4. The Schwinger's oscillator realization of a coupled tensor operator

At this stage, the way is cleared for constructing an SO(9) operator for general and special purposes.

In a Lie algebra, a tensor operator T^{ξ} with a number of components equal to the dimension of ξ is defined through commutation relations of its components with the Lie algebra generators [11],

$$[\vec{H}, T^{\xi}_{\xi_i}] = \xi_i T^{\xi}_{\xi_i}, \tag{24a}$$

$$[E^{\pm}_r, T^{\xi}_{\xi_i}] = <\xi, \xi_i \pm \alpha_r | E^{\pm}_r | \xi, \xi_i> T^{\xi}_{\xi_i \pm \alpha_r}. \tag{24b}$$

Similarly to the method of generating the other irreps from the tensor product of 9-vector and 16-spinor, a coupled tensor operator can be constructed from the 9-vector and 16-spinor operators. In practice, it is more efficient to realize the tensor operators and their components in terms of Schwinger's bosonic oscillators [12, 13].

4.1. The Schwinger's 9-vector operators

Let $a_0^{\pm}$ and $a_l^{\pm}$, where $l = \pm 1, \cdots, \pm 4$, be the Schwinger's bosonic oscillators associated with the 1-scalar and 8-vector, respectively. The non zero commutation relations of these oscillators are

$$[a_0^-, a_0^+] = 1, \quad [a_l^-, a_{l'}^+] = \delta_{l+l',0}. \tag{25}$$

The creation and annihilation oscillators are related to each other by an adjoint action such that

$$a_0^{\mp} = \left(a_0^{\pm}\right)^{\dagger}, \quad a_{-l}^{\mp} = \left(a_l^{\pm}\right)^{\dagger}, \tag{26}$$

and the actions of these oscillators on a vacuum state $|\Omega\rangle$ are defined as

$$\begin{aligned} \xi_0 &= a_0^+|\Omega\rangle, \quad a_0^-|\Omega\rangle = 0, \quad \xi_0^{\dagger} = \langle\Omega|a_0^-, \quad \langle\Omega|a_0^+ = 0, \\ \xi_l &= a_l^+|\Omega\rangle, \quad a_l^-|\Omega\rangle = 0, \quad \xi_l^{\dagger} = \langle\Omega|a_{-l}^-, \quad \langle\Omega|a_l^+ = 0. \end{aligned} \tag{27}$$

where ξ_0 and ξ_l are the weights labeled in Fig. 1(a). In other words, these oscillators create the states which correspond to the weights of the 9-vector irrep.

The negative simple root generators for the 9-vector irrep can be written in terms of these oscillators as:

$$\begin{aligned} E_1^- &= a_2^+ a_{-1}^- + a_{-1}^+ a_2^-, \quad E_2^- = a_3^+ a_{-2}^- + a_{-2}^+ a_3^-, \\ E_3^- &= a_4^+ a_{-3}^- + a_{-3}^+ a_4^-, \quad E_4^- = a_0^+ a_{-4}^- + a_{-4}^+ a_0^-. \end{aligned} \tag{28a}$$

The adjoints of (28a) are:

$$\begin{aligned} E_1^+ &= a_1^+ a_{-2}^- + a_{-2}^+ a_1^-, \quad E_2^+ = a_2^+ a_{-3}^- + a_{-3}^+ a_2^-, \\ E_3^+ &= a_3^+ a_{-4}^- + a_{-4}^+ a_3^-, \quad E_4^+ = a_4^+ a_0^- + a_0^+ a_4^-. \end{aligned} \tag{28b}$$

The Cartan subalgebra generators are as follows:

$$\begin{aligned} H_1 &= a_1^+ a_{-1}^- - a_2^+ a_{-2}^- + a_{-2}^+ a_2^- - a_{-1}^+ a_1^-, \\ H_2 &= a_2^+ a_{-2}^- - a_3^+ a_{-3}^- + a_{-3}^+ a_3^- - a_{-2}^+ a_2^-, \\ H_3 &= a_3^+ a_{-3}^- - a_4^+ a_{-4}^- + a_{-4}^+ a_4^- - a_{-3}^+ a_3^-, \\ H_4 &= 2(a_4^+ a_{-4}^- - a_{-4}^+ a_4^-). \end{aligned} \tag{28c}$$

The commutation relations among these generators are similar to (9a) and (9b) and those of the Cartan subalgebra and simple root generators with $a_l^{\pm}$ are as follows:

$$[\vec{H}, a_0^{\pm}] = \xi_0 a_0^{\pm} = 0, \quad [\vec{H}, a_l^{\pm}] = \xi_l a_l^{\pm}, \tag{29a}$$

$$\begin{aligned} &[E_4^+, a_{-4}^{\pm}] = \pm a_0^{\pm}, \quad [E_4^+, a_0^{\pm}] = \pm a_4^{\pm}, \\ &[E_r^+, a_{-r}^{\pm}] = \pm a_{-(r+1)}^{\pm}, \quad [E_r^+, a_{r+1}^{\pm}] = \pm a_r^{\pm}, \quad r = 1, 2, 3. \end{aligned} \tag{29b}$$

The adjoints of (29b) result in the actions of E_r^- on the oscillators $a_l^{\pm}$. Equation (29a) means that the oscillators $a_l^{\pm}$ behave like the 9-vector operator components, and (29b) and its adjoints describe how the oscillators or operator components $a_l^{\pm}$ are transformed under the actions of the positive and negative simple root generators.

4.2. The Schwinger's 16-spinor operators

Let $b_l^{\pm}$ and $c_l^{\pm}$, where $l = \pm 1, \cdots, \pm 4$, be the Schwinger's bosonic oscillators associated with the 8-spinor and 8-cospinor, respectively. The non zero commutation relations of these oscillators are

$$[b_l^-, b_{l'}^+] = [c_l^-, c_{l'}^+] = \delta_{l+l',0}, \tag{30}$$

and the actions of these oscillators on the vacuum state $|\Omega\rangle$ are defined as

$$\begin{aligned} \psi_{l(e)} &= b_l^+|\Omega\rangle, \quad b_l^-|\Omega\rangle = 0, \quad \psi_{l(e)}^\dagger = \langle\Omega|b_{-l}^-, \quad \langle\Omega|b_l^+ = 0, \\ \psi_{l(o)} &= c_l^+|\Omega\rangle, \quad c_l^-|\Omega\rangle = 0, \quad \psi_{l(o)}^\dagger = \langle\Omega|c_{-l}^-, \quad \langle\Omega|c_l^+ = 0, \end{aligned} \tag{31}$$

where $\psi_{l(e)}$ and $\psi_{l(o)}$ are the weights labeled in Fig. 1(b). This means that these oscillators create the states which correspond to the weights of the 16-spinor irrep.

The negative simple root generators for the 16-spinor irrep can be written in terms of these oscillators as:

$$\begin{aligned} E_1^- &= c_{-4}^+ c_{-8}^- + c_{-3}^+ c_{-4}^- + b_{-4}^+ b_{-3}^- + b_{-3}^+ b_{-4}^-, \\ E_2^- &= c_3^+ c_{-2}^- + c_{-2}^+ c_3^- + b_3^+ b_{-2}^- + b_{-2}^+ b_3^-, \\ E_3^- &= c_2^+ c_{-1}^- + c_{-1}^+ c_2^- + b_{-3}^+ b_4^- + b_4^+ b_{-3}^-, \\ E_4^- &= b_{-1}^+ c_1^- + b_2^+ c_{-2}^- + b_3^+ c_{-3}^- + b_{-4}^+ c_4^- + c_1^+ b_{-1}^- + c_{-2}^+ b_2^- + c_{-3}^+ b_3^- + c_4^+ b_{-4}^-. \end{aligned} \tag{32a}$$

The adjoints of (32*a*) are:

$$\begin{aligned} E_1^+ &= c_3^+ c_4^- + c_4^+ c_3^- + b_3^+ b_4^- + b_4^+ b_3^-, \\ E_2^+ &= c_2^+ c_{-3}^- + c_{-3}^+ c_2^- + b_2^+ b_{-3}^- + b_{-3}^+ b_2^-, \\ E_3^+ &= c_1^+ c_{-2}^- + c_{-2}^+ c_1^- + b_{-4}^+ b_3^- + b_3^+ b_{-4}^-, \\ E_4^+ &= c_{-1}^+ b_1^- + c_2^+ b_{-2}^- + c_3^+ b_{-3}^- + c_{-4}^+ b_4^- + b_1^+ c_{-1}^- + b_{-2}^+ c_2^- + b_{-3}^+ c_3^- + b_4^+ c_{-4}^-. \end{aligned} \tag{32b}$$

The Cartan subalgebra generators are:

$$\begin{aligned} H_1 &= c_3^+ c_{-3}^- - c_{-3}^+ c_3^- + c_4^+ c_{-4}^- - c_{-4}^+ c_4^- + b_3^+ b_{-3}^- - b_{-3}^+ b_3^- + b_4^+ b_{-4}^- - b_{-4}^+ b_4^-, \\ H_2 &= c_2^+ c_{-2}^- - c_{-2}^+ c_2^- + c_{-3}^+ c_3^- - c_3^+ c_{-3}^- + b_2^+ b_{-2}^- - b_{-2}^+ b_2^- + b_{-3}^+ b_3^- - b_3^+ b_{-3}^-, \\ H_3 &= c_1^+ c_{-1}^- - c_{-1}^+ c_1^- + c_{-2}^+ c_2^- - c_2^+ c_{-2}^- + b_3^+ b_{-3}^- - b_{-3}^+ b_3^- + b_{-4}^+ b_4^- - b_4^+ b_{-4}^-, \\ H_4 &= c_{-1}^+ c_1^- - c_1^+ c_{-1}^- + c_2^+ c_{-2}^- - c_{-2}^+ c_2^- + c_3^+ c_{-3}^- - c_{-3}^+ c_3^- + c_{-4}^+ c_4^- - c_4^+ c_{-4}^- \\ &\quad + b_1^+ b_{-1}^- - b_{-1}^+ b_1^- + b_{-2}^+ b_2^- - b_2^+ b_{-2}^- + b_{-3}^+ b_3^- - b_3^+ b_{-3}^- + b_4^+ b_{-4}^- - b_{-4}^+ b_4^-. \end{aligned} \tag{32c}$$

The commutation relations among these 16-spinor generators are similar to (9*a*) and (9*b*) and those of the Cartan subalgebra and simple root generators with $b_l^{\pm}$ are in the forms similar to (29*a*) and (29*b*).

4.3. The coupled tensor operators

The other SO(9) tensor operators can be constructed from products of 9-vector and 16-spinor operators such that their components are generally written as

$$T_{\zeta_k}^{\zeta} = \sum_{i,j} C_k^{ij} T_{\xi_i}^{\xi} T_{\eta_j}^{\eta}, \tag{33}$$

where C_k^{ij}'s are CGCs. Next, some coupled tensor operators are presented here. Only their top components are shown, the rest of their components can be obtained by an action of the type $[E_r^-, \cdot\ ]$.

- The scalar operator $C^{(0)} \equiv T^{(0000)}$:
For $C^{(0)}$, there is only one component. It can be constructed in general from a linear combination of 9-vector and 16-spinor operator components as,

$$\begin{aligned}C^{(0)\pm\mp} \equiv& \frac{1}{\sqrt{18}}(a_1^\pm a_{-1}^\mp - a_2^\pm a_{-2}^\mp + a_3^\pm a_{-3}^\mp - a_4^\pm a_{-4}^\mp + a_0^\pm a_0^\mp - a_{-4}^\pm a_4^\mp + a_{-3}^\pm a_3^\mp \\ & - a_{-2}^\pm a_2^\mp + a_{-1}^\pm a_1^\mp) + \frac{1}{4\sqrt{2}}(b_1^\pm b_{-1}^\mp + b_2^\pm b_{-2}^\mp - b_3^\pm b_{-3}^\mp - b_4^\pm b_{-4}^\mp \\ & - b_{-4}^\pm b_4^\mp - b_{-3}^\pm b_3^\mp + b_{-2}^\pm b_2^\mp + b_{-1}^\pm b_1^\mp - c_1^\pm c_{-1}^\mp + c_2^\pm c_{-2}^\mp - c_3^\pm c_{-3}^\mp \\ & + c_4^\pm c_{-4}^\mp + c_{-4}^\pm c_4^\mp - c_{-3}^\pm c_3^\mp + c_{-2}^\pm c_2^\mp - c_{-1}^\pm c_1^\mp). \qquad (34)\end{aligned}$$

Note that the operators $C^{(0)\pm\mp}$ are hermitian, but $C^{(0)\pm\pm}$ that are constructed from those pairs of creation and annihilation operators are not.

- The vector operator of the first kind $V \equiv T^{(1000)}$:
As can be seen from (18*b*), the 9-vector irrep is produced from the symmetric product of two 16-spinor irreps. In the same manner, from the products of 16-spinor oscillators $b_l^\pm$ with the same sign, one obtains the vector operators $V^\pm$, whose top components at the 4-th level are

$$V_1^\pm \equiv \frac{1}{2}(b_1^\pm c_4^\pm - c_1^\pm b_4^\pm + c_2^\pm b_3^\pm - c_3^\pm b_2^\pm). \qquad (35)$$

Note that the operators $V^\pm$ are not hermitian. When all V^+ components act on the vacuum, they create the 9-vector states.

- The vector operator of the second kind $A \equiv T^{(1000)}$:
From the products of 16-spinor oscillators $b_l^\pm$ with alternating signs, one obtains the vector operators $A^\pm$, whose top components at the 4-th level are

$$A_1^\pm \equiv \frac{1}{2\sqrt{2}}(b_1^\pm c_4^\mp - b_4^\pm c_1^\mp + c_4^\pm b_1^\mp - c_1^\pm b_4^\mp + b_3^\pm c_2^\mp - b_2^\pm c_3^\mp + c_2^\pm b_3^\mp - c_3^\pm b_2^\mp). \qquad (36)$$

Note that the operators $A^\pm$ are hermitian.

- The coupled spinor operator $\Psi^{(1/2)} \equiv T^{(0001)}$:
As can be seen from (15*b*), the coupled spinor operators can also be constructed from the vector-spinor product and nave four types according to the signs of $a^{(\pm)}$ and of $b^{(\pm)}$, their top components at the 5-th level are

$$\Psi_1^{(1/2)(\pm\pm)} \equiv \frac{1}{\sqrt{5}}(a_1^{(\pm)}c_{-4}^{(\pm)} - a_2^{(\pm)}c_3^{(\pm)} + a_3^{(\pm)}c_2^{(\pm)} - a_4^{(\pm)}c_1^{(\pm)} + a_0^{(\pm)}b_1^{(\pm)}). \qquad (37)$$

One can also use the operators either $A^\pm$ or $V^\pm$ instead of $a^\pm$. When all $\Psi^{(1/2)++}$ components act on the vacuum state, they create the 16-spinor states.

- The second rank symmetric and traceless tensor operator $G^{(2)} \equiv T^{(2000)}$:
The operators $G^{(2)(\pm\pm)}$ are constructed from the symmetric product of the oscillators $a^{(\pm)}$ with all possible combination of signs, their top components at the 8-th level are

$$G_1^{(2)(\pm\pm)} \equiv a_1^{(\pm)}a_1^{(\pm)}. \qquad (38)$$

Notice that the operators $G^{(2)\pm\mp}$ are by themselves the hermitian ones. While the operator $G^{(2)++}$ is not hermitian by itself, but is conjugate to $G^{(2)--}$ by the adjoint. However, if one constructs $G^{(2)}$ either from the symmetric product of $A^\pm$ or from the symmetric product of V^+ and V^-, one will obtain the hermitian $G^{(2)}$ operators. When all $G^{(2)++}$ components act on the vacuum state, they create the 44 graviton states.

- The second-rank antisymmetric tensor operator of the first kind $V^{[2]} \equiv T^{(0100)}$:
The operators $V^{[2]\pm}$ are constructed from the antisymmetric products of two copies of the oscillators $\mathrm{a}_l^{\pm}$ and $\mathrm{b}_l^{\pm}$, all with the same sign, and their top components at the 7-th level are

$$V_1^{[2]\pm} \equiv \frac{1}{2}(\mathrm{a}_1^{\pm}\mathrm{a}_2'^{\pm} - \mathrm{a}_2^{\pm}\mathrm{a}_1'^{\pm}) + \frac{1}{2\sqrt{2}}(\mathrm{b}_1^{\pm}\mathrm{b}_2'^{\pm} - \mathrm{b}_2^{\pm}\mathrm{b}_1'^{\pm} + \mathrm{c}_1^{\pm}\mathrm{c}_2'^{\pm} - \mathrm{c}_2^{\pm}\mathrm{c}_1'^{\pm}). \tag{39}$$

When all $V^{[2]+}$ components act on the vacuum state, they create 36 states of the adjoint irrep.

- The second-rank antisymmetric tensor operator of the second kind $A^{[2]} \equiv T^{(0100)}$:
The operators $A^{[2]\pm}$ are constructed from the antisymmetric products of the oscillators $\mathrm{a}_l^{\pm}$ and $\mathrm{b}_l^{\pm}$ with the alternating signs, their top components at the 7-th level are

$$A_1^{[2]\pm} \equiv \frac{1}{2}(\mathrm{a}_1^{\pm}\mathrm{a}_2^{\mp} - \mathrm{a}_2^{\pm}\mathrm{a}_1^{\mp}) + \frac{1}{2\sqrt{2}}(\mathrm{b}_1^{\pm}\mathrm{b}_2^{\mp} - \mathrm{b}_2^{\pm}\mathrm{b}_1^{\mp} + \mathrm{c}_1^{\pm}\mathrm{c}_2^{\mp} - \mathrm{c}_2^{\pm}\mathrm{c}_1^{\mp}). \tag{40}$$

Note that the operators $A^{[2]\pm}$ are hermitian.

- The third-rank antisymmetric tensor operator of the first kind $V^{[3]} \equiv T^{(0010)}$:
The operators $V^{[3]\pm}$ are constructed from the antisymmetric products of three copies of $\mathrm{a}_l^{\pm}$ and of two copies of $\mathrm{b}_l^{\pm}$, all with the same sign, and their top components at the 9-th level are

$$\begin{aligned} V_1^{[3]\pm} \equiv \frac{1}{2\sqrt{3}}(&a_1^{\pm}a_2'^{\pm}a_3''^{\pm} - a_1^{\pm}a_3'^{\pm}a_2''^{\pm} + a_2^{\pm}a_3'^{\pm}a_1''^{\pm} - a_2^{\pm}a_1^{\pm}a_3''^{\pm} \\ &+ a_3^{\pm}a_1'^{\pm}a_2''^{\pm} - a_3^{\pm}a_2'^{\pm}a_1''^{\pm}) + \frac{1}{2}(\mathrm{b}_1^{\pm}\mathrm{c}_1'^{\pm} - \mathrm{c}_1^{\pm}\mathrm{b}_1'^{\pm}). \end{aligned} \tag{41}$$

When all $V^{[3]+}$ components act on the vacuum state, they create 84 states of the three-form irrep.

- The third-rank antisymmetric tensor operator of the second kind $A^{[3]} \equiv T^{(0010)}$:
The operators $A^{[3]\pm}$ are constructed from the antisymmetric products of three copies of $\mathrm{a}_l^{\pm}$ and of two copies of $\mathrm{b}_l^{\pm}$, all with the alternating signs, and their top components at the 9-th level are

$$\begin{aligned} A_1^{[3]\pm} \equiv \frac{1}{2\sqrt{3}}(&a_1^{\pm}a_2^{\mp}a_3^{\pm} - a_1^{\pm}a_3^{\mp}a_2^{\pm} + a_2^{\pm}a_3^{\mp}a_1^{\pm} - a_2^{\pm}a_1^{\mp}a_3^{\pm} \\ &+ a_3^{\pm}a_1^{\mp}a_2^{\pm} - a_3^{\pm}a_2^{\mp}a_1^{\pm}) + \frac{1}{2}(\mathrm{b}_1^{\pm}\mathrm{c}_1^{\mp} - \mathrm{c}_1^{\pm}\mathrm{b}_1^{\mp}). \end{aligned} \tag{42}$$

Note that when using $A^{\pm}$ instead of $a^{\pm}$, the operators $A^{[3]\pm}$ are hermitian.

- The vector-spinor tensor operator $\Psi^{(3/2)} \equiv T^{(1001)}$:
The couples of $\mathrm{a}^{(\pm)}$ with $\mathrm{b}^{(\pm)}$ result in the operators $\Psi^{(3/2)(\pm\pm)}$, which have four types according to the signs of the coupled oscillators. Their top components at the 9-th level are

$$\Psi_1^{(3/2)(\pm\pm)} = \mathrm{a}_1^{(\pm)}\mathrm{b}_1^{(\pm)}. \tag{43}$$

One can use either $A^{(\pm)}$ or $V^{(\pm)}$ instead of $\mathrm{a}^{(\pm)}$. When all $\Psi^{(3/2)++}$ components act on the vacuum state, they create the 128 Rarita-Schwinger states.

4.4. The SO(9) operators for generating four-fold infinite triplet families

The SO(9) irreps have a remarkable property in that there are three irreps, $\lambda^{(1)}$, $\lambda^{(2)}$, and $\lambda^{(3)}$, such that the dimension of $\lambda^{(1)}$ is equal to the sum of the dimensions of $\lambda^{(2)}$ and $\lambda^{(3)}$, and all three irreps have the same quadratic Casimir value. The simplest one is the supergravity triplet (the lowest level triplet shown in Fig. 3), consisting of a Rarita-Schwinger spinor vector $\Psi^{(3/2)}$, a graviton $G^{(2)}$ and a third rank antisymmetric tensor $A^{[3]}$ as the physical degrees of freedom of N=1 supergravity in 11 dimensions. In [8], four-fold infinite triplet families are classified and are generated by four types of basic operators acting on the supergravity triplet. Next, the triplet generating operators, constructed by using the tensor operators from the previous section, are listed:

- $\Delta_1 \equiv V_1^{[2]+}(\lambda_1^{(1)}\lambda_1^{(1)\dagger} + \lambda_1^{(2)}\lambda_1^{(2)\dagger} + \lambda_1^{(3)}\lambda_1^{(3)\dagger})$:
  The operator Δ_1 increases the Dynkin labels of all three irreps by (0100) i.e.
  $$\Delta_1(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = (\lambda^{(1)} + \hat{\omega}_2, \lambda^{(2)} + \hat{\omega}_2, \lambda^{(3)} + \hat{\omega}_2). \tag{44}$$
- $\Delta_2 \equiv \frac{1}{\sqrt{2}}(V_1^{[2]+}V_2^{[2]+} - V_2^{[2]+}V_1^{[2]+})(\lambda_1^{(1)}\lambda_1^{(1)\dagger} + \lambda_1^{(2)}\lambda_1^{(2)\dagger} + \lambda_1^{(3)}\lambda_1^{(3)\dagger})$:
  The operator Δ_2 increases the Dynkin labels of all three irreps by (1010) i.e.
  $$\Delta_2(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = (\lambda^{(1)} + \hat{\omega}_1 + \hat{\omega}_3, \lambda^{(2)} + \hat{\omega}_1 + \hat{\omega}_3, \lambda^{(3)} + \hat{\omega}_1 + \hat{\omega}_3). \tag{45}$$
- $\Delta_3 \equiv \Psi_1^{(1/2)++}(\lambda_1^{(1)}\lambda_1^{(1)\dagger} + \lambda_1^{(3)}\lambda_1^{(3)\dagger}) + \frac{1}{\sqrt{2}}(a_1^+ + V_1^+)(\lambda_1^{(2)}\lambda_1^{(2)\dagger})$:
  The operator Δ_3 increases the Dynkin labels of $\lambda^{(1)}$ and $\lambda^{(3)}$ by (0001) and of $\lambda^{(2)}$ by (1000) i.e.
  $$\Delta_3(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = (\lambda^{(1)} + \hat{\omega}_4, \lambda^{(2)} + \hat{\omega}_1, \lambda^{(3)} + \hat{\omega}_4). \tag{46}$$
- $\Delta_4 \equiv \Psi_1^{(3/2)++}(\lambda_1^{(1)}\lambda_1^{(1)\dagger} + \lambda_1^{(2)}\lambda_1^{(2)\dagger}) + V_1^{[3]+}(\lambda_1^{(3)}\lambda_1^{(3)\dagger})$:
  The operator Δ_4 increases the Dynkin labels of $\lambda^{(1)}$ and $\lambda^{(2)}$ by (1001) and of $\lambda^{(3)}$ by (0010) i.e.
  $$\Delta_4(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) = (\lambda^{(1)} + \hat{\omega}_1 + \hat{\omega}_4, \lambda^{(2)} + \hat{\omega}_1 + \hat{\omega}_4, \lambda^{(3)} + \hat{\omega}_3). \tag{47}$$

The actions on the triplets of Δ_3 are represented by the up-right arrows and of Δ_4 by the up-left arrows as shown in Fig. 3. The operators $\Delta_{1,2}$ act on all triplets and yield two more triplet layers (not shown in the picture).