Fuzziness of n-ary Semigroups

John Patrick F. Solano

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
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Thesis Title	Fuzziness of n-ary Semigroups
Author	Mr. John Patrick F. Solano
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Major Advisor	**Examining Committee:**
....	....Chairperson
(Assoc. Prof. Dr. Ronnason Chinram)	(Asst. Prof. Dr. Samruam Baupradist)
	....Committee
	(Assoc. Prof. Dr. Ronnason Chinram)
	....Committee
	(Asst. Prof. Dr. Boonrod Yuttanan)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Mathematics

....

(Prof. Dr. Damrongsak Faroongsarng)

Dean of Graduate School

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.... Signature
(Assoc. Prof. Dr. Ronnason Chinram)
Major Advisor

.... Signature
(Mr. John Patrick F. Solano)
Candidate

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.... Signature
(Mr. John Patrick F. Solano)
Candidate

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Author Mr. John Patrick F. Solano

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ABSTRACT

A nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (S, f). The following sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $i > j$, it is $\emptyset$. We call an n-ary groupoid (S, f) as (i, j)-*associative* if the following holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $x_1, x_2, \ldots, x_{2n-1} \in S$. The operation f is *associative* if the above identity holds for every $1 \leq i \leq j \leq n$, and (S, f) is called an *n-ary semigroup*.

In this thesis, we study i-ideals and fuzzy i-ideals of n-ary semigroups. Moreover, we study almost i-ideals and fuzzy almost i-ideals of n-ary semigroups.

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John Patrick F. Solano

Contents

Chapter 1

Introduction

1.1 Background and significance

Zadeh [30] introduced the fundamental fuzzy subset concept in 1965. The applications of fuzzy subsets can now be seen in different disciplines. A fuzzy subset of S is a function from S to $[0, 1]$. In 1971, Rosenfeld introduced the notion of fuzzy groups and pioneered the study of fuzzy algebraic structures in [24]. Kuroki [17, 18, 19, 20] gave the definition of fuzzy semigroups and fuzzy ideals in semigroups. Considering the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, Kim [16] tackled the relation between fuzzy interior ideals of S and the subsets of $\underline{S}$. Hamouda [11] discussed the relation between some ideals of a semigroup S and the subsets of $\underline{S}$. Moreover, he later considered the ternary semigroup $\underline{S}$ of all fuzzy points of a ternary semigroup S and then studied the relation between some fuzzy ideals of a ternary semigroup S and the subsets of $\underline{S}$ in [12].

Grosek and Satko [7] presented the concept of a left almost ideal and a right almost ideal of a semigroup in 1980. They also studied minimal almost ideals of semigroups in [8] and smallest almost ideals of semigroups in [9]. Fuzzy almost bi-ideals of semigroups were discussed by Wattanatripop et al. in [28].

Kasner [13] initiated the generalization of the classical algebraic structures to n-ary structures in 1904. Sioson [25] gave some properties of regular n-ary semigroups. Dudek [2] extended Sioson's study on regular n-ary semigroups. In [3, 4, 5], he also proved some results of n-ary groups. Furthermore, he provided the properties of ideals of some elements of n-ary $(n \geq 3)$ semigroups that contains idempotent in [6]. Wang et al. [27] studied the relation between regular n-ary semigroups and soft regular n-ary semigroups. n-ary systems were applied in the following fields: physics in [22] and [26], automata theory in [10], to name a few.

In this thesis, we study the fuzziness of n-ary semigroups.

1.2 Objectives of study

1. To study i-ideals and fuzzy i-ideals of n-ary semigroups.

2. To study almost i-ideals and fuzzy almost i-ideals of n-ary semigroups.

1.3 Research plan

Task	2017	2018			2019	
	11-12	01-03	4-7	8	9-12	1-3	4
Literature review	*	*					
Write up the thesis proposal		*	*				
Present the thesis proposal				*			
Work on the problems					*	*	
Write up the thesis					*	*	
Present the thesis							*

1.4 Expected benefit of this study

We will give new definitions in n-ary semigroups and study i-ideals, fuzzy i-ideals, almost i-ideals, and fuzzy almost i-ideals of n-ary semigroups.

Chapter 2

Preliminaries

In this chapter, we introduce some basic definitions and examples of semigroups, ternary semigroups, and n-ary semigroups that will be useful in this thesis.

2.1 Fuzzy subsets in semigroups

Zadeh [30] initiated the concept of a fuzzy subset in 1965 which eventually opened up applications in different fields of science.

Definition 2.1.1. A nonempty set S is called a **semigroup** if there exists a binary operation $* : S \times S \to S$ satisfying $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

Example 2.1.2. $(\mathbb{N}, +), (\mathbb{N}, \cdot), (\mathbb{R}, +), (\mathbb{R}, \cdot)$ are semigroups.

Example 2.1.3. $(\mathbb{N}, -)$ is not a semigroup since $2 - 3 \notin \mathbb{N}$.

Throughout this section, let S be a semigroup.
Let A and B be nonempty subsets of S. Then

$$AB := \{ab \mid a \in A, b \in B\}.$$

Let $a \in S$ and B be a subset of S. Then

$$aB := \{a\}B = \{ab \mid b \in B\}.$$

Definition 2.1.4. Let A be a nonempty subset of S.

1. A is called a **subsemigroup** of S if $A^2 \subseteq A$.

2. A is called a **left ideal** of S if $SA \subseteq A$.

3. A is called a **right ideal** of S if $AS \subseteq A$.

4. A is called an **ideal** of S if it is both a left and a right ideal of S.

Definition 2.1.5. A function f is called a **fuzzy subset** in S if it is a function from S to the closed interval [0,1].

Let f and g be fuzzy subsets in S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$. $f \cap g$ and $f \cup g$ are fuzzy subsets in S defined by $(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$ and $(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$ for all $x \in S$.

Example 2.1.6. Let $S = \{a\}$. Define $f : S \to [0,1]$ by $f(a) = 0.1$ and $g : S \to [0,1]$ by $g(a) = 0.3$. Then $(f \cup g)(a) = 0.3$ and $(f \cap g)(a) = 0.1$.

The definition of fuzzy points was given by Pu and Liu [23] in 1980.

Definition 2.1.7. Let S be a nonempty set and $t \in (0,1]$, $x \in S$. A **fuzzy point** x_t of S is a fuzzy subset in S defined by

$$x_t(y) = \begin{cases} t, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in S$.

The fuzzy point x_t is said to be contained in a fuzzy subset f, denoted by $x_t \in f$, if and only if $f(x) \geq t$.

Definition 2.1.8. For all $x \in S$, the **characteristic function** C_A of a subset A of S is a fuzzy subset defined by

$$C_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Example 2.1.9. Let $S = \{a, b, c, d\}$ and $A = \{a, b\}$. Define a fuzzy subset $C_A : S \to [0,1]$ by

$$C_A(a) = 1 \text{ since } a \in A.$$
$$C_A(b) = 1 \text{ since } b \in A.$$
$$C_A(c) = 0 \text{ since } c \notin A.$$
$$C_A(d) = 0 \text{ since } d \notin A.$$

Definition 2.1.10. Let f be a nonzero fuzzy subset of S.

1. f is called a **fuzzy subsemigroup** of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

2. f is called a **fuzzy left ideal** of S if $f(xy) \geq f(y)$ for all $x, y \in S$.

3. f is called a **fuzzy right ideal** of S if $f(xy) \geq f(x)$ for all $x, y \in S$.

4. f is called a **fuzzy ideal** of S if $f(xy) \geq \max\{f(x), f(y)\}$ for all $x, y \in S$.

Example 2.1.11. Let $S = [0, 1]$. Clearly, $([0, 1], \cdot)$ is a semigroup.
Define $f : S \to [0, 1]$ by $f(x) = 1 - x$ for all $x \in S$. Then we have

$$f(xy) = 1 - xy \geq 1 - y = f(y)$$
$$f(xy) = 1 - xy \geq 1 - x = f(x)$$

for all $x, y \in S$. Hence, f is a fuzzy left ideal and a fuzzy right ideal of S and, thus, a fuzzy ideal of S.

2.2 Fuzzy subsets in ternary semigroups

The notion of ternary semigroups was first introduced by Lehmer in [21] in 1932.

Definition 2.2.1. A **ternary semigroup** is a nonempty set S together with a ternary operation $(a, b, c) \to abc$ satisfying $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in S$.

Example 2.2.2. $(\mathbb{Z}^-, \cdot)$ is a ternary semigroup, but is not a semigroup since $(-1)(-1) \notin \mathbb{Z}^-$.

Throughout this section, let S be a ternary semigroup.
Let A, B, C be nonempty subsets of S. Then

$$ABC := \{abc \mid a \in A, b \in B, c \in C\}.$$

Definition 2.2.3. Let A be a nonempty subset of S.

1. A is called a **ternary subsemigroup** of S if $A^3 \subseteq A$.

2. A is called a **left ideal** of S if $SSA \subseteq A$.

3. A is called a **lateral ideal** of S if $SAS \subseteq A$.

4. A is called a **right ideal** of S if $ASS \subseteq A$.

5. A is called an **ideal** of S if it is a left, a lateral, and a right ideal of S.

Definition 2.2.4. A function f from S to the closed interval [0,1] is called a **fuzzy subset** in S.

Let f and g be fuzzy subsets in S. Then $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$. $f \cap g$ and $f \cup g$ are fuzzy subsets in S defined by $(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$ and $(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$ for all $x \in S$.

Definition 2.2.5. Let $x \in S$ and $t \in (0, 1]$. A **fuzzy point** x_t of S is a fuzzy subset in S defined by

$$x_t(y) = \begin{cases} t, & \text{if } x = y, \\ 0, & \text{otherwise}, \end{cases}$$

for all $y \in S$.

The fuzzy point x_t is said to be contained in a fuzzy subset f, denoted by $x_t \in f$, if and only if $f(x) \geq t$.

Definition 2.2.6. For all $x \in S$, the **characteristic function** C_A of a subset A of S is a fuzzy subset defined by

$$C_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.2.7. Let f be a nonzero fuzzy subset of S.

1. f is called a **fuzzy ternary subsemigroup** of S if $f(xyz) \geq f(x) \wedge f(y) \wedge f(z)$ for all $x, y, z \in S$.

2. f is called a **fuzzy left ideal** of S if $f(xyz) \geq f(z)$ for all $x, y, z \in S$.

3. f is called a **fuzzy lateral ideal** of S if $f(xyz) \geq f(y)$ for all $x, y, z \in S$.

4. f is called a **fuzzy right ideal** of S if $f(xyz) \geq f(x)$ for all $x, y, z \in S$.

5. f is called a **fuzzy ideal** of S if it is a fuzzy left ideal, a fuzzy lateral ideal, and a fuzzy right ideal of S, i.e., $f(xyz) \geq f(x) \vee f(y) \vee f(z)$ for all $x, y, z \in S$.

2.3 Almost ideals in semigroups

In 1980, Grosek and Satko [7] introduced the concept of an almost ideal of a semigroup.
Throughout this section, let S be a semigroup.

Definition 2.3.1. Let A be a nonempty subset of S. For all $s \in S$,

1. A is called a **left almost ideal** of S if $sA \cap A \neq \emptyset$,
2. A is called a **right almost ideal** of S if $As \cap A \neq \emptyset$,
3. A is called an **almost ideal** of S if it is both a left almost ideal and a right almost ideal of S.

Example 2.3.2. Every left (right) ideal of S is a left (right) almost ideal of S. Similarly, every ideal of S is an almost ideal of S.

Proof. Assume A is a left ideal of S. Then $SA \subseteq A$. For all $s \in S, sA \subseteq SA \subseteq A$. This means $sA \cap A = sA \neq \emptyset$. Thus, A is a left almost ideal of S.
The proof is similar for every right ideal of S is a right almost ideal of S and every ideal of S is an almost ideal of S. □

Example 2.3.3. Consider $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ under the usual addition and $A = \{\overline{1}, \overline{2}, \overline{4}\}$. Then we have

$(\overline{0} + A) \cap A = \{\overline{1}, \overline{2}, \overline{4}\}$ and $(A + \overline{0}) \cap A = \{\overline{1}, \overline{2}, \overline{4}\}$.
$(\overline{1} + A) \cap A = \{\overline{2}\}$ and $(A + \overline{1}) \cap A = \{\overline{2}\}$.
$(\overline{2} + A) \cap A = \{\overline{4}\}$ and $(A + \overline{2}) \cap A = \{\overline{4}\}$.
$(\overline{3} + A) \cap A = \{\overline{1}, \overline{4}\}$ and $(A + \overline{3}) \cap A = \{\overline{1}, \overline{4}\}$.
$(\overline{4} + A) \cap A = \{\overline{2}\}$ and $(A + \overline{4}) \cap A = \{\overline{2}\}$.
$(\overline{5} + A) \cap A = \{\overline{1}\}$ and $(A + \overline{5}) \cap A = \{\overline{1}\}$.

Hence, A is both a left almost ideal and a right almost ideal of $\mathbb{Z}_6$. So, A is an almost ideal of $\mathbb{Z}_6$. But A is not an ideal of $\mathbb{Z}_6$ since $\overline{1} + \overline{2} = \overline{3} \notin A$.

2.4 Fuzzy subsets in n-ary semigroups

In 1904, Kasner [13] initiated the generalization of classical algebraic structures to n-ary structures.

Definition 2.4.1. A nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an n**-ary groupoid** and is denoted by (S, f). The following sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $i > j$, it is $\emptyset$. We call an n-ary groupoid (S, f) as (i, j)**-associative** if the following holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $x_1, x_2, \ldots, x_{2n-1} \in S$. The operation f is *associative* if the above identity holds for every $1 \leq i \leq j \leq n$, and (S, f) is called an n**-ary semigroup**. A nonempty subset T of S is called an n**-ary subsemigroup** of S if $f(a_1^n) \in T$ for all $a_1, a_2, \ldots, a_n \in T$.

Let $F(S)$ be the set of all fuzzy subsets in an n-ary semigroup S. For each $g_1, g_2, \ldots, g_n \in F(S)$, the product of $g_1, g_2, \ldots, g_n$ is a fuzzy subset $g_1 \circ g_2 \circ \cdots \circ g_n$ defined as follows:

$$(g_1 \circ g_2 \circ \cdots \circ g_n)(x) = \begin{cases} \bigvee\limits_{x=f(a_1,a_2,\ldots,a_n)} \{\bigwedge\limits_{i=1}^{n} g_i(a_i)\} & \text{if } x = f(a_1^n) \\ & a_1, a_2, \ldots, a_n \in S, \\ 0 & \text{otherwise.} \end{cases}$$

for all $x \in S$. Then $F(S)$ is an n-ary semigroup with the product $\circ$.

Example 2.4.2. Let $A = (\{2, 2^n, 2^{n+1}, 2^{n+2}, 2^{n+3}, \ldots\}, \cdot)$. Then A is an n-ary semigroup. If $n = 4$, then we have $A = (\{2, 2^4, 2^5, 2^6, 2^7, \ldots\}, \cdot)$ is a 4-ary semigroup. Notice that A is not a semigroup since $2^2 = 4 \notin A$, and is also not a ternary semigroup since $2^3 = 8 \notin A$.

Throughout this section, let S be an n-ary semigroup.

Definition 2.4.3. A function f from S to the closed interval [0,1] is called a **fuzzy subset** in S.

Let g and h be fuzzy subsets of S. The relation $g \subseteq h$ is defined by $g(x) \leq h(x)$ for all $x \in S$. The fuzzy subsets $g \cap h$ and $g \cup h$ are defined by $(g \cap h)(x) = \min\{g(x), h(x)\}$ and $(g \cup h)(x) = \max\{g(x), h(x)\}$ for all $x \in S$.

Definition 2.4.4. For any $\alpha \in (0, 1]$ and $x \in S$, a fuzzy subset x_α of S is defined by

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

for all $y \in S$. x_α is called a **fuzzy point** of S.

Definition 2.4.5. For any subset A of S, a fuzzy subset C_A of S is defined by

$$C_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for all $x \in S$. C_A is called the **characteristic function** of A.

Chapter 3

i-ideals and fuzzy i-ideals of n-ary semigroups

In this chapter, we consider the n-ary semigroup $\underline{S}$ of the fuzzy points of an n-ary semigroup S. We will also show the relation between i-ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$, and ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$.
Throughout this chapter, let (S, f) be an n-ary semigroup.

3.1 i-ideals

We get the definition of an i-ideal of (S, f) from [27].

Definition 3.1.1. A nonempty subset I of S is called an i**-ideal** of (S, f) if for all $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. A nonempty subset I of S is called an **ideal** of (S, f) if I is an i-ideal for every $1 \leq i \leq n$.

Example 3.1.2.

1. If $i = 1, n = 2$, then we have a 1-ideal of a semigroup. By Definition 3.1.1, we have $f(x_1^{1-1}, a, x_{1+1}^2) = f(a, x_2) = ax_2 \in I$ for all $a \in I, x \in S$, that is, I is a right ideal of (S, f).
2. If $i = 2, n = 2$, then we have a 2-ideal of a semigroup. By Definition 3.1.1, we get $f(x_1^{2-1}, a, x_{2+1}^2) = f(x_1, a) = x_1 a \in I$ for all $a \in I, x \in S$, that is, I is a left ideal of (S, f).

Example 3.1.3.

1. If $i = 1, n = 3$, then we have a 1-ideal of a ternary semigroup. By Definition 3.1.1, we have $f(x_1^{1-1}, a, x_{1+1}^3) = f(a, x_2 x_3) = ax_2 x_3 \in I$ for all $a \in I, x \in S$, that is, I is a right ideal of (S, f).

2. If $i = 2, n = 3$, then we have a 2-ideal of a ternary semigroup. By Definition 3.1.1, we get $f(x_1^{2-1}, a, x_{2+1}^3) = f(x_1, a, x_3) = x_1 a x_3 \in I$ for all $a \in I$, $x \in S$, that is, I is a lateral ideal of (S, f).

3. If $i = 3, n = 3$, then we have a 3-ideal of a ternary semigroup. By Definition 3.1.1, we obtain $f(x_1^{3-1}, a, x_{3+1}^3) = f(x_1 x_2, a) = x_1 x_2 a \in I$ for all $a \in I$, $x \in S$, that is, I is a left ideal of (S, f).

Definition 3.1.4. Let $\underline{S}$ be the set of all fuzzy points in (S, f). Then

$$(a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n} = (f(a_1^n))_{\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}.$$

Thus, $\underline{S}$ is an n-ary subsemigroup of $F(S)$. For any $g \in F(S)$, $\underline{g}$ denotes the set of all fuzzy points contained in g, that is,

$$\underline{g} = \{x_\alpha \in \underline{S} \mid g(x) \geq \alpha\}.$$

For any $\underline{g_1}, \underline{g_2}, \ldots, \underline{g_n} \subseteq \underline{S}$, we define the product of $\underline{g_1}, \underline{g_2}, \ldots, \underline{g_n}$ as

$$\underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n} = \{(a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n} \mid a_i \in (g_i)_{\alpha_i}\}.$$

Example 3.1.5. Consider the semigroup $([0, 1], \cdot)$ and a fuzzy subset g such that $g(1) = 1, g(0) = 0.5$, and $g(x) = 0$ for all $x \in (0, 1)$. Then

$$\underline{g} = \{1_\alpha \mid 1 \geq \alpha\} \cup \{0_\beta \mid 0.5 \geq \beta\}.$$

Theorem 3.1.6. *Let $g_1, g_2, \ldots, g_k$ be fuzzy subsets in S. Then*

(1) $\underline{\cup_{i=1}^k g_i} = \cup_{i=1}^k \underline{g_i}$.

(2) $\underline{\cap_{i=1}^k g_i} = \cap_{i=1}^k \underline{g_i}$.

Proof. (1) Let $x_\alpha \in \underline{\cup_{i=1}^k g_i}$. Then $(\cup_{i=1}^k g_i)(x) \geq \alpha$. So, $\max\{g_1(x), \ldots, g_k(x)\} \geq \alpha$. This means $g_i(x) \geq \alpha$ for some i. Hence, $x_\alpha \in \underline{g_i}$ for some i. Therefore, $x_\alpha \in \cup_{i=1}^k \underline{g_i}$. Conversely, let $x_\alpha \in \cup_{i=1}^k \underline{g_i}$. Then $x_\alpha \in \underline{g_i}$ for some i. So, $g_i(x) \geq \alpha$ for some i. Hence, $\max\{g_1(x), \ldots, g_k(x)\} \geq \alpha$. This implies $(\cup_{i=1}^k g_i)(x) \geq \alpha$. Therefore, $x_\alpha \in \underline{\cup_{i=1}^k g_i}$.

(2) Let $x_\alpha \in \underline{\cap_{i=1}^k g_i}$. Then $(\cap_{i=1}^k g_i)(x) \geq \alpha$. So, $\min\{g_1(x), \ldots, g_k(x)\} \geq \alpha$. This implies $g_i(x) \geq \alpha$ for all i. Hence, $x_\alpha \in \underline{g_i}$ for all i. Therefore, $x_\alpha \in \cap_{i=1}^k \underline{g_i}$. Conversely, let $x_\alpha \in \cap_{i=1}^k \underline{g_i}$. Then $x_\alpha \in \underline{g_i}$ for all i. So, $g_i(x) \geq \alpha$ for all i. Hence, $\min\{g_1(x), \ldots, g_k(x)\} \geq \alpha$. This implies $(\cap_{i=1}^k g_i)(x) \geq \alpha$. Therefore, $x_\alpha \in \underline{\cap_{i=1}^k g_i}$. $\square$

Theorem 3.1.7. *Let $g_1, g_2, \ldots, g_n$ be fuzzy subsets in S. Then $\underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n} \subseteq \underline{g_1 \circ g_2 \circ \cdots \circ g_n}$.*

Proof. Let $x_\alpha \in \underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n}$. Then $x_\alpha = (a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n}$ for some $(a_i)_{\alpha_i} \in \underline{g_i}$. This implies $x_\alpha = (f(a_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}$ and $(g_i)(a_i) \geq \alpha_i$ for all i. So, $x = f(a_1^n)$ and $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Therefore, $(g_i)(a_i) \geq \alpha_i \geq \alpha$ for all i. Hence, $(g_1 \circ g_2 \circ \cdots \circ g_n)(x) \geq \alpha$. So, $x_\alpha \in \underline{g_1 \circ g_2 \circ \cdots \circ g_n}$. □

Theorem 3.1.8. *Let A and B be nonempty subsets of S. Then $A \subseteq B$ if and only if $C_A \subseteq C_B$.*

Proof. Assume that $A \subseteq B$. We consider two cases:
Case 1: $x \notin A$. Then $C_A(x) = 0$. So, $C_A(x) = 0 \leq C_B(x)$.
Case 2: $x \in A$. Since $A \subseteq B, x \in B$. Then $C_B(x) = 1$. Hence, $C_A(x) \leq C_B(x)$.
Thus, $C_A \subseteq C_B$. Conversely, assume that $C_A \subseteq C_B$. Let $x \in A$. Then $C_A(x) = 1$. Since $C_A \subseteq C_B, 1 = C_A(x) \leq C_B(x)$. Then $C_B(x) = 1$. Hence, $x \in B$. Thus, $A \subseteq B$. □

Theorem 3.1.9. *Let A be a nonempty subset of S. Then $x_\alpha \in \underline{C_A}$ if and only if $x \in A$.*

Proof. Assume that $x_\alpha \in \underline{C_A}$. Then $C_A(x) \geq \alpha$. Hence, $C_A(x) = 1$. This implies $x \in A$. Conversely, assume that $x \in A$. Then $C_A(x) = 1 \geq \alpha$ for all $\alpha \in (0, 1]$. This implies $x_\alpha \in \underline{C_A}$. □

Theorem 3.1.10. *For any nonempty subsets A and B of S, $A \subseteq B$ if and only if $\underline{C_A} \subseteq \underline{C_B}$.*

Proof. Assume that $A \subseteq B$. Let $x_\alpha \in \underline{C_A}$. By Theorem 3.1.9, $x \in A$. Since $A \subseteq B$, $x \in B$. By Theorem 3.1.9, $x_\alpha \in \underline{C_B}$. Thus, $\underline{C_A} \subseteq \underline{C_B}$. Conversely, assume that $\underline{C_A} \subseteq \underline{C_B}$. Let $x \in A$. By Theorem 3.1.9, $x_\alpha \in \underline{C_A}$. Since $\underline{C_A} \subseteq \underline{C_B}$, $x_\alpha \in \underline{C_B}$. By Theorem 3.1.9, $x \in B$. Thus, $A \subseteq B$. □

Theorem 3.1.11. *For any fuzzy subsets g and h of S, $g \subseteq h$ if and only if $\underline{g} \subseteq \underline{h}$.*

Proof. Assume that $g \subseteq h$. Then $g(x) \leq h(x)$ for all $x \in S$. Let $x_\alpha \in \underline{g}$. Then $h(x) \geq g(x) \geq \alpha$. Hence, $x_\alpha \in \underline{h}$. Conversely, assume that $\underline{g} \subseteq \underline{h}$. Let $x \in S$. We consider two cases:
Case 1: $g(x) = 0$. Then $g(x) = 0 \leq h(x)$.
Case 2: $g(x) \neq 0$. Let $\alpha = g(x)$. Then $x_\alpha \in \underline{g}$. So, $x_\alpha \in \underline{h}$. Hence, $h(x) \geq \alpha = g(x)$. Thus, $g \subseteq h$. □

Definition 3.1.12. A fuzzy subset g of S is called a **fuzzy n-ary subsemigroup** of (S, f) if $g(f(a_1^n)) \geq \min\{g(a_1), g(a_2), \ldots, g(a_n)\}$ for all $a_1, a_2, \ldots, a_n \in S$.

Theorem 3.1.13. *Let g be a nonzero fuzzy subset of S. Then g is a fuzzy n-ary subsemigroup of (S, f) if and only if $\underline{g}$ is an n-ary subsemigroup of $(\underline{S}, f)$.*

Proof. Assume that g is a fuzzy n-ary subsemigroup of (S, f).
Let $(a_1)_{\alpha_1}$, $(a_2)_{\alpha_2}, \ldots, (a_n)_{\alpha_n} \in \underline{g}$. So, $g(a_i) \geq \alpha_i$ for all i. Then

$$g(f(a_1^n)) \geq \min\{g(a_1), g(a_2), \ldots, g(a_n)\} \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Hence, $(f(a_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}$. Thus, $\underline{g}$ is an n-ary subsemigroup of $(\underline{S}, f)$. Conversely, assume that $\underline{g}$ is an n-ary subsemigroup of $(\underline{S}, f)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in g$. We choose $\alpha_i = g(a_i)$ for all i. We consider two cases:
Case 1: $\alpha_i = 0$ for some i. Then $\min\{g(a_1), g(a_2), \ldots, g(a_n)\} = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = 0 \leq g(f(a_1^n))$.
Case 2: $\alpha_i \neq 0$ for all i. Then $\alpha_i = g(a_i)$ for all i. So, $(a_i)_{\alpha_i} \in \underline{g}$ for all i. Hence, $(f(a_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}$. Therefore,

$$g(f(a_1^n)) \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \min\{g(a_1), g(a_2), \ldots, g(a_n)\}.$$

Thus, g is a fuzzy n-ary subsemigroup of (S, f). □

3.2 Fuzzy i-ideals

Definition 3.2.1. A fuzzy subset g of S is called a **fuzzy i-ideal** of (S, f) if $g(f(a_1^n)) \geq g(a_i)$ for all $a_1, a_2, \ldots, a_n \in S$.

Example 3.2.2. Let $S = [0, 1]$. Clearly, $([0, 1], f)$ is an n-ary semigroup such that $f(x_1^n) = x_1 \cdots x_n$ for all $x_1, \ldots, x_n \in S$. Define $g : S \to [0, 1]$ by $g(x) = 1 - x$ for all $x \in S$. For all $i \in \{1, \ldots, n\}$, we have

$$g(f(x_1^n)) = 1 - f(x_1^n) = 1 - x_1 \cdots x_n \geq 1 - x_i \geq g(x_i).$$

Thus, g is a fuzzy i-ideal of (S, f).

Lemma 3.2.3. Let g be a nonzero fuzzy subset of S. Then g is a fuzzy i-ideal of (S, f) if and only if $\underline{g}$ is an i-ideal of $(\underline{S}, f)$.

Proof. Assume that g is a fuzzy i-ideal of (S, f). Let

$$a_{\alpha_i} \in \underline{g} \quad \text{and} \quad (x_1)_{\alpha_1}, \ldots, (x_{i-1})_{\alpha_{i-1}}, (x_{i+1})_{\alpha_{i+1}}, \ldots, (x_n)_{\alpha_n} \in \underline{S}.$$

Then $g(a) \geq \alpha_i$. Therefore, we have

$$g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq g(a) \geq \alpha_i \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Hence, $f(x_1^{i-1}, a, x_{i+1}^n)_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}$. Then $\underline{g}$ is an i-ideal of $(\underline{S}, f)$. Conversely, assume that $\underline{g}$ is an i-ideal of $(\underline{S}, f)$. Let $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, a \in S$. We consider two cases:

Case 1: $g(a) = 0$. Then $g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq 0 = g(a)$.

Case 2: $g(a) \neq 0$. Let $\alpha = g(a)$. This implies $a_\alpha \in \underline{g}$. Then $f(x_1^{i-1}, a, x_{i+1}^n)_\alpha \in \underline{g}$ by assumption. Therefore, we get $g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq \alpha = g(a)$. Thus, g is a fuzzy i-ideal of (S, f). □

Lemma 3.2.4. Let A be a nonempty subset of S. Then A is an i-ideal of (S, f) if and only if C_A is a fuzzy i-ideal of (S, f).

Proof. Assume that A is an i-ideal of (S, f). Let $a_1, a_2, \ldots, a_n \in S$.

Case 1: $a_i \in A$. Then $f(a_1^n) \in A$. Hence, $C_A(f(a_1^n)) = 1 \geq C_A(a_i)$.

Case 2: $a_i \notin A$. Then $C_A(a_i) = 0 \leq C_A(f(a_1^n))$.

Thus, C_A is a fuzzy i-ideal of (S, f). Conversely, assume that C_A is a fuzzy i-ideal of (S, f). Let $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in S$ and $a \in A$. So, $C_A(a) = 1$. Therefore, $C_A(f(x_1^{i-1}, a, x_{i+1}^n)) \geq C_A(a) = 1$. Hence, $f(x_1^{i-1}, a, x_{i+1}^n) \in A$. Thus, A is an i-ideal of (S, f). □

Theorem 3.2.5. *Let A be a nonempty subset of S. Then A is an i-ideal of (S, f) if and only if $\underline{C_A}$ is an i-ideal of $(\underline{S}, f)$.*

Proof. Assume that A is an i-ideal of (S, f). By Lemma 3.2.4, C_A is a fuzzy i-ideal of (S, f). Then by Lemma 3.2.3, $\underline{C_A}$ is an i-ideal of $(\underline{S}, f)$. Conversely, assume that $\underline{C_A}$ is an i-ideal of $(\underline{S}, f)$. By Lemma 3.2.3, C_A is a fuzzy i-ideal of (S, f). Then by Lemma 3.2.4, A is an i-ideal of (S, f). □

Theorem 3.2.6. *Let A be a nonempty subset of S. Then A is an ideal of (S, f) if and only if $\underline{C_A}$ is an ideal of $(\underline{S}, f)$.*

Proof. Assume that A is an ideal of (S, f). By Theorem 3.2.5, $\underline{C_A}$ is an ideal of $(\underline{S}, f)$ for every $1 \leq i \leq n$. Conversely, assume that $\underline{C_A}$ is an ideal of $(\underline{S}, f)$. By Theorem 3.2.5, A is an ideal of (S, f) for every $1 \leq i \leq n$. □

Chapter 4

Almost i-ideals and fuzzy almost i-ideals of n-ary semigroups

In this chapter, we introduce almost i-ideals and fuzzy almost i-ideals of n-ary semigroups and give some interesting properties.
Throughout this chapter, let (S, f) be an n-ary semigroup.

4.1 Almost i-ideals

We define an almost i-ideal of (S, f) by using the concept of an almost ideal in a semigroup.

Definition 4.1.1. A nonempty subset I of S is called an **almost i-ideal** of (S, f) if

$$f(x_1^{i-1}, I, x_{i+1}^n) \cap I \neq \emptyset \text{ for all } x_1^{i-1}, x_{i+1}^n \in S$$

where $f(x_1^{i-1}, I, x_{i+1}^n) = \{f(x_1^{i-1}, a, x_{i+1}^n) \mid a \in I\}$.

Example 4.1.2. Every i-ideal of (S, f) is an almost i-ideal of (S, f).

Proof. Let I be an i-ideal of (S, f). Then $f(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$. So, $f(x_1^{i-1}, I, x_{i+1}^n) \cap I \neq \emptyset$. Thus, I is an almost i-ideal of (S, f). □

Example 4.1.3. Consider $n = 2$, the n-ary semigroup $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ under the usual addition, and $I = \{\overline{1}, \overline{4}, \overline{5}\}$.

$(\overline{0} + I) \cap I = \{\overline{1}, \overline{4}, \overline{5}\}$ and $(I + \overline{0}) \cap I = \{\overline{1}, \overline{4}, \overline{5}\}$.
$(\overline{1} + I) \cap I = \{\overline{5}\}$ and $(I + \overline{1}) \cap I = \{\overline{5}\}$.
$(\overline{2} + I) \cap I = \{\overline{1}\}$ and $(I + \overline{2}) \cap I = \{\overline{1}\}$.

$(\overline{3}+I)\cap I=\{\overline{1},\overline{4}\}$ and $(I+\overline{3})\cap I=\{\overline{1},\overline{4}\}$.
$(\overline{4}+I)\cap I=\{\overline{5}\}$ and $(I+\overline{4})\cap I=\{\overline{5}\}$.
$(\overline{5}+I)\cap I=\{\overline{4}\}$ and $(I+\overline{5})\cap I=\{\overline{4}\}$.

Hence, $I=\{\overline{1},\overline{4},\overline{5}\}$ is both a left almost 1-ideal and a right almost 1-ideal of $\mathbb{Z}_6$. So, I is an almost 1-ideal of $\mathbb{Z}_6$. But I is not a 1-ideal of $\mathbb{Z}_6$ since $\overline{1}+\overline{5}=\overline{0}\notin I$.
I is also not an n-ary subsemigroup of $\mathbb{Z}_6$ since $\overline{4}+\overline{5}=\overline{9}=\overline{3}\notin I$.

Example 4.1.3 implies that, in general, an almost i-ideal of (S,f) need not be an n-ary subsemigroup of (S,f) nor an i-ideal of (S,f).

Theorem 4.1.4. *If I is an almost i-ideal of (S,f) and $I\subseteq H\subseteq S$, then H is an almost i-ideal of (S,f).*

Proof. Assume that I is an almost i-ideal of (S,f) with $I\subseteq H\subseteq S$. Then we have $\emptyset\neq f(x_1^{i-1},I,x_{i+1}^n)\cap I\subseteq f(x_1^{i-1},H,x_{i+1}^n)\cap H$ for all $x_1^{i-1},x_{i+1}^n\in S$. Therefore, H is an almost i-ideal of (S,f). □

Corollary 4.1.5. *The union of two almost i-ideals of (S,f) is an almost i-ideal of (S,f).*

Proof. Let I_1 and I_2 be almost i-ideals of (S,f). Then $I_1\subseteq I_1\cup I_2$. By Theorem 4.1.4, $I_1\cup I_2$ is an almost i-ideal of (S,f). □

Example 4.1.6. Consider $n=2$ and the n-ary semigroup $\mathbb{Z}_6$ under the usual addition. We have $I_1=\{\overline{1},\overline{4},\overline{5}\}$ and $I_2=\{\overline{1},\overline{2},\overline{5}\}$ are almost 1-ideals of $\mathbb{Z}_6$.
Consider $\overline{1}\in(\mathbb{Z}_6,+)$. Then we have

$$\begin{aligned}(\{\overline{1}\}+(I_1\cap I_2))\cap(I_1\cap I_2)&=(\{\overline{1}\}+\{\overline{1},\overline{5}\})\cap(\{\overline{1},\overline{5}\})\\&=(\{\overline{2},\overline{0}\})\cap(\{\overline{1},\overline{5}\})\\&=\emptyset.\end{aligned}$$

Hence, $I_1\cap I_2=\{\overline{1},\overline{5}\}$ is not an almost 1-ideal of $\mathbb{Z}_6$.

Example 4.1.6 implies that, in general, the intersection of two almost i-ideals of (S,f) need not be an almost i-ideal of (S,f).

4.2 Fuzzy almost i-ideals

Definition 4.2.1. A fuzzy subset g of S is called a **fuzzy almost i-ideal** of (S,f) if

$$((x_1)_{\alpha_1}\circ(x_2)_{\alpha_2}\circ\ldots\circ(x_{i-1})_{\alpha_{i-1}}\circ g\circ(x_{i+1})_{\alpha_{i+1}}\circ\ldots\circ(x_n)_{\alpha_n})\cap g\neq 0$$

for all fuzzy points $(x_k)_{\alpha_k}$ of S where $k\in\{1,2,\ldots,n\}\smallsetminus\{i\}$.

Theorem 4.2.2. *Let g be a fuzzy almost i-ideal of (S, f) and h be a fuzzy subset of S such that $g \subseteq h$. Then h is a fuzzy almost i-ideal of (S, f).*

Proof. Assume that g is a fuzzy almost i-ideal of (S, f) and h is a fuzzy subset of S such that $g \subseteq h$. For each $k \in \{1, 2, \ldots, n\} \setminus \{i\}$, let $(x_k)_{\alpha_k}$ be a fuzzy point in S. Let $A = ((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap g$ and $B = ((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ h \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap h$.
Since $A \neq 0$, then there exists $y \in S$ such that $A(y) \neq 0$. Since $g \subseteq h$, then $A \subseteq B$. So, $A(y) \leq B(y)$. This implies $B(y) \neq 0$. Hence, $B \neq 0$. Therefore, h is a fuzzy almost i-ideal of (S, f). □

Corollary 4.2.3. *Let g and h be fuzzy almost i-ideals of (S, f). Then $g \cup h$ is a fuzzy almost i-ideal of (S, f).*

Proof. Since $g \subseteq g \cup h$, by Theorem 4.2.2, $g \cup h$ is a fuzzy almost i-ideal of (S, f). □

Example 4.2.4. Consider $n = 2$ and the n-ary semigroup $\mathbb{Z}_6$ under the usual addition, $g : \mathbb{Z}_6 \to [0, 1]$ is defined by

$$g(\overline{0}) = 0, g(\overline{1}) = 0.3, g(\overline{2}) = 0, g(\overline{3}) = 0, g(\overline{4}) = 0.2, g(\overline{5}) = 0.1$$

and $h : \mathbb{Z}_6 \to [0, 1]$ defined by

$$h(\overline{0}) = 0, h(\overline{1}) = 0.3, h(\overline{2}) = 0.1, h(\overline{3}) = 0, h(\overline{4}) = 0, h(\overline{5}) = 0.3.$$

Then g and h are fuzzy almost 1-ideals of $\mathbb{Z}_6$, but $g \cap h$ is not a fuzzy almost 1-ideal of $\mathbb{Z}_6$.

Proof. (1) We will show that g is a fuzzy almost 1-ideal of $\mathbb{Z}_6$.

Case 1: $x_2 = \overline{0}$.

Choose $x = \overline{1}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{0})_{\alpha_2}) \cap g](\overline{1}) = \min\{(g \circ (\overline{0})_{\alpha_2})(\overline{1}), g(\overline{1})\}$.

Then

$$\begin{aligned}(g \circ (\overline{0})_{\alpha_2})(\overline{1}) &= \sup_{x = x_1 + x_2} \min\{g(x_1), (\overline{0})_{\alpha_2}(x_2)\} \\ &\geq \min\{g(\overline{1}), (\overline{0})_{\alpha_2}(\overline{0})\} \\ &= \min\{0.3, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{0})_{\alpha_2}) \cap g](\overline{1}) = \min\{(g \circ (\overline{0})_{\alpha_2})(\overline{1}), g(\overline{1})\} \neq 0$.

Case 2: $x_2 = \overline{1}$.

Choose $x = \overline{5}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{1})_{\alpha_2}) \cap g](\overline{5}) \quad = \min\{(g \circ (\overline{1})_{\alpha_2})(\overline{5}), g(\overline{5})\}$.

Then

$$\begin{aligned}(g \circ (\overline{1})_{\alpha_2})(\overline{5}) &= \sup_{x=x_1+x_2} \min\{g(x_1), (\overline{1})_{\alpha_2}(x_2)\} \\ &\geq \min\{g(\overline{4}), (\overline{1})_{\alpha_2}(\overline{1})\} \\ &= \min\{0.2, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{1})_{\alpha_2}) \cap g](\overline{5}) = \min\{(g \circ (\overline{1})_{\alpha_2})(\overline{5}), g(\overline{5})\} \neq 0$.

Case 3: $x_2 = \overline{2}$.

Choose $x = \overline{1}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{2})_{\alpha_2}) \cap g](\overline{1}) \quad = \min\{(g \circ (\overline{2})_{\alpha_2})(\overline{1}), g(\overline{1})\}$.

Then

$$\begin{aligned}(g \circ (\overline{2})_{\alpha_2})(\overline{1}) &= \sup_{x=x_1+x_2} \min\{g(x_1), (\overline{2})_{\alpha_2}(x_2)\} \\ &\geq \min\{g(\overline{5}), (\overline{2})_{\alpha_2}(\overline{2})\} \\ &= \min\{0.1, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{2})_{\alpha_2}) \cap g](\overline{1}) = \min\{(g \circ (\overline{2})_{\alpha_2})(\overline{1}), g(\overline{1})\} \neq 0$.

Case 4: $x_2 = \overline{3}$.

Choose $x = \overline{4}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{3})_{\alpha_2}) \cap g](\overline{4}) \quad = \min\{(g \circ (\overline{3})_{\alpha_2})(\overline{4}), g(\overline{4})\}$.

Then

$$\begin{aligned}(g \circ (\overline{3})_{\alpha_2})(\overline{4}) &= \sup_{x=x_1+x_2} \min\{g(x_1), (\overline{3})_{\alpha_2}(x_2)\} \\ &\geq \min\{g(\overline{1}), (\overline{3})_{\alpha_2}(\overline{3})\} \\ &= \min\{0.3, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{3})_{\alpha_2}) \cap g](\overline{4}) = \min\{(g \circ (\overline{3})_{\alpha_2})(\overline{4}), g(\overline{4})\} \neq 0.$

Case 5: $x_2 = \overline{4}$.

Choose $x = \overline{5}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{4})_{\alpha_2}) \cap g](\overline{5}) \ \ = \min\{(g \circ (\overline{4})_{\alpha_2})(\overline{5}), g(\overline{5})\}.$

Then

$$\begin{aligned}(g \circ (\overline{4})_{\alpha_2})(\overline{5}) &= \sup_{x=x_1+x_2} \min\{g(x_1), (\overline{4})_{\alpha_2}(x_2)\}\\ &\geq \min\{g(\overline{1}), (\overline{4})_{\alpha_2}(\overline{4})\}\\ &= \min\{0.3, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{4})_{\alpha_2}) \cap g](\overline{5}) = \min\{(g \circ (\overline{4})_{\alpha_2})(\overline{5}), g(\overline{5})\} \neq 0.$

Case 6: $x_2 = \overline{5}$.

Choose $x = \overline{4}$.

We have

$[(g \circ (x_2)_{\alpha_2}) \cap g](x) = \min\{(g \circ (x_2)_{\alpha_2})(x), g(x)\}$

$[(g \circ (\overline{5})_{\alpha_2}) \cap g](\overline{4}) \ \ = \min\{(g \circ (\overline{5})_{\alpha_2})(\overline{4}), g(\overline{4})\}.$

Then

$$\begin{aligned}(g \circ (\overline{5})_{\alpha_2})(\overline{4}) &= \sup_{x=x_1+x_2} \min\{g(x_1), (\overline{5})_{\alpha_2}(x_2)\}\\ &\geq \min\{g(\overline{5}), (\overline{5})_{\alpha_2}(\overline{5})\}\\ &= \min\{0.1, \alpha_2\} \neq 0.\end{aligned}$$

Hence, $[(g \circ (\overline{5})_{\alpha_2}) \cap g](\overline{4}) = \min\{(g \circ (\overline{5})_{\alpha_2})(\overline{4}), g(\overline{4})\} \neq 0.$

Therefore, g is a fuzzy almost 1-ideal of $\mathbb{Z}_6$. Similarly, h is a fuzzy almost 1-ideal of $\mathbb{Z}_6$.

(2) We will show that $g \cap h$ is not a fuzzy almost 1-ideal of $\mathbb{Z}_6$.

Case 1: $x = \overline{0}$.

Choose $x_2 = \overline{0}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{0}) \ \ = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{0}), (g \cap h)(\overline{0})\}.$

Then $\min\{(g \cap h)(\overline{0})\} = \min\{0, 0\} = 0.$

Hence, $[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{0}) = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{0}), (g \cap h)(\overline{0})\} = 0.$

Case 2: $x = \overline{1}$.

Choose $x_2 = \overline{1}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{1})_{\alpha_2}) \cap (g \cap h)](\overline{1}) \ = \min\{((g \cap h) \circ (\overline{1})_{\alpha_2})(\overline{1}), (g \cap h)(\overline{1})\}$.

Then $\min\{(g \cap h)(\overline{1})\} = \min\{0.3, 0.3\} = 0.3$.

Now,

$$\begin{aligned}((g \cap h) \circ (\overline{1})_{\alpha_2})(\overline{1}) &= \sup_{x = x_1 + x_2} \min\{(g \cap h)(x_1), (\overline{1})_{\alpha_2}(x_2)\} \\ &\geq \min\{\min\{g(\overline{0}), h(\overline{0})\}, (\overline{1})_{\alpha_2}(\overline{1})\} \\ &= \min\{0, \alpha_2\} = 0.\end{aligned}$$

Hence, $[((g \cap h) \circ (\overline{1})_{\alpha_2}) \cap (g \cap h)](\overline{1}) = \min\{((g \cap h) \circ (\overline{1})_{\alpha_2})(\overline{1}), (g \cap h)(\overline{1})\} = 0$.

Case 3: $x = \overline{2}$.

Choose $x_2 = \overline{0}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{2}) \ = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{2}), (g \cap h)(\overline{2})\}$.

Then $\min\{(g \cap h)(\overline{2})\} = \min\{0, 0.1\} = 0$.

Hence, $[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{2}) = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{2}), (g \cap h)(\overline{2})\} = 0$.

Case 4: $x = \overline{3}$.

Choose $x_2 = \overline{0}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{3}) \ = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{3}), (g \cap h)(\overline{3})\}$.

Then $\min\{(g \cap h)(\overline{3})\} = \min\{0, 0\} = 0$.

Hence, $[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{3}) = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{3}), (g \cap h)(\overline{3})\} = 0$.

Case 5: $x = \overline{4}$.

Choose $x_2 = \overline{0}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{4}) \ = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{4}), (g \cap h)(\overline{4})\}$.

Then $\min\{(g \cap h)(\overline{4})\} = \min\{0.2, 0\} = 0$.

Hence, $[((g \cap h) \circ (\overline{0})_{\alpha_2}) \cap (g \cap h)](\overline{4}) = \min\{((g \cap h) \circ (\overline{0})_{\alpha_2})(\overline{4}), (g \cap h)(\overline{4})\} = 0$.

Case 6: $x = \overline{5}$.

Choose $x_2 = \overline{2}$.

We have

$[((g \cap h) \circ (x_2)_{\alpha_2}) \cap (g \cap h)](x) = \min\{((g \cap h) \circ (x_2)_{\alpha_2})(x), (g \cap h)(x)\}$

$[((g \cap h) \circ (\overline{2})_{\alpha_2}) \cap (g \cap h)](\overline{5}) \ = \min\{((g \cap h) \circ (\overline{2})_{\alpha_2})(\overline{5}), (g \cap h)(\overline{5})\}$.

Then $\min\{(g \cap h)(\overline{5})\} = \min\{0.1, 0.3\} = 0.1$.

Now,

$$\begin{aligned}((g \cap h) \circ (\overline{2})_{\alpha_2})(\overline{5}) &= \sup_{x = x_1 + x_2} \min\{(g \cap h)(x_1), (\overline{2})_{\alpha_2}(x_2)\} \\ &\geq \min\{\min\{g(\overline{3}), h(\overline{3})\}, (\overline{2})_{\alpha_2}(\overline{2})\} \\ &= \min\{0, \alpha_2\} = 0.\end{aligned}$$

Hence, $[((g \cap h) \circ (\overline{2})_{\alpha_2}) \cap (g \cap h)](\overline{5}) = \min\{((g \cap h) \circ (\overline{2})_{\alpha_2})(\overline{5}), (g \cap h)(\overline{5})\} = 0$.

Therefore, $g \cap h$ is not a fuzzy almost 1-ideal of $\mathbb{Z}_6$.

Example 4.2.4 implies that, in general, the intersection of two fuzzy almost i-ideals of (S, f) need not be a fuzzy almost i-ideal of (S, f). □

Theorem 4.2.5. *Let A be a nonempty subset of S. Then A is an almost i-ideal of (S, f) if and only if C_A is a fuzzy almost i-ideal of (S, f).*

Proof. Assume that A is an almost i-ideal of (S, f). Then $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$ for all $x_1^{i-1}, x_{i+1}^n \in S$. Thus, there exists $x \in f(x_1^{i-1}, A, x_{i+1}^n) \cap A$. So,

$$[((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_A \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap C_A](x) \neq 0$$

for all $\alpha_1, \ldots, \alpha_n \in (0, 1]$. Hence, C_A is a fuzzy almost i-ideal of (S, f).

Conversely, assume C_A is a fuzzy almost i-ideal of (S, f). Let $x_1^{i-1}, x_{i+1}^n \in S$. Hence,

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_A \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap C_A \neq 0.$$

Then there exists $x \in S$ such that

$$[((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_A \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap C_A](x) \neq 0.$$

So, $x \in f(x_1^{i-1}, A, x_{i+1}^n) \cap A$. Hence, $f(x_1^{i-1}, A, x_{i+1}^n) \cap A \neq \emptyset$. Thus, A is an almost i-ideal of (S, f). □

Definition 4.2.6. For a fuzzy subset g of S, the **support of g** is defined by $supp\ (g) = \{x \in S \mid g(x) \neq 0\}$.

Theorem 4.2.7. *Let g be a nonzero fuzzy subset of S. Then g is a fuzzy almost i-ideal of (S, f) if and only if $supp\ (g)$ is an almost i-ideal of (S, f).*

Proof. Assume that g is a fuzzy almost i-ideal of (S, f). Let $x_1^{i-1}, x_{i+1}^n \in S$. Then for any $\alpha_k \in (0, 1]$ where $k \in \{1, 2, \ldots, n\} \setminus \{i\}$, we have

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap g \neq 0.$$

Thus, there exists $x \in S$ such that

$$[((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap g](x) \neq 0.$$

So, $g(x) \neq 0$ and there exists $z \in S$ such that $g(z) \neq 0$ and $x = f(x_1^{i-1}, z, x_{i+1}^n)$, which implies $x, z \in supp\ (g)$. Thus,

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_{supp\ (g)} \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n})(x) \neq 0$$

and $C_{supp\ (g)}(x) \neq 0$. Hence,

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_{supp\ (g)} \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n} \cap C_{supp\ (g)})(x) \neq 0.$$

So, $C_{supp\ (g)}$ is a fuzzy almost i-ideal of (S, f). By Theorem 4.2.5, $supp\ (g)$ is an almost i-ideal of (S, f).

Conversely, assume that $supp\ (g)$ is an almost i-ideal of (S, f). By Theorem 4.2.5, $C_{supp\ (g)}$ is a fuzzy almost i-ideal of (S, f). For each $k \in \{1, 2, \ldots, n\} \setminus \{i\}$, let $(x_k)_{\alpha_k}$ be a fuzzy point in S. Then

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_{supp\ (g)} \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap C_{supp\ (g)}) \neq 0.$$

Then there exists $x \in S$ such that

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_{supp\ (g)} \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap C_{supp\ (g)})(x) \neq 0.$$

Hence,

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ C_{supp\ (g)} \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n})(x) \neq 0.$$

and $C_{supp\ (g)}(x) \neq 0$. Then there exists $z \in S$ such that $x = f(x_1^{i-1}, z, x_{i+1}^n)$, and $g(z) \neq 0$. Therefore,

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n})(x) \neq 0.$$

This implies

$$((x_1)_{\alpha_1} \circ (x_2)_{\alpha_2} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n} \cap g)(x) \neq 0.$$

Consequently, g is a fuzzy almost i-ideal of (S, f). □

4.3 Minimal almost i-ideals and minimal fuzzy almost i-ideals

We define a minimal almost i-ideal of (S, f) by using the concept of a minimal ideal in a semigroup and minimal fuzzy almost i-ideal of (S, f) by using the concept of a minimal fuzzy ideal in a semigroup in [15].

Definition 4.3.1. An almost i-ideal I of (S, f) is called **minimal** if for all almost i-ideal H of (S, f) such that $H \subseteq I$, we have $H = I$.

Definition 4.3.2. A nonzero fuzzy almost i-ideal g of (S, f) is called **minimal** if for all nonzero fuzzy almost i-ideal h of (S, f) such that $h \subseteq g$, we have $supp\,(h) = supp\,(g)$.

Example 4.3.3. Let $S = [0, 1]$. Clearly, $([0, 1], f)$ is an n-ary semigroup such that $f(x_1^n) = x_1 \cdots x_n$ for all $x_1, \ldots, x_n \in S$. Define a fuzzy subset $g : S \to [0, 1]$ by

$$g(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in S$. Then g is a minimal fuzzy almost i-ideal of (S, f).

Proof. We will show that g is a fuzzy almost i-ideal of (S, f).
We have $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$. Then $((x_1)_{\alpha_1} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n})(0) = (f(x_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}(0) = (0)_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}(0) \neq 0$ and $g(0) = 1$. So,

$$((x_1)_{\alpha_1} \circ \ldots \circ (x_{i-1})_{\alpha_{i-1}} \circ g \circ (x_{i+1})_{\alpha_{i+1}} \circ \ldots \circ (x_n)_{\alpha_n}) \cap g \neq 0.$$

Hence, g is a fuzzy almost i-ideal of (S, f).
Now, let g and h be nonzero fuzzy almost i-ideals of (S, f) such that $h \subseteq g$.
Then $h(x) \leq g(x)$ for all $x \in S$. Since $h \subseteq g$, then $supp\,(h) \subseteq supp\,(g)$.
Let $x \in supp\,(g)$. Then $g(x) \neq 0$. So, $g(x) = 1$. Hence, $x = 0$. So, $supp\,(g) = \{0\}$. Since $h \neq 0$ and $supp\,(h) \subseteq supp\,(g)$, then $supp\,(h) \neq \emptyset$ and $supp\,(h) = \{0\} = supp\,(g)$. Hence, $supp\,(g) = supp\,(h)$. Therefore, g is a minimal fuzzy almost i-ideal of (S, f). □

Theorem 4.3.4. *Let A be a nonempty subset of S. Then A is a minimal almost i-ideal of (S, f) if and only if C_A is a minimal fuzzy almost i-ideal of (S, f).*

Proof. Assume that A is a minimal almost i-ideal of (S, f). By Theorem 4.2.5, C_A is a fuzzy almost i-ideal of (S, f). Let g be a nonzero fuzzy almost i-ideal of (S, f) such that $g \subseteq C_A$. So, $supp\,(g) \subseteq supp\,(C_A) = A$. By Theorem 4.2.7, $supp\,(g)$ is an almost i-ideal of (S, f). Since A is minimal, $supp\,(g) = A = supp\,(C_A)$. Therefore, C_A is a minimal fuzzy almost i-ideal of (S, f). Conversely, assume that C_A is a minimal fuzzy almost i-ideal of (S, f). Let I be an almost i-ideal of (S, f) such that $I \subseteq A$. By Theorem 4.2.5, C_I is a fuzzy almost i-ideal of (S, f) such that $C_I \subseteq C_A$. Hence, $I = supp\,(C_I) = supp\,(C_A) = A$. Therefore, A is a minimal almost i-ideal of (S, f). □

4.4 Prime almost i-ideals and prime fuzzy almost i-ideals

We derive the definition of a prime almost i-ideal of (S, f) by using the concept of a prime ideal in a semigroup in [1] and the definition of a prime fuzzy almost i-ideal of (S, f) by using the concept of a prime fuzzy ideal in a semigroup in [14].

Definition 4.4.1. An almost i-ideal A of (S, f) is called **prime** if for all $x_1, \ldots, x_n \in S$, $f(x_1^n) \in A$ implies $x_i \in A$ for some i.

Definition 4.4.2. A fuzzy almost i-ideal g of (S, f) is called **prime** if for all $x_1, \ldots, x_n \in S$, $g(f(x_1^n)) \leq \max\{g(x_1), \ldots, g(x_n)\}$.

Example 4.4.3. Let $S = [0, 1]$. Clearly, $([0, 1], f)$ is an n-ary semigroup such that $f(x_1^n) = x_1 \cdots x_n$ for all $x_1, \ldots, x_n \in S$. Define a prime fuzzy subset $g : S \to [0, 1]$ by

$$g(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in S$. Then g is a prime fuzzy almost i-ideal of (S, f).

Proof. By Example 4.3.3., g is a fuzzy almost i-ideal of (S, f). Let $x_1, \ldots, x_n \in S$. Then we consider two cases:

Case 1: $g(x_i) = 0$ for all i. Then $x_i \neq 0$ for all i. So, $f(x_1^n) \neq 0$. Hence, $\max\{g(x_1), \ldots, g(x_n)\} = 0 = g(f(x_1^n))$.

Case 2: $g(x_i) = 1$ for some i. Then $x_i = 0$ for all i. So, $f(x_1^n) = 0$. Hence, $\max\{g(x_1), \ldots, g(x_n)\} = 1 = g(f(x_1^n))$.

Thus, g is a prime fuzzy almost i-ideal of (S, f). □

Theorem 4.4.4. *Let A be a nonempty subset of S. Then A is a prime almost i-ideal of (S, f) if and only if C_A is a prime fuzzy almost i-ideal of (S, f).*

Proof. Assume that A is a prime almost i-ideal of (S, f). By Theorem 4.2.5, C_A is a fuzzy almost i-ideal of (S, f). Let $x_1, \ldots, x_n \in S$. We consider two cases:
Case 1: $f(x_1^n) \in A$. So, $x_i \in A$ for some i. Then $\max\{C_A(x_1), \ldots, C_A(x_n)\} = 1 \geq C_A(f(x_1^n))$.
Case 2: $f(x_1^n) \notin A$. Then $C_A(f(x_1^n)) = 0 \leq \max\{C_A(x_1), \ldots, C_A(x_n)\}$.
Thus, C_A is a prime fuzzy almost i-ideal of (S, f). Conversely, assume that C_A is a prime fuzzy almost i-ideal of (S, f). By Theorem 4.2.5, A is an almost i-ideal of (S, f). Let $x_1, \ldots, x_n \in S$ such that $f(x_1^n) \in A$. Then $C_A(f(x_1^n)) = 1$. By assumption, $C_A(f(x_1^n)) \leq \max\{C_A(x_1), \ldots, C_A(x_n)\}$. So, $\max\{C_A(x_1), \ldots, C_A(x_n)\} = 1$. Therefore, $x_i \in A$ for some i. Thus, A is a prime almost i-ideal of (S, f). □

4.5 Semiprime almost i-ideals and semiprime fuzzy almost i-ideals

We derive the definition of a semiprime almost i-ideal of (S, f) by using the concept of a semiprime ideal in a semigroup in [1] and the definition of a semiprime fuzzy almost i-ideal of (S, f) by using the concept of a semiprime fuzzy ideal in a semigroup in [29].

Definition 4.5.1. An almost i-ideal A of (S, f) is called **semiprime** if for all $x \in S$, $f(x^n) \in A$ implies $x \in A$.

Definition 4.5.2. A fuzzy almost i-ideal g of (S, f) is called **semiprime** if for all $x \in S, \;\; g(f(x^n)) \leq g(x)$.

Example 4.5.3. Let $S = [0, 1]$. Clearly, $([0, 1], f)$ is an n-ary semigroup such that $f(x_1^n) = x_1 \cdots x_n$ for all $x_1, \ldots, x_n \in S$. Define a semiprime fuzzy subset $g : S \to [0, 1]$ by

$$g(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in S$. Then g is a semiprime fuzzy almost i-ideal of (S, f).

Proof. By Example 4.3.3., g is a fuzzy almost i-ideal of (S, f). Let $x \in S$. Then we consider two cases:

Case 1: $g(f(x^n)) = 1$. Then $x^n = 0$. So, $x = 0$. Hence, $1 = g(x) \geq g(f(x^n))$.
Case 2: $g(f(x^n)) = 0$. Then $x^n \neq 0$. So, $x \neq 0$. Hence, $g(f(x^n)) = 0 = g(x)$.
Thus, g is a semiprime fuzzy almost i-ideal of (S, f). □

Theorem 4.5.4. *Let A be a nonempty subset of S. Then A is a semiprime almost i-ideal of (S, f) if and only if C_A is a semiprime fuzzy almost i-ideal of (S, f).*

Proof. Assume that A is a semiprime almost i-ideal of (S, f). By Theorem 4.2.5, C_A is a fuzzy almost i-ideal of (S, f). Let $x \in S$. We consider two cases:
Case 1: $f(x^n) \in A$. Then $x \in A$. So, $C_A(x) = 1$. Hence, $C_A(x) \geq C_A(f(x^n))$.
Case 2: $f(x^n) \notin A$. Then $C_A(f(x^n)) = 0 \leq C_A(x)$.
Thus, C_A is a semiprime fuzzy almost i-ideal of (S, f). Conversely, assume that C_A is a semiprime fuzzy almost i-ideal of (S, f). By Theorem 4.2.5, A is an almost i-ideal of (S, f). Let $x \in S$ such that $f(x^n) \in A$. Then $C_A(f(x^n)) = 1$. By assumption, we have $C_A(f(x^n)) \leq C_A(x)$. Since $C_A(f(x^n)) = 1$, $C_A(x) = 1$. Hence, $x \in A$. Thus, A is a semiprime almost i-ideal of (S, f). □

Chapter 5

Conclusions and suggestions

In this thesis, we studied about the fuzziness of n-ary semigroups. We also showed the relation between i-ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$, and ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$. Furthermore, we introduced and studied the properties of almost i-ideals and fuzzy almost i-ideals of n-ary semigroups. We defined minimal almost i-ideals, minimal fuzzy almost i-ideals, prime almost i-ideals, prime fuzzy almost i-ideals, semiprime almost i-ideals, and semiprime fuzzy almost i-ideals in n-ary semigroups, and studied their properties in n-ary semigroups.

Suggestions

1. Study i-ideals and fuzzy i-ideals in other algebras.

2. Study almost i-ideals and fuzzy almost i-ideals in other algebras.

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VITAE

Name Mr. John Patrick F. Solano

Student ID 6010220006

Educational Attainment

Degree	Name of Institution	Year of Graduation
Bachelor of Science (Applied Mathematics)	Polytechnic University of the Philippines	2013

Scholarship Awards during Enrolment

Thailand's Education Hub for ASEAN Countries (TEH-AC) Scholarship, 2017-2019.

List of Publications and Proceeding

Solano, J. P. F., Suebsung, S. and Chinram, R. 2018. On ideals of fuzzy points n-ary semigroups, International Journal of Mathematics and Computer Science, 13(2), 179-186. Retrieved from http://ijmcs.future-in-tech.net/13.2/R-Chinram3.pdf.

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