Quasi – ideals of Γ -semigroups

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ชื่อวิทยานิพนธ์ ควอซี-ไอดีลของแกมมากึ่งกรุป

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บทคัดย่อ

วิทยานิพนธ์ฉบับนี้ เราศึกษาควอซี-ไอดีล ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่ม และความสัมพันธ์ระหว่างควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มกับความสัมพันธ์กรีนของแกมมากึ่งกรุป เราศึกษา m-ไอดีลซ้าย n-ไอดีลขวา และ (m, n)-ควอซี-ไอดีลของแกมมากึ่งกรุป เราได้ขยายทฤษฎีสำหรับไอดีลซ้าย ไอดีลขวา และควอซี-ไอดีล ของแกมมากึ่งกรุป เราแสดงได้ว่า ทุก (m, n)-ควอซี-ไอดีลของแกมมากึ่งกรุป S สามารถเขียนได้เป็นอินเตอร์เซกชันของ m-ไอดีลซ้าย และ n-ไอดีลขวาของ S และทุก (m, n)-ควอซี-ไอดีลเล็กสุดเฉพาะกลุ่มของกึ่งกรุป S สามารถเขียนได้เป็นอินเตอร์เซกชันของ m-ไอดีลซ้ายเล็กสุดเฉพาะกลุ่ม และ n-ไอดีลขวาเล็กสุดเฉพาะกลุ่ม S ยิ่งไปกว่านั้น เราได้ศึกษาประมาณค่าขอบเขตล่างหยาบและประมาณค่าขอบเขตบนหยาบของควอซี-ไอดีล และประมาณค่าขอบเขตล่างหยาบและประมาณค่าขอบเขตบนหยาบของ (m, n)-ควอซี-ไอดีลของแกมมากึ่งกรุป

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ABSTRACT

In this thesis, we study quasi-ideals, minimal quasi-ideals and relationship between minimal quasi-ideals and Green's relations of Γ-semigroups. We study m-left ideals, n-right ideals and (m, n)-quasi-ideals of Γ-semigroups. We generalize some theorems for left ideals, right ideals and quasi-ideals of Γ-semigroups. We show that every (m, n)-quasi-ideal of a Γ-semigroup S can be written as the intersection of an m-left ideal and an n-right ideal of S and every minimal (m, n)-quasi-ideal of a Γ-semigroup S can be written as the intersection of a minimal m-left ideal and a minimal n-right ideal of S. Moreover, we investigate Θ-lower and Θ-upper rough quasi-ideals and (m, n)-quasi-ideals of Γ-semigroups.

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CONTENTS

CHAPTER 1

Introduction and Preliminaries

The notion of quasi-ideals for semigroups was introduced by O. Steinfeld (Steinfeld, 1956) in 1956. The class of quasi-ideals in semigroups is a generalization of one-sided ideals in semigroups.

The notion of a Γ-semigroup was introduced by M. K. Sen (Sen, 1981) in 1981. Γ-semigroups generalize semigroups. Many classical notions of semigroups have been extended to Γ-semigroups.

In this thesis, we study quasi-ideals and minimal quasi-ideals of Γ-semigroups. We study (m, n)-quasi-ideals and minimal (m, n)-quasi-ideals of Γ-semigroups. Moreover, we investigate Θ-lower and Θ-upper rough quasi-ideals and (m, n)-quasi-ideals of Γ-semigroups.

1.1 Semigroups

We will use the notation and terminology of Howie (Howie, 1976) to introduce the notion of semigroups.

Definition 1.1. Let S be a nonempty set and $*$ be a binary operation on S. Then $(S, *)$ is called a *semigroup* if $*$ is associative, i.e.,

$$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in S.$$

We give some examples of semigroups.

Example 1.1. $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}, \times)$ and $(\mathbb{R}, \times)$ are semigroups where $+$ is the usual addition and $\times$ is the usual multiplication.

Example 1.2. $(\mathbb{Z}, -)$ is not a semigroup since for $a, b, c \in \mathbb{Z}$ such that $c \neq 0$, we have

$$a - (b - c) = a - b + c \neq a - b - c = (a - b) - c.$$

Definition 1.2. Let S be a semigroup. A nonempty subset K of S is called a *subsemigroup* of S if it is closed under the binary operation of S, i.e.

$$ab \in K \quad \text{for all } a, b \in K.$$

Example 1.3. $(\mathbb{N}, +)$ is a subsemigroup of $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \times)$ is a subsemigroup of $(\mathbb{R}, \times)$ where $+$ is the usual addition and $\times$ is the usual multiplication.

In 1956, O. Steinfeld (Steinfeld, 1956) introduced the notion of quasi-ideals of semigroups as follows:

Definition 1.3. Let S be a semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$.

Let Q be a quasi-ideal of S. Then $Q^2 \subseteq SQ \cap QS \subseteq Q$. Hence Q is a subsemigroup of S.

Example 1.4. Let $S = [0, 1]$. Then S is a semigroup under the usual multiplication. Let $Q = [0, \frac{1}{2}]$. Thus $SQ \cap QS = [0, \frac{1}{2}] \subseteq Q$. Therefore, Q is a quasi-ideal of S.

Definition 1.4. Let S be a semigroup. A nonempty subset L of S is called a *left ideal* of S if $SL \subseteq L$. A nonempty subset R of S is called a *right ideal* of S if $RS \subseteq R$. A nonempty subset I of S is called an *ideal* of S if I is a left and right ideals of S.

Clearly, every left ideal, right ideal and ideal of a semigroup S is a subsemigroup of S.

Theorem 1.1. *Let S be a semigroup. Let L and R be a left ideal and a right ideal of S, respectively. Then $L \cap R$ is a quasi-ideal of S.*

Proof. Let L and R be any left-ideal and any right-ideal of a semigroup S, respectively. By properties of L and R, we have $RL \subseteq L \cap R$. This implies that $L \cap R$ is a nonempty set. We have that

$$S(L \cap R) \cap (L \cap R)S \subseteq SL \cap RS \subseteq L \cap R.$$

Hence, $L \cap R$ is a quasi-ideal of S. □

Theorem 1.2. *Every quasi-ideal Q of a semigroup S is the intersection of a left ideal and a right ideal of S.*

Proof. Let Q be any quasi-ideal of a semigroup S.
Let $L = Q \cup SQ$ and $R = Q \cup QS$. Then

$$SL = S(Q \cup SQ) = SQ \cup S^2Q \subseteq SQ \subseteq L$$

and

$$RS = (Q \cup QS)S = QS \cup QS^2 \subseteq QS \subseteq R.$$

Then L and R is a left-ideal and a right-ideal of S, respectively.

Next, we claim that $Q = L \cap R$. We have

$$L \cap R = (Q \cup SQ) \cap (Q \cup QS) = Q \cup (SQ \cap QS) = Q.$$

Hence, $Q = L \cap R$. □

Example 1.5. Let $\mathbb{Z}$ be the set of all integers and $M_2(\mathbb{Z})$ the set of all 2×2 matrices over $\mathbb{Z}$. We have known that $M_2(\mathbb{Z})$ is a semigroup under the usual multiplication. Let

$$L = \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in \mathbb{Z} \}$$

and

$$R = \{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{Z} \}.$$

Then L is a left ideal of $M_2(\mathbb{Z})$, R is a right ideal of $M_2(\mathbb{Z})$ and

$$L \cap R = \{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \in \mathbb{Z} \}$$

is a quasi-ideal of $M_2(\mathbb{Z})$.

Definition 1.5. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S were introduced by J. A. Green (Green, 1951) as the following rules:

(i) $a\mathcal{L}b$ if and only if $S^1a = S^1b$, where $S^1a = Sa \cup \{a\}$;

(ii) $a\mathcal{R}b$ if and only if $aS^1 = bS^1$, where $aS^1 = aS \cup \{a\}$;

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$;

(iv) $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

We have the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S are equivalence relations. The equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ are called *Green's relations.* An alternative characterization is given in the following remark:

Remark 1.1. Let a, b be elements of a semigroup S. We have

(i) $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$;

(ii) $a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$ and $by = a$;

(iii) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$;

(iv) $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$;

where

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

1.2 Γ-semigroups

In 1981, M. K. Sen (Sen, 1981) introduced the definition of a Γ-semigroup as follows:

Definition 1.6. Let S and Γ be nonempty sets. S is called a Γ-*semigroup* if
(i) $a\alpha b \in S$, for all $a, b \in S$ and $\alpha \in \Gamma$;
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Let S be an arbitrary semigroup and Γ be any nonempty set. Define $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to prove that S is a Γ-semigroup.

Thus a semigroup can be considered to be a Γ-semigroup.

Example 1.6. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n \mid n \in \mathbb{N}\}$. Define $a\alpha b = a + \alpha + b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $+$ is the usual addition. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.7. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n \mid n \in \mathbb{N}\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.8. Let $\mathbb{R}$ be the set of all real numbers and $\Gamma = \left\{\frac{1}{n} \middle| \ n \in \mathbb{N}\right\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{R}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have $\mathbb{R}$ is a Γ-semigroup.

Example 1.9. Let S be a set of all negative rational numbers and $\Gamma = \left\{-\frac{1}{p} \middle| \ p \text{ is prime}\right\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in S$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have S is a Γ-semigroup.

Example 1.10. Let $S = \{4z + 3 \mid z \in \mathbb{Z}\}$ and $\Gamma = \{4n + 1 \mid n \in \mathbb{N}\}$. Define $a\alpha b = a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ where $+$ is the usual addition. We have S is a Γ-semigroup.

Definition 1.7. Let S be a Γ-semigroup. A nonempty subset K of S is said to be a *subΓ-semigroup* of S if $K\Gamma K \subseteq K$ where $K\Gamma K = \{a\alpha b \mid a, b \in K \text{ and } \alpha \in \Gamma\}$.

Example 1.11. Consider the Γ-semigroup in Example 1.7. Let $\mathbb{N}$ be the set of all natural numbers. We have $\mathbb{N}$ is a subΓ-semigroup of $\mathbb{Z}$ since $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{N}\Gamma\mathbb{N} \subseteq \mathbb{N}$.

Example 1.12. Consider the Γ-semigroup S in Example 1.10. Let $K = \{4n-1 \mid n \in \mathbb{N}\}$. We have K is a Γ-subsemigroup of S since $K \subseteq S$ and $K\Gamma K \subseteq K$.

Example 1.13. Let $S = [0, 1]$ and

$$\Gamma = \{\frac{1}{n} \mid n \text{ is a positive integer}\}.$$

Then S is a Γ-semigroup under the usual multiplication.

Next, let $K = [0, \frac{1}{2}]$. We have that K is a nonempty subset of S and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a subΓ-semigroup of S.

Definition 1.8. Let S be a Γ-semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S if $S\Gamma Q \cap Q\Gamma S \subseteq Q$.

Let Q be a quasi-ideal of S. Then $Q\Gamma Q \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$. This implies that Q is a subΓ-semigroup of S.

Example 1.14. Let S be an arbitrary semigroup and Γ any nonempty set. Define $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to prove that S is a Γ-semigroup. Let Q be a quasi-ideal of a semigroup S. Thus $SQ \cap QS \subseteq Q$. We have that $S\Gamma Q \cap Q\Gamma S = SQ \cap QS \subseteq Q$. Hence, Q is a quasi-ideal of a Γ-semigroup S.

Example 1.14 implies that the class of quasi-ideals of Γ-semigroups is a generalization of quasi-ideals of semigroups.

Theorem 1.3. *Let S be a Γ-semigroup and Q_i a quasi-ideal of S for each $i \in I$. If $\bigcap_{i \in I} Q_i$ is a nonempty set, then $\bigcap_{i \in I} Q_i$ is a quasi-ideal of S.*

Proof. Let S be a Γ-semigroup and Q_i a quasi-ideal of S for each $i \in I$. Assume that $\bigcap_{i \in I} Q_i$ is a nonempty set. Take any $a, b \in \bigcap_{i \in I} Q_i$, $s_1, s_2 \in S$ and $\gamma, \mu \in \Gamma$ such that $s_1\mu b = a\gamma s_2$. Then $a, b \in Q_i$ for all $i \in I$. Since Q_i is a quasi-ideal of S for all $i \in I$,

$$s_1\mu b = a\gamma s_2 \in S\Gamma Q_i \cap Q_i\Gamma S \subseteq Q_i \text{ for all } i \in I.$$

Therefore

$$s_1\mu b = a\gamma s_2 \in \bigcap_{i \in I} Q_i. \text{ Thus } S\Gamma \bigcap_{i \in I} Q_i \cap \bigcap_{i \in I} Q_i\Gamma S \subseteq \bigcap_{i \in I} Q_i.$$

Hence, $\bigcap_{i \in I} Q_i$ is a quasi-ideal of S. □

In Theorem 1.3, the condition $\bigcap_{i \in I} Q_i$ is a nonempty set is necessary. For example, let $\mathbb{N}$ be the set of all positive integers and $\Gamma = \{1, 2\}$. Then $\mathbb{N}$ is a Γ-semigroup. For $n \in \mathbb{N}$, let $Q_n = \{n+1, n+2, n+3, ...\}$. It is easy to show that each Q_n is a quasi-ideal of $\mathbb{N}$ for all $n \in \mathbb{N}$ but $\bigcap_{n\in\mathbb{N}} Q_n$ is a empty set.

Let A be a nonempty subset of a Γ-semigroup S and

$$\Im = \{Q \mid Q \text{ is a quasi-ideal of S containing } A\}.$$

Then $\Im$ is a nonempty set because $S \in \Im$. Let $(A)_q = \bigcap_{Q\in\Im} Q$. It is easy to see that $A \subseteq (A)_q$. By Theorem 1.3, $(A)_q$ is a quasi-ideal of S. Moreover, $(A)_q$ is the smallest quasi-ideal of S containing A. $(A)_q$ is called *the quasi-ideal of S generated by A*.

Theorem 1.4. *Let A be a nonempty subset of a Γ-semigroup S. Then*

$$(A)_q = A \cup (S\Gamma A \cap A\Gamma S).$$

Proof. Let A be a nonempty subset of a Γ-semigroup S. Let $Q = A\cup(S\Gamma A\cap A\Gamma S)$. It is easy to see that $A \subseteq Q$. We have that

$$\begin{aligned} S\Gamma Q \cap Q\Gamma S &= S\Gamma[A \cup (S\Gamma A \cap A\Gamma S)] \cap [A \cup (S\Gamma A \cap A\Gamma S)]\Gamma S \\ &\subseteq S\Gamma[A \cup (S\Gamma A)] \cap [A \cup (A\Gamma S)]\Gamma S \\ &\subseteq S\Gamma A \cap A\Gamma S \\ &\subseteq Q. \end{aligned}$$

Therefore, Q is a quasi-ideal of S.

Let C be any quasi-ideal of S containing A. Since C is a quasi-ideal of S and $A \subseteq C$, $S\Gamma A \cap A\Gamma S \subseteq Q$. Therefore,

$$Q = A \cup (S\Gamma A \cap A\Gamma S) \subseteq C.$$

Hence,Q is the smallest quasi-ideal-ideal of S containing A.
Therefore, $(A)_q = A \cup (S\Gamma A \cap A\Gamma S)$, as required. □

Example 1.15. Let $\mathbb{N}$ be the set of natural integers and $\Gamma = \{5\}$. Then $\mathbb{N}$ is a Γ-semigroup under the usual addition.

(i) Let $A = \{2\}$. We have that $(A)_q = \{2\} \cup \{8, 9, 10, ...\}$.

(ii) Let $A = \{3, 4\}$. We have that $(A)_q = \{3, 4\} \cup \{9, 10, 11, ...\}$.

Definition 1.9. Let S be a Γ-semigroup. A nonempty subset L of S is called a *left ideal* of S if $S\Gamma L \subseteq L$. A nonempty subset R of S is called a *right ideal* of S if $R\Gamma S \subseteq R$. A nonempty subset of S is called an *ideal* of S if it is both a left and a right ideal of S.

Clearly, every left-ideal, right-ideal and ideal of a Γ-semigroup S is a subΓ-semigroup of S.

Example 1.16. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.7. Let $A = \{0\} \subseteq \mathbb{Z}$. We have A is a left and a right ideal of $\mathbb{Z}$ since $\mathbb{Z}\Gamma A \subseteq A$ and $A\Gamma\mathbb{Z} \subseteq A$, respectively. Therefore A is an ideal of $\mathbb{Z}$.

Theorem 1.5. *Let S be a Γ-semigroup. Let L and R be a left ideal and a right ideal of S, respectively. Then $L \cap R$ is a quasi-ideal of S.*

Proof. Let L and R be any left ideal and any right ideal of a Γ-semigroup S, respectively. By properties of L and R, we have $R\Gamma L \subseteq L \cap R$. This implies that $L \cap R$ is a nonempty set. We have that

$$S\Gamma(L \cap R) \cap (L \cap R)\Gamma S \subseteq S\Gamma L \cap R\Gamma S \subseteq L \cap R.$$

Hence, $L \cap R$ is a quasi-ideal of S. □

Theorem 1.6. *Every quasi-ideal Q of a Γ-semigroup S is the intersection of a left ideal and a right ideal of S.*

Proof. Let Q be any quasi-ideal of a Γ-semigroup S. Let $L = Q \cup S\Gamma Q$ and $R = Q \cup Q\Gamma S$. Then

$$S\Gamma L = S\Gamma(Q \cup S\Gamma Q) = S\Gamma Q \cup S\Gamma S\Gamma Q \subseteq S\Gamma Q \subseteq L$$

and

$$R\Gamma S = (Q \cup Q\Gamma S)\Gamma S = Q\Gamma S \cup Q\Gamma S\Gamma S \subseteq Q\Gamma S \subseteq R.$$

Then L and R is a left ideal and a right ideal of S, respectively.

Next, we claim that $Q = L \cap R$. We have Conversely,

$$L \cap R = (Q \cup S\Gamma Q) \cap (Q \cup Q\Gamma S) = Q \cup (S\Gamma Q \cap Q\Gamma S) = Q.$$

Hence, $Q = L \cap R$. □

Definition 1.10. Let S be a Γ-semigroup. S is called a *quasi-simple* Γ*-semigroup* if S is a unique quasi-ideal of S.

Definition 1.11. Let S be a Γ-semigroup. A quasi-ideal Q of a Γ-semigroup S is called a *minimal quasi-ideal of* S if Q does not properly contain any quasi-ideals of S.

Theorem 1.7. *Let S be a Γ-semigroup. Then S is a quasi-simple Γ-semigroup if and only if $S\Gamma s \cap s\Gamma S = S$ for all $s \in S$.*

Proof. Let S be a Γ-semigroup. Assume that S is a quasi-simple Γ-semigroup. Take any $s \in S$. First, we claim that $S\Gamma s \cap s\Gamma S$ is a quasi-ideal of S. We have that $s\Gamma s \in S\Gamma s \cap s\Gamma S$, this implies $S\Gamma s \cap s\Gamma S$ is a nonempty set. Moreover,

$$\begin{aligned} S\Gamma(S\Gamma s \cap s\Gamma S) \cap (S\Gamma s \cap s\Gamma S)\Gamma S &\subseteq S\Gamma(S\Gamma s) \cap (s\Gamma S)\Gamma S \\ &= (S\Gamma S)\Gamma s \cap s\Gamma(S\Gamma S) \\ &\subseteq S\Gamma s \cap s\Gamma S. \end{aligned}$$

Therefore, $S\Gamma s \cap s\Gamma S$ is a quasi-ideal of S. Since S is a quasi-simple Γ-semigroup, $S\Gamma s \cap s\Gamma S = S$.

Conversely, assume that $S\Gamma s \cap s\Gamma S = S$ for all $s \in S$. Let Q be a quasi-ideal of S and $q \in Q$. By assumption, $S = S\Gamma q \cap q\Gamma S$. Since Q is a quasi-ideal of S,

$$S = S\Gamma q \cap q\Gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q.$$

Therefore $Q = S$. Hence, S is a quasi-simple Γ-semigroup. □

Example 1.17. Let G be a group and $\Gamma = \{e_G\}$. It is easy to see that G is a unique quasi-ideal of G under the usual binary operation. Then G is a quasi-simple Γ-semigroup.

Theorem 1.8. *Let S be a Γ-semigroup and Q be a quasi-ideal of S. If Q is a quasi-simple Γ-semigroup, then Q is a minimal quasi-ideal of S.*

Proof. Let S be a Γ-semigroup and Q be a quasi-ideal of S. Assume that Q is a quasi-simple Γ-semigroup. Let C be a quasi-ideal of S such that $C \subseteq Q$. Then

$$Q\Gamma C \cap C\Gamma Q \subseteq S\Gamma C \cap C\Gamma S \subseteq C.$$

Therefore, C is a quasi-ideal of Q. Since Q is a quasi-simple Γ-semigroup, $C = Q$. Then Q is a minimal quasi-ideal of S. □

Definition 1.12. Let S be a Γ-semigroup. A Γ-semigroup S is called *left simple* if S is the unique left ideal of S, *right simple* if S is the unique right ideal of S, and *simple* if S is the unique ideal of S.

Example 1.18. Consider the Γ-semigroups $\mathbb{Z}$ in Example 1.6. We have $\mathbb{Z}$ is left and right simple since $\mathbb{Z}$ is the unique left and right ideal of $\mathbb{Z}$, respectively. Also Γ-semigroup $\mathbb{Z}$ is simple since $\mathbb{Z}$ is the unique ideal of $\mathbb{Z}$.

Example 1.19. Consider the Γ-semigroups $\mathbb{Z}$ in Example 1.7. By Example 1.16, we see that $\mathbb{Z}$ is not left and right simple. Also $\mathbb{Z}$ is not simple.

CHAPTER 2

Minimal quasi-ideals in Γ-semigroups

In this chapter, we give some other properties of minimal quasi-ideals in Γ-semigroups. We show that every minimal quasi-ideal of a Γ-semigroup S can be the written as intersection of a minimal left ideal and a minimal right ideal of S. Moreover, we show that relationship between minimal quasi-ideals and Green's relations of Γ-semigroups. We can see some properties of Green's relations of Γ-semigroups (Chinram and Siammai, 2008) in 2008.

2.1 Minimal quasi-ideals

Theorem 2.1 shows that every minimal quasi-ideal of a Γ-semigroup S can be the written as intersection of some minimal left ideal and some minimal right ideal of S.

Theorem 2.1. *Let S be a Γ-semigroup and Q be a quasi-ideal of S. Then Q is a minimal quasi-ideal of S if and only if Q is the intersection of a minimal left ideal L and a minimal right ideal R of S.*

Proof. Assume that $Q = L \cap R$ for some a minimal left ideal L and a minimal right ideal R of S. So $Q \subseteq L$ and $Q \subseteq R$. Let Q' be a quasi-ideal of S contained in Q. Then

$$S\Gamma Q' \subseteq S\Gamma Q \subseteq S\Gamma L \subseteq L \text{ and } Q'\Gamma S \subseteq Q\Gamma S \subseteq R\Gamma S \subseteq R.$$

It is easy to prove that $S\Gamma Q'$ and $Q'\Gamma S$ is a left ideal and a right ideal of S, respectively. By the minimality of L and R, we have $S\Gamma Q' = L$ and $Q'\Gamma S = R$. Hence $Q = L \cap R = S\Gamma Q' \cap Q'\Gamma S \subseteq Q'$. Then $Q' = Q$. Therefore Q is a minimal quasi-ideal of S.

Conversely, assume Q is a minimal quasi-ideal of S. Let $a \in Q$. Since $a\Gamma S$ and $S\Gamma a$ are a right ideal and left ideal of S, respectively, $S\Gamma a \cap a\Gamma S$ is a quasi-ideal of S. Since $S\Gamma a \cap a\Gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$, by the minimality of Q, $S\Gamma a \cap a\Gamma S = Q$.

Now we shall show that $S\Gamma a$ is a minimal left ideal of S. Let L be a left ideal of S contained in $S\Gamma a$. Then $L \cap a\Gamma S \subseteq S\Gamma a \cap a\Gamma S = Q$. Since $L \cap a\Gamma S$ is a quasi-ideal of S, therefore the minimality of Q implies that $L \cap a\Gamma S = Q$. Then $Q \subseteq L$. So $S\Gamma a \subseteq S\Gamma Q \subseteq S\Gamma L \subseteq L$. This implies $L = S\Gamma a$. Thus the left ideal $S\Gamma a$ is minimal. The minimality of the right ideal $a\Gamma S$ can be proved dually. □

It is remarkable that not every Γ-semigroup contains a minimal quasi-ideal, for example we can see this example.

Example 2.1. Let $\Gamma = \{1, 2\}$. Define $a\gamma b = a + \gamma + b$ for all $a, b \in \mathbf{N}$ and $\gamma \in \Gamma$ where $+$ is a usual addition on $\mathbf{N}$. Then $\mathbf{N}$ is a Γ-semigroup. Suppose that $\mathbf{N}$ contains a minimal quasi-ideal Q. Let k be a minimum positive integer in Q. Let $Q' = Q \setminus \{k\}$. It is easy to see that Q' is a quasi-ideal of $\mathbf{N}$ and Q' is a proper subset of Q, a contradiction. Hence $\mathbf{N}$ does not contain a minimal quasi-ideal.

Now, we give necessary and sufficient for the existence of a minimal quasi-ideal of a Γ-semigroup. The following corollary is obtained directly from Theorem 2.1.

Corollary 2.2. *Let S be a Γ-semigroup. Then S has at least one minimal quasi-ideal if and only if S has at least one minimal left ideal and at least one minimal right ideal.*

Theorem 2.3. *Let Q be a quasi-ideal of a Γ-semigroup S. Then the following statements are equivalent.*

(i) Q is a minimal quasi-ideal of S.

(ii) $Q = S\Gamma a \cap a\Gamma S$ for all $a \in Q$.

(iii) $Q = S^1\Gamma a \cap a\Gamma S^1$ for all $a \in Q$.

Proof. The proof of $(i) \rightarrow (ii)$: Assume Q is a minimal quasi-ideal of S. Let $a \in Q$. So $S\Gamma a \cap a\Gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$. Since $S\Gamma a \cap a\Gamma S$ is a quasi-ideal of S, by the minimality of $Q, Q = S\Gamma a \cap a\Gamma S$.

The proof of $(ii) \rightarrow (iii)$: Assume $Q = S\Gamma a \cap a\Gamma S$ for all $a \in Q$. Let $a \in Q$. Thus $S^1\Gamma a \cap a\Gamma S^1 = \{a\} \cup (S\Gamma a \cap a\Gamma S) = \{a\} \cup Q = Q$.

The proof of $(iii) \rightarrow (i)$: Assume $Q = S^1\Gamma a \cap a\Gamma S^1$ for all $a \in Q$. To show Q is minimal, let Q' be a quasi-ideal of S contained in Q. Let $x \in Q'$. So $x \in Q$. By assumption, $Q = S^1\Gamma x \cap x\Gamma S^1$. Therefore $Q = S^1\Gamma x \cap x\Gamma S^1 \subseteq S^1\Gamma Q' \cap Q'\Gamma S^1 = Q'$. Hence $Q' = Q$. Thus Q is a minimal quasi-ideal of S. □

2.2 Relationship between Minimal quasi-ideals and Green's relations

The Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ on a Γ-semigroup S are defined by the following rules :

(i) $a\mathcal{L}b$ if and only if $S^1\Gamma a = S^1\Gamma b$ where $S^1\Gamma a = S\Gamma a \cup \{a\}$.

(ii) $a\mathcal{R}b$ if and only if $a\Gamma S^1 = b\Gamma S^1$ where $a\Gamma S^1 = a\Gamma S \cup \{a\}$.

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

We can see some properties of Green's relations of Γ-semigroups (Chinram and Siammai, 2008) in 2008.

Let S be a Γ-semigroup. Define the relation $\mathcal{Q}$ on S by

$$a\mathcal{Q}b \text{ if and only if } a\Gamma S^1 \cap S^1\Gamma a = b\Gamma S^1 \cap S^1\Gamma b$$

where $S^1\Gamma a \cap a\Gamma S^1 = (S\Gamma a \cap a\Gamma S) \cup \{a\}$. We have the relation $\mathcal{Q}$ is an equivalence relation on S, moreover we have

Theorem 2.4. $\mathcal{H} = \mathcal{Q}$.

Proof. Let $a, b \in S$ be such that $a\mathcal{H}b$. Hence $a\mathcal{L}b$ and $a\mathcal{R}b$. This implies that $S^1\Gamma a = S^1\Gamma b$ and $a\Gamma S^1 = b\Gamma S^1$. So $a\Gamma S^1 \cap S^1\Gamma a = b\Gamma S^1 \cap S^1\Gamma b$. Therefore $a\mathcal{Q}b$.

Conversely, let $a, b \in S$ be such that $a\mathcal{Q}b$. Then $a\Gamma S^1 \cap S^1\Gamma a = b\Gamma S^1 \cap S^1\Gamma b$. If $a = b$, then $a\mathcal{H}b$. Suppose that $a \neq b$. So $a \in S\Gamma b \cap b\Gamma S$ and

$b \in S\Gamma a \cap a\Gamma S$. Since $a \in S\Gamma b$ and $b \in S\Gamma a, S^1\Gamma a = S^1\Gamma b$. Since $a \in b\Gamma S$ and $b \in a\Gamma S, a\Gamma S^1 = b\Gamma S^1$. So $a\mathcal{L}b$ and $a\mathcal{R}b$. Therefore $a\mathcal{H}b$. □

Let Q be a quasi-ideal of a Γ-semigroup S. Then $S^1\Gamma Q \cap Q\Gamma S^1 = Q \cup (S\Gamma Q \cap Q\Gamma S) = Q$. The following theorem give the necessary and sufficient conditions for a quasi-ideal of S to be a minimal quasi-ideal of S.

Theorem 2.5. *Let Q be a quasi-ideal of a Γ-semigroup S. Then Q is a minimal quasi-ideal of S if and only if Q is a $\mathcal{H}$-class.*

Proof. Assume Q is a minimal quasi-ideal of S. Let $a, b \in S$. By Theorem 2.4, $S^1\Gamma a \cap a\Gamma S^1 = Q = S^1\Gamma b \cap b\Gamma S^1$. Thus $a\mathcal{H}b$. This implies Q is a $\mathcal{H}$-class.

Conversely, assume Q is a $\mathcal{H}$-class. We have that

$$S^1\Gamma a \cap a\Gamma S^1 = S^1\Gamma b \cap b\Gamma S^1 \text{ for all } a, b \in Q. \tag{2.5.1}$$

Let x be a fixed element in Q and let $y \in Q$. Then $y \in S^1\Gamma y \cap y\Gamma S^1 = S^1\Gamma x \cap x\Gamma S^1$. Hence $Q \subseteq S^1\Gamma x \cap x\Gamma S^1$. Since $x \in Q$, $S^1\Gamma x \cap x\Gamma S^1 \subseteq S^1\Gamma Q \cap Q\Gamma S^1 \subseteq Q$. Hence $Q = S^1\Gamma x \cap x\Gamma S^1$. By (2.5.1), $Q = S^1\Gamma a \cap a\Gamma S^1$ for all $a \in Q$. By Theorem 2.4, Q is a minimal quasi-ideal of S. □

The following corollary holds.

Corollary 2.6. *Let S be a Γ-semigroup. Then S is a quasi-simple Γ-semigroup if and only if S is a $\mathcal{H}$-class.*

Proof. Assume S is a quasi-simple Γ-semigroup. Then S is a minimal quasi-ideal of S. By Theorem 2.5, S is a $\mathcal{H}$-class.

Conversely, assume S is a $\mathcal{H}$-class. By Theorem 2.5, S is a minimal quasi-ideal of S. Hence S is a quasi-simple Γ-semigroup. □

CHAPTER 3

(m, n)-quasi-ideals and minimal (m, n)-quasi-ideals in Γ-semigroups

In this chapter, we study m-left ideals, n-right ideals and (m, n)-quasi-ideals of Γ-semigroups. We have that a quasi-ideal Q of a Γ-semigroup S is a $(1, 1)$-quasi-ideal of S. We show that every (m, n)-quasi-ideal of a Γ-semigroup S can be written as the intersection of an m-left ideal and an n-right ideal of S.

Moreover, every minimal (m, n)-quasi-ideal of a Γ-semigroup S can be written as the intersection of a minimal m-left ideal and a minimal n-right ideal of S.

3.1 (m, n)-quasi-ideals

In this section, we study (m, n)-quasi-ideals of Γ-semigroups. We show that the intersection of an m-left ideal and an n-right ideal of a Γ-semigroup S is an (m, n)-quasi-ideal of S and every (m, n)-quasi-ideal can be obtiaｎed in this way.

Definition 3.1. Let A be a nonempty subset of a Γ-semigroup S and $n \in \mathbb{N}$. A set A^n is defined to be the set

$$A^n = \underbrace{A\Gamma A\Gamma ... \Gamma A}_{n \text{ numbers of } A}.$$

For example, $A^2 = A\Gamma A$ and $A^3 = A\Gamma A\Gamma A$.

Definition 3.2. Let S be a Γ-semigroup. A subΓ-semigroup Q of S is called an (m, n)-*quasi-ideal* of S if $S^m\Gamma Q \cap Q\Gamma S^n \subseteq Q$ where m and n are positive integers.

We have that a quasi-ideal Q of a Γ-semigroup S is a $(1, 1)$-quasi-ideal of S.

Definition 3.3. Let S be a Γ-semigroup. A subΓ-semigroup L of S is called an *m-left ideal* of S if $S^m\Gamma L \subseteq L$ where m is a positive integer. A subΓ-semigroup R of S is called an *n-right ideal* of S if $R\Gamma S^n \subseteq R$ where n is a positive integer.

The following lemma is ture.

Lemma 3.1. *Let S be a Γ-semigroup and T_i be a subΓ-semigroup of S for all $i \in I$. If $\bigcap_{i\in I} T_i \neq \emptyset$, then $\bigcap_{i\in I} T_i$ is a subΓ-semigroup of S.*

Proof. Assume $\bigcap_{i\in I} T_i$ is nonempty. Let $a, b \in \bigcap_{i\in I} T_i$. Then $a, b \in T_i$ for all $i \in I$. Since T_i is a subΓ-semigroup of S for all $i \in I$, $a\gamma b \in T_i$ for all $i \in I$,$\gamma \in \Gamma$. Thus $a\gamma b \in \bigcap_{i\in I} T_i$ for all $a, b \in \bigcap_{i\in I} T_i$, $\gamma \in \Gamma$. Therefore $\bigcap_{i\in I} T_i$ is a subΓ-semigroup of S. □

The following theorem is true.

Theorem 3.2. *Let S be a Γ-semigroup.*

(i) Let Q_i be an (m,n)-quasi-ideal of S for all $i \in I$. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-ideal of S.

(ii) Let L_i be an m-left ideal of S for all $i \in I$. If $\bigcap_{i\in I} L_i \neq \emptyset$, then $\bigcap_{i\in I} L_i$ is an m-left ideal of S.

(iii) Let R_i be an n-right ideal of S for all $i \in I$. If $\bigcap_{i\in I} R_i \neq \emptyset$, then $\bigcap_{i\in I} R_i$ is an n-right ideal of S.

Proof. (i) Assume $\bigcap_{i\in I} Q_i$ is nonempty. By Lemma 3.1, we have $\bigcap_{i\in I} Q_i$ is a subΓ-semigroup of S. Next , let $c \in S^m\Gamma(\bigcap_{i\in I} Q_i) \cap (\bigcap_{i\in I} Q_i)\Gamma S^n$. Then $c = x\gamma q = p\mu y$ for some $x \in S^m, y \in S^n$ and $p, q \in \bigcap_{i\in I} Q_i$ and $\gamma, \mu \in \Gamma$. Thus $p, q \in Q_i$ for all $i \in I$. Then $c \in S^m\Gamma Q_i \cap Q_i\Gamma S^n$ for all $i \in I$. Since Q_i is a (m,n)-quasi-ideal of S for all $i \in I, c \in Q_i$ for all $i \in I$. Then $c \in \bigcap_{i\in I} Q_i$. Therefore $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-ideal of S.

(ii) and (iii) can be proved similarly to (i). □

Let A be a nonempty subset of a Γ-semigroup S and $\mathcal{I} = \{Q \mid Q$ is an (m,n)-quasi-ideal of S containing $A\}$. Then $\mathcal{I}$ is not empty because $S \in \mathcal{I}$. Let $(A)_{q(m,n)} = \bigcap_{Q \in \mathcal{I}} Q$. It is clearly see that $(A)_{q(m,n)}$ is nonempty because $A \subseteq (A)_{q(m,n)}$. By Theorem 3.2, $(A)_{q(m,n)}$ is an (m,n)-quasi-ideal of S. Moreover, $(A)_{q(m,n)}$ is the smallest (m,n)-quasi-ideal of S containing A. The (m,n)-quasi-ideal $(A)_{q(m,n)}$ is called the *(m,n)-quasi-ideal of S generated by A*. The *m-left ideal $(A)_{l(m)}$ of S generated by A* and *n-right ideal $(A)_{r(n)}$ of S generated by A* are defined analogously.

Theorem 3.3. *Let A be a nonempty subset of a Γ-semigroup S. Then*

(i) $(A)_{q(m,n)} = \left(\bigcup_{i=1}^{\max\{m,n\}} A^i\right) \cup (S^m\Gamma A \cap A\Gamma S^n)$.

(ii) $(A)_{l(m)} = \left(\bigcup_{i=1}^{m} A^i\right) \cup S^m\Gamma A$.

(iii) $(A)_{r(n)} = \left(\bigcup_{i=1}^{n} A^i\right) \cup A\Gamma S^n$.

Proof. (i) Let $k = \max\{m,n\}$ and $Q = \left(\bigcup_{i=1}^{k} A^i\right) \cup (S^m\Gamma A \cap A\Gamma S^n)$. Clearly, $A \subseteq Q$. Next, let $x, y \in Q$.

Case 1 : $x, y \in \bigcup_{i=1}^{k} A^i$. Then $xy \in A^t$ for some $t \in \mathbb{N}$.

Case 1.1 : $t \leq k$. Then $xy \in \bigcup_{i=1}^{k} A^i$.

Case 1.2 : $t > k$. Then $xy \in S^m\Gamma A \cap A\Gamma S^n$.

Case 2 : $x \in S^m\Gamma A \cap A\Gamma S^n$ or $y \in S^m\Gamma A \cap A\Gamma S^n$. Then $xy \in S^m\Gamma A \cap A\Gamma S^n$.

Then Q is a subΓ-semigroup of S containing A. Next, we have

$$\begin{aligned} S^m\Gamma Q\cap Q\Gamma S^n &= S^m\Gamma((\bigcup_{i=1}^{k}A^i)\cup(S^m\Gamma A\cap A\Gamma S^n))\cap((\bigcup_{i=1}^{k}A^i)\cup(S^m\Gamma A\cap A\Gamma S^n))\Gamma S^n \\ &\subseteq S^m\Gamma((\bigcup_{i=1}^{k}A^i)\cup S^m\Gamma A)\cap((\bigcup_{i=1}^{k}A^i)\cup A\Gamma S^n)\Gamma S^n \\ &\subseteq S^m\Gamma A\cap A\Gamma S^n \\ &\subseteq Q. \end{aligned}$$

Now, we have Q is an (m,n)-quasi-ideal of S containing A.

To show Q is smallest, let Q' be any (m,n)-quasi-ideal of S containing A. Thus $A^i\subseteq Q'$ for all $i\in\mathbf{N}$ and $S^m\Gamma A\cap A\Gamma S^n\subseteq S^m\Gamma Q'\cap Q'\Gamma S^n\subseteq Q'$. Therefore $Q=(\bigcup_{i=1}^{k}A^i)\cup(S^m\Gamma A\cap A\Gamma S^n)\subseteq Q'$. Then $Q\subseteq Q'$.

Hence, Q is the smallest (m,n)-quasi-ideal of S containing A. Therefore $(A)_{q(m,n)}=(\bigcup_{i=1}^{\max\{m,n\}}A^i)\cup(S^m\Gamma A\cap A\Gamma S^n)$, as required.

(ii) Let $k=m$ and $L=(\bigcup_{i=1}^{k}A^i)\cup S^m\Gamma A$. Clearly, $A\subseteq L$. Next, let $x,y\in L$.

Case 1 : $x,y\in\bigcup_{i=1}^{k}A^i$. Then $xy\in A^t$ for some $t\in\mathbf{N}$.

Case 1.1 : $t\leq k$. Then $xy\in\bigcup_{i=1}^{k}A^i$.

Case 1.2 : $t>k$. Then $xy\in S^m\Gamma A$.

Case 2 : $x\in S^m\Gamma A$ or $y\in S^m\Gamma A$. Then $xy\in S^m\Gamma A$.

Then L is a subΓ-semigroup of S containing A. Next, we have

$$\begin{aligned} S^m\Gamma L &= S^m\Gamma((\bigcup_{i=1}^{k}A^i)\cup(S^m\Gamma A)) \\ &\subseteq S^m\Gamma A \\ &\subseteq L. \end{aligned}$$

Now, we have L is an m-left ideal of S containing A.

To show L is smallest, let L' be any m-left ideal of S containing A. Thus $A^i \subseteq L'$ for all $i \in \mathbf{N}$ and $S^m\Gamma A \subseteq S^m\Gamma L' \subseteq L'$. Therefore $L = (\bigcup_{i=1}^{k} A^i) \cup S^m\Gamma A \subseteq L'$. Then $L \subseteq L'$.

Hence, L is the smallest m-left ideal of S containing A. Therefore $(A)_{l(m)} = (\bigcup_{i=1}^{m} A^i) \cup S^m\Gamma A$, as required.

(iii) can be proved similarly to (ii). □

The following theorem is true.

Theorem 3.4. *Let L and R be an m-left ideal and an n-right ideal of a Γ-semigroup S, respectively. Then $L \cap R$ is an (m,n)-quasi-ideal of S.*

Proof. By properties of L and R, we have $R^m\Gamma L^n \subseteq S^m\Gamma L \cap R\Gamma S^n \subseteq L \cap R$. Then $L \cap R$ is nonempty. By Lemma 3.1, we have $L \cap R$ is a subΓ-semigroup of S. Next, we have

$$(S^m\Gamma(L \cap R)) \cap ((L \cap R)\Gamma S^n) \subseteq S^m\Gamma L \cap R\Gamma S^n \subseteq L \cap R.$$

Hence $L \cap R$ is an (m,n)-quasi-ideal of S. □

The next theorem shows that every (m,n)-quasi-ideal of a Γ-semigroup S can be written as the intersection of an m-left ideal and an n-right ideal of S.

Theorem 3.5. *Every (m,n)-quasi-ideal Q of a Γ-semigroup S is the intersection of some m-left ideal and some n-right ideal of S.*

Proof. Let Q be an (m,n)-quasi-ideal of S. Let $L = Q \cup S^m\Gamma Q$ and $R = Q \cup Q\Gamma S^n$. To show L is a subΓ-semigroup of S, let $a, b \in L$ and $\gamma \in \Gamma$.

Case 1 : $a, b \in Q$. Since Q is a sub Γ-semigroup of S , $a\gamma b \in Q \subseteq L$.

Case 2 : $a \in Q$ and $b \in S^m\Gamma Q$. Then $a\gamma b \in Q\Gamma S^m\Gamma Q \subseteq S^m\Gamma Q \subseteq L$.

Case 3 : $a \in S^m\Gamma Q$ and $b \in Q$. Then $a\gamma b \in S^m\Gamma Q\Gamma Q \subseteq S^m\Gamma Q \subseteq L$.

Case 4 : $a \in S^m\Gamma Q$ and $b \in S^m\Gamma Q$. Then $a\gamma b \in S^m\Gamma Q S^m\Gamma Q \subseteq S^m\Gamma Q \subseteq L$.

Then L is a subΓ-semigroup of S. Next, we have

$$S^m\Gamma L = S^m\Gamma(Q \cup S^m\Gamma Q) = S^m\Gamma Q \cup S^m\Gamma Q \subseteq S^m\Gamma Q \subseteq L.$$

Hence L is an m-left ideal of S. Similarly, R is an n-right ideal of S.

Finally, we claim that $Q = L \cap R$. It is easy to see that

$$Q \subseteq (Q \cup S^m\Gamma Q) \cap (Q \cup Q\Gamma S^n) = L \cap R.$$

Conversely, we have $L \cap R = (Q \cup S^m\Gamma Q) \cap (Q \cup Q\Gamma S^n) = Q \cup (S^m\Gamma Q \cap Q\Gamma S^n) \subseteq Q$. Hence $Q = L \cap R$. □

3.2 Minimal (m, n)-quasi-ideals

In this section, we study minimal (m, n)-quasi-ideals of Γ-semigroups. We have that every minimal (m, n)-quasi-ideal of a Γ-semigroup S can be written as the the intersection of a minimal m-left ideal and a minimal n-right ideal of S.

Definition 3.4. Let S be a Γ-semigroup. An (m, n)-quasi-ideal of a Γ-semigroup S is called a *minimal (m, n)-quasi-ideal* of S if Q does not properly contain any (m, n)-quasi-ideal of S. *Minimal m-left ideals* and *minimal n-right ideals* are defined analogously.

The following lemma holds.

Lemma 3.6. *Let S be a Γ-semigroup and $a \in S$. The following statements are true.*

(i) $S^m\Gamma a$ is an m-left ideal of S.

(ii) $a\Gamma S^n$ is an n-right ideal of S.

(iii) $S^m\Gamma a \cap a\Gamma S^n$ is an (m, n)-quasi-ideal of S.

Proof. (i) We have $(S^m\Gamma a)(S^m\Gamma a) \subseteq S^m\Gamma a$ and $S^m\Gamma(S^m\Gamma a) \subseteq S^m\Gamma a$, (i) holds.

(ii) It is similar to (i).

(iii) It follows by (i), (ii) and Theorem 3.4. □

The following theorem is true.

Theorem 3.7. *Let S be a Γ-semigroup and Q be an (m,n)-quasi-ideal of S. Then Q is minimal if and only if Q is the intersection of some minimal m-left ideal L and some minimal n-right ideal R of S.*

Proof. Assume that $Q = L \cap R$ for some minimal m-left ideal L and some minimal n-right ideal R of S. So $Q \subseteq L$ and $Q \subseteq R$. Let Q' be an (m,n)-quasi-ideal of S contained in Q. Then

$$S^m\Gamma Q' \subseteq S^m\Gamma Q \subseteq S^m\Gamma L \subseteq L \text{ and } Q'\Gamma S^n \subseteq Q\Gamma S^n \subseteq R\Gamma S^n \subseteq R.$$

It is easy to prove that $S^m\Gamma Q'$ and $Q'\Gamma S^n$ is an m-left ideal and an n-right ideal of S, respectively. By the minimality of L and R, we have $S^m\Gamma Q' = L$ and $Q'\Gamma S^n = R$. Hence $Q = L \cap R = S^m\Gamma Q' \cap Q'\Gamma S^n \subseteq Q'$. Thus $Q' = Q$. Therefore Q is a minimal (m,n)-quasi-ideal of S.

Conversely, assume Q is a minimal (m,n)-quasi-ideal of S. Let $a \in Q$. By Lemma 3.6, we have known that $S^m\Gamma a, a\Gamma S^n$ and $S^m\Gamma a \cap a\Gamma S^n$ are an m-left ideal, an n-right ideal and an (m,n)-quasi-ideal of S, respectively. Since $S^m\Gamma a \cap a\Gamma S^n \subseteq S^m\Gamma Q \cap Q\Gamma S^n \subseteq Q$, by the minimality of Q, $S^m\Gamma a \cap a\Gamma S^n = Q$.

Now, we shall show that $S^m\Gamma a$ is a minimal m-left ideal of S. Let L be an m-left ideal of S contained in $S^m\Gamma a$. Then $L \cap a\Gamma S^n \subseteq S^m\Gamma a \cap a\Gamma S^n = Q$. Since $L \cap a\Gamma S^n$ is an (m,n)-quasi-ideal of S, therefore the minimality of Q implies that $L \cap a\Gamma S^n = Q$. Then $Q \subseteq L$. Therefore $S^m\Gamma a \subseteq S^m\Gamma Q \subseteq S^m\Gamma L \subseteq L$. This implies $L = S^m\Gamma a$. Thus the m-left ideal $S^m\Gamma a$ is minimal. The minimality of the n-right ideal $a\Gamma S^n$ can be proved dually. □

It is remarkable that not every Γ-semigroup contains a minimal (m,n)-quasi-ideal. Now, we give necessary and sufficient for the existence of a minimal (m,n)-quasi-ideal of a Γ-semigroup. The following corollary is obtained directly from Theorem 3.7.

Corollary 3.8. *Let S be a Γ-semigroup. Then S has at least one minimal (m,n)-quasi-ideal if and only if S has at least one minimal m-left ideal and at least one minimal n-right ideal.*

The following theorem holds.

Theorem 3.9. *Let S be a Γ-semigroup. The following statements are true.*

(i) An m-left ideal L is minimal if and only if $S^m\Gamma a = L$ for all $a \in L$.

(ii) An n-right ideal R is minimal if and only if $a\Gamma S^n = R$ for all $a \in R$.

(iii) An (m,n)-quasi-ideal Q is minimal if and only if $S^m\Gamma a \cap a\Gamma S^n = Q$ for all $a \in Q$.

Proof. (i) Assume L is minimal. Let $a \in L$. Then $S^m\Gamma a \subseteq S^m\Gamma L \subseteq L$. By Lemma 3.6(i), we have known that $S^m\Gamma a$ is an m-left ideal of S. Since L is a minimal m-left ideal of S, $S^m\Gamma a = L$.

Conversely, assume $S^m\Gamma a = L$ for all $a \in L$. To show L is minimal, let L' be an m-left ideal of S contained in L. Let $x \in L'$. Then $x \in L$, by assumption. $S^m\Gamma x = L$. Now, we have $L = S^m\Gamma x \subseteq S^m\Gamma L' \subseteq L'$. Then $L' = L$. This implies that L is minimal.

(ii) and (iii) can be proved similarly to (i). □

Definition 3.5. A Γ-semigroup S is called an (m,n)*-quasi-simple* Γ*-semigroup* if S is a unique (m,n)-quasi-ideal of S. An m*-left simple* Γ*-semigroup* and an n*-right simple* Γ*-semigroup* are defined analogously.

The following theorem follows by Theorem 3.9.

Theorem 3.10. *Let S be a Γ-semigroup. The following statements are true.*

(i) S is an m-left simple Γ-semigroup if and only if $S^m\Gamma a = S$ for all $a \in S$.

(ii) S is an n-right simple Γ-semigroup if and only if $a\Gamma S^n = S$ for all $a \in S$.

(iii) S is an (m,n)-quasi simple Γ-semigroup if and only if $S^m\Gamma a \cap a\Gamma S^n = S$ for all $a \in S$.

Proof. (i) Since S is an m-left simple semigroup, S is a minimal m-left ideal of S. By Theorem 3.9(i), $S^m\Gamma a = S$ for all $a \in S$.

Conversely, assume $S^m\Gamma a = S$ for all $a \in S$. By Theorem 3.9(i), S is a minimal m-left ideal of S. Then S is an m-left simple Γ-semigroup.

(ii) and (iii) can be proved similarly to (i). □

Theorem 3.11. *Let S be a Γ-semigroup. The following statements are true.*

(i) Let L be an m-left ideal of S. If L is an m-left simple Γ-semigroup, then L is a minimal m-left ideal of S.

(ii) Let R be an n-right ideal of S. If R is an n-right simple Γ-semigroup, then R is a minimal n-right ideal of S.

(iii) Let Q be an (m,n)-quasi-ideal of S. If Q is an (m,n)-quasi-simple Γ-semigroup, then Q is a minimal (m,n)-quasi-ideal of S.

Proof. (i) Assume L is an m-left simple Γ-semigroup. By Theorem 3.10(i), $L^m\Gamma a = L$ for all $a \in L$. For each $a \in L$, we have

$$L = L^m\Gamma a \subseteq S^m\Gamma a \subseteq S^m\Gamma L \subseteq L.$$

Then $S^m\Gamma a = L$ for all $a \in L$. By Theorem 3.9(i), L is minimal.

(ii) and (iii) can be proved similarly to (i). □

Remark 3.1. We have known that a left ideal, a right ideal and a quasi-ideal of a Γ-semigroup are a 1-left ideal, a 1-right ideal and a $(1,1)$ quasi-ideal of S. Then all theorems of this chapter is true for left ideals, right ideals and quasi-ideals of Γ-semigroups where $m = n = 1$.

CHAPTER 4

Roughness of quasi-ideals in Γ-semigroups

The notion of rough sets was introduced by Z. Pawlak (Pawlak, 1982) in the year 1982. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. In the year 2003, Young Bae Jun (Jun, 2003) studied the lower and upper approximations with respect to congruences in Γ-semigroups. In this chapter, we study Θ-lower and Θ-upper rough quasi-ideals and (m,n)-quasi-ideals in Γ-semigroups.

4.1 The lower and upper approximations with respect to congruences

In 2003, Y. B. Jun (Jun, 2003) studied the lower and upper approximations with respect to congruence in Γ-semigroups as follows:

Definition 4.1. Let S be a Γ-semigroup. An equivalence relation Θ on S is called a *congruence* on S if

$$(a,b) \in \Theta, \text{ implies } (a\gamma x, b\gamma x) \in \Theta \text{ and } (x\gamma a, x\gamma b) \in \Theta$$

for all $a,b,x \in S$ and $\gamma \in \Gamma$.

Definition 4.2. Let $[a]_\Theta$ denote the congruence class containing the element $a \in S$. A congruence Θ on S is said to be *complete* if

$$[a]_\Theta \gamma [b]_\Theta = [a\gamma b]_\Theta$$

for all $a,b \in S$ and $\gamma \in \Gamma$.

Definition 4.3. Let A be a nonempty subset of a Γ-semigroup S and Θ be a congruence on S. The Θ-*lower approximation* and Θ-*upper approximation* of A are defined to be the sets

$$\underline{\Theta}(A) = \{x \in S \mid [x]_\Theta \subseteq A\}$$

and

$$\overline{\Theta}(A) = \{x \in S \mid [x]_\Theta \cap A \neq \emptyset\},$$

respectively.

Example 4.1. Let $\Gamma = \{5, 7\}$. For $x, y \in \mathbb{N}$ and $\gamma \in \Gamma$, define $x\gamma y = x \times \gamma \times y$ where $\cdot$ is the usual multiplication on $\mathbb{N}$. Then $\mathbb{N}$ is a Γ-semigroup.

Define a relation Θ on $\mathbb{N}$ by

$$a\Theta b \Longleftrightarrow 3 \mid a - b \text{ for all } a, b \in \mathbb{N}.$$

Let $A_1 = \{3, 12\}$. We have $\underline{\Theta}(A_1) = \emptyset$ and $\overline{\Theta}(A_1) = \{3n \mid n \in \mathbb{N}\}$.
Let $A_2 = \{3n \mid n \in \mathbb{N}\} \cup \{1\}$. We have $\underline{\Theta}(A_2) = \{3n \mid n \in \mathbb{N}\}$ and $\overline{\Theta}(A_2) = \{3n \mid n \in \mathbb{N}\} \cup \{3n - 2 \mid n \in \mathbb{N}\}$.
Let $A_3 = \{3n \mid n \in \mathbb{N}\} \cup \{1, 2\}$. We have $\underline{\Theta}(A_3) = \{3n \mid n \in \mathbb{N}\}$ and $\overline{\Theta}(A_3) = \mathbb{N}$.

The next theorem 4.1 is well-known (Jun, 2003).

Theorem 4.1. *Let Θ be a congruence on a Γ-semigroup S and let A and B be nonempty subsets of S. The following statements are true.*

(*i*) $\underline{\Theta}(A) \subseteq A \subseteq \overline{\Theta}(A)$.

(*ii*) $\underline{\Theta}(S) = S = \overline{\Theta}(S)$.

(*iii*) $\overline{\Theta}(A \cup B) = \overline{\Theta}(A) \cup \overline{\Theta}(B)$.

(*iv*) $\underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B)$.

(*v*) $A \subseteq B$ *implies* $\underline{\Theta}(A) \subseteq \underline{\Theta}(B)$ *and* $\overline{\Theta}(A) \subseteq \overline{\Theta}(B)$.

(*vi*) $\underline{\Theta}(\underline{\Theta}(A)) = \underline{\Theta}(A)$ *and* $\overline{\Theta}(\overline{\Theta}(A)) = \overline{\Theta}(A)$.

(vii) $\underline{\Theta}(A) \cup \underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$.

(viii) $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$.

(ix) $\overline{\Theta}(A)\Gamma\overline{\Theta}(B) \subseteq \overline{\Theta}(A\Gamma B)$.

(x) If Θ is complete, then $\underline{\Theta}(A)\Gamma\underline{\Theta}(B) \subseteq \underline{\Theta}(A\Gamma B)$.

Proof. (i) If $x \in \underline{\Theta}(A)$, then $x \in [x]_\Theta \subseteq A$. Hence $\underline{\Theta}(A) \subseteq A$. Now let $x \in A$. Since $x \in [x]_\Theta$, we have $[x]_\Theta \cap A \neq \emptyset$, and so $x \in \overline{\Theta}(A)$. Thus $A \subseteq \overline{\Theta}(A)$.

(ii) By(i), we have $\underline{\Theta}(S) \subseteq (S)$. Let $x \in S$. Obviously, we have $S \subseteq \underline{\Theta}(S)$. Then $\underline{\Theta}(S) = (S)$. By(i), we have $S \subseteq \overline{\Theta}(S)$. It is cleary that $\overline{\Theta}(S) \subseteq (S)$. Hence $\underline{\Theta}(S) = S = \overline{\Theta}(S)$.

(iii) and (iv) Note that

$$\begin{aligned} x \in \overline{\Theta}(A \cup B) &\Leftrightarrow [x]_\Theta \cap (A \cup B) \neq \emptyset \\ &\Leftrightarrow ([x]_\Theta \cap A) \cup ([x]_\Theta \cap B) \neq \emptyset \\ &\Leftrightarrow [x]_\Theta \cap A \neq \emptyset \text{ or } [x]_\Theta \cap B \neq \emptyset \\ &\Leftrightarrow x \in \overline{\Theta}(A) \text{ or } x \in \overline{\Theta}(B) \\ &\Leftrightarrow x \in \overline{\Theta}(A) \cup \overline{\Theta}(B), \end{aligned}$$

and

$$\begin{aligned} x \in \underline{\Theta}(A \cap B) &\Leftrightarrow [x]_\Theta \subseteq (A \cap B) \\ &\Leftrightarrow [x]_\Theta \subseteq A \text{ and } [x]_\Theta \subseteq B \\ &\Leftrightarrow x \in \underline{\Theta}(A) \text{ and } x \in \underline{\Theta}(B) \\ &\Leftrightarrow x \in \underline{\Theta}(A) \cap \underline{\Theta}(B). \end{aligned}$$

Hence (iii) and (iv) and valid.

(v) Since $A \subseteq B$ if and only if $A \cap B = A$, it follows from (iv) that

$$\underline{\Theta}(A) = \underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B)$$

so that $\underline{\Theta}(A) \subseteq \underline{\Theta}(B)$. Note also that $A \subseteq B$ if and only if $A \cup B = B$. Thus, by (v), we have

$$\overline{\Theta}(B) = \overline{\Theta}(A \cup B) = \overline{\Theta}(A) \cup \overline{\Theta}(B),$$

which implies that $\overline{\Theta}(A) \subseteq \overline{\Theta}(B)$.

(vi) By (i) and (vi), we get $\underline{\Theta}(\underline{\Theta}(A)) \subseteq \underline{\Theta}(A)$. Now let $x \in \underline{\Theta}(A)$, then $x \in [x]_\Theta \subseteq A$. Thus for all $y \in [x]_\Theta$, $[y]_\Theta = [x]_\Theta \subseteq A$. Then $[y]_\Theta \subseteq A$ for all $y \in [x]_\Theta$. Hence $y \in \underline{\Theta}(A)$ for all $y \in [x]_\Theta$. If follow that $[x]_\Theta \subseteq \underline{\Theta}(A)$, which implies that $x \in \underline{\Theta}(\underline{\Theta}(A))$. Then $\underline{\Theta}(\underline{\Theta}(A)) = \underline{\Theta}(A)$. By (i) and (vi), we get $\overline{\Theta}(A) \subseteq \overline{\Theta}(\overline{\Theta}(A))$. Now let $x \in \overline{\Theta}(\overline{\Theta}(A)$, then $[x]_\Theta \cap \overline{\Theta}(A) \neq \emptyset$. Therefore there exist $y \in [x]_\Theta \cap \overline{\Theta}(A)$. Thus $y \in \overline{\Theta}(A)$. So $[y]_\Theta \cap A \neq \emptyset$. Since $y \in [x]_\Theta$, $[y]_\Theta = [x]_\Theta$. This implies that $[x]_\Theta \cap A \neq \emptyset$. Then $x \in \overline{\Theta}(A)$. Hence (vi) valid.

(vii) and (viii) Using (vi), we get $\underline{\Theta}(A) \subseteq \underline{\Theta}(A \cup B)$ and $\underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$. Hence $\underline{\Theta}(A) \cup \underline{\Theta}(B) \subseteq \underline{\Theta}(A \cup B)$. Also $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A)$ and $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(B)$, which yields $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$.

(ix) Let $w \in \overline{\Theta}(A)\Gamma\overline{\Theta}(B)$. Then $w = x\gamma y$ with $x \in \overline{\Theta}(A), y \in \overline{\Theta}(B)$ and $\gamma \in \Gamma$, and therefore there exist $a, b \in S$ such that $a \in [x]_\Theta \cap A$ and $b \in [y]_\Theta \cap B$. Since Θ is a congruence, it follows that

$$a\gamma b \in [x]_\Theta \gamma [y]_\Theta \subseteq [x\gamma y]_\Theta.$$

On the other hand, since $a\gamma b \in A\Gamma B$, we have $a\gamma b \in [x\gamma y]_\Theta \cap A\Gamma B$, and so $w = x\gamma y \in \overline{\Theta}(A\Gamma B)$. This shown that $\overline{\Theta}(A)\Gamma\overline{\Theta}(B) \subseteq \overline{\Theta}(A\Gamma B)$.

(x) Assme that Θ is complete and let $u \in \underline{\Theta}(A)\Gamma\underline{\Theta}(B)$. Then $u = x\gamma y$ with $x \in \underline{\Theta}(A), y \in \underline{\Theta}(B)$ and $\gamma \in \Gamma$. It follows that $[x]_\Theta \subseteq A$ and $[y]_\Theta \subseteq B$. Since Θ is complete, we have

$$[x\gamma y]_\Theta = [x]_\Theta \gamma [y]_\Theta \subseteq A\Gamma B,$$

and so $u = x\gamma y \in \underline{\Theta}(A\Gamma B)$. Hence $\underline{\Theta}(A)\Gamma\underline{\Theta}(B) \subseteq \underline{\Theta}(A\Gamma B)$. □

4.2 The Θ-lower and Θ-upper rough quasi-ideals

In this section, we study Θ-lower and Θ-upper rough quasi-ideals in Γ-semigroups. We have the intersection of a Θ-lower rough left ideal and a Θ-lower rough right ideal of a Γ-semigroup S is a Θ-lower rough quasi-ideal of S and if Θ is complete, then we have every quasi-ideal of a Γ-semigroup is the intersection of a Θ-lower rough left ideal and a Θ-lower rough right ideal of S.

Definition 4.4. Let Θ be a congruence on a Γ-semigroup S. A nonempty subset A of S is called a *Θ-lower (Θ-upper, respectively) rough subΓ-semigroup* of S if the Θ-lower (Θ-upper, respectively) approximation of A is a subΓ-semigroup of S. The *Θ-lower* and *Θ-upper rough right ideal (left ideal, ideal)* of S are defined analogously.

Definition 4.5. Let S be a Γ-semigroup and Θ be a congruence on S. A nonempty subset Q of S is called a *Θ-lower (Θ-upper, respectively) rough quasi-ideal* of S if the Θ-lower (Θ-upper, respectively) approximation of Q is a quasi-ideal of S.

Theorem 4.2. *Let Θ be a congruence on a Γ-semigroup S and Q be a quasi-ideal of S. If Θ is complete and $\underline{\Theta}(Q) \neq \emptyset$, then Q is a Θ-lower rough quasi-ideal of S.*

Proof. Let Q be a quasi-ideal of S. Then $S\Gamma Q \cap Q\Gamma S \subseteq Q$. Assume that Θ is complete and $\underline{\Theta}(Q) \neq \emptyset$. We have

$$S\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma S = \underline{\Theta}(S)\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma\underline{\Theta}(S) \subseteq \underline{\Theta}(S\Gamma Q \cap Q\Gamma S) \subseteq \underline{\Theta}(Q).$$

Then $\underline{\Theta}(Q)$ is a quasi-ideal of S. Therefore Q is a Θ-lower rough quasi-ideal of S. □

The next theorem shows that the intersection of a Θ-lower rough left ideal and a Θ-lower rough right ideal of a Γ-semigroup S is a Θ-lower rough quasi-ideal of S.

Theorem 4.3. *Let S be a Γ-semigroup and Θ be a congruence on S. Let L and R be a Θ-lower rough left ideal and a Θ-lower rough right ideal of S, respectively. Then $L \cap R$ is a Θ-lower rough quasi-ideal of S.*

Proof. Let L and R be a Θ-lower rough left ideal and a Θ-lower rough right ideal of S, respectively. Then $S\Gamma\underline{\Theta}(L) \subseteq \underline{\Theta}(L)$ and $\underline{\Theta}(R)\Gamma S \subseteq \underline{\Theta}(R)$. We have

$$\underline{\Theta}(R)\Gamma\underline{\Theta}(L) \subseteq S\Gamma\underline{\Theta}(L) \cap \underline{\Theta}(R)\Gamma S \subseteq \underline{\Theta}(L) \cap \underline{\Theta}(R) = \underline{\Theta}(L \cap R).$$

Then $\underline{\Theta}(L \cap R)$ is nonempty. We have

$$\begin{aligned} S\Gamma\underline{\Theta}(L \cap R) \cap \underline{\Theta}(L \cap R)\Gamma S &\subseteq S\Gamma\underline{\Theta}(L) \cap \underline{\Theta}(R)\Gamma S \\ &\subseteq \underline{\Theta}(L) \cap \underline{\Theta}(R) \\ &= \underline{\Theta}(L \cap R). \end{aligned}$$

Then $\underline{\Theta}(L \cap R)$ is a quasi-ideal of S. Hence $L \cap R$ is a Θ-lower rough quasi-ideal of S. □

The next theorem shows that if Θ is complete, then we have every quasi-ideal of a Γ-semigroup is the intersection of a Θ-lower rough left ideal and a Θ-lower rough right ideal of S.

Theorem 4.4. *Let S be a Γ-semigroup and Θ be a complete congruence on S. Let Q be a quasi-ideal of S. Then there exist a Θ-lower rough left ideal L and a Θ-lower rough right ideal R of S such that $Q = L \cap R$.*

Proof. Let Q be a quasi-ideal of S. Then $S\Gamma Q \cap Q\Gamma S \subseteq Q$. Let $L = Q \cup \underline{\Theta}(S\Gamma Q)$ and $R = Q \cup \underline{\Theta}(Q\Gamma S)$. It is easy to see that $Q \subseteq L \cap R$. We have

$$\begin{aligned} L \cap R &= (Q \cup \underline{\Theta}(S\Gamma Q)) \cap (Q \cup \underline{\Theta}(Q\Gamma S)) \\ &= Q \cup (\underline{\Theta}(S\Gamma Q) \cap \underline{\Theta}(Q\Gamma S)) \\ &= Q \cup \underline{\Theta}(S\Gamma Q \cap Q\Gamma S) \\ &\subseteq Q \cup \underline{\Theta}(Q) \\ &= Q. \end{aligned}$$

So $Q = L \cap R$.

Next, we show that L is a Θ-lower rough left ideal of S. We have

$$\begin{aligned} S\Gamma\underline{\Theta}(L) &= S\Gamma\underline{\Theta}(Q \cup \underline{\Theta}(S\Gamma Q)) \\ &= \underline{\Theta}(S)\Gamma\underline{\Theta}(Q \cup \underline{\Theta}(S\Gamma Q)) \\ &\subseteq \underline{\Theta}(S\Gamma(Q \cup \underline{\Theta}(S\Gamma Q))) \text{ since } \Theta \text{ is complete} \\ &= \underline{\Theta}(S\Gamma Q \cup S\Gamma\underline{\Theta}(S\Gamma Q)) \\ &\subseteq \underline{\Theta}(S\Gamma Q \cup \underline{\Theta}(S\Gamma S\Gamma Q)) \\ &\subseteq \underline{\Theta}(S\Gamma Q \cup S\Gamma S\Gamma Q) \\ &= \underline{\Theta}(S\Gamma Q) \\ &= \underline{\Theta}(\underline{\Theta}(S\Gamma Q)) \\ &\subseteq \underline{\Theta}(L). \end{aligned}$$

Then L is a Θ-lower rough left ideal of S. Similarly, R is a Θ-lower rough right ideal of S. □

Let Θ be a congruence on a Γ-semigroup S. The Θ-lower approximation and Θ-upper approximation can be presented in an equivalent from as shown below:

$$\underline{\Theta}(A)/\Theta = \{[x]_\Theta \in S/\Theta \mid [x]_\Theta \subseteq A\}$$

and

$$\overline{\Theta}(A)/\Theta = \{[x]_\Theta \in S/\Theta \mid [x]_\Theta \cap A \neq \emptyset\},$$

respectively.

Theorem 4.5. *Let Θ be complete congruence on a Γ-semigroup S. The following statements are true.*

(i) If Q is a Θ-lower rough quasi-ideal of S, then $\underline{\Theta}(Q)/\Theta$ is a quasi-ideal of S/Θ.

(ii) If Q is a Θ-upper rough quasi-ideal of S, then $\overline{\Theta}(Q)/\Theta$ is a quasi-ideal of S/Θ.

Proof. (i) Let $[a]_\Theta \in (S/\Theta)\Gamma(\underline{\Theta}(Q)/\Theta) \cap (\underline{\Theta}(Q)/\Theta)\Gamma(S/\Theta)$. Then there exist $[x_1]_\Theta, [x_2]_\Theta \in \underline{\Theta}(Q)/\Theta$, $\alpha, \beta \in \Gamma$ and $[y_1]_\Theta, [y_2]_\Theta \in S/\Theta$ such that

$$[a]_\Theta = [y_1]_\Theta \alpha [x_1]_\Theta = [x_2]_\Theta \beta [y_2]_\Theta.$$

Then $[x_1]_\Theta, [x_2]_\Theta \subseteq Q$. This implies $x \in \underline{\Theta}(Q)$ for all $x \in [x_1]_\Theta \cup [x_2]_\Theta$. Thus $[x_1]_\Theta \subseteq \underline{\Theta}(Q)$ and $[x_2]_\Theta \subseteq \underline{\Theta}(Q)$. Since $\underline{\Theta}(Q)$ is a quasi-ideal of S,

$$[a]_\Theta \subseteq S\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma S \subseteq \underline{\Theta}(Q) \subseteq Q.$$

So $[a]_\Theta \in \underline{\Theta}(Q)/\Theta$. Therefore $\underline{\Theta}(Q)/\Theta$ is a quasi-ideal of S/Θ.

(ii) Let $[a]_\Theta \in (S/\Theta)\Gamma(\underline{\Theta}(Q)/\Theta) \cap (\underline{\Theta}(Q)/\Theta)\Gamma(S/\Theta)$. Then there exist $[x_1]_\Theta, [x_2]_\Theta \in \underline{\Theta}(Q)/\Theta$, $\alpha, \beta \in \Gamma$ and $[y_1]_\Theta, [y_2]_\Theta \in S/\Theta$ such that

$$[a]_\Theta = [y_1]_\Theta \alpha [x_1]_\Theta = [x_2]_\Theta \beta [y_2]_\Theta.$$

Then $[x_1]_\Theta \cap Q \neq \emptyset$ and $[x_2]_\Theta \cap Q \neq \emptyset$. This implies $x \in \overline{\Theta}(Q)$ for all $x \in [x_1]_\Theta \cup [x_2]_\Theta$. We have

$$[a]_\Theta \subseteq S\Gamma\overline{\Theta}(Q) \cap \overline{\Theta}(Q)\Gamma S \subseteq \overline{\Theta}(Q).$$

Then $a \in \overline{\Theta}(Q)$. Thus $[a]_\Theta \cap Q \neq \emptyset$. This implies that $[a]_\Theta \in \overline{\Theta}(Q)/\Theta$. Therefore $\overline{\Theta}(Q)/\Theta$ is a quasi-ideal of S/Θ. □

Definition 4.6. Let S be a Γ-semigroup. A subΓ-semigroup B of S is called a *bi-ideal* of S if $B\Gamma S\Gamma B \subseteq B$.

We have known that every quasi-ideal of S is a bi-ideal of S.

Definition 4.7. Let S be a Γ-semigroup and Θ be a congruence on S. A nonempty subset A of S is called a *Θ-lower (Θ-upper, respectively) rough bi-ideal* of S if the Θ-lower (Θ-upper, respectively) approximation of A is a bi-ideal of S.

Theorem 4.6. *Let S be a Γ-semigroup and Θ be a congruence on S. The following statements are true.*

(i) Eevery Θ-lower rough quasi-ideal of S is a Θ-lower rough bi-ideal of S.

(ii) Every Θ-upper rough quasi-ideal of S is a Θ-upper rough bi-ideal of S.

Proof. (i) Let Q be a Θ-lower rough quasi-ideal of S. Then $\underline{\Theta}(Q)$ is a subΓ-semigroup of S and $S\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma S \subseteq \underline{\Theta}(Q)$. We have that

$$\underline{\Theta}(Q)\Gamma S\Gamma\underline{\Theta}(Q) \subseteq S\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma S \subseteq \underline{\Theta}(Q).$$

Hence $\underline{\Theta}(Q)$ is a bi-ideal of S. Hence Q is a Θ-lower rough bi-ideal of S.

(ii) It is similar to (i). □

Theorem 4.7. *Let S be a Γ-semigroup and Θ be a congruence on S. If Q is a quasi-ideal of S, then Q is a Θ-upper rough bi-ideal of S.*

Proof. Let Q be a quasi-ideal of S. Then $S\Gamma Q \cap Q\Gamma S \subseteq Q$. Since $Q \subseteq \overline{\Theta}(Q)$, $\overline{\Theta}(Q) \neq \emptyset$. We have

$$\overline{\Theta}(Q)\Gamma\overline{\Theta}(Q) \subseteq \overline{\Theta}(Q\Gamma Q) \subseteq \overline{\Theta}(Q).$$

Then $\overline{\Theta}(Q)$ is a subΓ-semigroup of S. Next, we have

$$\overline{\Theta}(Q)\Gamma S\Gamma\overline{\Theta}(Q) = \overline{\Theta}(Q)\Gamma\overline{\Theta}(S)\Gamma\overline{\Theta}(Q) \subseteq \overline{\Theta}(Q\Gamma S\Gamma Q) \subseteq \overline{\Theta}((S\Gamma Q)\cap(Q\Gamma S)) \subseteq \overline{\Theta}(Q).$$

Therefore $\overline{\Theta}(Q)$ is a bi-ideal of S. Then Q is a Θ-upper rough bi-ideal of S. □

4.3 The Θ-lower and Θ-upper rough (m,n)-quasi-ideals

In this section, we study Θ-lower and Θ-upper rough (m,n)-quasi-ideals in Γ-semigroups. We have the intersection of a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of a Γ-semigroup S is a Θ-lower rough quasi-ideal of S and if Θ is complete, then we have every (m,n)-quasi-ideal of a Γ-semigroup is the intersection of a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of S.

Definition 4.8. Let S be a Γ-semigroup and Θ be a congruence on S. A nonempty subset Q of S is called a Θ-*lower* (Θ-*upper, respectively*) *rough* (m,n)-*quasi-ideal* of S if the Θ-lower (Θ-upper, respectively) approximation of Q is a (m,n)-quasi-ideal of S. The *Θ-lower* and *Θ-upper rough n-right ideal (m-left ideal)* of S are defined analogously.

Theorem 4.8. *Let Θ be a congruence on a Γ-semigroup S and Q be an (m,n) quasi-ideal of S. If Θ is complete and $\underline{\Theta}(Q) \neq \emptyset$, then Q is a Θ-lower rough (m,n)-quasi-ideal of S.*

Proof. Let Q be an (m,n)-quasi-ideal of S. Then $S^m\Gamma Q \cap Q\Gamma S^n \subseteq Q$. Assume that Θ is complete and $\underline{\Theta}(Q) \neq \emptyset$. We have

$$S^m\Gamma\underline{\Theta}(Q)\cap\underline{\Theta}(Q)\Gamma S^n = \underline{\Theta}(S)^m\Gamma\underline{\Theta}(Q)\cap\underline{\Theta}(Q)\Gamma\underline{\Theta}(S)^n \subseteq \underline{\Theta}(S^m\Gamma Q\cap Q\Gamma S^n) \subseteq \underline{\Theta}(Q).$$

Then $\underline{\Theta}(Q)$ is an (m,n)-quasi-ideal of S. Therefore Q is a Θ-lower rough (m,n)-quasi-ideal of S. □

The next theorem shows that the intersection of a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of a Γ-semigroup S is a Θ-lower rough quasi-ideal of S.

Theorem 4.9. *Let S be a Γ-semigroup and Θ be a congruence on S. Let L and R be a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of S, respectively. Then $L \cap R$ is a Θ-lower rough (m,n)-quasi-ideal of S.*

Proof. Let L and R be a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of S, respectively. Then $S^m\Gamma\underline{\Theta}(L) \subseteq \underline{\Theta}(L)$ and $\underline{\Theta}(R)\Gamma S^n \subseteq \underline{\Theta}(R)$. We have

$$\underline{\Theta}(R)^m\Gamma\underline{\Theta}(L)^n \subseteq S^m\Gamma\underline{\Theta}(L) \cap \underline{\Theta}(R)\Gamma S^n \subseteq \underline{\Theta}(L) \cap \underline{\Theta}(R) = \underline{\Theta}(L \cap R).$$

Then $\underline{\Theta}(L \cap R)$ is nonempty. We have

$$\begin{aligned} S^m\Gamma\underline{\Theta}(L \cap R) \cap \underline{\Theta}(L \cap R)\Gamma S^n &\subseteq S^m\Gamma\underline{\Theta}(L) \cap \underline{\Theta}(R)\Gamma S^n \\ &\subseteq \underline{\Theta}(L) \cap \underline{\Theta}(R) \\ &= \underline{\Theta}(L \cap R). \end{aligned}$$

Then $\underline{\Theta}(L \cap R)$ is an (m, n)-quasi-ideal of S. Hence $L \cap R$ is a Θ-lower rough (m, n)-quasi-ideal of S. □

The next theorem shows that if Θ is complete, then we have every (m, n)-quasi-ideal of a Γ-semigroup is the intersection of a Θ-lower rough m-left ideal and a Θ-lower rough n-right ideal of S.

Theorem 4.10. *Let S be a Γ-semigroup and Θ be a complete congruence on S. Let Q be an (m, n)-quasi-ideal of S. Then there exist a Θ-lower rough m-left ideal L and a Θ-lower rough n-right ideal R of S such that $Q = L \cap R$.*

Proof. Let Q be an (m, n)-quasi-ideal of S. Then $S^m\Gamma Q \cap Q\Gamma S^n \subseteq Q$. Let $L = Q \cup \underline{\Theta}(S^m\Gamma Q)$ and $R = Q \cup \underline{\Theta}(Q\Gamma S^n)$. It is easy to see that $Q \subseteq L \cap R$. We have

$$\begin{aligned} L \cap R &= (Q \cup \underline{\Theta}(S^m\Gamma Q)) \cap (Q \cup \underline{\Theta}(Q\Gamma S^n)) \\ &= Q \cup (\underline{\Theta}(S^m\Gamma Q) \cap \underline{\Theta}(Q\Gamma S^n)) \\ &= Q \cup \underline{\Theta}(S^m\Gamma Q \cap Q\Gamma S^n) \\ &\subseteq Q \cup \underline{\Theta}(Q) \\ &= Q. \end{aligned}$$

So $Q = L \cap R$.

Next, we show that L is a Θ-lower rough m-left ideal of S. We have

$$\begin{aligned}
S^m\Gamma\underline{\Theta}(L) &= S^m\Gamma\underline{\Theta}(Q \cup \underline{\Theta}(S^m\Gamma Q)) \\
&= \underline{\Theta}(S)^m\Gamma\underline{\Theta}(Q \cup \underline{\Theta}(S^m\Gamma Q)) \\
&\subseteq \underline{\Theta}(S^m\Gamma(Q \cup \underline{\Theta}(S^m\Gamma Q))) \text{ since } \Theta \text{ is complete} \\
&= \underline{\Theta}(S^m\Gamma Q \cup S^m\Gamma\underline{\Theta}(S^m\Gamma Q)) \\
&\subseteq \underline{\Theta}(S^m\Gamma Q \cup \underline{\Theta}(S^m\Gamma S^m\Gamma Q)) \\
&\subseteq \underline{\Theta}(S^m\Gamma Q \cup S^m\Gamma S^m\Gamma Q) \\
&= \underline{\Theta}(S^m\Gamma Q) \\
&= \underline{\Theta}(\underline{\Theta}(S^m\Gamma Q)) \\
&\subseteq \underline{\Theta}(L).
\end{aligned}$$

Then L is a Θ-lower rough m-left ideal of S. Similarly, R is a Θ-lower rough n-right ideal of S. □

Theorem 4.11. *Let Θ be a complete congruence on a Γ-semigroup S. The following statements are true.*

(i) If Q is a Θ-lower rough (m,n)-quasi-ideal of S, then $\underline{\Theta}(Q)/\Theta$ is an (m,n)-quasi-ideal of S/Θ.

(ii) If Q is a Θ-upper rough (m,n)-quasi-ideal of S, then $\overline{\Theta}(Q)/\Theta$ is an (m,n)-quasi-ideal of S/Θ.

Proof. (i) Let $[a]_\Theta \in (S/\Theta)^m\Gamma(\underline{\Theta}(Q)/\Theta) \cap (\underline{\Theta}(Q)/\Theta)\Gamma(S/\Theta)^n$. Then there exist $[x_1]_\Theta, [x_2]_\Theta \in \underline{\Theta}(Q)/\Theta$, $\alpha, \beta \in \Gamma$ and $[y_1]_\Theta, [y_2]_\Theta \in (S/\Theta)^m, (S/\Theta)^n$ such that $[a]_\Theta = [y_1]_\Theta\alpha[x_1]_\Theta = [x_2]_\Theta\beta[y_2]_\Theta$. Then $[x_1]_\Theta, [x_2]_\Theta \subseteq Q$. This implies $x \in \underline{\Theta}(Q)$ for all $x \in [x_1]_\Theta \cup [x_2]_\Theta$. Thus $[x_1]_\Theta \subseteq \underline{\Theta}(Q)$ and $[x_2]_\Theta \subseteq \underline{\Theta}(Q)$. Since $\underline{\Theta}(Q)$ is an (m,n)-quasi-ideal of S,

$$[a]_\Theta \subseteq S^m\Gamma\underline{\Theta}(Q) \cap \underline{\Theta}(Q)\Gamma S^n \subseteq \underline{\Theta}(Q) \subseteq Q.$$

So $[a]_\Theta \in \underline{\Theta}(Q)/\Theta$. Therefore $\underline{\Theta}(Q)/\Theta$ is an (m,n)-quasi-ideal of S/Θ.

(ii) Let $[a]_\Theta \in (S/\Theta)^m\Gamma(\underline{\Theta}(Q)/\Theta) \cap (\underline{\Theta}(Q)/\Theta)\Gamma(S/\Theta)^n$. Then there exist $[x_1]_\Theta, [x_2]_\Theta \in \underline{\Theta}(Q)/\Theta$, $\alpha, \beta \in \Gamma$ and $[y_1]_\Theta, [y_2]_\Theta \in (S/\Theta)^m, (S/\Theta)^n$ such that $[a]_\Theta = [y_1]_\Theta\alpha[x_1]_\Theta = [x_2]_\Theta\beta[y_2]_\Theta$. Then $[x_1]_\Theta \cap Q \neq \emptyset$ and $[x_2]_\Theta \cap Q \neq \emptyset$. This implies $x \in \overline{\Theta}(Q)$ for all $x \in [x_1]_\Theta \cup [x_2]_\Theta$. We have

$$[a]_\Theta \subseteq S^m\Gamma\overline{\Theta}(Q) \cap \overline{\Theta}(Q)\Gamma S^n \subseteq \overline{\Theta}(Q).$$

Then $a \in \overline{\Theta}(Q)$. Thus $[a]_\Theta \cap Q \neq \emptyset$. This implies that $[a]_\Theta \in \overline{\Theta}(Q)/\Theta$. Therefore $\overline{\Theta}(Q)/\Theta$ is an (m, n) quasi-ideal of S/Θ. □

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