3. The explicit calculation of the SO(9) tensor products

In order to generate the SO(9) irreps by means of an explicit calculation from tensor products of 9-vector and 16-spinor irreps, the following approach is effective, and can also be used to obtain similar tensor products of more than two irreps.

3.1. Vector-vector product

The tensor product of two 9-vector irreps can be decomposed in terms of dimensions as

$$9 \otimes 9 = 44 \oplus 36 \oplus 1, \tag{11a}$$

or in terms of Dynkin labels as

$$\overset{\xi}{(1000)} \otimes \overset{\eta}{(1000)} = \overset{\zeta}{(2000)} \oplus \overset{\zeta'}{(0100)} \oplus \overset{\zeta''}{(0000)} \,. \tag{11b}$$

Firstly, one needs to consider the (2000) subspace. Acting on the top level (the eighth level) of the highest weight in the vector-vector product, $\zeta_1 = \xi_1\eta_1$, by a series of four negative simple root generators, $E_r^- \equiv E_{\zeta_r}^- = I \otimes E_{\eta r}^- + E_{\xi_r}^- \otimes I$, the lower levels of weight vectors are obtained:

At the seventh level:

$$\zeta_2 \equiv \frac{1}{\sqrt{2}} E_1^- \zeta_1 = \frac{1}{\sqrt{2}} \left(\xi_1 (E_{\eta 1}^- \eta_1) + (E_{\xi 1}^- \xi_1)\eta_1 \right)$$
$$= \frac{1}{\sqrt{2}} (\xi_1\eta_2 + \xi_2\eta_1) \,, \tag{12a}$$

$$E_2^- \zeta_1 = E_3^- \zeta_1 = E_4^- \zeta_1 = 0. \tag{12b}$$

At the sixth level:

$$\zeta_3 \equiv E_2^- \zeta_2 = \frac{1}{\sqrt{2}} (\xi_1\eta_3 + \xi_3\eta_1) \,, \tag{12c}$$

$$\zeta_4 \equiv \frac{1}{\sqrt{2}} E_1^- \zeta_2 = \xi_2\eta_2, \tag{12d}$$

$$E_3^- \zeta_2 = E_4^- \zeta_2 = 0. \tag{12e}$$

At the fifth level:

$$\zeta_5 \equiv E_3^- \zeta_3 = \frac{1}{\sqrt{2}} (\xi_1\eta_4 + \xi_4\eta_1) \,, \tag{12f}$$

$$\zeta_6 \equiv E_1^- \zeta_3 = \frac{1}{\sqrt{2}} E_2^- \zeta_4 = \frac{1}{\sqrt{2}} (\xi_2\eta_3 + \xi_3\eta_2) \,, \tag{12g}$$

$$E_2^- \zeta_3 = E_4^- \zeta_3 = E_1^- \zeta_4 = E_2^- \zeta_4 = E_3^- \zeta_4 = 0. \tag{12h}$$

At the fourth level:

$$\zeta_7 \equiv E_4^- \zeta_5 = \frac{1}{\sqrt{2}} (\xi_1\eta_0 + \xi_0\eta_1) \,, \tag{12i}$$

$$\zeta_8 \equiv E_1^- \zeta_5 = E_3^- \zeta_6 = \frac{1}{\sqrt{2}} (\xi_2\eta_4 + \xi_4\eta_2) \,, \tag{12j}$$

$$\zeta_9 \equiv \frac{1}{\sqrt{2}} E_2^- \zeta_6 = \xi_3\eta_3, \tag{12k}$$

$$E_2^- \zeta_5 = E_3^- \zeta_5 = E_1^- \zeta_6 = E_4^- \zeta_6 = 0. \tag{12l}$$

At the zero level:

$$\zeta_{21} \equiv E_1^- E_2^- E_3^- E_4^- \zeta_7 = \frac{1}{2} \left(\xi_1 \eta_{-1} + \xi_2 \eta_{-2} + \xi_{-2} \eta_2 + \xi_{-1} \eta_1\right), \tag{12m}$$

$$\zeta_{22} \equiv \frac{1}{\sqrt{2}} E_2^- E_1^- E_3^- E_4^- \zeta_7 = \frac{1}{2} \left(\xi_2 \eta_{-2} + \xi_3 \eta_{-3} + \xi_{-3} \eta_3 + \xi_{-2} \eta_2\right), \tag{12n}$$

$$\zeta_{23} \equiv \frac{1}{\sqrt{2}} E_3^- E_2^- E_1^- E_4^- \zeta_7 = \frac{1}{2} \left(\xi_3 \eta_{-3} + \xi_4 \eta_{-4} + \xi_{-4} \eta_4 + \xi_{-3} \eta_3\right), \tag{12o}$$

$$\zeta_{24} \equiv \frac{1}{\sqrt{3}} E_4^- E_3^- E_2^- E_1^- \zeta_7 = \frac{1}{\sqrt{6}} \left(\xi_4 \eta_{-4} + 2\xi_0 \eta_0 + \xi_{-4} \eta_4\right). \tag{12p}$$

The weight elements of ζ_i are found by letting $H_{\zeta r} \equiv I \otimes H_{\eta r} + H_{\xi r} \otimes I$ act directly on the right hand side of ζ_i.

Note that ζ_7 at the fourth level is the non-degenerate dominant weight |1000> and $\zeta_{21,22,23,24}$ at the zero level are the degenerated dominant weights |0000>. An upper half of the action series of the negative simple root generators in the (2000) subspace is sŭmmarized and shown as the upper-half weight diagram in Fig. 2.

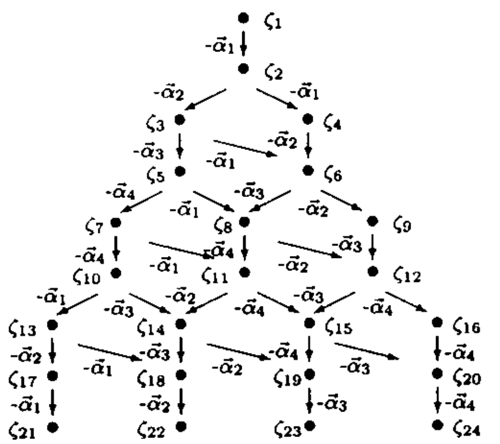

Figure 2. The upper-half weight diagram for the 44-dimensional irrep.

In the linear combinations of weight vectors of the vector-vector product, there are 81 (= 9^2) CGCs in the (2000), (0100), and (0000) subspaces. As can be seen from the above computation in the (2000) subspace, many of CGCs are zero and the rest has only a few distinct non zero values. Therefore, it suffices to compute only the dominant weights in the tensor product [3, 4, 5].

For the (0100) subspace, its highest weight $\zeta_1' \equiv$ |0100> at the seventh level is orthonormal to ζ_2. Up to an overall phase, one can choose it to be

$$\zeta_1' \equiv \frac{1}{\sqrt{2}} \left(\xi_1 \eta_2 - \xi_2 \eta_1\right). \tag{13a}$$

For the remaining dominant weight vectors in the (0100) subspace, one obtains at the fourth level:

$$\zeta_5' \equiv E_4^- E_3^- E_2^- \zeta_1' = \frac{1}{\sqrt{2}} \left(\xi_1 \eta_0 - \xi_0 \eta_1\right). \tag{13b}$$

and at the zero level:

$$\zeta'_{17} \equiv E_1^- E_2^- E_3^- E_4^- \zeta'_5 = \frac{1}{2}\left(\xi_1\eta_{-1} + \xi_2\eta_{-2} - \xi_{-2}\eta_2 - \xi_{-1}\eta_1\right), \qquad (13c)$$

$$\zeta'_{18} \equiv \frac{1}{\sqrt{2}} E_2^- E_1^- E_3^- E_4^- \zeta'_5 = \frac{1}{2}\left(\xi_2\eta_{-2} + \xi_3\eta_{-3} - \xi_{-3}\eta_3 - \xi_{-2}\eta_2\right), \qquad (13d)$$

$$\zeta'_{19} \equiv \frac{1}{\sqrt{2}} E_3^- E_2^- E_1^- E_4^- \zeta'_5 = \frac{1}{2}\left(\xi_3\eta_{-3} + \xi_4\eta_{-4} - \xi_{-4}\eta_4 - \xi_{-3}\eta_3\right), \qquad (13e)$$

$$\zeta'_{20} \equiv E_4^- E_3^- E_2^- E_1^- \zeta'_5 = \frac{1}{\sqrt{2}}\left(\xi_4\eta_{-4} - \xi_{-4}\eta_4\right). \qquad (13f)$$

Note that $\zeta'_5 \equiv |1000\rangle$ in the (0100) subspace is orthonormal to $\zeta_7 \equiv |1000\rangle$ in the (2000) subspace. By removing ζ_1, ζ_4, ζ_9 and ζ_{16} from Fig. 2 and relabeling its weight vectors, one obtains the upper-half weight diagram of (0100) subspace.

For the (0000) subspace, its highest weight $\zeta'' \equiv |0000\rangle$ at the zero level is orthonormal to $\zeta_{21,22,23,24}$ and to $\zeta'_{17,18,19,20}$. Up to an overall phase, the weight vector ζ'' is

$$\zeta'' \equiv \frac{1}{\sqrt{9}}\left(\xi_1\eta_{-1} - \xi_2\eta_{-2} + \xi_3\eta_{-3} - \xi_4\eta_{-4} + \xi_0\eta_0 - \xi_{-4}\eta_4 + \xi_{-3}\eta_3 - \xi_{-2}\eta_2 + \xi_{-1}\eta_1\right). \qquad (14)$$

Note that the weights in ζ and ζ'' are symmetric under the interchange between ξ and η, whereas the weights in ζ' are antisymmetric.

3.2. Vector-spinor product

The $9 \otimes 16$ can be decomposed as

$$9 \otimes 16 = 128 \oplus 16, \qquad (15a)$$

or in terms of Dynkin label

$$\overset{\xi}{(1000)} \otimes \overset{\psi}{(0001)} = \overset{\zeta}{(1001)} \oplus \overset{\zeta'}{(0001)}. \qquad (15b)$$

Acting on the top level (the ninth level) of the highest weight $\zeta_1 = \xi_1\psi_{1(e)}$ by a series of those four simple root generators, one gets the four-fold degenerate dominant weights $|0001\rangle$ at the fifth level,

$$\zeta_{12} \equiv \frac{1}{\sqrt{2}} E_1^- E_2^- E_3^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}}\left(\xi_1\psi_{-4(o)} + \xi_2\psi_{3(o)}\right), \qquad (16a)$$

$$\zeta_{13} \equiv \frac{1}{\sqrt{2}} E_2^- E_1^- E_3^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}}\left(\xi_2\psi_{3(o)} + \xi_3\psi_{2(o)}\right), \qquad (16b)$$

$$\zeta_{14} \equiv \frac{1}{\sqrt{2}} E_3^- E_2^- E_1^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}}\left(\xi_3\psi_{2(o)} + \xi_4\psi_{1(o)}\right), \qquad (16c)$$

$$\zeta_{15} \equiv \frac{1}{\sqrt{2}} E_4^- E_3^- E_2^- E_1^- \zeta_1 = \frac{1}{\sqrt{2}}\left(\xi_4\psi_{1(o)} + \xi_0\psi_{1(e)}\right). \qquad (16d)$$

These are the only dominant weights in the (1001) subspace.

For the (0001) subspace, its highest weight ζ'_1 at the fifth level is orthonormal to $\zeta_{12,13,14,15}$. Up to an overall phase, it is

$$\zeta'_1 = \frac{1}{\sqrt{5}}\left(\xi_1\psi_{-4(o)} - \xi_2\psi_{3(o)} + \xi_3\psi_{2(o)} - \xi_4\psi_{1(o)} + \xi_0\psi_{1(e)}\right). \qquad (17)$$

3.3. Spinor-spinor product

The $16 \otimes 16$ can be decomposed as

$$16 \otimes 16 = 126 \oplus 84 \oplus 36 \oplus 9 \oplus 1, \tag{18a}$$

or in terms of Dynkin labels as

$$\overset{\psi}{(0001)} \otimes \overset{\chi}{(0001)} = \overset{\zeta}{(0002)} \oplus \overset{\zeta'}{(0010)} \oplus \overset{\zeta''}{(0100)} \oplus \overset{\zeta'''}{(1000)} \oplus \overset{\zeta''''}{(0000)} . \tag{18b}$$

The dominant weights in the spinor-spinor product can be computed explicitly as above. Their results for the top level in each subspace are as follow:
For the (0002) subspace:

$$\zeta_1 = \psi_{1(e)}\chi_{1(e)}. \tag{19}$$

For the (0010) subspace:

$$\zeta_1' = \frac{1}{\sqrt{2}}(\psi_{1(e)}\chi_{1(o)} - \psi_{1(o)}\chi_{1(e)}). \tag{20}$$

For the (0100) subspace:

$$\zeta_1'' = \frac{1}{2}(\psi_{1(e)}\chi_{2(e)} - \psi_{2(e)}\chi_{1(e)} + \psi_{2(o)}\chi_{1(o)} - \psi_{1(o)}\chi_{2(o)}). \tag{21}$$

For the (1000) subspace:

$$\begin{aligned}\zeta_1''' = \frac{1}{2\sqrt{2}}(&\psi_{1(e)}\chi_{4(o)} + \psi_{4(o)}\chi_{1(e)} + \psi_{2(o)}\chi_{3(e)} + \psi_{3(e)}\chi_{2(o)} \\ &- \psi_{1(o)}\chi_{4(e)} - \psi_{4(e)}\chi_{1(o)} - \psi_{2(e)}\chi_{3(o)} - \psi_{3(o)}\chi_{2(e)}).\end{aligned} \tag{22}$$

For the (0000) subspace:

$$\begin{aligned}\zeta_1'''' = \frac{1}{4}(&\psi_{1(e)}\chi_{-1(e)} + \psi_{-1(e)}\chi_{1(e)} + \psi_{2(e)}\chi_{-2(e)} + \psi_{-2(e)}\chi_{2(e)} \\ &- \psi_{3(e)}\chi_{-3(e)} - \psi_{-3(e)}\chi_{3(e)} - \psi_{4(e)}\chi_{4(e)} - \psi_{-4(e)}\chi_{4(e)} \\ &- \psi_{1(o)}\chi_{-1(o)} - \psi_{-1(o)}\chi_{1(o)} + \psi_{2(o)}\chi_{-2(o)} + \psi_{-2(o)}\chi_{2(o)} \\ &- \psi_{3(o)}\chi_{-3(o)} - \psi_{-3(o)}\chi_{3(o)} + \psi_{4(o)}\chi_{4(o)} + \psi_{-4(o)}\chi_{-4(o)}).\end{aligned} \tag{23}$$

Notice that the (0002), (1000) and (0000) subspaces are symmetric in the interchange between ψ and χ whereas the (0010) and (0100) subspaces are antisymmetric.