text{ran}\,\alpha$ such that $(X_i, x_j) \cap \text{ran}\,\alpha \neq \emptyset$, we can choose $y_{x_j} = \min\Big((X_i, x_j) \cap \text{ran}\,\alpha\Big) = \min A_j$ for all $j \geq 4$ and define a function β in $T_E(X, P)$ by $x\beta = x$ for all $x \in A_0 \cup A_1$, $2\beta = 3\beta = 1$, $4\beta = 6\beta = 2$, $5\beta = 3$ and

$$x\beta = \begin{cases} x & \text{if } x \neq \max A_j, \\ y_{x_j} & \text{if } x = \max A_j \end{cases}$$

for all $x \in A_j$ such that $j \geq 4$. For convenience, we write β as

$$\beta = \begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 1 & 1 & 2 & 3 & 2 & 7 & 8 & 9 & 7 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\alpha\beta = \begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 2 & 4 & 5 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 1 & 1 & 2 & 3 & 2 & 7 & 8 & 9 & 7 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & \cdots \end{pmatrix} = \gamma.$$

Note that $\max A_i \notin \text{ran}\,\alpha$ for all A_i such that $i \geq 1$. The main ideas behind the concept are as follows: We illustrate the idea by considering $1, 3, 6,$ and $10 \notin \text{ran}\,\alpha$. Since $1 \in (X_2, 1)$ and $(X_2, 1) \cap \text{ran}\,\alpha = \emptyset, 1\beta = 1$. Consider $3 \in (X_2, 3)$ and $(X_2, 3) \cap \text{ran}\,\alpha = \{2\}$. We can see that $1\alpha = 2$. Hence $y_2 = 1$ and $y_2\gamma = 1$. Therefore, $3\beta = 1$. Consider $6 \in (X_2, 6)$ and $(X_2, 6) \cap \text{ran}\,\alpha = \{4, 5\}$. Then we choose $4 \in (X_3, 6) \cap \text{ran}\,\alpha$. We can see that $2\alpha = 4$. Hence $y_4 = 2$ and $y_4\gamma = 2$. Therefore, $6\beta = 2$. Consider $10 \in (X_2, 10)$ and $(X_2, 10) \cap \text{ran}\,\alpha = \{7, 8, 9\}$. Then we choose $7 \in (X_2, 10) \cap \text{ran}\,\alpha$. We can see that $4\alpha = 7$. Hence $y_7 = 4$ and $y_7\gamma = 7$. Therefore, $10\beta = 7$.

Theorem 3.1.12. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that X_i is infinite for some $i \in \Lambda$ and for each $x \in X$,* *there is exactly one $X_i \in \mathcal{P}$ with $[x]_E \subseteq X_i$. A function $\alpha \in T_E(X,\mathcal{P})$ is a left magnifying element if and only if α is one-to-one but not onto and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$.*

Proof. It follows by Corollary 3.1.5 and Lemma 3.1.9. □

Theorem 3.1.13. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that for each $x \in X$, there is exactly one $X_i \in \mathcal{P}$ with $[x]_E \subseteq X_i$. There exists a left magnifying element in $T_E(X,\mathcal{P})$ if and only if at least one element of $\mathcal{P}$ is infinite.*

Proof. The necessity is obtained by Lemma 3.1.6. On the other hand, suppose that there exists $X_i \in \mathcal{P}$ such that X_i is infinite.

Case 1: There exists $t \in X_i$ such that (X_i, t) is infinite. Then there is a proper subset A of (X_i, t) such that $|A| = |(X_i, t)| = |(X_i, t) \setminus A|$. So there is a bijection γ from (X_i, t) to A. Define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in (X_i, t), \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T_E(X,\mathcal{P})$, α is one-to-one. Hence ran $\alpha \subseteq X \setminus ((X_i, t) \setminus A) \neq X$. Then α is one-to-one but not onto. It is easy to see that for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Theorem 3.1.12, α is a left magnifying element.

Case 2: (X_i, t) is finite for all $t \in X_i$.

Case 2.1: There exists $n \in \mathbb{N}$ such that $K = \{(X_i, t) \mid t \in X_i \text{ and } |(X_i, t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijection λ from K to K'. So $|A| = |A\lambda| = n$ for all $A \in K$. Hence for all $A \in K$, there exists a bijective function η_A from A to $A\lambda$. Let $\eta = \bigcup\limits_{A \in K} \eta_A$. Then η is a bijection from $\bigcup\limits_{A \in K} A$ to $\bigcup\limits_{A \in K'} A$. Define a function α by

$$x\alpha = \begin{cases} x\eta & \text{if } x \in \bigcup\limits_{A \in K} A, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T_E(X,\mathcal{P})$ and α is one-to-one. Since ran $\alpha = X \setminus (\bigcup\limits_{A \in K} A \setminus \bigcup\limits_{A \in K'} A) \neq X$, α is not onto. Then α is one-to-one but not onto. It is easy to see that for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Theorem 3.1.12, α is a left magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_i, t) \mid t \in X_i \text{ and } |(X_i, t)| = n\}$ is finite. Then for each $t \in X_i$, there exists $t' \in X_i$ such that $|(X_i, t)| < |(X_i, t')|$. Let $A = \{(X_i, t) \mid [t]_E \subseteq X_i\}$. In this case, A is an infinite set. Let $n_1 = \min\limits_{(X_i,t)\in A} |(X_i, t)|$ and $K_1 = \{(X_i, t) \mid |(X_i, t)| = n_1\}$. Choose $(X_i, t_1) \in K_1$. Let $n_2 = \min\limits_{(X_i,t)\in A_1} |(X_i, t)|$ where $A_1 = A \setminus K_1$ and $K_2 = \{(X_i, t) \mid |(X_i, t)| = n_2\}$. Choose $(X_i, t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_i, t_1), (X_i, t_2), \ldots, (X_i, t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min\limits_{(X_i,t)\in A_k} |(X_i, t)|$ where $A_k = A \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_i, t_k) \in K_k$, where $K_k = \{(X_i, t) \mid |(X_i, t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < ... < n_k < ...$

Next, we let $B = \{(X_i, t_l) \mid l \geq 1\}$. Then $|(X_i, t_l)| < |(X_i, t_{l+1})|$ for all $l \geq 1$. Hence there exists an injection $\gamma_l : (X_i, t_l) \to (X_i, t_{l+1})$. Let $\gamma = \bigcup\limits_{l\geq 1} \gamma_l$. Then γ is an injection from $\bigcup\limits_{C\in B} C$ to itself. Next, define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup\limits_{C\in B} C, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T_E(X, \mathcal{P})$ and α is one-to-one. Since ran $\alpha \subseteq X \setminus (X_i, t_1) \neq X$, α is not onto. Then α is one-to-one but not onto. It is easy to see that for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Theorem 3.1.12, α is a left magnifying element. □

3.1.2 Right magnifying elements in $T_E(X, \mathcal{P})$

In this subsection, we provide the necessary and sufficient conditions for elements in $T_E(X, \mathcal{P})$ to be a right magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 3.1.14. If α is a right magnifying element in $T_E(X, \mathcal{P})$, then α is onto.

Proof. Assume that α is a right magnifying element in $T_E(X, \mathcal{P})$. By definition, there exists a proper subset M of $T_E(X, \mathcal{P})$ such that $M\alpha = T_E(X, \mathcal{P})$. Since the identity map id_X on X belongs to $T_E(X, \mathcal{P})$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. This implies that α is onto. □

Lemma 3.1.15. Let α be a right magnifying element in $T_E(X, \mathcal{P})$. For any $x, y \in X$, if $(x, y) \in E$, then there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.

Proof. Assume that α is a right magnifying element in $T_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $T_E(X,\mathcal{P})$ such that $M\alpha = T_E(X,\mathcal{P})$. Since $id_X \in T_E(X,\mathcal{P})$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. Let $x, y \in X$ be such that $(x,y) \in E$. It follows that $x\beta\alpha = xid_X = x$ and $y\beta\alpha = yid_X = y$. Choose $a = x\beta$ and $b = y\beta$. Clearly, $(a,b) = (x\beta, y\beta) \in E$ since $\beta \in T_E(X,\mathcal{P})$. Therefore, the proof is completed. □

Lemma 3.1.16. If α is a right magnifying element in $T_E(X,\mathcal{P})$, then $M\alpha = T_E(X,\mathcal{P})\alpha$ for some proper subset M of $T_E(X,\mathcal{P})$.

Proof. Assume that α is a right magnifying element in $T_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $T_E(X,\mathcal{P})$ such that $M\alpha = T_E(X,\mathcal{P})$. It is clear that $M\alpha \subseteq T_E(X,\mathcal{P})\alpha$ and $T_E(X,\mathcal{P})\alpha \subseteq T_E(X,\mathcal{P}) = M\alpha$. This shows that $M\alpha = T_E(X,\mathcal{P})\alpha$. □

Lemma 3.1.17. If $\alpha \in T_E(X,\mathcal{P})$ is bijective, then α is not a right magnifying element.

Proof. Assume that $\alpha \in T_E(X,\mathcal{P})$ is bijective. So α^{-1} is also bijective. Suppose to the contrary that α is a right magnifying element. By definition, there is a proper subset M of $T_E(X,\mathcal{P})$ such that $M\alpha = T_E(X,\mathcal{P})$. By Lemma 3.1.16, we have $M\alpha = T_E(X,\mathcal{P})\alpha$. Then $M = M\alpha\alpha^{-1} = T_E(X,\mathcal{P})\alpha\alpha^{-1} = T_E(X,\mathcal{P})$, which is a contradiction. Therefore, α is not a right magnifying element. □

The next corollary follows by Lemma 3.1.14, Lemma 3.1.15 and Lemma 3.1.17.

Corollary 3.1.18. *If α is a right magnifying element in $T_E(X,\mathcal{P})$, then α is onto but not one-to-one and for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Lemma 3.1.19. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If X_i is finite for all $i \in \Lambda$, then there exists no right magnifying element in $T_E(X,\mathcal{P})$.

Proof. Suppose to the contrary that there is a right magnifying element $\alpha \in T_E(X,\mathcal{P})$. By assumption and Lemma 3.1.14, we have $\alpha|_{X_i}$ is one-to-one for all $i \in \Lambda$. Since $X\alpha = \big(\bigcup\limits_{i\in\Lambda} X_i\big)\alpha = \bigcup\limits_{i\in\Lambda} X_i\alpha$, α is one-to-one, which is a contradiction. □

It is noticeable in Lemma 3.1.19 that if a right magnifying element exists in $T_E(X,\mathcal{P})$, then X_i is infinite for some $i \in \Lambda$. However, the converse of this statement is not true in general. It is illustrated by the following counterexample.

Example 3.1.20. Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N} \cup \{0\}\}$ where $X_0 = \{0, -1, -2, \ldots\}$ and $X_i = \{2i-1, 2i\}$ for all $i \in \mathbb{N}$, that is, $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{5, 6\}, \ldots$ Define a relation E on X by $E = \bigcup_{j=1}^{\infty} (A_j \times A_j)$ where $A_1 = \{0, \pm 1, \pm 2\}$ and $A_j = \{\pm(2j-1), \pm 2j\}$ for all possitive integer $j \geq 2$, that is, $A_2 = \{\pm 3, \pm 4\}$, $A_3 = \{\pm 5, \pm 6\}$, $A_4 = \{\pm 7, \pm 8\}, \ldots$ Clearly, (X_i, x) is finite for all $i \in \mathbb{N} \cup \{0\}$ and $x \in X$. Moreover, $A_i = (X_0, x_i) \dot{\cup} X_i$ where $[x_i]_E = A_i$ for all $i \in \mathbb{N}$. Note that $(X_0, x_i) \subseteq A_i$ where $[x_i]_E = A_i$ and $X_i \subseteq A_i$ for all $i \in \mathbb{N}$. Let $\alpha \in T_E(X, \mathcal{P})$ be onto. Then $X_i\alpha = X_i \subseteq A_i$ for all $i \in \mathbb{N}$. This forces that $(X_0, x_i)\alpha = (X_0, x_i) \subseteq A_i$ where $[x_i]_E = A_i$ for all $i \in \mathbb{N}$. Then α is one-to-one. This implies that there exists no right magnifying element in $T_E(X, \mathcal{P})$.

Corollary 3.1.21. *If X is a finite set, then $T_E(X, \mathcal{P})$ has no right magnifing elements.*

Lemma 3.1.22. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in T_E(X, \mathcal{P})$ is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in T_E(X, \mathcal{P})$ is onto but not one-to-one and and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $\gamma \in T_E(X, \mathcal{P})$ and $M = \{\beta \in T_E(X, \mathcal{P}) \mid \beta \text{ is not onto }\}$. Clearly, M is a proper subset of $T_E(X, \mathcal{P})$ since the identity map id_X on X does not belong to M. Let $x_j \in X$ and $x \in (X_i, x_j)$. So $x\gamma \in X_i$. Since α is onto, we can choose $y_x \in X_i$ such that $y_x\alpha = x\gamma$ and $(y_x, y_z) \in E$ for all $z \in (X_i, x_j)$. Then for all $x, z \in (X_i, x_j)$, there are $y_x, y_z \in X_i$ such that $y_x\alpha = x\gamma$, $y_z\alpha = z\gamma$ and $(y_x, y_z) \in E$ because $(x, z) \in E$. Define a function β by $x\beta = y_x$ for all $x \in X$. Obviously, β is a function on X. To show that $\beta \in T_E(X, P)$, let $a, b \in X$ be such that $(a, b) \in E$. Clearly, $(a\beta, b\beta) = (y_a, y_b) \in E$. For each $X_i \in \mathcal{P}$, if $a \in X_i$, then $a\beta = y_a$. Since $y_a\alpha = a$, $a\beta \in X_i$. Therefore, $\beta \in T_E(X, \mathcal{P})$. Since α is not one-to-one, there are distinct elements $x, y \in X_i$ for some $i \in \Lambda$ such that $x\alpha = y\alpha = z$ for some $z \in X_i$. Hence $y_z = x$ or $y_z = y$. Therefore, either $x \notin \operatorname{ran} \beta$ or $y \notin \operatorname{ran} \beta$. So $\beta \in M$. For all $x \in X, x\beta\alpha = y_x\alpha = x\gamma$. This shows that $\beta\alpha = \gamma$, which implies $M\alpha = T_E(X, \mathcal{P})$. Therefore, α is a right magnifying element. □

The following examples illustrate the ideas of the proof given in Lemma 3.1.22.

Example 3.1.23. Let $X = \mathbb{Z}$ and $\mathcal{P}$ be a partition on X where $\mathcal{P} = \{\mathbb{Z}^-, \mathbb{Z}^+ \cup \{0\}\}$. Define a relation E on X by $E = \bigcup_{j=1}^{\infty} (A_j \times A_j)$ where $A_1 = \{-1, -3, -5, \ldots,\}$,

$A_2 = \{\pm 2, \pm 4, \pm 6, \ldots\}, A_3 = \{0\}, A_4 = \{1, 3\}, A_5 = \{5, 7\}, A_6 = \{9, 11\}, \ldots$ Clearly, E is an equivalence relation on X and $X/E = \{A_1, A_2, A_3, A_4, \ldots\}$. Let $\alpha \in T_E(X, \mathcal{P})$ be defined by $x\alpha = 4$ for all $x \in A_4$ and

$$x\alpha = \begin{cases} x - 4 & \text{if } x \in \mathbb{Z}^+ \setminus (2\mathbb{Z}^+ \cup A_4), \\ x + 4 & \text{if } x \in 4\mathbb{Z}^+, \\ x & \text{otherwise.} \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \ldots & -3 & -2 & -1 & 0 & 4 & 2 & 4 & 8 & 1 & 6 & 3 & 12 & 5 & 10 & \cdots \end{pmatrix}.$$

It is easy to see that α belongs to $T_E(X, \mathcal{P})$ and it is onto but not one-to-one. However, we can not construct the function $\beta \in T_E(X, P)$ such that $\beta\alpha = id_X$ since $(4, 8) \in E$ but there exist no $a, b \in X$ such that $(a, b) \in E$ satisfying $a\alpha = 4$ and $b\alpha = 8$.

Example 3.1.24. Let $X = \mathbb{N}$ and $\mathcal{P}$ be a partition on X such that $\mathcal{P} = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6, 7, 8, 9, \ldots\}\}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } x \equiv y \mod 2.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}\}$. It is easy to see that $\{6, 7, 8, 9, \ldots\} \in \mathcal{P}$ is infinite. Let $\alpha \in T_E(X, \mathcal{P})$ be a function defined by $1\alpha = 1, 2\alpha = 2, 3\alpha = 5, 4\alpha = 4, 5\alpha = 3, 6\alpha = 6, 7\alpha = 7$ and $x\alpha = x - 2$ for all $x \geq 8$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 5 & 4 & 3 & 6 & 7 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdots \end{pmatrix}.$$

It is obvious that α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 3.1.22, the function α is a right magnifying element. Let $M = \{\beta \in T_E(X, \mathcal{P}) \mid \beta \text{ not onto}\}$ and let γ be any fucntion in $T_E(X, \mathcal{P})$. Then there exists a function $\beta \in M$ such that $\beta\alpha = \gamma$.

We illustrate the ideas by considering the element γ of $T_E(X, \mathcal{P})$, which is defined by $1\gamma = 1, 2\gamma = 2, 3\gamma = 5, 4\gamma = 4, 5\gamma = 3$, and $x\gamma = x + 2$ for all $x \geq 6$. For convenience, we write β as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 5 & 4 & 3 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 13 & \cdots \end{pmatrix}.$$

To get the desired result, define a function β in $T_E(X, \mathcal{P})$ by

$$x\beta = \begin{cases} x & \text{if } x \leq 5, \\ x+4 & \text{if } x > 5. \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 5 & 4 & 3 & 6 & 7 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 5 & 4 & 3 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 13 & \cdots \end{pmatrix} = \gamma.$$

Theorem 3.1.25. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. A function $\alpha \in T_E(X, \mathcal{P})$ is a right magnifying element if and only if α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Proof. It follows by Corollary 3.1.18 and Lemma 3.1.22. □

Theorem 3.1.26. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that for each $x \in X$, there is exactly one $X_i \in \mathcal{P}$ with $[x]_E \subseteq X_i$. There exists a right magnifying element in $T_E(X, \mathcal{P})$ if and only if at least one element of $\mathcal{P}$ is infinite.*

Proof. The necessity is obtained by Lemma 3.1.19. On the other hand, suppose that there exists $X_i \in \mathcal{P}$ such that X_i is infinite.

Case 1: There exists $t \in X_i$ such that (X_i, t) is infinite. Then there is a proper subset A of (X_i, t) such that $|A| = |(X_i, t)| = |(X_i, t) \setminus A|$. So there is a bijection γ from A to (X_i, t). Define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in A, \\ x & \text{otherwise.} \end{cases}$$

Clearly, α belongs to $T_E(X,\mathcal{P})$ and α is onto. Since $(X \setminus (X_i \setminus A))\alpha = X$, α is not one-to-one. Then α is onto but not one-to-one. It is clear that for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.1.25, α is a right magnifying element.

Case 2: (X_i, t) is finite for all $t \in X_i$.

Case 2.1: There exists $n \in \mathbb{N}$ such that $K = \{(X_i,t) \mid t \in X_i \text{ and } |(X_i,t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijection λ from K' to K. So $|A| = |A\lambda| = n$ for all $A \in K'$. Hence for all $A \in K'$, there exists a bijective function γ_A from A to $A\lambda$. Let $\gamma = \bigcup_{A\in K'} \gamma_A$. Then γ is a bijection from $\bigcup_{A\in K'} A$ to $\bigcup_{A\in K} A$. Define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup_{A\in K'} A, \\ x & \text{otherwise.} \end{cases}$$

Clearly, α belongs to $T_E(X,\mathcal{P})$ and α is onto. Since $(X \setminus (\bigcup_{A\in K} A \setminus \bigcup_{A\in K'} A))\alpha = X$, α is not one-to-one. Then α is onto but not one-to-one. It is clear that for any $(x,y) \in E$, there exists $(a,b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.1.25, α is a right magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_i,t) \mid t \in X_i \text{ and } |(X_i,t)| = n\}$ is finite. Then for each $t \in X_i$, there exists $t' \in X_i$ such that $|(X_i,t)| < |(X_i,t')|$. Let $A = \{(X_i,t) \mid [t]_E \subseteq X_i\}$. In this case, A is an infinite set. Let $n_1 = \min_{(X_i,t)\in A} |(X_i,t)|$ and $K_1 = \{(X_i,t) \mid |(X_i,t)| = n_1\}$. Choose $(X_i,t_1) \in K_1$. Let $n_2 = \min_{(X_i,t)\in A_1} |(X_i,t)|$ where $A_1 = A \setminus K_1$ and $K_2 = \{(X_i,t) \mid |(X_i,t)| = n_2\}$. Choose $(X_i,t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_i,t_1), (X_i,t_2), \ldots, (X_i,t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min_{(X_i,t)\in A_k} |(X_i,t)|$ where $A_k = A \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_i,t_k) \in K_k$, where $K_k = \{(X_i,t) \mid |(X_i,t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < ... < n_k < ....$

Next, we let $B = \{(X_i,t_j) \mid j \geq 1\}$. Then $|(X_i,t_j)| < |(X_i,t_{j+1})|$ for all $j \geq 1$. Hence there exists a surjection $\gamma_j : (X_i,t_j) \to (X_i,t_{j-1})$ for all $j \geq 2$. Let $\gamma = \bigcup_{j\geq 2} \gamma_j$. Then γ is a surjection from $\bigcup_{C\in B} C \setminus (X_i,t_1)$ to $\bigcup_{C\in B} C$. Next, define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup_{C\in B} C \setminus (X_i,t_1), \\ x & \text{otherwise.} \end{cases}$$

Clearly, α belongs to $T_E(X,\mathcal{P})$ and α is onto. Since $(X_i,t_1)\alpha = (X_i,t_1) = (X_i,t_2)\alpha$, α is not one-to-one. Then α is onto but not one-to-one. It is clear that for any $(x,y)\in E$, there exists $(a,b)\in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.1.25, α is a right magnifying element. □

3.2 Magnifying elements in $P_E(X,\mathcal{P})$

Recall that, for an equivalence relation E on a nonempty set X, $[x]_E = \{y\in X \mid (x,y)\in E\}$ denotes the equivalence class of an element $x\in X$ determined by E, and we set $X/E = \{[x]_E \mid x\in X\}$. Let $\mathcal{P} = \{X_i \mid i\in\Lambda\}$ be a partition of $X = \{x_j \mid j\in\Lambda'\}$. Put $(X_i,x_j) = X_i\cap[x_j]_E$ for $i\in\Lambda$ and $j\in\Lambda'$. The semigroup $P_E(X,\mathcal{P}) = P_E(X)\cap P(X,\mathcal{P})$, i.e.,

$$P_E(X,\mathcal{P}) = \{\alpha\in P_E(X) \mid X_i\alpha\subseteq X_i \text{ for all } i\in\Lambda\},$$

which is a semigroup under the composition of functions. A function $\alpha\in P_E(X,\mathcal{P})$ is called a left (right) magnifying element if there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$ $(M\alpha = P_E(X,\mathcal{P}))$.

3.2.1 Left magnifying elements in $P_E(X,\mathcal{P})$

In this subsection, we provide the necessary and sufficient conditions for elements in $P_E(X,\mathcal{P})$ to be a left magnifying element and also illustrate the ideas of the main theorem and lemmas by giving the examples.

Lemma 3.2.1. If α is a left magnifying element in $P_E(X,\mathcal{P})$, then α is one-to-one and dom $\alpha = X$.

Proof. Assume that α is a left magnifying element in $P_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$. Since the identity map id_X on X belongs to $P_E(X,\mathcal{P})$, there exists $\beta\in M$ such that $\alpha\beta = id_X$. This implies that α is one-to-one and dom $\alpha = X$. □

However, the converse of Lemma 3.2.1 is not true in general since there exists no proper subset M of $P_E(X,\mathcal{P})$ such that $id_X M = P_E(X,\mathcal{P})$.

Lemma 3.2.2. Let α be a left magnifying element in $P_E(X,\mathcal{P})$. For any $x,y\in X$, $(x,y)\in E$ if and only if $(x\alpha,y\alpha)\in E$.

Proof. The necessity is obvious. Conversely, assume that α is a left magnifying element in $P_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$. Since $id_X \in P_E(X,\mathcal{P})$, there exists $\beta \in M$ such that $\alpha\beta = id_X$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. It follows that $x = xid_X = x\alpha\beta$ and $y = yid_X = y\alpha\beta$. Therefore, $(x, y) = (x\alpha\beta, y\alpha\beta) \in E$ since $\beta \in P_E(X,\mathcal{P})$. □

Lemma 3.2.3. If α is a left magnifying element in $P_E(X,\mathcal{P})$, then $\alpha M = \alpha P_E(X,\mathcal{P})$ for some proper subset M of $P_E(X,\mathcal{P})$.

Proof. Assume that α is a left magnifying element in $P_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$. It is clear that $\alpha M \subseteq \alpha P_E(X,\mathcal{P})$ and $\alpha P_E(X,\mathcal{P}) \subseteq P_E(X,\mathcal{P}) = \alpha M$. This shows that $\alpha M = \alpha P_E(X,\mathcal{P})$. □

Lemma 3.2.4. If $\alpha \in P_E(X,\mathcal{P})$ is bijective on X, then α is not a left magnifying element.

Proof. Assume that $\alpha \in P_E(X,\mathcal{P})$ is bijective on X. So α^{-1} is also bijective on X. Suppose to the contrary that α is a left magnifying element. By definition, there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $\alpha M = P_E(X,\mathcal{P})$. By Lemma 3.2.3, we have $\alpha M = \alpha P_E(X,\mathcal{P})$. Then $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha P_E(X,\mathcal{P}) = P_E(X,\mathcal{P})$, which is a contradiction. Therefore, α is not a left magnifying element. □

The next corollary follows by Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.4.

Corollary 3.2.5. *If α is a left magnifying element in $P_E(X,\mathcal{P})$, then α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Lemma 3.2.6. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If $X_i \in \mathcal{P}$ is finite for all $i \in \Lambda$, then there exists no left magnifying element in $P_E(X,\mathcal{P})$.

Proof. Suppose to the contrary that there is a left magnifying element α in $P_E(X,\mathcal{P})$. By assumption and Lemma 3.2.1, we have $\alpha|_{X_i}$ is bijective for all $i \in \Lambda$. Since dom $\alpha = X$ and $X\alpha = \big(\bigcup_{i\in\Lambda} X_i\big)\alpha = \bigcup_{i\in\Lambda} X_i\alpha = \bigcup_{i\in\Lambda} X_i = X$, α is onto which is a contradiction. □

It is noticeable in Lemma 3.2.6 that if a left magnifying element exists in $P_E(X,\mathcal{P})$, then $X_i \in \mathcal{P}$ is infinite for some $i \in \Lambda$. However, the converse of this statement is not true in general. It is illustrated by the following counterexample.

Example 3.2.7. Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N} \cup \{0\}\}$ be a partition on X where $X_0 = \{\pm(2n-1), \mid n \in \mathbb{N}\} \cup \{0\}$, that is, $X_0 = \{0, \pm1, \pm3, \pm5, \ldots\}$ and $X_i = \{\pm 2i\}$ for all $i \in \mathbb{N}$, that is, $X_1 = \{\pm 2\}$, $X_2 = \{\pm 4\}$, $X_3 = \{\pm 6\}, \ldots$. Define a relation E on X by $E = \bigcup_{j=1}^{\infty}(A_j \times A_j)$ where $A_1 = \{0, \pm1, \pm2\}$ and $A_j = \{\pm(2j-1), \pm 2j\}$ for all possitive integers $j \geq 2$, that is, $A_2 = \{\pm3, \pm4\}$, $A_3 = \{\pm5, \pm6\}$, $A_4 = \{\pm7, \pm8\}, \ldots$. Clearly, $X_0 \in \mathcal{P}$ is infinite and E is an equivalence relation on X. We can see that every injection on X in $P_E(X, \mathcal{P})$ is surjective on X. Therefore, there exists no left magnifying element in $P_E(X, \mathcal{P})$.

Corollary 3.2.8. *If X is a finite set, then $P_E(X, \mathcal{P})$ has no left magnifying element.*

Lemma 3.2.9. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in P_E(X, \mathcal{P})$ is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$, then α is a left magnifying element.

Proof. Let $\alpha \in P_E(X, \mathcal{P})$ be one-to-one but not onto, dom $\alpha = X$ and for $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$ and $M = \{\beta \in P_E(X, \mathcal{P}) \mid \text{dom } \beta \subseteq \text{ran } \alpha\}$. Since α is not onto, ran $\alpha \neq X$ and hence dom $\beta \neq X$ for all $\beta \in M$. So M is a proper subset of $P_E(X, \mathcal{P})$ since id_X does not belong to M. To show that $\alpha M = P_E(X, \mathcal{P})$, let $\gamma \in P_E(X, \mathcal{P})$. For each $x \in (\text{dom}\gamma)\alpha$, there exists $y_x \in$ dom γ such that $y_x\alpha = x$. Define $\beta \in P(X)$ by $x\beta = y_x\gamma$ if $x \in (\text{dom}\gamma)\alpha$. To claim that $\beta \in P_E(X)$, let $a, b \in (\text{dom}\gamma)\alpha$ such that $(a, b) \in E$. Then there exist $y_a, y_b \in$ dom γ such that $y_a\alpha = a$ and $y_b\alpha = b$. By assumption, $(y_a, y_b) \in E$. Then $(a\beta, b\beta) = (y_a\gamma, y_b\gamma) \in E$ since $\gamma \in P_E(X, \mathcal{P})$. Next, let $x \in (\text{dom } \gamma)\alpha$ be such that $x \in X_i$ for some $X_i \in \mathcal{P}$. Then there exists $y_x \in$ dom γ and $y_x \in X_i$ such that $y_x\alpha = x$. Thus $x\beta = y_x\gamma \in X_i$ since $\gamma \in P_E(X, \mathcal{P})$. Thus $\beta \in P_E(X, \mathcal{P})$. Clearly, dom $\beta \subseteq$ ran α. So $\beta \in M$. Since $y_{x\alpha}\alpha = x\alpha$ and α is one-to-one, $y_{x\alpha} = x$. For any $x \in$ dom γ, we have $x\alpha\beta = y_{x\alpha}\gamma = x\gamma$. This shows that $\alpha\beta = \gamma$, which implies $M\alpha = T_E(X, \mathcal{P})$. Therefore, α is a left magnifying element. □

The next examples illustrates the ideas of the proof given in Lemma 3.2.9.

Example 3.2.10. Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6, 7, 8, 9, \ldots\}\}$ be a partition on X. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1,2\},\{3,4,5\},\{6,7,8\},\ldots\}$. It is easy to see that $\{6,7,8,9,\ldots\} \in \mathcal{P}$ is infinite. Let $\alpha \in P_E(X,\mathcal{P})$ be defined by

$$x\alpha = \begin{cases} x & \text{if } x \leq 5, \\ x+3 & \text{if } x > 5. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \cdots \end{pmatrix}.$$

It is obvious that α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, if $(x\alpha, y\alpha) \in E$, then $(x, y) \in E$. By Lemma 3.2.9, the function α is a left magnifying element. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \text{dom}\,\beta \subseteq \mathbb{N}\backslash\{6,7,8\}\}$ and let γ be any function in $P_E(X,\mathcal{P})$. Then there exists an element $\beta \in M$ such that $\alpha\beta = \gamma$. Consider the element $\gamma \in P_E(X,\mathcal{P})$, which is defined by

$$x\gamma = \begin{cases} x & \text{if } x \leq 4, \\ x-3 & \text{if } x > 8. \end{cases}$$

For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & - & - & - & - & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix}.$$

We illustrate the ideas by considering $9, 10 \in \text{dom}\,\gamma$. Hence $12, 13 \in (\text{dom}\,\gamma)\alpha$ such that $y_{12} = 9$ and $y_{13} = 10$. Therefore, $12\beta = y_{12}\gamma = 9\gamma = 6$ and $13\beta = y_{13}\gamma = 10\gamma = 7$. To get the desired result, define a function β in $P_E(X,\mathcal{P})$ by

$$x\beta = \begin{cases} x & \text{if } x \leq 4, \\ x-6 & \text{if } x > 11. \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & - & - & - & - & - & - & - & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

Clearly, $\beta \in M$ and we have

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & - & - & - & - & - & - & - & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & - & - & - & - & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix} = \gamma.$$

Theorem 3.2.11. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. A function $\alpha \in P_E(X, \mathcal{P})$ is a left magnifying element if and only if α is one-to-one but not onto, dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.*

Proof. It follows by Corollary 3.2.5 and Lemma 3.2.9. □

Theorem 3.2.12. *Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition and E be an equivalence relation on a set X such that for each $x \in X$, there is exactly one $X_i \in \mathcal{P}$ with $[x]_E \subseteq X_i$. There exists a left magnifying element in $P_E(X, \mathcal{P})$ if and only if at least one element of $\mathcal{P}$ is infinite.*

Proof. The necessity is obtained by Lemma 3.2.6. On the other hand, suppose that there exists $X_i \in \mathcal{P}$ such that X_i is infinite.

Case 1: There exists $t \in X_i$ such that (X_i, t) is infinite. Then there is a proper subset A of (X_i, t) such that $|A| = |(X_i, t)| = |(X_i, t) \setminus A|$. So there is a bijection γ from (X_i, t) to A. Define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in (X_i, t), \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X, \mathcal{P})$ and α is one-to-one. Hence ran $\alpha \subseteq X \setminus ((X_i, t) \setminus A) \neq X$. Then α is one-to-one but not onto. It is clear that dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.2.11, α is a left magnifying element.

Case 2: (X_i, t) is finite for all $t \in X$.

Case 2.1: There exists $n \in \mathbb{N}$ such that $K = \{(X_i, t) \mid t \in X_i \text{ and } |(X_i, t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijection λ from K to K'. So $|A| = |A\lambda| = n$ for all $A \in K$. Hence for all $A \in K$, there exists a bijection η_A from A to $A\lambda$. Let $\eta = \bigcup_{A \in K} \eta_A$. Then η is a bijection from $\bigcup_{A \in K} A$ to $\bigcup_{A \in K'} A$. Define a function α by

$$x\alpha = \begin{cases} x\eta & \text{if } x \in \bigcup_{A \in K} A, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X,\mathcal{P})$ and α is one-to-one. Since ran $\alpha = X \setminus \bigcup\limits_{A\in K\setminus K'} A \neq X$, α is not onto. Then α is one-to-one but not onto. It is clear that dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.2.11, α is a left magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_i, t) \mid t \in X_i$ and $|(X_i, t)| = n\}$ is finite. Then for each $t \in X_i$, there exists $t' \in X_i$ such that $|(X_i, t)| < |(X_i, t')|$. Let $A = \{(X_i, t) \mid [t]_E \subseteq X_i\}$. In this case, A is an infinite set. Let $n_1 = \min\limits_{(X_i,t)\in A} |(X_i, t)|$ and $K_1 = \{(X_i, t) \mid |(X_i, t)| = n_1\}$. Choose $(X_i, t_1) \in K_1$. Let $n_2 = \min\limits_{(X_i,t)\in A_1} |(X_i, t)|$ where $A_1 = A \setminus K_1$ and $K_2 = \{(X_i, t) \mid |(X_i, t)| = n_2\}$. Choose $(X_i, t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_i, t_1), (X_i, t_2), \ldots, (X_i, t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min\limits_{(X_i,t)\in A_k} |(X_i, t)|$ where $A_k = A \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_i, t_k) \in K_k$, where $K_k = \{(X_i, t) \mid |(X_i, t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < ... < n_k < ....$

Next, we let $B = \{(X_i, t_j) \mid j \geq 1\}$. Then $|(X_i, t_j)| < |(X_i, t_{j+1})|$ for all $j \geq 1$. Hence there exists an injection $\gamma_j : (X_i, t_j) \to (X_i, t_{j+1})$. Let $\gamma = \bigcup\limits_{j\geq 1} \gamma_j$. Then γ is one-to-one on $\bigcup\limits_{C\in B} C$. Next, define a function α by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup\limits_{C\in B} C, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in P_E(X,\mathcal{P})$ and α is one-to-one. Since ran $\alpha \subseteq X \setminus (X_i, t_1) \neq X$, α is not onto. Then α is one-to-one but not onto. It is clear that dom $\alpha = X$ and for any $x, y \in X$, $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$. By Theorem 3.2.11 , α is a left magnifying element. □

3.2.2 Right magnifying elements in $P_E(X,\mathcal{P})$

In this subsection, we provide the necessary and sufficient conditions for elements in $P_E(X,\mathcal{P})$ to be a right magnifying element and also illustrste the ideas of the main theorem and lemmas by giving the examples.

Lemma 3.2.13. If α is a right magnifying element in $P_E(X,\mathcal{P})$, then α is onto.

Proof. Assume that α is a right magnifying element in $P_E(X,\mathcal{P})$. By definition, there exists a proper subset M of $P_E(X,\mathcal{P})$ such that $M\alpha = P_E(X,\mathcal{P})$. Since the identity map id_X on X belongs to $P_E(X, \mathcal{P})$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. This implies that α is onto. $\square$

Lemma 3.2.14. Let α be a right magnifying element in $P_E(X, \mathcal{P})$. For any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.

Proof. Assume that α is a right magnifying element of $P_E(X, \mathcal{P})$. By definition, there exists a proper subset M of $P_E(X, \mathcal{P})$ such that $M\alpha = P_E(X, \mathcal{P})$. Since $id_X \in P_E(X, \mathcal{P})$, there exists $\beta \in M$ such that $\beta\alpha = id_X$. Let $x, y \in X$ be such that $(x, y) \in E$. It follows that $x = xid_X = x\beta\alpha$ and $y = yid_X = y\beta\alpha$. Choose $a = x\beta$ and $b = y\beta$. Clearly, $(a, b) = (x\beta, y\beta) \in E$ since $\beta \in P_E(X, \mathcal{P})$. Therefore, the proof is completed. $\square$

Lemma 3.2.15. If $\alpha \in P_E(X)$ is a right magnifying element, then $M\alpha = P_E(X, \mathcal{P})\alpha$ for some proper subset M of $P_E(X, \mathcal{P})$.

Proof. Assume that α is a right magnifying element in $P_E(X, \mathcal{P})$. By definition, there exists a proper subset M of $P_E(X, \mathcal{P})$ such that $M\alpha = P_E(X, \mathcal{P})$. It is clear that $M\alpha \subseteq P_E(X, \mathcal{P})\alpha$ and $P_E(X, \mathcal{P})\alpha \subseteq P_E(X, \mathcal{P}) = M\alpha$. This shows that $M\alpha = P_E(X, \mathcal{P})\alpha$. $\square$

Lemma 3.2.16. If $\alpha \in P_E(X, \mathcal{P})$ is bijective on X, then α is not a right magnifying element.

Proof. Assume that $\alpha \in P_E(X, \mathcal{P})$ is bijective on X. So α^{-1} is also bijective on X. Suppose to the contrary that α is a right magnifying element. By definition, there is a proper subset M of $P_E(X, \mathcal{P})$ such that $M\alpha = P_E(X, \mathcal{P})$. By Lemma 3.2.15, we have $M\alpha = P_E(X, \mathcal{P})\alpha$. Then $M = M\alpha\alpha^{-1} = P_E(X, \mathcal{P})\alpha\alpha^{-1} = P_E(X, \mathcal{P})$, which is a contradiction. Therefore, α is not a right magnifying element. $\square$

The next corollary follows by Lemma 3.2.13, Lemma 3.2.14 and Lemma 3.2.16.

Corollary 3.2.17. *If α is a right magnifying element in $P_E(X, \mathcal{P})$ and dom $\alpha = X$, then α is onto but not one-to-one and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$.*

Lemma 3.2.18. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X. If X_i is finite for all $i \in \Lambda$, then there exists no right magnifying element in $P_E(X, \mathcal{P})$.

Proof. Suppose to the contrary that there is a right magnifying element $\alpha \in P_E(X, \mathcal{P})$. By Lemma 3.2.13, α is onto and hence dom $\alpha = X$ since $X_i\alpha \subseteq X_i$ and X_i is finite for all $i \in \Lambda$. So $\alpha|_{X_i}$ is onto X_i and hence $\alpha|_{X_i}$ is bijective on X_i. Since $X\alpha = \big(\bigcup_{i\in\Lambda} X_i\big)\alpha = \bigcup_{i\in\Lambda} X_i\alpha = X$, α is one-to-one on X which is a contradiction. □

It is noticeable in Lemma 3.2.18 that if a right magnifying element exists in $P_E(X,\mathcal{P})$, then X_i is infinite for some $i \in \Lambda$. However, the converse of this statement is not true in general. It is illustrated by a following counterexample.

Example 3.2.19. Let $X = \mathbb{Z}$ and $\mathcal{P} = \{X_i \mid i \in \mathbb{N} \cup \{0\}\}$ be a partition on X where $X_0 = \{0, -1, -2, \ldots\}$ and $X_i = \{2i-1, 2i\}$ for all $i \in \mathbb{N}$, that is, $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $X_3 = \{5, 6\}, \ldots$. Define a relation E on X by $E = \bigcup_{j=1}^{\infty} (A_j \times A_j)$ where $A_1 = \{0, \pm 1, \pm 2\}$ and $A_j = \{\pm(2j-1), \pm 2j\}$ for all possitive integers $j \geq 2$, that is, $A_2 = \{\pm 3, \pm 4\}$, $A_3 = \{\pm 5, \pm 6\}$, $X_4 = \{\pm 7, \pm 8\}, \ldots$. It is easy to verify that every surjection in $P_E(X,\mathcal{P})$ is bijctive on X. Therefore, there exists no right magnifying element in $P_E(X,\mathcal{P})$.

Corollary 3.2.20. *If X is a finite set, then $P_E(X,\mathcal{P})$ has no right magnifying element.*

Lemma 3.2.21. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in P_E(X,\mathcal{P})$ is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in P_E(X,\mathcal{P})$ is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $T_E(X,\mathcal{P})$ since the identity map id_X on X does not belong to M. Let γ be a function in $P_E(X,\mathcal{P})$. Since α is onto, for each $x \in$ dom γ such that $x \in X_i$ for some $X_i \in \mathcal{P}$, there exists $y_x \in X_i$ such that $y_x\alpha = x\gamma$ (if $x_1\gamma = x_2\gamma$, then choose $y_{x_1} = y_{x_2}$ and if $(a\gamma, a\gamma) \in E$, then choose $(y_a, y_a) \in E$). Define $\beta \in P(X)$ by $x\beta = y_x$ for all $x \in$ dom γ. To show that $\beta \in P_E(X,\mathcal{P})$, let $a, b \in X$ be such that $(a, b) \in E$. Since $\gamma \in P_E(X,\mathcal{P})$, $(a\gamma, b\gamma) \in E$. By assumption, we can choose $(y_a, y_b) \in E$ such that $y_a\alpha = a\gamma$ and $y_b\alpha = b\gamma$. Then $(a\beta, b\beta) = (y_a, y_b) \in E$. Since α is not one-to-one, there are distinct elements $x, y \in X$ such that $x\alpha = y\alpha$. Thus at least one of x and y does not belong to ran β. So $\beta \in M$. For all $x \in$ dom γ, we see that $x\beta\alpha = y_x\alpha = x\gamma$. Therefore, α is a right magnifying element. □

Example 3.2.22. Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1, 3, 5, 7, 9, \ldots\}, \{2, 4, 6, 8, 10, \ldots\}\}$ be a partition on X. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1,2\},\{3,4,5\},\{6,7,8\},\ldots\}$. We now see that $\{1,3,5,7,9,\ldots\} \in \mathcal{P}$ is infinite. Let $\alpha \in P_E(X,\mathcal{P})$ be defined by

$$x\alpha = \begin{cases} x & \text{if } x \leq 8, \\ x - 6 & \text{if } x > 8. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

It is obvious that α is onto but not one-to-one, dom $\alpha = X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 3.2.21, the function α is a right magnifying element. Let $M = \{\beta \in P_E(X,\mathcal{P}) \mid \beta \text{ is not onto}\}$ and let γ be any function in $P_E(X,\mathcal{P})$. Then there exists an element $\beta \in M$ such that $\beta\alpha = \gamma$. Consider the element $\gamma \in P_E(X,\mathcal{P})$, which is defined by

$$x\gamma = \begin{cases} x & \text{if } x \leq 5, \\ x - 12 & \text{if } x \geq 15. \end{cases}$$

For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & \cdots & - & - & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix}.$$

We illustrate the idea by considering $3, 4, 15, 16 \in$ dom γ. It is easy to see that $3\gamma = 15\gamma = 3$ and $4\gamma = 16\gamma = 4$. Now we have 2 choices of each y_3 and y_4, i.e., $y_3 = 3$ or 9 and $y_4 = 4$ or 10. If we follow the proof of Lemma 3.2.21, then we choose $y_3 = y_{15} = 9$. Since $(3\gamma, 4\gamma) \in E$, we must choose $y_4 = y_{16} = 10$. To get the desired result, define a function β in $P_E(X,\mathcal{P})$ by $3\beta = 15\beta = 9$, $4\beta = 16\beta = 10$, $5\beta = 17\beta = 11$ and

$$x\beta = \begin{cases} x & \text{if } x \leq 2, \\ x - 6 & \text{if } x \geq 18. \end{cases}$$

For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & \cdots \\ 1 & 2 & 9 & 10 & 11 & - & \cdots & - & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & \cdots \\ 1 & 2 & 9 & 10 & 11 & - & \cdots & - & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & \cdots & - & - & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix} = \gamma.$$

Lemma 3.2.23. Let $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on a set X such that X_i is infinite for some $i \in \Lambda$. If $\alpha \in P_E(X, \mathcal{P})$ is onto, dom $\alpha \neq X$ and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$, then α is a right magnifying element.

Proof. Assume that $\alpha \in P_E(X, \mathcal{P})$ is onto, dom $\alpha \neq X$ and for any $(x, y) \in E$, there is $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $T_E(X, \mathcal{P})$ since the identity map id_X on X does not belong to M. Let γ be a function in $P_E(X, \mathcal{P})$. Since α is onto, for each $x \in$ dom γ, there exists $y_x \in X$ such that $y_x\alpha = x\gamma$ (if $x_1\gamma = x_2\gamma$, we must choose $y_{x_1} = y_{x_2}$ and if $(a\gamma, b\gamma) \in E$, we must choose $(y_a, y_b) \in E$). Define a function $\beta \in P(X)$ by $x\beta = y_x$ for all $x \in$ dom γ. To show that $\beta \in P_E(X)$, let $a, b \in$ dom γ be such that $(a, b) \in E$. Then $(a\gamma, b\gamma) \in E$ since $\gamma \in P_E(X, \mathcal{P})$. By assumption, there exists $(y_a, y_b) \in E$ such that $y_a\alpha = a\gamma$ and $y_b\alpha = b\gamma$. Let $a \in$ dom γ be such that $a \in X_i$ and hence $a\gamma \in X_i$. Then there exists $y_a \in X_i$ such that $y_{a\gamma}\alpha = a\gamma$. So $a\beta = y_a \in X_i$. Since ran $\beta \subseteq$ dom $\alpha \neq X$, β is not onto. Thus $\beta \in M$. For all $x \in$ dom γ, we see that $x\beta\alpha = y_x\alpha = x\gamma$. This shows that $\beta\alpha = \gamma$, which implies $M\alpha = P_E(X, \mathcal{P})$. Therefore, α is a right magnifying element. $\square$

Example 3.2.24. Let $X = \mathbb{N}$ and $\mathcal{P} = \{\{1, 3, 5, 7, 9, \ldots\}, \{2, 4, 6, 8, 10, \ldots\}\}$ be a partition on X. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \ldots\}$. We now see that $\{1, 3, 5, 7, 9, \ldots\} \in \mathcal{P}$ is infinite. Let α be a function defined by

$$x\alpha = \begin{cases} x & \text{if } x \leq 5, \\ x - 6 & \text{if } x \geq 9. \end{cases}$$

For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & - & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

It is obvious that $\alpha \in P_E(X, \mathcal{P})$ is onto, dom $\alpha \neq X$, and and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 3.2.23, the function α is a right magnifying element. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta \text{ is not onto}\}$ and let γ be any function in $P_E(X, \mathcal{P})$. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element $\gamma \in P_E(X, \mathcal{P})$, which is defined by $x\gamma = x$ for all odd positive integers. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & - & 3 & - & 5 & - & 7 & - & 9 & - & 11 & - & 13 & - & 15 & \cdots \end{pmatrix}.$$

To get the desired result, define a function β in $P_E(X, \mathcal{P})$ by $1\beta = 1$ and $x\beta = x + 6$ for all odd positive integers. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & - & 9 & - & 11 & - & 13 & - & 15 & - & 17 & - & 19 & - & 21 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & - & 9 & - & 11 & - & 13 & - & 15 & - & 17 & - & 19 & - & 21 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & - & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & - & 3 & - & 5 & - & 7 & - & 9 & - & 11 & - & 13 & - & 15 & \cdots \end{pmatrix} = \gamma.$$

The next example shows that α is a right magnifying element such that dom $\alpha \neq X$ and α is bijective.

Example 3.2.25. Let $X = \mathbb{N}$ and $\mathcal{P} = \{X_1, X_2\}$ be a partition on X such that $X_1 = \{1, 3, 5, 7, 9, \ldots\}$ and $X_2 = \{2, 4, 6, 8, 10, \ldots\}$. Define a relation E on X by

$$(x, y) \in E \text{ if and only if } \lfloor \frac{x}{3} \rfloor = \lfloor \frac{y}{3} \rfloor.$$

Clearly, E is an equivalence relation on X and $X/E = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \ldots\}$. We now see that $X_1 \in \mathcal{P}$ is infinite. Let α be a function defined by $3\alpha = 1$, $4\alpha = 2$ and

$$x\alpha = \begin{cases} x & \text{if } |(X_1, x)| = 1, \\ x - 6 & \text{if } |(X_1, x)| = 2. \end{cases}$$

for all $x > 5$. For convenience, we write α as

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ - & - & 1 & 2 & - & 6 & 7 & 8 & 3 & 4 & 5 & 12 & 13 & 14 & 9 & \cdots \end{pmatrix}.$$

It is obvious that $\alpha \in P_E(X, \mathcal{P})$ is bijective, dom $\alpha \neq X$, and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Lemma 3.2.23, the function α is a right magnifying element. Let $M = \{\beta \in P_E(X, \mathcal{P}) \mid \beta \text{ is not onto}\}$ and let γ be any function in $P_E(X, \mathcal{P})$. Then there exists $\beta \in M$ such that $\beta\alpha = \gamma$.

We will illustrate the ideas by considering the element $\gamma \in P_E(X, \mathcal{P})$, which is defined by $x\gamma = x$ if $x \leq 5$ and $x\gamma = x - 6$ if $x \geq 12$. For convenience, we write γ as

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & - & - & - & - & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

To get the desired result, define a function $\beta \in P_E(X, \mathcal{P})$ by $x\beta = x + 2$ if $x = 1, 2$, $x\beta = x + 6$ if $x = 3, 4, 5$ and

$$x\beta = \begin{cases} x & \text{if } |(X_1, x)| = 2, \\ x - 6 & \text{if } |(X_1, x)| = 1. \end{cases}$$

for all $x > 11$. For convenience, we write β as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 3 & 4 & 9 & 10 & 11 & - & - & - & - & - & - & 6 & 7 & 8 & 15 & \cdots \end{pmatrix}.$$

So $\beta \in M$ and we have

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 3 & 4 & 9 & 10 & 11 & - & - & - & - & - & - & 6 & 7 & 8 & 15 & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ - & - & 1 & 2 & - & 6 & 7 & 8 & 3 & 4 & 5 & 12 & 13 & 14 & 9 & \cdots \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\ 1 & 2 & 3 & 4 & 5 & - & - & - & - & - & - & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} = \gamma.$$

Theorem 3.2.26. *Let E be an equivalence relation on a set X and $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ be a partition on X such that X_i is infinite for some $i \in \Lambda$. A function $\alpha \in P_E(X, \mathcal{P})$ is a right magnifying element if and only if α is onto, for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$ and either*

1. *dom* $\alpha \neq X$ *or*

2. *dom* $\alpha = X$ *and* α *is not one-to-one.*

Proof. If follows by Corollary 3.2.17, Lemma 3.2.21 and Lemma 3.2.23. $\square$

Theorem 3.2.27. *Let* $\mathcal{P} = \{X_i \mid i \in \Lambda\}$ *be a partition and* E *be an equivalence relation on a set* X *such that for each* $x \in X$*, there is exactly one* $X_i \in \mathcal{P}$ *with* $[x]_E \subseteq X_i$*. There exists a right magnifying element in* $P_E(X, \mathcal{P})$ *if and only if at least one element of* $\mathcal{P}$ *is infinite.*

Proof. The necessity is obtained by Lemma 3.2.18. On the other hand, suppose that there exists $X_i \in \mathcal{P}$ such that X_i is infinite.
Case 1: There exists $t \in X_i$ such that (X_i, t) is infinite. Then there is a proper subset A of (X_i, t) such that $|A| = |(X_i, t)| = |(X_i, t) \setminus A|$. So there is a bijective function γ from A to (X_i, t). Define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in A, \\ x & \text{if } x \in X \setminus (X_i, t). \end{cases}$$

Clearly, dom $\alpha \neq X$, α is onto and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.2.26, α is a right magnifying element.
Case 2: (X_i, t) is finite for all $t \in X_i$.

Case 2.1: There exists $n \in \mathbb{N}$ such that $K = \{(X_i, t) \mid t \in X_i \text{ and } |(X_i, t)| = n\}$ is infinite. Then there exists a proper subset K' of K such that $|K'| = |K| = |K \setminus K'|$. There is a bijective function λ from K' to K. So $|A| = |A\lambda| = n$ for all $A \in K'$. Hence for all $A \in K'$, there exists a bijection γ_A from A to $A\lambda$. Let $\gamma = \bigcup_{A \in K'} \gamma_A$. Then γ is a bijection from $\bigcup_{A \in K'} A$ to $\bigcup_{A \in K} A$. Define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup_{A \in K'} A, \\ x & \text{if } x \notin \bigcup_{A \in K} A. \end{cases}$$

Clearly, dom $\alpha \neq X$, α is onto and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.2.26 , α is a right magnifying element.

Case 2.2: For all $n \in \mathbb{N}$, the set $K = \{(X_i, t) \mid t \in X_i \text{ and } |(X_i, t)| = n\}$ is finite. Then for each $t \in X_i$, there exists $t' \in X_i$ such that $|(X_i, t)| < |(X_i, t')|$. Let $A = \{(X_i, t) \mid [t]_E \subseteq X_i\}$. In this case, A is an infinite set. Let $n_1 = \min_{(X_i, t) \in A} |(X_i, t)|$

and $K_1 = \{(X_i, t) \mid |(X_i, t)| = n_1\}$. Choose $(X_i, t_1) \in K_1$. Let $n_2 = \min\limits_{(X_i,t)\in A_1} |(X_i, t)|$ where $A_1 = A \setminus K_1$ and $K_2 = \{(X_i, t) \mid |(X_i, t)| = n_2\}$. Choose $(X_i, t_2) \in K_2$. Proceeding in this way, we obtain the sets $(X_i, t_1), (X_i, t_2), \ldots, (X_i, t_k), \ldots$ and positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $n_k = \min\limits_{(X_i,t)\in A_k} |(X_i, t)|$ where $A_k = A \setminus \bigcup_{l=1}^{k-1} K_l$ and $(X_i, t_k) \in K_k$, where $K_k = \{(X_i, t) \mid |(X_i, t)| = n_k\}$ for all $k \geq 2$. Clearly, $n_1 < n_2 < ... < n_k < ....$

Next, we let $B = \{(X_i, t_j) \mid j \geq 1\}$. Then $|(X_i, t_j)| < |(X_i, t_{j+1})|$ for all $j \geq 1$. Hence there exists a surjection $\gamma_j : (X_i, t_j) \to (X_i, t_{j-1})$ for all $j \geq 2$. Let $\gamma = \bigcup\limits_{j\geq 2} \gamma_j$. Then γ is a surjection from $\bigcup\limits_{C\in B} C \setminus (X_i, t_1)$ to $\bigcup\limits_{C\in B} C$. Next, define a function $\alpha \in P_E(X, \mathcal{P})$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in \bigcup\limits_{C\in B} C \setminus (X_i, t_1), \\ x & \text{if } x \in X \setminus \bigcup\limits_{C\in B} C. \end{cases}$$

Clearly, dom $\alpha \neq X$, α is a onto and for any $(x, y) \in E$, there exists $(a, b) \in E$ such that $x = a\alpha$ and $y = b\alpha$. By Theorem 3.2.26 , α is a right magnifying element. $\square$

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