2. Matrix representations of SO(9) simple root generators

2.1. SO(9) simple roots

SO(9) is the rank-four Lie algebra and has 36 generators, four of which are mutually commuting and form a Cartan subalgebra. The commutator of each generator with the Cartan subalgebra generates a 36-root structure, of which four are zero. Another four of those, called (positive) simple roots $\vec{\alpha}_{r=1,2,3,4}$, are linearly independent and these are encoded (in the Dynkin- or $\hat{\omega}-$basis) in the Cartan matrix A [10]

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{1a}$$

One can read off these four simple roots from each row of the Cartan matrix as follows:

$$\begin{aligned} \vec{\alpha}_1 &= 2\hat{\omega}_1 - \hat{\omega}_2 = \hat{e}_1 - \hat{e}_2, \\ \vec{\alpha}_2 &= -\hat{\omega}_1 + 2\hat{\omega}_2 - \hat{\omega}_3 = \hat{e}_2 - \hat{e}_3, \\ \vec{\alpha}_3 &= -\hat{\omega}_2 + 2\hat{\omega}_3 - 2\hat{\omega}_4 = \hat{e}_3 - \hat{e}_4, \\ \vec{\alpha}_4 &= -\hat{\omega}_3 + 2\hat{\omega}_4 = \hat{e}_4, \end{aligned} \tag{1b}$$

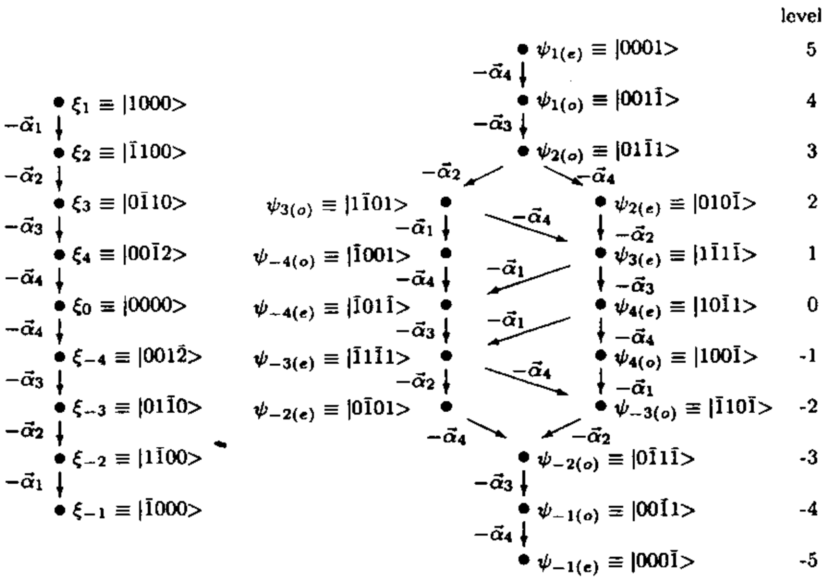
where the last column shows the simple roots in the orthornormal basis $\hat{e}_{r=1,2,3,4}$ such that $(\hat{e}_r, \hat{e}_s) = \delta_{rs}$. Notice that an element of the Cartan matrix, A_{rs}, is equal to $2(\vec{\alpha}_r, \vec{\alpha}_s)/(\vec{\alpha}_s, \vec{\alpha}_s)$.

2.2. Irreducible representations of SO(9)

An irreducible representation (irrep) ξ of SO(9) having a finite dimension is represented in the Dynkin basis as (a_1, a_2, a_3, a_4), where $a_{r=1,2,3,4}$ are zero or positive integers. Its highest weight vector is defined as $|\xi, \xi_1\rangle \equiv |a_1, a_2, a_3, a_4\rangle$, the other weight vectors as $|\xi, \xi_i\rangle \equiv |a_{i1}, a_{i2}, a_{i3}, a_{i4}\rangle$ (where $i = 2$ up to the dimension of ξ) can be obtained from the highest one by subtracting a sequence of the SO(9) simple roots. A level number ℓ defined as

$$\ell = 4a_{i1} + 7a_{i2} + 9a_{i3} + 5a_{i4}. \tag{2}$$

is used to keep track of the i-th weight $|\xi, \xi_i\rangle$ in a subtraction. For instance, the weight vectors of the 9−dimensional vector and 16−dimensional spinor irreps can be simply worked out as shown in Fig. 1. Labeling the weight vectors of SO(9) is restricted to those of SO(8). In the orthonormal basis, the SO(9) weights, ξ_0, ξ_l, $\psi_{l(e)}$ and $\psi_{l(o)}$, where $l = \pm1, \cdots, \pm4$, exactly form the 1-dimensional scalar, 8-dimensional vector, 8-dimensional spinor and 8-dimensional cospinor irreps of SO(8), respectively, and the weights $\psi_{l(e)}$ have an even number of plus signs, whereas the weights $\psi_{l(o)}$ have an odd number of plus signs.



(a) (b)

Figure 1. Weight diagrams (a) of a 9-vector irrep and (b) of a 16-spinor irrep. A number with a bar in a weight vector means that it has a minus sign.

For convenience in the calculation and to shorten the formulas, $\xi_i \equiv |\xi, \ \xi_i\rangle$ and $\xi_i^\dagger \equiv \langle\xi, \ \xi_i|$, such that the inner product equals $\xi_i^\dagger \xi_j \equiv \langle\xi, \ \xi_i|\xi, \ \xi_j\rangle = \delta_{ij}$ and the outer product $\xi_i \xi_j^\dagger = |\xi, \ \xi_i\rangle\langle\xi, \ \xi_j|$.

2.3. Matrix representation of the SO(9) generators

Suppose the dimension of irrep ξ is n. The 36 SO(9) generators of the irrep can be represented by $n \times n$ matrices. Eight of these are associated with the positive and negative simple roots. The generators for the positive and negative simple roots are denoted respectively as $E_r^+ \equiv E_{\alpha_r}$ and $E_r^- \equiv E_{-\alpha_r} = (E_r^+)^\dagger$, where $r = 1, 2, 3, 4$. It is a coincidence that in computing the tensor product, only the negative root generators are used and their matrix representations can be read from a weight diagram of the irrep ξ.

For the 9-vector irrep, the matrix representations of these four negative simple root generators are

$$
\begin{aligned}
E_1^- &= \xi_2\xi_1^\dagger + \xi_{-1}\xi_{-2}^\dagger, \quad & E_2^- &= \xi_3\xi_2^\dagger + \xi_{-2}\xi_{-3}^\dagger, \\
E_3^- &= \xi_4\xi_3^\dagger + \xi_{-3}\xi_{-4}^\dagger, \quad & E_4^- &= \xi_0\xi_4^\dagger + \xi_{-4}\xi_0^\dagger.
\end{aligned} \tag{3}
$$

For the 16-spinor, they are

$$
\begin{aligned}
E_1^- &= \psi_{-4(o)}\psi_{3(o)}^\dagger + \psi_{-3(o)}\psi_{4(o)}^\dagger + \psi_{-4(e)}\psi_{3(e)}^\dagger + \psi_{-3(e)}\psi_{4(e)}^\dagger, \\
E_2^- &= \psi_{3(o)}\psi_{2(o)}^\dagger + \psi_{-2(o)}\psi_{-3(o)}^\dagger + \psi_{3(e)}\psi_{2(e)}^\dagger + \psi_{-2(e)}\psi_{-3(e)}^\dagger, \\
E_3^- &= \psi_{2(o)}\psi_{1(o)}^\dagger + \psi_{-1(o)}\psi_{-2(o)}^\dagger + \psi_{4(e)}\psi_{3(e)}^\dagger + \psi_{-3(e)}\psi_{-4(e)}^\dagger, \\
E_4^- &= \psi_{1(o)}\psi_{1(e)}^\dagger + \psi_{4(o)}\psi_{4(e)}^\dagger + \psi_{-3(o)}\psi_{-3(e)}^\dagger + \psi_{-2(o)}\psi_{-2(e)}^\dagger \\
&\quad + \psi_{-1(e)}\psi_{-1(o)}^\dagger + \psi_{2(e)}\psi_{2(o)}^\dagger + \psi_{3(e)}\psi_{3(o)}^\dagger + \psi_{-4(e)}\psi_{-4(o)}^\dagger.
\end{aligned} \tag{4}
$$

The generator $E_4^\pm$ are two of the eight SO(9) generators which are missed in SO(8). For the 9-vector irrep, the generators $E_4^\pm$ intertwine the one-dimensional scalar weight to the middle weight of the 8-dimensional vector. For the 16-spinor irrep, they transmute the 8-spinor (8-cospinor) weights into the other 8-cospinor (8-spinor) ones. Also notice that the generators $E_{1,2,3}^\pm$ transform the 8-spinor (8-cospinor) weights into the other 8-spinor (8-cospinor) ones.

It is also useful to know the Cartan subalgebra generators in the Dynkin basis. When the Cartan generator $\vec{H} \equiv (H_1, H_2, H_3, H_4)$ acts on a weight ξ_i in the corresponding irrep, it results in

$$
\vec{H}\xi_i = (a_{i1}, a_{i2}, a_{i4}, a_{i4})\xi_i. \tag{5}
$$

To obtain the weight elements in the orthonormal basis, the Cartan generator $\vec{h} \equiv (h_1, h_2, h_3, h_4)$, which is related to $\vec{H}$ as,

$$
\begin{aligned}
h_1 &= H_1 + H_2 + H_3 + \frac{1}{2}H_4 \\
h_2 &= H_2 + H_3 + \frac{1}{2}H_4 \\
h_3 &= H_3 + \frac{1}{2}H_4 \\
h_4 &= \frac{1}{2}H_4,
\end{aligned} \tag{6}
$$

is used to act on the weight ξ_i.

For the 9-vector irrep, the matrix representations of these four Cartan generators are

$$\begin{aligned} H_1 &= \xi_1\xi_1^\dagger - \xi_2\xi_2^\dagger + \xi_{-2}\xi_{-2}^\dagger - \xi_{-1}\xi_{-1}^\dagger, \\ H_2 &= \xi_2\xi_2^\dagger - \xi_3\xi_3^\dagger + \xi_{-3}\xi_{-3}^\dagger - \xi_{-2}\xi_{-2}^\dagger, \\ H_3 &= \xi_3\xi_3^\dagger - \xi_4\xi_4^\dagger + \xi_{-4}\xi_{-4}^\dagger - \xi_{-3}\xi_{-3}^\dagger, \\ H_4 &= 2\xi_4\xi_4^\dagger - 2\xi_{-4}\xi_{-4}^\dagger, \end{aligned} \tag{7}$$

and for the 16-spinor irrep they are

$$\begin{aligned} H_1 = {} & \psi_{3(o)}\psi_{3(o)}^\dagger - \psi_{-3(o)}\psi_{-3(o)}^\dagger + \psi_{4(o)}\psi_{4(o)}^\dagger - \psi_{-4(o)}\psi_{-4(o)}^\dagger \\ & + \psi_{3(e)}\psi_{3(e)}^\dagger - \psi_{-3(e)}\psi_{-3(e)}^\dagger + \psi_{4(e)}\psi_{4(e)}^\dagger - \psi_{-4(e)}\psi_{-4(e)}^\dagger, \\ H_2 = {} & \psi_{2(o)}\psi_{2(o)}^\dagger - \psi_{-2(o)}\psi_{-2(o)}^\dagger + \psi_{-3(o)}\psi_{-3(o)}^\dagger - \psi_{3(o)}\psi_{3(o)}^\dagger \\ & + \psi_{2(e)}\psi_{2(e)}^\dagger - \psi_{-2(e)}\psi_{-2(e)}^\dagger + \psi_{-3(e)}\psi_{-3(e)}^\dagger - \psi_{3(e)}\psi_{3(e)}^\dagger, \\ H_3 = {} & \psi_{1(o)}\psi_{1(o)}^\dagger - \psi_{-1(o)}\psi_{-1(o)}^\dagger + \psi_{-2(o)}\psi_{-2(o)}^\dagger - \psi_{2(o)}\psi_{2(o)}^\dagger \\ & + \psi_{3(e)}\psi_{3(e)}^\dagger - \psi_{-3(e)}\psi_{-3(e)}^\dagger + \psi_{-4(e)}\psi_{-4(e)}^\dagger - \psi_{4(e)}\psi_{4(e)}^\dagger, \\ H_4 = {} & \psi_{-1(o)}\psi_{-1(o)}^\dagger - \psi_{1(o)}\psi_{1(o)}^\dagger + \psi_{2(o)}\psi_{2(o)}^\dagger - \psi_{-2(o)}\psi_{-2(o)}^\dagger \\ & + \psi_{3(o)}\psi_{3(o)}^\dagger - \psi_{-3(o)}\psi_{-3(o)}^\dagger + \psi_{-4(o)}\psi_{-4(o)}^\dagger - \psi_{4(o)}\psi_{4(o)}^\dagger \\ & + \psi_{4(e)}\psi_{4(e)}^\dagger - \psi_{-4(e)}\psi_{-4(e)}^\dagger + \psi_{-3(e)}\psi_{-3(e)}^\dagger - \psi_{3(e)}\psi_{3(e)}^\dagger \\ & + \psi_{-2(e)}\psi_{-2(e)}^\dagger - \psi_{2(e)}\psi_{2(e)}^\dagger + \psi_{1(e)}\psi_{1(e)}^\dagger - \psi_{-1(e)}\psi_{-1(e)}^\dagger. \end{aligned} \tag{8}$$

The Cartan generators H_r and the simple root generators $E_r^\pm$ satisfy the following commutation relations

$$[E_r^+, E_r^-] = N_{r,-r} H_r, \tag{9a}$$

where in general $N_{r,-r} = 1$ except for the 9-vector irrep, as then $N_{4,-4} = 1/2$, and

$$[\vec{H}, E_r^\pm] = \pm\vec{\alpha}_r E_r^\pm. \tag{9b}$$

The equations (3) to (8) are the expressions used to compute a tensor product of SO(9) irreps. This can be done directly by means of a straightforward algebraic method. Before doing so, it is opportune to briefly recall a general form of the tensor product decomposition.

2.4. Tensor products of SO(9) representations

From the 9-vector and 16-spinor irreps, one can produce a higher dimensional irrep by means of generating a tensor product. Here again one has the advantage that most tensor products of any two irreps, ξ and η, are reducible and can be decomposed into a sum of irreps, i.e.,

$$\xi \otimes \eta = \zeta \oplus \zeta' \oplus \zeta'' \oplus \dots. \tag{10a}$$

Conversely, a weight vector in each irrep on the right hand side of (10*a*) can be written as a sum of weight products,

$$\zeta_k = \sum_{i,j} C_k^{ij}\xi_i\eta_j, \quad \zeta'_k = \sum_{i,j} C'^{ij}_k\xi_i\eta_j, \quad \zeta''_k = \sum_{i,j} C''^{ij}_k\xi_i\eta_j, \quad \dots \tag{10b}$$

where C_k^{ij}'s are the Clebsch-Gordan coefficients (CGCs). The CGCs are the purely geometrical factors that describe how the other irreps are made from the weight products.