A Common Fixed Point Iterative Process with Errors for Quasi-Nonexpansive Mappings in Banach Spaces

Pattraporn Kaonein

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Author	Miss Pattraporn Kaonein
Major Program	Mathematics and Statistics

Major Advisor:	**Examining Committee:**
……………………………..……………………	………………………………….Chairperson
(Assoc. Prof. Dr. Jantana Ayaragarnchanakul)	(Dr. Orawan Tripak)
	……………………..……………………..…
	(Assoc. Prof. Dr. Jantana Ayaragarnchanakul)
	……………………..…………………….…
	(Dr. Suwicha Imnang)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Mathematics and Statistics

………………………….……..……

(Prof. Dr. Amornrat Phongdara)

Dean of Graduate School

ชื่อวิทยานิพนธ์	ขบวนการทำซ้ำที่มีค่าคลาดเคลื่อนกำกับ เพื่อหาจุดตรึงร่วม สำหรับการแปลงชนิดควอไซนอนเอกซ์แพนซีฟในปริภูมิบานาค
ผู้เขียน	นางสาว พัสตราภรณ์ เก้าเอี้ยน
สาขาวิชา	คณิตศาสตร์และสถิติ
ปีการศึกษา	2553

บทคัดย่อ

ให้ X เป็นปริภูมิบานาคเชิงจริง และให้ C เป็นสับเซตคอนเวกซ์ปิดและไม่ เป็นเซตว่าง ของ X สำหรับ $i=1,2$ ให้ $T_i : C \to C$ เป็น ฟังก์ชัน ชนิดควอไซนอนเอกซ์แพนซีฟ ซึ่ง $F(T_1) \cap F(T_2) \neq \varnothing$ ใน C เรามีความสนใจลำดับในกระบวนการต่อไปนี้ สำหรับ $x_1 \in C$ และ $n \geq 1$ นิยามลำดับ $\{x_n\}$ และ $\{y_n\}$ โดย

$$y_n = \beta_n T_2 x_n + (1-\beta_n - a_n) x_n + a_n v_n$$

$$x_{n+1} = \alpha_n T_1 y_n + (1-\alpha_n - b_n) y_n + b_n u_n \qquad (1)$$

เมื่อ $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ และ $\{\beta_n\}$ เป็นลำดับของจำนวนจริง ในช่วง $[0,1]$ และ $\{u_n\}$ และ $\{v_n\}$ เป็นลำดับใน C เราพิสูจน์การลู่เข้าของลำดับ $\{x_n\}$ ไปสู่จุดตรึงร่วมของ T_1 และ T_2 ภายใต้เงื่อนไขที่เหมาะสม

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ABSTRACT

Let X be a real Banach space and let C be a nonempty closed convex subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \varnothing$ in C. We are interested in sequences in the following process. For $x_1 \in C$ and $n \geq 1$, define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1 - \beta_n - a_n)x_n + a_n v_n, \\ x_{n+1} &= \alpha_n T_1 y_n + (1 - \alpha_n - b_n)y_n + b_n u_n, \end{aligned} \tag{1}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[0, 1]$ and $\{u_n\}$ and $\{v_n\}$ are sequences in C. We prove some convergence theorems of the sequence $\{x_n\}$ to a common fixed point of T_1 and T_2 under appropriate conditions.

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Pattraporn Kaonein

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CHAPTER 1

Introduction

Let X be a real Banach space, C a closed subset of X, and $T : C \to X$ such that T has a nonempty set of fixed points $F(T) \subset C$. T is called *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|$$

for all x in C and p in $F(T)$. It is introduced by Tricomi for real functions and further studied by Diaz and Metcalf.

In 1972, Petryshyn and Williamson had presented two new theorems which provided necessary and sufficient conditions for the convergence of the successive approximation method Theorem and of the convex combination iteration method Theorem for quasi-nonexpansive mappings defined on suitable subset of the Banach space and with nonempty set of fixed points as follows.

Theorem 1.1. *Let X be a real Banach space, C a closed subset of X, and T a quasi-nonexpansive mapping of C into C with nonempty set $F(T)$ of fixed points. Suppose there exists a point x_0 in C such that the sequence $\{x_n\}$ of iterates lies in C, where $x_n \,(= T^n(x_0))$ is given by*

$$(S1) \qquad x_n = T(x_{n-1}), \qquad n = 1, 2, 3, 4, ...$$

Then $\{x_n\}$ converges to a fixed point of T in C if and only if the following condition (M) holds:

$$(M) \qquad \lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Theorem 1.2. *Let X be a Banach space, C a closed convex subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such* *that, for some* λ *in* $(0,1)$*, the sequence* $\{x_n\} = \{T_\lambda^n(x_0)\}$ *given by* $(S2)$ *lies in* C *,where*

$$(S2) \qquad x_n = T_\lambda(x_{n-1}), \quad x_0 \in C, \quad T_\lambda = \lambda T + (1-\lambda)I, \quad \lambda \in (0,1).$$

Then $\{x_n\}$ *converges to a fixed point of* T *in* C *if and only if*

$$(\acute{C}) \qquad \lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0.$$

They also indicated briefly how these theorems were used to deduce a number of known, as well as some new, convergence results for various special classes of mappings of nonexpansive, P-compact, and 1-set-contractive type which recently have been extensively studied by a number of authors.

In this thesis, we create new iterative process with errors for quasi-nonexpansive mappings in Banach space and prove some convergence theorems as follows.

Theorem 1.3. *Let* X *be a real Banach space and let* C *be a nonempty closed convex subset of* X*. For* $i = 1, 2$*, let* $T_i : C \to C$ *be a quasi-nonexpansive mapping such that* $F(T_1) \cap F(T_2) \neq \emptyset$*. Let* $\{a_n\}$*,* $\{b_n\}$*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$ *and* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* C*, where define the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1-\beta_n-a_n)x_n + a_n v_n \\ x_{n+1} &= \alpha_n T_1 y_n + (1-\alpha_n-b_n)y_n + b_n u_n. \end{aligned} \tag{1.1}$$

Assume that

(i) $\{a_n+\beta_n\}$ *and* $\{b_n+\alpha_n\}$ *are sequences in* $[0,1]$ *and*

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty;$$

(ii) $\{u_n\}$ *and* $\{v_n\}$ *are bounded.*

Then the iterative sequence $\{x_n\}$ *defined in* (1.1) *converges strongly to a common fixed point of* T_1 *and* T_2 *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0.$$

Theorem 1.4. *Let X, C, $T_i (i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem* 1.3*. Suppose that conditions* (*i*) *and* (*ii*) *in Theorem* 1.3 *hold. Assume further that the mapping $T_i (i = 1, 2)$ is asymptotically regular in x_n, and there exists an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(r) > 0$ for all $r > 0$ and for $i = 1, 2$, we have*

$$\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2))) \text{ for all } n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

CHAPTER 2

Preliminaries

In this chapter, we first collect fundamental knowledge in mathematical analysis (p 4-7) and basic knowledge about metric spaces and normed spaces (p 11-13). Then we study in detail on the classical Banach fixed point theorem, and finally, some fixed point theorems on quasi-nonexpansive mappings.

2.1 Fundamental knowledge without proof

In this section, we give some well-known definitions and theorems without proof. Definition 2.1 - 2.9, Axiom 2.1, Theorem 2.2 - 2.4, 2.6 - 2.9, 2.11 - 2.13 and Corollary 2.5, 2.10 are from [7], Definition 2.10 - 2.19, 2.21 Theorem 2.17 - 2.18 are from [5], Definition 2.22 is form [6] and Definition 2.23 is from [4].

Definition 2.1. (Upper Bound and Lower Bound). Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound of* S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound of* S and the set S is said to be *bounded below*.

Definition 2.2. (Supremum and Infimum). Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum of* S and denote it by $supS$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum of* S and denote it by $infS$.

Axiom 2.1. ***(Completeness Axiom)***. *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other word,* $\sup S$ *exists and is a real number.*

Definition 2.3. (Convergent Sequence). A sequence $\{s_n\}$ of real numbers is said to *converge* to the real number s provided that

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$n > N \text{ implies } |s_n - s| < \epsilon.$$

If $\{s_n\}$ converges to s, then we will write $\lim_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. The number s is called the *limit* of the sequence $\{s_n\}$. A sequence that does not converge to some real number is said to *diverge*.

Definition 2.4. (Bounded Sequence). A sequence $\{s_n\}$ of real numbers is said to be *bounded* if there exists a constant M such that $|s_n| \leq M$ for all n.

Theorem 2.2. *Convergent sequences are bounded.*

Definition 2.5. (Monotone Sequence). A sequence $\{s_n\}$ of real numbers is called a *nondecreasing sequence* if $s_n \leq s_{n+1}$ for all n and $\{s_n\}$ is called a *nonincreasing sequence* if $s_n \geq s_{n+1}$ for all n. Note that if $\{s_n\}$ is nondecreasing then $s_n \leq s_m$ whenever $n < m$. A sequence that is nondecreasing or nonincreasing will be called a *monotone sequence* or a *monotonic sequence*.

Theorem 2.3. ***(Monotone Convergence Theorem)***. *All bounded monotone sequences converge.*

Theorem 2.4.

(1) *If $\{s_n\}$ is an unbounded nondecreasing sequence, then* $\lim_{n\to\infty} s_n = +\infty$.

(2) *If $\{s_n\}$ is an unbounded nonincreasing sequence, then* $\lim_{n\to\infty} s_n = -\infty$.

Corollary 2.5. *If $\{s_n\}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim_{n\to\infty} s_n$ is always meaningful for monotone sequences.*

Definition 2.6. (lim sup and lim inf). Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$

Theorem 2.6. *Let $\{s_n\}$ be a sequence in $\mathbb{R}$.*

(1) *If $\lim_{n\to\infty} s_n$ is defined [as a real number, $+\infty$, $-\infty$], then*

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

(2) *If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$, then $\lim_{n\to\infty} s_n$ is defined and*

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Definition 2.7. (Cauchy Sequence). A sequence $\{s_n\}$ of real numbers is called a *Cauchy sequence* if

for each $\epsilon > 0$ there exists a number N such that

$m, n > N$ implies $|s_n - s_m| < \epsilon$.

Theorem 2.7. ***(Cauchy Completeness Theorem).*** *A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

Theorem 2.8. ***(Sandwich Theorem).*** *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences and $a_n \le b_n \le c_n$ for all n. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ (defined) then $\lim_{n\to\infty} b_n = L$.*

Definition 2.8. (Subsequence). Suppose that $\{s_n\}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $\{t_k\}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

Thus $\{t_k\}$ is just a selection of some [possibly all] of the s_n's, taken in order.

Theorem 2.9. *If the sequence $\{s_n\}$ converges, then every subsequence converges to the same limit.*

Corollary 2.10. *Let $\{s_n\}$ be any sequence. There exists a monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$ and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.*

Definition 2.9. (The Cauchy Criterion for Series). We say that a series $\sum_{n=1}^{\infty} a_n$ satisfies the *Cauchy criterion* if its sequence $\{s_n\}$ of *partial sum* is a Cauchy sequence:

$$\textit{for each } \epsilon > 0 \textit{ there exists a number } N \textit{ such that}$$
$$m, n > N \textit{ implies } |s_n - s_m| < \epsilon. \tag{2.1}$$

Nothing is lost in this definition if we impose the restriction $n > m$. Moreover, it is only a natural matter to work with $m - 1$ where $m \leq n$ instead of m where $m < n$. Therefore (2.1) is equivalent to

$$\textit{for each } \epsilon > 0 \textit{ there exists a number } N \textit{ such that}$$
$$n \geq m > N \textit{ implies } |s_n - s_{m-1}| < \epsilon. \tag{2.2}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, condition (2.2) can be written

$$\textit{for each } \epsilon > 0 \textit{ there exists a number } N \textit{ such that}$$
$$n \geq m > N \textit{ implies } \left|\sum_{k=m}^{n} a_k\right| < \epsilon. \tag{2.3}$$

Theorem 2.11. *A series converges if and only if it satisfies the Cauchy critertion.*

Theorem 2.12. $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ *converges if and only if $p > 1$.*

Theorem 2.13. ***(Mean Value Theorem)***. *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that*

$$f'(x) = \tfrac{f(b) - f(a)}{b - a}.$$

2.2 Basic knowledge with proof

In this section, we give some basic knowledge which known, but the proof cannot be found. Some are very old results while the other have proof but we want to give more detail here so that those who are interested in this area may study and understand more easily.

The following are useful lemmas we will use to obtain Theorem 3.1.

Lemma 2.14. *Let* $\{a_n\}$ *be a sequence of real numbers. Then* $\lim_{m\to\infty}\sup_{n\geq m} a_{n+m} = \lim_{m\to\infty}\sup_{n\geq m} a_n$.

Proof. Let $L_1 = \lim_{m\to\infty}\sup_{n\geq m} a_{n+m}$ and $L_2 = \lim_{m\to\infty}\sup_{n\geq m} a_n$. We will prove that $L_1 = L_2$. Since $\{a_{n+m} : n \geq m\} \subset \{a_n : n \geq m\}$, we see that

$$\sup\{a_{n+m} : n \geq m\} \leq \sup\{a_n : n \geq m\}.$$

That is $\lim_{m\to\infty}\sup\{a_{n+m} : n \geq m\} \leq \lim_{m\to\infty}\sup\{a_n : n \geq m\}$, i.e., $L_1 \leq L_2$. Next, we will show that $L_1 \geq L_2$. Suppose not, i.e., $L_1 < L_2$.

Since $\lim_{m\to\infty}\sup_{n\geq m} a_{n+m} = L_1$, $\exists N \in \mathbb{N}$ such that $m > N$ implies

$$|\sup\{a_{n+m} : n \geq m\} - L_1| < L_2 - L_1.$$

Thus

$$a_{n+m} < L_2 \qquad , \forall n \geq m > N,$$

which implies that

$$a_r < L_2 \qquad , \forall r \geq m > N$$

$$\sup\{a_r : r \geq m\} < L_2 \qquad , \forall m > N.$$

Taking $m \to \infty$, we get $\limsup_{r\to\infty} a_r < L_2$, i.e., $L_2 < L_2$, a contradiction. Hence $L_1 \geq L_2$, as desired. □

Lemma 2.15. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \textit{ for all } n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(1) $\lim_{n\to\infty} a_n < \infty$ *exists.*

(2) $\lim_{n\to\infty} a_n = 0$ *if* $\{a_n\}$ *has a subsequence converging to zero.*

Proof. Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \text{ for all } n.$$

where $\{b_n\}$ and $\{\delta_n\}$ converges. We will show that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ which implies $\lim_{n\to\infty} a_n$ exists. Since we know that $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$, we need only prove that $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n$. Since $a_{n+1} \leq (1+\delta_n)a_n + b_n$, we have

$$\begin{aligned}
a_{n+m} &\leq (1+\delta_{n+m-1})a_{n+m-1} + b_{n+m-1} \\
&\leq (e^{\delta_{n+m-1}})a_{n+m-1} + b_{n+m-1} \\
&\leq e^{\delta_{n+m-1}}\{(1+\delta_{n+m-2})a_{n+m-2} + b_{n+m-2}\} + b_{n+m-1} \\
&\leq e^{\delta_{n+m-1}}\{(e^{\delta_{n+m-2}})a_{n+m-2} + b_{n+m-2}\} + b_{n+m-1} \\
&= (e^{\delta_{n+m-1}+\delta_{n+m-2}})a_{n+m-2} + e^{\delta_{n+m-1}}b_{n+m-2} + b_{n+m-1} \\
&\vdots \\
&\leq a_n e^{\left(\sum_{k=n}^{n+m-1} \delta_k\right)} + \left(\sum_{k=n}^{n+m-1} b_k\right) e^{\left(\sum_{k=n}^{n+m-1} \delta_k\right)}, \forall n, m \in \mathbb{N}. \qquad (2.4)
\end{aligned}$$

Let $\epsilon > 0$. Since $\sum_{k=1}^{n+m-1} \delta_k$ and $\sum_{k=1}^{n+m-1} b_k$ converge, $\exists N \in \mathbb{N}$ such that

$$\sum_{k=n}^{n+m-1} \delta_k < \epsilon \textit{ and } \sum_{k=n}^{n+m-1} b_k < \epsilon \textit{ for all } n > N. \qquad (2.5)$$

From (2.4) and (2.5), for all $n, m \geq N$, we have

$$a_{n+m} \leq a_n e^{\epsilon} + \epsilon e^{\epsilon}$$

Thus $\sup_{n\geq m} a_{n+m} \leq e^{\epsilon} \inf_{n\geq m} a_n + \epsilon e^{\epsilon}, \ \forall m \geq N.$
We see that

$$\begin{aligned}\lim_{m\to\infty} \sup_{n\geq m} a_{n+m} &\leq e^{\epsilon} \lim_{m\to\infty} \inf_{n\geq m} a_n + \epsilon e^{\epsilon} \\ &= e^{\epsilon} \liminf_{n\to\infty} a_n + \epsilon e^{\epsilon}.\end{aligned}$$

By Lemma 2.14, we see that

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty} \sup_{n\geq m} a_{n+m} \leq e^{\epsilon} \liminf_{n\to\infty} a_n + \epsilon e^{\epsilon}.$$

Taking $\epsilon \to 0$, we get

$$\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n.$$

Hence $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Therefore $\lim_{n\to\infty} a_n$ exists.
If $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to zero, then we have

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_{n_k} = 0,$$

by Theorem 2.9. □

Lemma 2.16. *Let $\{x_n\}$ be a sequence in a normed space X. Assume that for any $\epsilon > 0$ there exists an N such that*

$$\|x_{n+N} - x_N\| < \epsilon \text{ for all } n.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $\epsilon > 0$. By assumption, there exists N such that

$$\|x_{n+N} - x_N\| < \frac{\epsilon}{2} \quad for\ all\ n.$$

For $m, n > N$, we have

$$\|x_n - x_N\| < \tfrac{\epsilon}{2} \text{ and } \|x_m - x_N\| < \tfrac{\epsilon}{2}.$$

Thus

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - x_N\| + \|x_m - x_N\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. □

2.3 Basic knowledge about metric spaces and normed spaces

Definition 2.10. (Metric space, Metric). Let X be a nonempty set. A function d defined on $X \times X$ is called a *metric* on X if it satisfies the following properties:

(1) $\forall x, y \in X,\ d(x, y) \geq 0$.

(2) $\forall x, y \in X,\ d(x, y) = 0 \Leftrightarrow x = y$.

(3) $\forall x, y \in X,\ d(x, y) = d(y, x)$. (Symmetry)

(4) $\forall x, y \in X,\ d(x, y) \leq d(x, z) + d(z, y)$. (Triangle inequality)

In this case, (X, d) is called a *metric space.*

Definition 2.11. (Distance). The *distance* $d(x, A)$ from a point x to a nonempty subset A of a metric space (X, d) is defined to be

$$d(x, A) = \inf_{a \in A} d(x, a).$$

This infimum certainly exists in $\mathbb{R}$ and is nonnegative. If x is already in A, then, of course, $d(x, A) = 0$.

Definition 2.12. (Ball and Sphere). Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:

(1) $B(x_0; r) = \{x \in X | d(x, x_0) < r\}$. (Open ball)

(2) $\tilde{B}(x_0; r) = \{x \in X | d(x, x_0) \leq r\}$. (Closed ball)

(3) $S(x_0; r) = \{x \in X | d(x, x_0) = r\}$. (Sphere)

In all three cases, x_0 is called the *center* and r is called the *radius.*

Definition 2.13. (Open Set, Closed Set). A subset M of a metric space X is said to be *open* if it contains a ball about each of its points. A subset K of X is said to be *closed* if its complement (in X) is open, that is, $K^c = X - K$ is open.

Definition 2.14. (Convergence of a Sequence, Limit). A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to *converge* or to be *convergent* if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x$$

or, simply, $$x_n \to x.$$

We say that $\{x_n\}$ *converge* to x or has the *limit* x. If $\{x_n\}$ is not convergent, it is said to be *divergent*.

Definition 2.15. (Cauchy sequence, completeness). A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* (or *fundamental*) if for every $\epsilon > 0$ there is an N such that

$$d(x_m, x_n) < \epsilon \text{ for every } m, n > N.$$

The space X is said to be *complete* if every Cauchy sequence in X converges. (That is, has a limit which is an element of X).

Theorem 2.17. ***(Closed set)***. *Let M be a nonempty subset of a matric space (X, d). Then M is closed if and only if the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.*

Definition 2.16. (Normed Space, Banach Space). Let X be a vector space. A norm $\|\cdot\|$ defined on X is called a *norm* on X if it satisfies the following properties:

(1) $\|x\| \geq 0$.

(2) $\|x\| = 0$ if and only if $x = 0$.

(3) $\|\alpha x\| = |\alpha|\|x\|$. (Absolute homogeneity)

(4) $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

In this case, $(X, \|\cdot\|)$ is called a normed space. Note that a complete normed space is called a *Banach space*.

Definition 2.17. (Convex set). A subset C of a vector space X is said to be *convex* if $x, y \in C$ implies

$$M = \{z \in X | z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subset C.$$

Definition 2.18. (Fixed Point). Let X be a set and $T : X \to X$ be a self-mapping. A *fixed point* of T is an $x \in X$ such that

$$Tx = x.$$

Example 2.1. Let $X = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 - 3x + 4.$$

To show that f has a fixed point, we solve $f(x) = x$ and get that $x = 2$. Thus 2 is the only fixed point of f.

Example 2.2. Let $X = \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x^2 - 2.$$

To show that g has a fixed point, we solve $g(x) = x$ and get that $x = -1, 2$. Thus -1 and 2 are fixed point of g.

Example 2.3. Let $X = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x + 1.$$

We see that $f(x) \neq x$. Therefore f has no fixed point.

Theorem 2.18. ***(Subspace of a Banach space)***. *A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Convergence of sequences and related concepts in normed spaces follow readily from the corresponding Definition 2.15 for metric spaces and the fact that now $d(x, y) = \|x - y\|$

A sequence $\{x_n\}$ in a normed space X is *Cauchy* if for every $\epsilon > 0$ there is an N such that

$$\|x_m - x_n\| < \epsilon \qquad \text{for all } m, n > N.$$

Definition 2.19. (Strong Convergence). A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in the norm*) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

This is written

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x$$

x is called the *strong limit* of $\{x_n\}$, and we say that $\{x_n\}$ *converges strongly to* x.

2.4 Banach fixed point theorem

Definition 2.20. (Picard Iteration). Let $X = (X, d)$ be a metric space and $T : X \to X$. *Picard iteration* of T is a recursive sequence $x_0, x_1, x_2, ...$ from a relation of the form

$$x_{n+1} = Tx_n \qquad\qquad n = 0, 1, 2, ...$$

with arbitrary $x_0 \in X$.

We see that

$$\begin{aligned} x_1 &= Tx_0 \\ x_2 &= Tx_1 = T(Tx_0) = T^2x_0 \\ &\vdots \\ x_n &= T^n x_0. \end{aligned}$$

This shows that $x_n = Tx_{n-1} = T^n x_0, \qquad n = 1, 2, 3, \ldots$

Definition 2.21. (Contraction Mapping). Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a *contraction mapping* on X if there is a positive real number $0 < \alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Geometrically this means that any point x and y have images that are closer together than those points x and y; more precisely, the ratio $d(Tx, Ty)/d(x, y)$ dose not exceed a constant α which is strictly less than 1.

Example 2.4. Let $X = \mathbb{R}$ with the usual norm $|\cdot|$. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \tfrac{x}{2}, \qquad x \in \mathbb{R}.$$

Consider

$$|Tx - Ty| \;=\; \left|\frac{x}{2} - \frac{y}{2}\right| = \left|\frac{1}{2}(x-y)\right| = \frac{1}{2}\,|x-y|\,.$$

Therefore T is a contraction mapping on $\mathbb{R}$.

Example 2.5. Let $X = [a, 1]$, $0 < a < 1$ with the usual norm $|\cdot|$. Define $f : [a, 1] \to [a, 1]$ by

$$f(x) = \sin x, \qquad x \in [a, 1].$$

By mean-value theorem: for any differentiable function f, $f'(t) = \frac{f(x)-f(y)}{(x-y)}$ for some t between x and y. So that, we get

$$\begin{aligned} f(x) - f(y) &= f'(t)(x-y) \\ |f(x) - f(y)| &= |f'(t)||x-y| \\ |\sin x - \sin y| &= |\cos t||x-y|. \end{aligned}$$

Since $\cos x$ is decreasing on $[a, 1]$, $|\cos t| = \cos t \leq \cos a < 1$. Therefore $|\sin x - \sin y| < |x - y|$. Hence f is a contraction mapping on $[a, 1]$.

We now ready to prove the very first fixed point theorem. The Banach fixed point theorem is important as a source of existence and uniqueness theorems in different branches of analysis, as Erwin Kreyzig said in [5]. So the generalization of Banach fixed point theorem is very important and is interesting.

Theorem 2.19. ***(Banach Fixed Point Theorem or Contraction Theorem)***. *Consider a metric space $X = (X, d)$, where $X \neq \varnothing$. Suppose that X is complete and let $T : X \to X$ be a* contraction mapping *on X. Then T has precisely one fixed point.*

Proof. We will show that the sequence $\{x_n\}$ of Picard iteration of T is Cauchy, so that it converges in the complete space X, and then we prove that its limit x is a fixed point of T and T has no further fixed points.

Let $x_0 \in X$ and define the $\{x_n\}$ to be a sequence of Picard iteration that is $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ Let $\epsilon > 0$ and $m, n \in \mathbb{N} \cup \{0\}$. Consider

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1}) \\ &\leq \alpha d(x_m, x_{m-1}) \\ &= \alpha d(Tx_{m-1}, Tx_{m-2}) \\ &\leq \alpha^2 d(x_{m-1}, x_{m-2}) \\ &\vdots \\ &\leq \alpha^m d(x_1, x_0). \end{aligned} \tag{2.6}$$

Assume that $n > m$. By the triangle inequality, the formula for the sum of a geometric progression and (2.6), we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{n-1}, x_n) \\ &\leq \alpha^m d(x_1, x_0) + \alpha^{m+1} d(x_1, x_0) + ... + \alpha^{n-1} d(x_1, x_0) \\ &= (\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}) d(x_1, x_0) \\ &\leq \alpha^m (1 + \alpha + \alpha^2 + ...) d(x_1, x_0) \\ &= \frac{\alpha^m}{1-\alpha} d(x_1, x_0). \end{aligned} \tag{2.7}$$

Since $0 < \alpha < 1$, we have $\lim_{n\to\infty} \alpha^m = 0$. Since $\dfrac{\epsilon(1-\alpha)}{d(x_0, x_1) + 1} > 0$, there exists N_0^* such that

$$\alpha^m < \frac{\epsilon(1-\alpha)}{d(x_0, x_1) + 1} \qquad \text{for all } m > N_0^*.$$

From this and the inequality (2.7), we get

$$\begin{aligned} d(x_n, x_m) &\leq \alpha^m \frac{1}{1-\alpha} d(x_0, x_1) \\ &< \frac{\epsilon(1-\alpha)}{d(x_0,x_1)+1} \cdot \frac{1}{1-\alpha} d(x_0, x_1) \\ &= \frac{\epsilon d(x_0,x_1)}{d(x_0,x_1)+1} < \epsilon \quad \text{for all} \quad m, n > N_0^*. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges, say $x_n \to x \in X$. That is, $\lim_{n\to\infty} d(x_n, x) = 0$.

We next show that this limit x is a fixed point of the mapping T. By the triangle inequality and definition of contraction mapping we have

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(x_n, Tx) \\ &= d(x, x_n) + d(Tx_{n-1}, Tx) \\ &\leq d(x, x_n) + \alpha d(x_{n-1}, x). \end{aligned}$$

Taking $n \to \infty$, we have

$$d(x, Tx) \leq 0.$$

By property (M1) of the metric d we obtain that $d(x, Tx) \geq 0$ and so $d(x, Tx) = 0$. By property (M2) of the metric d we get $Tx = x$. This show that x is a fixed point of T.

Next we show that x is the only fixed point of T. Let x and x^* be fixed points of T. Thus $Tx = x$ and $Tx^* = x^*$. Then

$$d(x, x^*) = d(Tx, Tx^*) \leq \alpha d(x, x^*)$$

and so

$$\begin{aligned} d(x, x^*) - \alpha d(x, x^*) &\leq 0 \\ (1-\alpha) d(x, x^*) &\leq 0 \end{aligned}$$

Since $1 - \alpha > 0$, we have $d(x, x^*) = 0$. By property (M2) of the metric d, we get $x = x^*$ and the theorem is proved. □

2.5 Some fixed point theorems on quasi-nonexpansive mappings

Definition 2.22. (Quasi-Nonexpansive Mapping). Let X be a real Banach space, C a closed subset of X, and $T : C \to X$ such that T has a nonempty set of fixed points $F(T) \subset C$. T is called *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|$$

for all x in C and p in $F(T)$. If the range of T is C, i.e., $T : C \to C$, we called T a self-mapping.

Example 2.6. Let $X = \mathbb{R}$ with the usual norm $|\cdot|$. Define $T : [0, 1] \to [0, 1]$ by

$$Tx = \tfrac{x}{2}, \quad x \in [0, 1].$$

To show that T has a fixed point, we solve $Tx = x$ and get that $x = 0$. Thus 0 is the only fixed point of T. Next, we want to show that T is a quasi-nonexpansive mapping. Also, we get

$$|Tx - 0| = |\frac{x}{2} - 0| = |\frac{x}{2}| \leq |x| = |x - 0|.$$

Hence T is a quasi-nonexpansive mapping on $[0, 1]$.

Example 2.7. Let $X = \mathbb{R}$ with the usual norm $|\cdot|$. Define $T : [0, 2] \to [0, 2]$ by

$$Tx = x^2 - 3x + 4, \quad x \in [0, 2].$$

To show that T has a fixed point, we solve $Tx = x$ and get that $x = 2$. Thus 2 is the only fixed point of T. Next, we want to show that T is a quasi-nonexpansive mapping. Also, we get

$$\begin{aligned}|Tx-2| &= |x^2-3x+4-2| \\ &= |x^2-3x+2| \\ &= |x^2-2x-x+2| \\ &= |x(x-2)-(x-2)| \\ &= |(x-1)(x-2)| \\ &= |x-1||x-2| \\ &\leq |x-2|.\end{aligned}$$

Hence T is a quasi-nonexpansive mapping on $[0,2]$.

Definition 2.23. (Asymptotically regular). Let X be a real Banach space. Let C be a nonempty subset of X. A mapping $T: C \to C$ is said to be

(1) *asymptotically regular* in x_0, if

$$\lim_{n\to\infty} \|T^n(x_0) - T^{n+1}(x_0)\| = 0.$$

(2) *asymptotically regular* in x_n, if

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The following are useful lemmas we will use to obtain the Theorem 2.23, 2.26 and 3.1 . Lemma 2.20 is from [5]. Lemma 2.21 and Lemma 2.22 are exercises in some text. We give proof here for those who are interested in this field.

Lemma 2.20. *If C is a nonempty closed subset of a normed space X, $x \in X$ and $d(x,C) = 0$ then $x \in C$.*

Proof. Let $C \subseteq X$ be closed, $x \in X$ and $d(x,C) = 0$ i.e. $\inf_{y\in C} d(x,y) = 0$. We will show that $x \in C$. For $n \in N$ we have

$$\inf_{y\in C} d(x,y) < \inf_{y\in C} d(x,y) + \frac{1}{n}.$$

By definition of infimum we obtain that for $n \in N$ there exists $y_n \in C$ such that

$$0 = \inf_{y \in C} d(x, y) < d(x, y_n) < \inf_{y \in C} d(x, y) + \frac{1}{n}.$$

From this and by Sandwich Theorem we get

$$\lim_{n \to \infty} d(x, y_n) = 0.$$

That is $y_n \to x$. Since C is closed, $y_n \in C$ and $y_n \to x$, by Theorem 2.17 (Closed set) we have $x \in C$. □

Lemma 2.21. *Let C be a nonempty closed subset of a Banach space X and $T : C \to C$ be a quasi-nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Then $F(T)$ is a closed subset of C.*

Proof. We will show that $F(T)$ is closed. For this we let $y_n \in F(T)$ and $y_n \to y$. If we can show that $y \in F(T)$, then by Theorem 2.17 (Closed set) $F(T)$ is closed. From $y_n \to y$, it means

$$\lim_{n \to \infty} \|y_n - y\| = 0.$$

We will show that $y \in F(T)$, i.e., $Ty = y$ or $\|Ty - y\| = 0$.
Since T is quasi-nonexpansive, by the Triangle inequality, we have

$$\begin{aligned} 0 \leq \|Ty - y\| &= \|Ty - y_n + y_n - y\| \\ &\leq \|Ty - y_n\| + \|y_n - y\| \\ &\leq \|y - y_n\| + \|y_n - y\| \\ &= 2\|y_n - y\|. \end{aligned}$$

By Sandwich Theorem we get

$$\|Ty - y\| = 0.$$

By property of norm we get $Ty = y$. Thus $y \in F(T)$.
Hence $F(T)$ is closed. □

Note that, for quasi-nonexpansive mapping $T_i : C \to C$ (i=1,2) with the common fixed point set $F(T_1) \cap F(T_2) \neq \emptyset$, this shows that $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. From above we get $F(T_1)$ and $F(T_2)$ are closed. Thus $F(T_1) \cap F(T_2)$ is closed.

Lemma 2.22. *Let X be a metric space and C a nonempty subset of X. If $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} d(x_n, C) = d(x, C)$.*

Proof. Let $x_n \to x$. We want to show that $d(x_n, C) \to d(x, C)$. We will show that $\lim_{n\to\infty} |d(x_n, C) - d(x, C)| = 0$. Since $x_n \to x$, $\lim_{n\to\infty} d(x_n, x) = 0$. By the Triangle inequality, we have for $n \in N$, that

$$d(x_n, C) \leq d(x_n, x) + d(x, C)$$

$$d(x_n, C) - d(x, C) \leq d(x_n, x) \tag{2.8}$$

Similarly, we can show that

$$d(x_n, C) - d(x, C) \geq -d(x_n, x). \tag{2.9}$$

From (2.8) and (2.9) we have $-d(x_n, x) \leq d(x_n, C) - d(x, C) \leq d(x_n, x)$ for all n, then $|d(x_n, C) - d(x, C)| \leq d(x_n, x)$, for all n. By Sandwich Theorem and $\lim_{n\to\infty} d(x_n, x) = 0$. We get that $\lim_{n\to\infty} |d(x_n, C) - d(x, C)|$=0. □

Note that, for a quasi-nonexpansive mapping $T_i : C \to C$ (i=1,2) with the common fixed point set $F(T_1) \cap F(T_2) \neq \emptyset$ and $\{x_n\}$ is a sequence in C such that $x_n \to x$. From above we get $\lim_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = d(x, F(T_1) \cap F(T_2))$.

We found Theorem 2.23, Proposition 2.24, Lemma 2.25 and Theorem 2.26 in [6]. There they gave only rough explanation about the proof. We gave a more detailed proof for those who are interested in this field and need a further study.

Theorem 2.23. *Let X be a real Banach space, C a closed subset of X, and T a quasi-nonexpansive mapping of C into C with nonempty set $F(T)$ of fixed points. Suppose there exists a point x_0 in C such that the sequence $\{x_n\}$ of iterates lies in C, where $x_n\,(= T^n(x_0))$ is given by*

$$(S1) \qquad x_n = T(x_{n-1}), \qquad n = 1, 2, 3, 4, ...$$

Then $\{x_n\}$ converges to a fixed point of T in C if and only if the following condition (M) holds:

$$(M) \qquad \lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. Let $\{x_n\}$ converge to a fixed point of T in C. We will show that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Let x be a fixed point of T in C such that $\{x_n\}$ converges to x, we have $\lim_{n\to\infty} d(x_n, x) = 0$. By definition of infimum and $x \in F(T)$ we obtain that

$$0 \leq d(x_n, F(T)) = \inf_{z\in F(T)} d(x_n, z) \leq d(x_n, x).$$

By Sandwich Theorem we get

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Conversely, let $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We want to show that $\{x_n\}$ converges to a fixed point of T in C. We will show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists N such that

$$\begin{aligned} n > N \quad &\Rightarrow \quad d(x_n, F(T)) < \frac{\epsilon}{2} \\ &\Rightarrow \quad \inf_{y\in F(T)} d(x_n, y) < \frac{\epsilon}{2} \\ &\Rightarrow \quad \inf_{y\in F(T)} \|x_n - y\| < \frac{\epsilon}{2} \end{aligned}$$

By definition of infimum, for $n > N$, there exists $y_n \in F(T)$ such that

$$\|x_n - y_n\| < \tfrac{\epsilon}{2}.$$

Since T is quasi-nonexpansive and $x_n = Tx_{n-1}$, for all n, we have

$$\begin{aligned}
\|x_{m+k} - y_m\| &= \|Tx_{m+k-1} - y_m\| \\
&\leq \|x_{m+k-1} - y_m\| \\
&\;\vdots \\
&\leq \|x_m - y_m\|.
\end{aligned}$$

For $n, m > N$ such that $n = m + k > m > N$ we have

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - y_m\| + \|y_m - x_m\| \\
&\leq \|x_m - y_m\| + \|y_m - x_m\| \\
&= 2\|y_m - x_m\| \\
&< 2 \cdot \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in C. Since X is complete and $C \subset X$ is closed, by Theorem 2.18 (Subspace of a Banach space) we have C is complete. So that $x_n \to x \in C$.

Next, we will show that x is a fixed point. By Lemma 2.22 we have $\lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T))$. So that

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T)).$$

By Lemma 2.21 we have $F(T) \subset C$ is closed and by Lemma 2.20 since $x \in C$ and $d(x, F(T)) = 0$, $x \in F(T)$. Hence $\{x_n\}$ converges to a fixed point of T in C. □

Proposition 2.24. *Suppose X, C, T and x_0 satisfy the conditions of Theorem* 2.23 . *Suppose further that*

(1) *T is asymptotically regular at x_0.*

(2) *If $\{y_n\}$ is any sequence in C such that $\|(I - T)(y_n)\| \to 0$ as $n \to \infty$, then* $\liminf_{n\to\infty} d(y_n, F(T)) = 0$.

Then $\{x_n\}$ determined by the process $(S1)$ in Theorem 2.23 *converges to a fixed point of T in C.*

Proof. To show that $\{x_n\}$ converges to a fixed point of T in C, we apply Theorem 2.23. So that we will show that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Let $p \in F(T)$. Since T is quasi-nonexpansive, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|T^{n+1}x_0 - p\| \\ &\leq \|T^n x_0 - p\| \\ &= \|x_n - p\|. \end{aligned}$$

So that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$. This implies that the sequence $\{d(x_n, F(T))\}$ is nonincreasing. Since T is asymptotically regular at x_0, we have

$$0 = \lim_{n\to\infty} \|T^n x_0 - T^{n+1}x_0\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|(I - T)x_n\|.$$

Hence, by (2), we have that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

By Corollary 2.10, there exists a monotone subsequence whose limit is $\liminf_{n\to\infty} d(x_n, F(T))$. Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, by Theorem 2.3 we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus from Theorem 2.23, it follows that $\{x_n\}$ converges to a fixed point of T in C. □

Let X be a Banach space, C a closed convex subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such that, for some λ in $(0, 1)$, the sequence $\{x_n\}$ given by $(S2)$ lies in C.

(S2) $\quad x_n = T_\lambda(x_{n-1}), \quad x_0 \in C, \quad T_\lambda = \lambda T + (1-\lambda)I, \quad \lambda \in (0, 1),$

We see that

$$\begin{aligned} x_1 &= T_\lambda(x_0) \\ x_2 &= T_\lambda(x_1) = T_\lambda(T_\lambda x_0) = T_\lambda^2(x_0) \\ &\vdots \\ x_n &= T_\lambda^n(x_0) \end{aligned}$$

Thus we conclude that $x_n = T_\lambda(x_{n-1}) = T_\lambda^n(x_0), \qquad n = 1, 2, 3, \ldots$

Lemma 2.25. *Let X be a Banach space, C a subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such that, for some λ in $(0,1)$, the sequence $\{x_n\}$ given by $(S2)$ lies in C, where $T_\lambda = \lambda T + (1-\lambda)I$ (the identity mapping on C). Then T_λ is a quasi-nonexpansive mapping.*

Proof. Let $x_0 \in C$ and $p \in F(T)$. We want to show that $\|T_\lambda(x_0) - p\| \leq \|x_0 - p\|$. Since T is quasi-nonexpansive, by the Triangle inequality and absolute homogeneity, we have

$$\begin{aligned}
\|T_\lambda(x_0) - p\| &= \|\lambda T x_0 + (1-\lambda)Ix_0 - (1-\lambda)p - \lambda p\| \\
&= \|\lambda T x_0 + (1-\lambda)x_0 - (1-\lambda)p - \lambda p\| \\
&\leq (1-\lambda)\|x_0 - p\| + \lambda\|Tx_0 - p\| \\
&\leq (1-\lambda)\|x_0 - p\| + \lambda\|x_0 - p\| \\
&= \|x_0 - p\|.
\end{aligned}$$

So that $\|T_\lambda(x_0) - p\| \leq \|x_0 - p\|$ for all $x_0 \in C$ and $p \in F(T)$. Hence T_λ is a quasi-nonexpansive mapping. □

Theorem 2.26. *Let X be a Banach space, C a closed convex subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such that, for some λ in $(0,1)$, the sequence $\{x_n\} = \{T_\lambda^n(x_0)\}$ given by $(S2)$ lies in C. Then $\{x_n\}$ converges to a fixed point of T in C if and only if*

$(\acute{C})$
$$\lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0.$$

Proof. We first assume that $\{x_n\}$ converges to a fixed point of T in C. We will show that $\lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0$. Let x be a fixed point of T in C such that $\{x_n\}$ converges to x and $\{x_n\} = \{T_\lambda^n(x_0)\}$, we have $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(T_\lambda^n(x_0), x) = 0$. By definition of infimum and $x \in F(T)$ we obtain that

$$0 \leq d(T_\lambda^n(x_0), F(T)) = \inf_{y\in F(T)} d(T_\lambda^n(x_0), y) \leq d(T_\lambda^n(x_0), x).$$

By Sandwich Theorem we get

$$\lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0.$$

Conversely, let $\lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0$. We want to show that $\{x_n\}$ converges to a fixed point of T in C. We will show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$. Since $\{x_n\} = \{T_\lambda^n(x_0)\}$ and $\lim_{n\to\infty} d(T_\lambda^n(x_0), F(T)) = 0$, there exists N such that

$$\begin{aligned} n > N \quad &\Rightarrow \quad d(T_\lambda^n(x_0), F(T)) < \frac{\epsilon}{2} \\ &\Rightarrow \quad \inf_{y\in F(T)} d(T_\lambda^n(x_0), y) < \frac{\epsilon}{2} \\ &\Rightarrow \quad \inf_{y\in F(T)} \|T_\lambda^n(x_0) - y\| < \frac{\epsilon}{2} \\ &\Rightarrow \quad \inf_{y\in F(T)} \|x_n - y\| < \frac{\epsilon}{2}. \end{aligned}$$

By definition of infimum for $n > N$ there exists $y_n \in F(T)$ such that

$$\|x_n - y_n\| < \tfrac{\epsilon}{2}.$$

Since T is quasi-nonexpansive and $x_n = Tx_{n-1}$, for all n, we have

$$\begin{aligned} \|x_{m+k} - y_m\| &= \|Tx_{m+k-1} - y_m\| \\ &\leq \|x_{m+k-1} - y_m\| \\ &\vdots \\ &\leq \|x_m - y_m\|. \end{aligned}$$

For $n, m > N$ such that $n = m + k > m > N$ we have

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - y_m\| + \|y_m - x_m\| \\ &\leq \|x_m - y_m\| + \|y_m - x_m\| \\ &= 2\|y_m - x_m\| \\ &< 2 \cdot \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in C. Since X is complete and $C \subset X$ is closed, by Theorem 2.18 (Subspace of a Banach space) we have C is complete. So that $x_n \to x \in C$.

Next we will show that x is a fixed point. By Lemma 2.22 we have $\lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T))$. So that

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T)).$$

By Lemma 2.21 we have $F(T) \subset C$ is closed and by Lemma 2.20 since $x \in C$ and $d(x, F(T)) = 0$, $x \in F(T)$. Hence $\{x_n\}$ converges to a fixed point of T in C. $\square$

CHAPTER 3

Main Results

Let X be a real Banach space and let C be a nonempty closed convex subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$ in C. We are interested in sequences in the following process. For $x_1 \in C$ and $n \geq 1$, define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1 - \beta_n - a_n)x_n + a_n v_n \\ x_{n+1} &= \alpha_n T_1 y_n + (1 - \alpha_n - b_n)y_n + b_n u_n, \end{aligned} \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$ and $\{v_n\}$ are sequences in C. Note that since C is a nonempty convex subset of X and the sequences $\{x_n\}$ and $\{y_n\}$ are convex combinations of elements in C, we conclude that $\{x_n\}$ and $\{y_n\}$ are sequences in C.

If $T_1 = T_2 = T$, (3.1) becomes

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n - a_n)x_n + a_n v_n \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n - b_n)y_n + b_n u_n, \end{aligned} \tag{3.2}$$

3.1 Main Theorems

We have the following theorems.

Theorem 3.1. *Let X be a real Banach space and let C be a nonempty closed convex subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. Assume that*

(i) $\{a_n + \beta_n\}$ and $\{b_n + \alpha_n\}$ are sequences in $[0, 1]$ and

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty;$$

(ii) $\{u_n\}$ *and* $\{v_n\}$ *are bounded.*

Then the iterative sequence $\{x_n\}$ *defined in* (3.1) *converges strongly to a common fixed point of* T_1 *and* T_2 *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

Proof. We first prove the necessity.

Assume that $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 i.e. there exists $p \in F(T_1) \cap F(T_2)$ such that $\lim_{n\to\infty} \|x_n - p\| = 0$. That is $\liminf_{n\to\infty} \|x_n - p\| = 0$ by Theorem 2.6. By Definition 2.11,

$$d(x_n, F(T_1) \cap F(T_2)) = \inf_{p^* \in F(T_1)\cap F(T_2)} \|x_n - p^*\| \leq \|x_n - p\|.$$

Taking limit infimum as $n \to \infty$, we have $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$, as desired.

Now, we prove the sufficiency. Let $p \in F(T_1) \cap F(T_2)$. By the boundedness of $\{u_n\}$ and $\{v_n\}$, we let

$$M = \max\left\{\sup_{n\geq 1} \|u_n - p\|, \sup_{n\geq 1} \|v_n - p\|\right\}.$$

Since $T_i : C \to C$ is a quasi-nonexpansive mapping for $i = 1, 2$, (3.1) and by the triangle inequality, we have

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n T_2 x_n + (1 - \beta_n - a_n)x_n + a_n v_n - p\| \\
&= \|\beta_n T_2 x_n + (1 - \beta_n - a_n)x_n + a_n v_n - (1 - \beta_n - a_n)p - \beta_n p - a_n p\| \\
&\leq \beta_n \|T_2 x_n - p\| + (1 - \beta_n - a_n)\|x_n - p\| + a_n \|v_n - p\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n - a_n)\|x_n - p\| + a_n M \\
&= (1 - a_n)\|x_n - p\| + a_n M
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n T_1 y_n + (1-\alpha_n-b_n)y_n + b_n u_n - p\| \\
&= \|\alpha_n T_1 y_n + (1-\alpha_n-b_n)y_n + b_n u_n - (1-\alpha_n-b_n)p - \alpha_n p - b_n p\| \\
&\leq \alpha_n\|T_1 y_n - p\| + (1-\alpha_n-b_n)\|y_n-p\| + b_n\|u_n-p\| \\
&\leq \alpha_n\|y_n-p\| + (1-\alpha_n-b_n)\|y_n-p\| + b_n M \\
&= (1-b_n)\|y_n-p\| + b_n M \\
&\leq (1-b_n)\{(1-a_n)\|x_n-p\| + a_n M\} + b_n M \\
&= (1-b_n)(1-a_n)\|x_n-p\| + (1-b_n)a_n M + b_n M \\
&= (1-a_n-b_n+a_nb_n)\|x_n-p\| + (a_n+b_n-a_nb_n)M \\
&\leq \|x_n-p\| + (a_n+b_n)M \\
&= \|x_n-p\| + d_n, \qquad (3.3)
\end{aligned}$$

where $d_n = (a_n+b_n)M$. Now by the assumptions that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, we have that $\sum_{n=1}^{\infty} d_n < \infty$. Then Lemma 2.15 implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. From (3.3) and by induction, for $m, n \geq 1$ and $p \in F(T_1) \cap F(T_2)$, we have

$$\|x_{n+m} - p\| \leq \|x_n - p\| + \sum_{i=n}^{n+m-1} d_i. \qquad (3.4)$$

From (3.3) and taking the infimum over $p \in F(T_1) \cap F(T_2)$, we obtain

$$d(x_{n+1}, F(T_1) \cap F(T_2)) \leq \; d(x_n, F(T_1) \cap F(T_2)) + d_n.$$

By Corollary 2.10, there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2))$. By the assumption $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$ and Lemma 2.15 tells us that

$$\lim_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0. \qquad (3.5)$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $\epsilon > 0$ From (3.5) and $\sum_{n=1}^{\infty} d_n < \infty$, there exists $n_0 \in N$ such that, for $n \geq n_0$, we have

$$d(x_n, F(T_1) \cap F(T_2)) < \frac{\epsilon}{4}, \qquad \sum_{n=n_0}^{\infty} d_n < \frac{\epsilon}{2}. \tag{3.6}$$

By the first inequality in (3.6) and the definition of infimum, there exists $p_0 \in F(T_1) \cap F(T_2)$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{4}. \tag{3.7}$$

Combining (3.4), (3.6) and (3.7), for any positive integer m, we have

$$\begin{aligned}
\|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p\| + \|x_{n_0} - p\| \\
&\leq \|x_{n_0} - p\| + \sum_{i=n_0}^{n_0+m-1} d_i + \|x_{n_0} - p\| \\
&= 2\|x_{n_0} - p\| + \sum_{i=n_0}^{n_0+m-1} d_i \\
&< 2(\frac{\epsilon}{4}) + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Which implies that $\{x_n\}$ is a Cauchy sequence in X. But X is a Banach space, so there must exist $z \in X$ such that $x_n \to z$. Since C is closed and $\{x_n\}$ is a sequence in C converging to z, we have that $z \in C$. Also, by Lemma 2.21, we have that $F(T_1)$ and $F(T_2)$ are closed. Thus $F(T_1) \cap F(T_2)$ is closed. From the continuity of $d(x, F(T_1) \cap F(T_2))$ by Lemma 2.22 with $x_n \to z$ as $n \to \infty$, we have

$$d(x_n, F(T_1) \cap F(T_2)) \to d(z, F(T_1) \cap F(T_2)).$$

From (3.5), we have $d(x_n, F(T_1) \cap F(T_2)) \to 0$ So that

$$d(z, F(T_1) \cap F(T_2)) = 0.$$

Since $F(T_1) \cap F(T_2)$ is closed, $z \in F(T_1) \cap F(T_2)$ by lemma 2.20. This completes the proof. $\square$

Corollary 3.2. *Let X, C, $T_i(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that conditions (i) and (ii) in Theorem 3.1 hold and*

(1) *the mapping $T_i(i = 1, 2)$ is asymptotically regular in x_n and*

(2) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ *implies that* $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. Since $T_i(i = 1, 2)$ is asymptotically regular in x_n, by Definition 2.23 we have

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0;\ i = 1, 2.$$

From (2), $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$. By Theorem 3.1, we see that the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2. □

Theorem 3.3. *Let X, C, $T_i(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that conditions (i) and (ii) in Theorem 3.1 hold. Assume further that the mapping $T_i(i = 1, 2)$ is asymptotically regular in x_n, and there exists an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(r) > 0$ for all $r > 0$ and for $i = 1, 2$, we have*

$$\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2))) \text{ for all } n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. From $\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2)))$ for all $n \geq 1$, we see that

$$\begin{aligned} \|x_n - T_1 x_n\| &\geq f(d(x_n, F(T_1) \cap F(T_2))) \\ \|x_n - T_2 x_n\| &\geq f(d(x_n, F(T_1) \cap F(T_2))). \end{aligned}$$

From these, we have

$$\begin{aligned} \|x_n - T_1 x_n\| + \|x_n - T_2 x_n\| &\geq 2f(d(x_n, F(T_1) \cap F(T_2))) \\ \frac{1}{2}\left(\|x_n - T_1 x_n\| + \|x_n - T_2 x_n\|\right) &\geq f(d(x_n, F(T_1) \cap F(T_2))). \end{aligned}$$

By the assumption that T_i is asymptotically regular in x_n for $i = 1, 2$,

$$0 \geq \liminf_{n\to\infty} f(d(x_n, F(T_1) \cap F(T_2))).$$

Since $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we have that

$$\liminf_{n\rightarrow\infty} f(d(x_n, F(T_1) \cap F(T_2))) = 0. \tag{3.8}$$

We claim that $\liminf_{n\rightarrow\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$. Suppose not

$$\liminf_{n\rightarrow\infty}(d(x_n, F(T_1) \cap F(T_2)) \neq 0.$$

So that

$$\liminf_{n\rightarrow\infty} d(x_n, F(T_1) \cap F(T_2)) = L > 0.$$

Since $\liminf_{n\rightarrow\infty} d(x_n, F(T_1) \cap F(T_2)) = L > 0$, we see that $\exists N_1 \in \mathbb{N}$ such that $N > N_1$ implies

$$\left| \inf_{n\geq N} d(x_n, F(T_1) \cap F(T_2)) - L \right| < \frac{L}{2}$$

$$\begin{aligned} \frac{L}{2} &< \inf_{n\geq N} d(x_n, F(T_1) \cap F(T_2)) < \frac{3L}{2} && \forall N > N_1. \\ \frac{L}{2} &< d(x_n, F(T_1) \cap F(T_2)) && ,\forall n \geq N > N_1 \end{aligned}$$

Since f is increasing, we have

$$f\left(\frac{L}{2}\right) \leq f(d(x_n, F(T_1) \cap F(T_2))) \qquad ,\forall n \geq N > N_1.$$

We get

$$\begin{aligned} f\left(\frac{L}{2}\right) &\leq \inf\{f(d(x_n, F(T_1) \cap F(T_2))); n \geq N\} \qquad ,\forall N > N_1. \\ &\leq \liminf_{n\rightarrow\infty}\{f(d(x_n, F(T_1) \cap F(T_2))); n \geq N\} \end{aligned}$$

Since $f(r) > 0$ if $r > 0$, we obtain

$$\therefore 0 < f\left(\frac{L}{2}\right) \leq \liminf_{n\rightarrow\infty} f(d(x_n, F(T_1) \cap F(T_2))),$$

contradiction with (3.8). Hence,

$$\liminf_{n\rightarrow\infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

We see that $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2, by Theorem 3.1, as desired. $\square$

If $T_1 = T_2 = T$, we have the following result.

Corollary 3.4. *Let X be a real Banach space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a quasi-nonexpansive mapping with nonempty fixed point set $F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. Assume that*

(i) $\{a_n + \beta_n\}$ and $\{b_n + \alpha_n\}$ are sequences in $[0, 1]$ and

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty;$$

(ii) $\{u_n\}$ and $\{v_n\}$ are bounded.

Then the iterative sequence $\{x_n\}$ defined in (3.2) converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Corollary 3.5. *Let X, C, T and the iterative sequence $\{x_n\}$ be as in Corollary 3.4. Suppose that conditions (i) and (ii) in Corollary 3.4 hold. Assume further that*

(1) the mapping T is asymptotically regular in x_n and

(2) $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ implies that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 3.6. *Let X, C, T and the iterative sequence $\{x_n\}$ be as in Corollary 3.4. Suppose that conditions (i) and (ii) in Corollary 3.4 hold. Assume further that mapping T is asymptotically regular in x_n, and there exists an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(r) > 0$ for all $r > 0$ and*

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))) \text{ for all } n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

3.2 An Example

The following is an example that give an application of our main results.

Let $X = \mathbb{R}$ and $C = [0,1]$. Then X is a Banach space with C as a closed convex subset. For $i = 1, 2$, define $T_i : [0,1] \to [0,1]$ by

$$T_1 x = \tfrac{3x}{10} \text{ and } T_2 x = \tfrac{x}{2}.$$

Then $T_1 x = x \Leftrightarrow x = 0$ and $T_2 x = x \Leftrightarrow x = 0$. Thus 0 is the only common fixed point of T_1 and T_2. That is $F(T_1) \cap F(T_2) = \{0\}$.
Consider, for all $x \in [0,1]$, we get

$$\begin{aligned} |T_1 x - 0| &= |\frac{3x}{10} - 0| = |\frac{3x}{10}| \leq |x| = |x - 0| \\ |T_2 x - 0| &= |\frac{x}{2} - 0| = |\frac{x}{2}| \leq |x| = |x - 0|. \end{aligned}$$

Hence T_1 and T_2 are quasi-nonexpansive mapping on $[0,1]$. Let $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{(n+1)^2}$, $\alpha_n = 1 - \frac{1}{(n+1)^2}$, $\beta_n = 1 - \frac{1}{n^2}$. Consider the condition (i), $\{a_n + \beta_n\} = \{1\}$ and $\{b_n + \alpha_n\} = \{1\}$ are sequences in $[0,1]$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \infty$ by Theorem 2.12. Therefore the condition (i) holds. Next, we consider the condition (ii). Since $\{u_n\} = \{\frac{1}{2n}\}$ and $\{v_n\} = \{\frac{1}{n}\}$, $\frac{1}{2n} \to 0$ and $\frac{1}{n} \to 0$. We have $\{u_n\}$ and $\{v_n\}$ are bounded by Theorem 2.2. Therefore the condition (ii) holds. Choose $x_1 = 1$. Then the iteration in (3.1) becomes

$$\begin{aligned} y_n &= \left(1 - \frac{1}{n^2}\right) \cdot \left(\frac{x}{2}\right) + \left(\frac{1}{n^2}\right) \cdot \left(\frac{1}{n}\right) \\ x_{n+1} &= \left(1 - \frac{1}{(n+1)^2}\right) \cdot \left(\frac{3x}{10}\right) + \left(\frac{1}{(n+1)^2}\right) \cdot \left(\frac{1}{2n}\right). \end{aligned}$$

We show that with $x_1 = 1$, $\{x_n\}$ and $\{y_n\}$ are convex combinations of elements in $[0,1]$ by calculation using microsoft office excel. See Appendix A.

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APPENDIX A

The table below shows the calculation for

$$\alpha_n = 1 - \frac{1}{(n+1)^2},\ b_n = \frac{1}{(n+1)^2},\ u_n = \frac{1}{2n},\ \beta_n = 1 - \frac{1}{n^2},\ a_n = \frac{1}{n^2},\ v_n = \frac{1}{n}$$

Table 3.1 : Value of α_n, b_n, u_n, β_n, a_n and v_n

n	α_n	b_n	u_n	β_n	a_n	v_n
1	0.75	0.25	0.5	0	1	1
2	0.888888889	0.111111111	0.25	0.75	0.25	0.5
3	0.9375	0.0625	0.166666667	0.888888889	0.111111111	0.333333333
4	0.96	0.04	0.125	0.9375	0.0625	0.25
5	0.972222222	0.027777778	0.1	0.96	0.04	0.2
6	0.979591837	0.020408163	0.083333333	0.972222222	0.027777778	0.166666667
7	0.984375	0.015625	0.071428571	0.979591837	0.020408163	0.142857143
8	0.987654321	0.012345679	0.0625	0.984375	0.015625	0.125
9	0.99	0.01	0.055555556	0.987654321	0.012345679	0.111111111
10	0.991735537	0.008264463	0.05	0.99	0.01	0.1

The table below shows the calculation for

$$y_n = \beta_n T_2 x_n + (1-\beta_n - a_n)x_n + a_n v_n$$

$$x_{n+1} = \alpha_n T_1 y_n + (1-\alpha_n - b_n)y_n + b_n u_n$$

$$T_2 x = \frac{x}{2} \quad \text{and} \quad T_1 x = \frac{3x}{10}$$

Table 3.2 : Value of y_n and x_{n+1}

n	y_n	x_{n+1}
1	1	0.35
2	0.25625	0.096111111
3	0.079753086	0.032847222
4	0.031022135	0.013934375
5	0.0146885	0.007061924
6	0.008062509	0.004070071
7	0.004908956	0.002565747
8	0.003215954	0.00172448
9	0.002223337	0.001215887
10	0.001601864	0.000889811

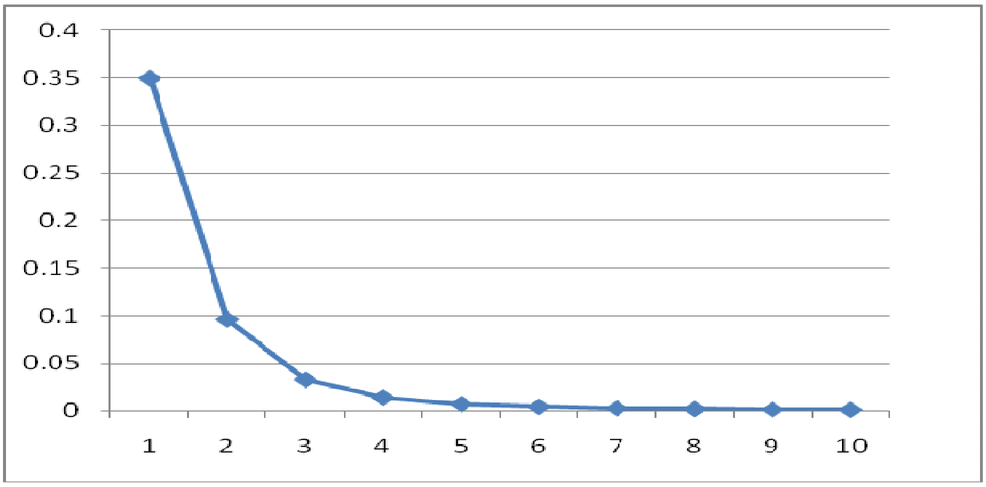

Figure 3.1 : Graph of $\{x_n\}$ in our iteration

The table below shows the calculation for

$$y_n = \beta_n T x_n + (1 - \beta_n - a_n) x_n + a_n v_n$$

$$x_{n+1} = \alpha_n T y_n + (1 - \alpha_n - b_n) y_n + b_n u_n$$

$$x_{n+1} = T x_n + b_n u_n$$

$$Tx = \frac{x}{2}$$

Table 3.3 : The comparison $\{x_{n+1}\}$ of our iteration and Picard iteration

n	Our iteration		Picard iteration
	y_n	x_{n+1}	$x_{n=1}$
1	1	0.5	0.625
2	0.3125	0.166666667	0.527777778
3	0.111111111	0.0625	0.510416667
4	0.044921875	0.0265625	0.505
5	0.02075	0.012864583	0.502777778
6	0.010883247	0.00703125	0.50170068
7	0.006359329	0.004246054	0.501116071
8	0.00404298	0.002768138	0.500771605
9	0.002738724	0.001911224	0.500555556
10	0.001946056	0.001378209	0.500413223

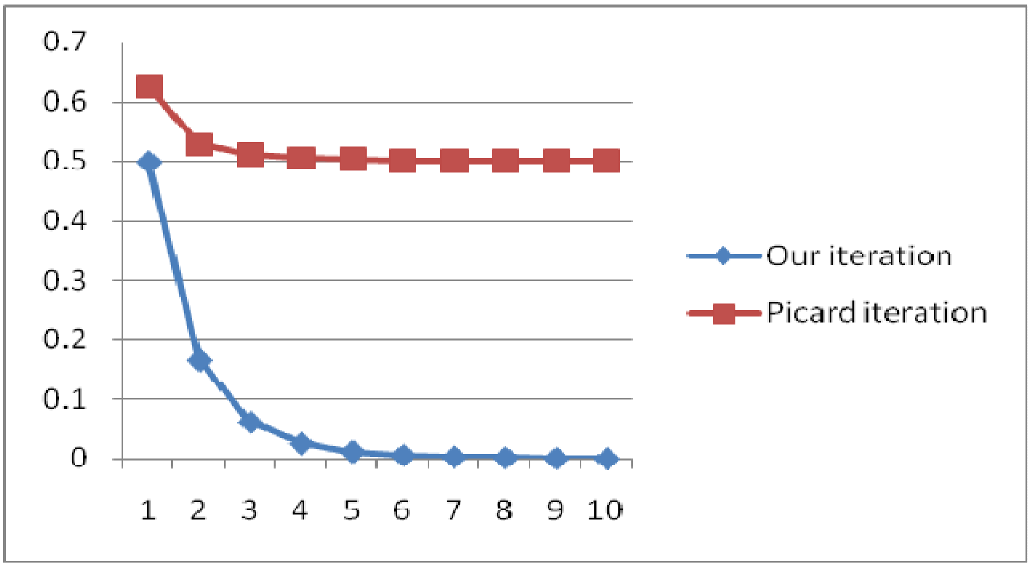

Figure 3.2. The comparison of our iteration and Picard iteration

This shows that our iteration is better than Picard iteration. Thus, our iteration is an alternative iteration for approximation a fixed point of quasi-nonexpansive mapping.

The table below shows the calculation for

$$y_n = \beta_n T x_n + \left(1 - \beta_n - a_n\right) x_n + a_n v_n$$

$$x_{n+1} = \alpha_n T y_n + \left(1 - \alpha_n - b_n\right) y_n + b_n u_n$$

$$x_{n+1} = T x_n + b_n u_n$$

$$Tx = \frac{x}{2}$$

Table 3.3 : The comparison $\{x_{n+1}\}$ of our iteration and Picard iteration

n	Our iteration		Picard iteration
	y_n	x_{n+1}	$x_{n=1}$
1	1	0.5	0.625
2	0.3125	0.166666667	0.527777778
3	0.111111111	0.0625	0.510416667
4	0.044921875	0.0265625	0.505
5	0.02075	0.012864583	0.502777778
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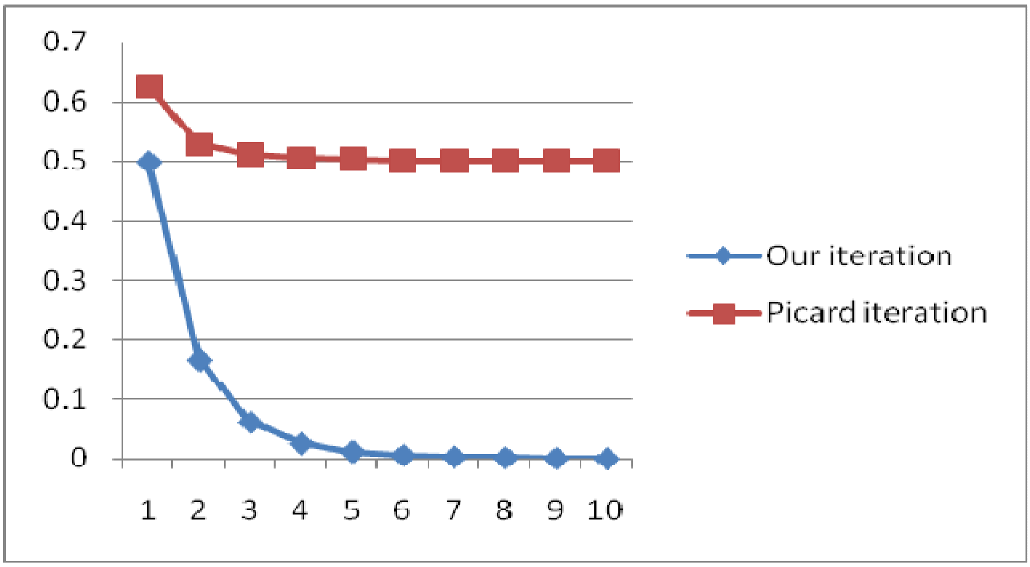

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This shows that our iteration is better than Picard iteration. Thus, our iteration is an alternative iteration for approximation a fixed point of quasi-nonexpansive mapping.

VITAE

Name Miss Pattraporn Kaonein

Student ID 5110220049

Educational Attainment

Degree	Name of Institution	Year of Graduation
B.Ed. (Mathematics)	Songkla Rajabhat University	2006

Scholarship Awards during Enrolment

Teaching Assistant ship from Faculty of Science, Prince of Songkla University, 2009-2010.

List of Publication and Proceedings

Kaonein, P. and Ayaragarnchanakul, J. 2010. A Common Fixed Point Iterative Process with Errors for Quasi-Nonexpansive Mappings in Banach Spaces. *The 20th Thaksin University Annual Conference : Thai Society Development with Creative Research 2010*, September 16-18, 2010 : 406-411.