บทที่ 1
แกมมา-กึ่งกรุป
(Gamma-semigroups)

In 1981, the notion of Γ-semigroups was introduced by M. K. Sen (See [5], [6] and [7]). Let M and Γ be any two nonempty sets. If there exists a mapping $M \times \Gamma \times M \to M$, written (a, γ, b) by $a\gamma b$, M is called a *Γ-semigroup* if M satisfies the identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$. Let K be a nonempty subset of M. Then K is called a *subΓ-semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Example 1.1. Let S be a semigroup and Γ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup.

Example 1.2. Let $M = [0,1]$ and

$$\Gamma = \{ \frac{1}{n} \mid n \text{ is a positive integer} \}.$$

Then M is a Γ-semigroup under the usual multiplication. Next, let $K = [0, \frac{1}{2}]$. We have that K is a nonempty subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a subΓ-semigroup.

From example 1.1, we have that every semigroup is a Γ-semigroup. Therefore, Γ-semigroups are generalizations of semigroups.

1.1 Quasi-gamma-ideals

Let S be a semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. Let Q be a quasi-ideal of S. Then $Q^2 \subseteq SQ \cap QS \subseteq Q$. Hence Q is a subsemigroup of S. The concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld (see [1]). The author have studied some properties of quasi-ideals in semigroups (See [2] and [3]).

Example 1.3. Let $S = [0, 1]$. Then S is a semigroup under usual multiplication. Let $Q = [0, \frac{1}{2}]$. Thus $SQ \cap QS = [0, \frac{1}{2}] \subseteq Q$. Therefore, Q is a quasi-ideal of S.

A nonempty subset L of S is called a *left ideal* of S if $SL \subseteq L$ and a nonempty subset R of S is called a *right ideal* of S if $RS \subseteq R$. Clearly, every left ideal and every right ideal of a semigroup S is a subsemigroup of S. Next, let L and R be a left ideal and a right ideal of a semigroup S. By the definition of quasi-ideals of semigroups, it is easy to prove that $L \cap R$ is a quasi-ideal of S (See [4]). Let Q be a quasi-ideal of a semigroup. Then $Q = (Q \cup SQ) \cap (Q \cup QS)$. It is easy to show that $(Q \cup SQ)$ is a left ideal of S and $Q \cup QS$ is a right ideal of S.

Then every quasi-ideal Q of S can be written as the intersection of a left ideal and a right ideal of S.

Example 1.4. Let **Z** be the set of all integers and $M_2(\mathbf{Z})$, the set of all 2 × 2 matrices over **Z**. We have known that $M_2(\mathbf{Z})$ is a semigroup under the usual multiplication. Let

$$L = \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in \mathbf{Z} \}$$

and

$$R = \{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbf{Z} \}.$$

Then L is a left ideal of $M_2(\mathbf{Z})$, R is a right ideal of $M_2(\mathbf{Z})$ and

$$L \cap R = \{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \in \mathbf{Z} \}$$

is a quasi-ideal of $M_2(\mathbf{Z})$.

In this section, we generalize some properties of quasi-ideals of semigroups to some properties of quasi-Γ-ideals in Γ-semigroups.

Let M be a Γ-semigroup. A nonempty subset Q of M is called a *quasi-Γ-ideal* of M if $M\Gamma Q \cap Q\Gamma M \subseteq Q$. Let Q be a quasi-Γ-ideal of M. Then $Q\Gamma Q \subseteq M\Gamma Q \cap Q\Gamma M \subseteq Q$. This implies that Q is a subΓ-semigroup of M.

Example 1.5. Let S be a semigroup and Γ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. From example 1.3, S is a Γ-semigroup. Let Q be a quasi-ideal of S. Thus $SQ \cap QS \subseteq Q$. We have that $S\Gamma Q \cap Q\Gamma S = SQ \cap QS \subseteq Q$. Hence, Q is a quasi-Γ-ideal of S.

Example 1.5 implies that the class of quasi-Γ-ideals in Γ-semigroups is a generalization of quasi-ideals in semigroups.

Theorem 1.1. Let M be a Γ-semigroup and Q_i a quasi-Γ-ideal of M for each $i \in I$. If $\bigcap_{i \in I} Q_i$ is a nonempty set, then $\bigcap_{i \in I} Q_i$ is a quasi-Γ-ideal of M.

Proof. Let M be a Γ-semigroup and Q_i a quasi-Γ-ideal of M for each $i \in I$. Assume that $\bigcap_{i \in I} Q_i$ is a nonempty set. Take any $a, b \in \bigcap_{i \in I} Q_i$, $m_1, m_2 \in M$ and $\gamma, \mu \in \Gamma$ such that $m_1 \mu b = a\gamma m_2$. Then $a, b \in Q_i$ for all $i \in I$. Since Q_i is a quasi-Γ-ideal of M for all $i \in I$, $m_1 \mu b = a\gamma m_2 \in M\Gamma Q_i \cap Q_i \Gamma M \subseteq Q_i$ for all $i \in I$. Therefore $m_1 \mu b = a\gamma m_2 \in \bigcap_{i \in I} Q_i$. Thus $M\Gamma \bigcap_{i \in I} Q_i \cap \bigcap_{i \in I} Q_i \Gamma M \subseteq \bigcap_{i \in I} Q_i$. Hence, $\bigcap_{i \in I} Q_i$ is a quasi-Γ-ideal of M. □

In Theorem 1.1, the condition $\bigcap_{i\in I} Q_i$ is a nonempty set is necessary. For example, let **N** be the set of all positive integers and $\Gamma = \{1\}$. Then M is a Γ-semigroup. For $n \in$ **N**, let $Q_n =$ $\{n+1, n+2, n+3, \ldots\}$. It is easy to show that each Q_n is a quasi-Γ-ideal of M for all $n \in$ **N** but $\bigcap_{n\in N} Q_n$ is a empty set.

Let A be a nonempty subset of a Γ-semigroup M and $\Im = \{ Q \mid Q$ is a quasi-Γ-ideal of M containing $A\}$. Then $\Im$ is a nonempty set because $M \in \Im$. Let $(A)_q = \bigcap_{Q\in\Im} Q$. It is clear to see that $A \subseteq (A)_q$. By Theorem 2.1, $(A)_q$ is a quasi-Γ-ideal of M. Moreover, $(A)_q$ is the smallest quasi-Γ-ideal of M containing A. $(A)_q$ is called *the quasi-Γ-ideal of M generated by A.*

Theorem 1.2. Let A be a nonempty subset of a Γ-semigroup M. Then

$$(A)_q = A \cup (M\Gamma A \cap A\Gamma M).$$

Proof. Let A be a nonempty subset of a Γ-semigroup M. Let $Q = A \cup (M\Gamma A \cap A\Gamma M)$. It is easy to see that $A \subseteq Q$. We have that

$$M\Gamma Q \cap Q\Gamma M = M\Gamma [A \cup (M\Gamma A \cap A\Gamma M)] \cap [A \cup (M\Gamma A \cap A\Gamma M)]\Gamma M \subseteq$$
$$M\Gamma (A \cup M\Gamma A) \cap [A \cup (A\Gamma M)]\,\Gamma M \subseteq M\Gamma A \cap A\Gamma M \subseteq Q.$$

Therefore, Q is a quasi-Γ-ideal of M.

Let C be any quasi-Γ-ideal of M containing A. Since C is a quasi-Γ-ideal of M and $A \subseteq C$, $M\Gamma A \cap A\Gamma M \subseteq C$. Therefore, $Q = A \cup (M\Gamma A \cap A\Gamma M) \subseteq C$.

Hence, Q is the smallest quasi-Γ-ideal of M containing A. Therefore,

$$(A)_q = A \cup (M\Gamma A \cap A\Gamma M),$$

as required. □

Example 1.6. Let **N** be the set of natural integers and $\Gamma = \{5\}$. Then **N** is a Γ-semigroup under usual addition.

(i) Let $A = \{2\}$. We have that

$$(A)_q = \{2\} \cup \{8, 9, 10, \ldots\}.$$

(ii) Let $A = \{3, 4\}$. We have that

$$(A)_q = \{3, 4\} \cup \{9, 10, 11, \ldots\}.$$

Let M be a Γ-semigroup. A subΓ-semigroup L of M is called *a left Γ-ideal* of M if $M\Gamma L \subseteq L$ and a subΓ-semigroup R of M is called a *right Γ-ideal* of M if $R\Gamma M \subseteq R$. The following theorem is true.

Theorem 1.3. Let M be a Γ-semigroup. Let L and R be a left Γ-ideal and a right Γ-ideal of M, respectively. Then $L \cap R$ is a quasi-Γ-ideal of M.

Proof. Let L and R be any left Γ-ideal and any right Γ-ideal of a Γ-semigroup M, respectively. By properties of L and R, we have $RL \subseteq L \cap R$. This implies that $L \cap R$ is a nonempty set. We have that

$$M\Gamma (L \cap R) \cap (L \cap R)\Gamma M \subseteq M\Gamma L \cap R\Gamma M \subseteq L \cap R.$$

Hence, $L \cap R$ is a quasi-Γ-ideal of M. □

Theorem 1.4. Every quasi-Γ-ideal Q of a Γ-semigroup M is the intersection of a left Γ-ideal and a right Γ-ideal of M.

Proof. Let Q be any quasi-Γ-ideal of a Γ-semigroup M. Let

$$L = Q \cup M\Gamma Q \text{ and } R = Q \cup Q\Gamma M.$$

Then $M\Gamma L = M\Gamma(Q \cup M\Gamma Q) = M\Gamma Q \cup M\Gamma M\Gamma Q \subseteq M\Gamma Q \subseteq L$ and $R\Gamma M = (Q \cup Q\Gamma M)\Gamma M = Q\Gamma M \cup Q\Gamma M\Gamma M \subseteq Q\Gamma M \subseteq R$. Then L and R is a left Γ-ideal and a right Γ-ideal of M, respectively.

Next, we claim that $Q = L \cap R$. It is easy to see that $Q \subseteq (Q \cup M\Gamma Q) \cap (Q \cup Q\Gamma M) \subseteq L \cap R$. Conversely, $L \cap R = Q \cup M\Gamma Q) \cap (Q \cup Q\Gamma M) \subseteq Q \cup (M\Gamma Q \cap Q\Gamma M) \subseteq Q$. Hence, $Q = L \cap R$. □

Let M be a Γ-semigroup. M is called a *quasi-simple* Γ*-semigroup* if M is a unique quasi-Γ-ideal of M. A quasi-Γ-ideal Q of M is called a *minimal quasi-*Γ*-ideal* of M if Q does not properly contain any quasi-Γ-ideals of M.

Example 1.7. Let G be a group and $\Gamma = \{e_G\}$. It is easy to see that G is a unique quasi-Γ-ideal of G under the usual binary operation. Then G is a quasi-simple Γ-semigroup.

Theorem 1.5. Let M be a Γ-semigroup. Then M is a quasi-simple Γ-semigroup if and only if $M\Gamma m \cap m\Gamma M = M$ for all $m \in M$.

Proof. Let M be a Γ-semigroup.

The proof of $(\rightarrow)$: Assume that M is a quasi-simple Γ-semigroup. Take any $m \in M$. First, we claim that $M\Gamma m \cap m\Gamma M$ is a quasi-ideal of M. We have that $m\Gamma m \in M\Gamma m \cap m\Gamma M$, this implies $M\Gamma m \cap m\Gamma M$ is a nonempty set. Moreover, $M\Gamma(M\Gamma m \cap m\Gamma M) \cap (M\Gamma m \cap m\Gamma M)\Gamma M \subseteq M\Gamma(M\Gamma m) \cap (m\Gamma M)\Gamma M = (M\Gamma M)\Gamma m \cap m\Gamma(M\Gamma M) \subseteq M\Gamma m \cap m\Gamma M$. Therefore, $M\Gamma m \cap m\Gamma M$ is a quasi-Γ-ideal of M. Since M is a quasi-simple Γ-semigroup, $M\Gamma m \cap m\Gamma M = M$.

The proof of $(\leftarrow)$: Assume that $M\Gamma m \cap m\Gamma M = M$ for all $m \in M$. Let Q be a quasi-Γ-ideal of M and $q \in Q$. By assumption, $M = M\Gamma q \cap q\Gamma M$. Since Q is a quasi-Γ-ideal of M, $M = M\Gamma q \cap q\Gamma M \subseteq M\Gamma Q \cap Q\Gamma M \subseteq Q$. Therefore $Q = M$. Hence, M is a quasi-simple Γ-semigroup. □

Theorem 1.6. Let M be a Γ-semigroup and Q a quasi-Γ-ideal of M. If Q is a quasi-simple Γ-semigroup, then Q is a minimal quasi-Γ-ideal of M.

Proof. Suppose M be a Γ-semigroup and Q a quasi-Γ-ideal of M. Assume that Q is a quasi-simple Γ-semigroup. Let C be a quasi-Γ-ideal of M such that $C \subseteq Q$. Then $Q\Gamma C \cap C\Gamma Q \subseteq M\Gamma C \cap C\Gamma M \subseteq C$. Therefore, C be a quasi-Γ-ideal of Q. Since Q is a quasi-simple Γ-semigroup, $C = Q$. Then Q is a minimal quasi-Γ-ideal of M. □

1.2 Bi-gamma-ideals

Let M be a Γ-semigroup. A subΓ-semigroup B of M is called a *bi-Γ-ideal* of M if $B\Gamma M\Gamma B \subseteq B$.

Example 1.8. Let S be a semigroup, and $\Gamma = \{1\}$. Define a mapping $S\times \Gamma \times S \to S$ by $a1b = ab$ for all $a, b \in S$. From Example 1.4, we have known that S is a Γ-semigroup. Let B be a bi-ideal of a semigroup S. Thus $BSB \subseteq B$. Since $\Gamma = \{1\}$, $B\Gamma S\Gamma B = BSB \subseteq B$. Hence B is a bi-Γ-ideal of S.

Example 1.8 implies that bi-Γ-ideals in Γ-semigroups are a generalization of bi-ideals in semigroups (for a suitable Γ).

Theorem 1.7. Let M be a Γ-semigroup and B_i a bi-Γ-ideal of M for all $i \in I$. If $\bigcap_{i\in I} B_i \neq \varnothing$, then $\bigcap_{i\in I} B_i$ is a bi-Γ-ideal of M.

Proof. Let M be a Γ-semigroup and B_i a bi-Γ-ideal of M for all $i \in I$. Assume that $\bigcap_{i\in I} B_i \neq \varnothing$. Let $a, b \in \bigcap_{i\in I} B_i$, $m \in M$ and $\gamma, \mu \in \Gamma$. Then $a, b \in B_i$ for all $i \in I$. Since B_i is a bi-Γ-ideal of M for all $i \in I$, $a\gamma b \in B_i$ and $a\gamma m\mu b \in B_i\Gamma M\Gamma B_i \subseteq B_i$ for all $i \in I$. Therefore $a\gamma b \in \bigcap_{i\in I} B_i$ and $a\gamma m\mu b \in \bigcap_{i\in I} B_i$. Hence $\bigcap_{i\in I} B_i$ is a bi-Γ-ideal of M. □

In Theorem 1.7, $\bigcap_{i\in I} B_i \neq \varnothing$ is a necessary condition. Let $M = (0, 1)$ and $\Gamma = \{1\}$. Then M is a Γ-semigroup under the usual multiplication. Let **N** be the set of all positive integers. For $n \in$ **N**, let $B_n = (0,\frac{1}{n})$. It is easy to prove that B_n is a bi-Γ-ideal of M for all $n \in$ **N** but $\bigcap_{n\in N} B_n = \varnothing$.

Let A be a nonempty subset of a Γ-semigroup M. Let $\Im = \{ B \mid B$ is a bi-Γ-ideal of M containing $A\}$. Then $\Im \neq \varnothing$ because $M \in \Im$. Let $(A)_b = \bigcap_{B\in\Im} B$. It is clearly seen that $A \subseteq (A)_b$. By Theorem 1.7, $(A)_b$ is a bi-Γ-ideal of M. Moreover, $(A)_b$ is the smallest bi-Γ-ideal of M containing A. $(A)_b$ is called *the bi-Γ-ideal of M generated by A.*

Theorem 1.8. Let A be a nonempty subset of a Γ-semigroup M. Then

$$(A)_b = A \cup A\Gamma A \cup A\Gamma M\Gamma A.$$

Proof. Let A be a nonempty subset of a Γ-semigroup M. Let $B = A \cup A\Gamma A \cup A\Gamma M\Gamma A$. Clearly, $A \subseteq B$. We have that $B\Gamma B = (A \cup A\Gamma A \cup A\Gamma M\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma M\Gamma A) \subseteq A\Gamma A \cup A\Gamma M\Gamma A \subseteq B$. Hence B is a subΓ-semigroup of M. Since M is a Γ-semigroup, all elements in

$B\Gamma M\Gamma B = (A \cup A\Gamma A \cup A\Gamma M\Gamma A)\Gamma M\Gamma(A \cup A\Gamma A \cup A\Gamma M\Gamma A)$ are in the form of $a_1\gamma m\mu a_2$ for some $a_1, a_2 \in A$, $\gamma, \mu \in \Gamma$ and $m \in M$. Thus $B\Gamma M\Gamma B \subseteq A\Gamma M\Gamma A \subseteq B$. Therefore B is a bi-Γ-ideal of M.

Let C be any bi-Γ-ideal of M containing A. Since C is a sub-Γ-semigroup of M and $A \subseteq C$, $A\Gamma A \subseteq C$. Since C is a bi-Γ-ideal of M and $A \subseteq C$, $A\Gamma M\Gamma A \subseteq C$. Therefore $B = A \cup A\Gamma A \cup A\Gamma M\Gamma A \subseteq C$.

Hence B is the smallest bi-Γ-ideal of M containing A. Therefore $(A)_b = B = A \cup A\Gamma A \cup A\Gamma M\Gamma A$, as required. □

Example 1.9. Let **N** be the set of all positive integers and $\Gamma = \{5\}$. Then **N** is a Γ-semigroup under usual addition.

(i) Let $A = \{2\}$. We have that $(A)_b = \{2\} \cup \{9\} \cup \{15, 16, 17, \ldots\}$.

(ii) Let $A = \{3, 4\}$. We have that $(A)_b = \{3, 4\} \cup \{11, 12, 13\} \cup \{17, 18, 19, \ldots\}$.

Theorem 1.9. Let M be a Γ-semigroup. Let B be a bi-Γ-ideal of M and A a nonempty subset of M. Then the following statements are true.

(i) $B\Gamma A$ is a bi-Γ-ideal of M.

(ii) $A\Gamma B$ is a bi-Γ-ideal of M.

Proof. (i) We have that $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A$ and $(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma A\Gamma M\Gamma B)\Gamma A$. Since B is a bi-Γ-ideal of M, $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A \subseteq B\Gamma A$ and $(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma A\Gamma M\Gamma B)\Gamma A \subseteq (B\Gamma M\Gamma B)\Gamma A \subseteq B\Gamma A$. Therefore $B\Gamma A$ is a bi-Γ-ideal of M.

The proof of (ii) is similar to the proof of (i). □

Corollary 1.10. Let M be a Γ-semigroup. For a positive integer n, let $B_1, B_2, \ldots, B_n$ be bi-Γ-ideals of M. Then $B_1\Gamma B_2\Gamma\ldots\Gamma B_n$ is a bi-Γ-ideal of M.

Proof. We will prove the corollary by mathematical induction. By Theorem 1.8, $B_1\Gamma B_2$ is a bi-Γ-ideal of M. Next, let n be any positive integer such that $k < n$ and assume $B_1\Gamma B_2\Gamma\ldots\Gamma B_k$ is a bi-Γ-ideal of M. We have that $B_1\Gamma B_2\Gamma\ldots\Gamma B_k\Gamma B_{k+1} = (B_1\Gamma B_2\Gamma\ldots\Gamma B_k)\Gamma B_{k+1}$ is a bi-Γ-ideal of M by Theorem 1.8. □

Let M be a Γ-semigroup. M is called a *bi-simple Γ-semigroup* if M is the unique bi-Γ-ideal of M. A bi-Γ-ideal B of M is called a *minimal bi-Γ-ideal* of M if B does not properly contain any bi-Γ-ideal of M.

Example 1.10. Let G be a group and $\Gamma = G$. Then $G^n = G$ and $gG = G = Gg$ for all $g \in G$. Then G is a Γ-semigroup under the usual binary operation. It is easy to see that G is the unique bi-Γ-ideal of G. Then G is a bi-simple Γ-semigroup.

Theorem 1.11. Let M be a Γ-semigroup. Then M is a bi-simple Γ-semigroup if and only if $M = m\Gamma M\Gamma m$ for all $m \in M$, where $m\Gamma M\Gamma m$ means $\{m\}\Gamma M\Gamma\{m\}$.

Proof. Let M be a Γ-semigroup.

Assume that M is a bi-simple Γ-semigroup. Let $m \in M$. By Theorem 2.3, $m\Gamma M\Gamma m$ is a bi-Γ-ideal of M. Then $M = m\Gamma M\Gamma m$.

Assume that $M = m\Gamma M\Gamma m$ for all $m \in M$. Let B be a bi-Γ-ideal of M. Let $b \in B$. By assumption, $M = b\Gamma M\Gamma b \subseteq B\Gamma M\Gamma B \subseteq B$. Hence $M = B$. Therefore M is a bi-simple Γ-semigroup. □

Theorem 1.12. Let M be a Γ-semigroup and B a bi-Γ-ideal of M. Then B is a minimal bi-Γ-ideal of M if and only if B is a bi-simple Γ-semigroup.

Proof. Let M be a Γ-semigroup and B a bi-Γ-ideal of M.

Assume that B is a minimal bi-Γ-ideal of M. Let C be a bi-Γ-ideal of B. Then $C\Gamma B\Gamma C \subseteq C$. Since B is a bi-Γ-ideal of M, by Theorem 2.3, $C\Gamma B\Gamma C$ is a bi-Γ-ideal of M. Since B is a minimal bi-Γ-ideal of M and $C\Gamma B\Gamma C \subseteq B$, $C\Gamma B\Gamma C = B$. Hence $B = C\Gamma B\Gamma C \subseteq C$, this implies $B = C$. Then B is a bi-simple Γ-semigroup.

Assume that B is a bi-simple Γ-semigroup. Let C be a bi-Γ-ideal of M such that $C \subseteq B$. Then $C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C \subseteq C$. Therefore C is a bi-Γ-ideal of B. Since B is a bi-simple Γ-semigroup, $C = B$. Hence B is a minimal bi-Γ-ideal of M, as required. □