Sequences Generated by Polynomials over Integral Domain

Veasna Kim

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics
Prince of Songkla University
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I hereby certify that this work has not been accepted in substance for any degree and is not being currently submitted in candidature for any degree.

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ABSTRACT

In the first part of this dissertation, let D be an integral domain. For sequences $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $I = (i_1, i_2, \ldots, i_n)$ in D^n with distinct i_j, call $\bar{a}$ a (D^n, I)-polynomial sequence if there exists $f(x) \in D[x]$ such that $f(i_j) = a_j$ foe all $1 \leq j \leq n$. Criteria for a sequence to be a (D^n, I)-polynomial sequence are established, and explicit structures of $D^n/P_{n,I}$ are determined.

In the second part of this dissertation, let $f(x) \in \mathbb{Z}[x]$, call $\Delta_F f(x) = f(x+1) - f(x)$ a difference polynomial of $f(x)$. Let $\bar{c} = (c_1, c_2, \ldots, c_{n-1})$ in $\mathbb{Z}^{n-1}$. If there exists $f(x) \in \mathbb{Z}[x]$ such that $\Delta_F f(i) = c_i$ for all $1 \leq i \leq n-1$, then we call $\bar{c}$, a difference polynomial sequence of length $n-1$. Denote by ΔP_n the set of all difference polynomial sequences. Criteria for a difference polynomial sequences are established, and explicit structures of $\mathbb{Z}^{n-1}/\Delta P_n$ and $P_{n-1}/\Delta P_n$ are determined.

In the third part of this dissertation, let D be an integral domain, $I = (i_1, i_2, \ldots, i_n) \in D^n$ with $i_j \neq i_k$ if $j \neq k$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right) \right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. If there exists $f(x)$ in $D[x]$ such that $f^{(m)}(i_j) = a_j^m$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$ where $f^{(m)}(i_j) = a_j^m$ denotes the $m^{(th)}$ derivative of $f(x)$ evaluated at the point i_j, call $\mathscr{A}$ a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. Criteria for a sequence to be a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. We also investigate the case where $r_j = k$ for all j and $(n, k) = (1, k), (2, 1), (3, 1)$ and $(2, 2)$.

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Veasna Kim

CONTENTS

CHAPTER 1

Introduction

This chapter consists of five sections: background and motivation, objectives of research, outcomes, plan of study, and outline of the dissertation.

1.1 Background and motivation

Let $\bar{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of integers of length n. Is there a polynomial $f(x) \in \mathbb{Z}[x]$ such that $(f(1), f(2), \ldots, f(n))$ equals the given sequence? This interesting question has been studied by [1] in 2008. They used the Lagrange interpolation polynomial and the Newton interpolation polynomial to show the necessary and sufficient conditions for which integral sequence of length n is the polynomial sequence and show that if $\bar{a} \in \mathbb{Z}^n$ then $(n-1)!\bar{a}$ is a polynomial sequence of length n. Moreover, they study the structure of $\mathbb{Z}^n/P_n$ and show that

$$\mathbb{Z}/P_1 \cong \mathbb{Z},\ \mathbb{Z}^2/P_2 \cong \mathbb{Z}^2 \text{ and } \mathbb{Z}^n/P_n \cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}$$

for $n \geq 3$ where

$$P_n = \{\bar{a} \in \mathbb{Z}^n | \text{ there exists } f(x) \in \mathbb{Z}[x] \text{ such that } f(i) = a_i \text{ for all } 1 \leq i \leq n\}.$$

In this thesis, we extend the result above from the set of integer to an integral domain. Let D be an integral domain. Given a sequence $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct for all $1 \leq j \leq n$ and $(a_1, a_2, \ldots, a_n)$ of length n, is there a polynomial $f(x) \in D[x]$ such that $f(i_j) = a_j$ for all $j = 1, 2, \ldots, n$? By using the same method in [1], we obtain theorems on sequences generated by polynomial on a finite set I of integral domains. General criteria for a polynomial sequence is provided. Denote the set of all polynomials sequences with respective to $I = (i_1, i_2, \ldots, i_n)$ by $P_{n,I}$. We study the structure of $D^n/P_{n,I}$ and finally we show that if $a_{j,k} = \frac{\prod_{m=1}^{k}(i_{j+1}-i_m)}{\prod_{m=1}^{k}(i_{k+1}-i_m)}$ for

$1 \leq j \leq n-1$ and $1 \leq k \leq j+1$ is an element in D then

$$D^n/P_{n,I} \cong D/(i_2-i_1)D \oplus D/(i_3-i_1)(i_3-i_2)D \oplus \cdots \oplus D/(i_n-i_1)\cdots(i_n-i_{n-1})D.$$

In the second part of this dissertation, we consider the difference of the polynomial and the difference of the sequence over the set of integers. For polynomial $f(x) \in \mathbb{Z}[x]$, call $\Delta_F f(x) = f(x+1) - f(x)$ a difference polynomial of $f(x)$. For sequence $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, call $\Delta \bar{a} = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}) \in \mathbb{Z}^{n-1}$ a difference sequence of $\bar{a}$. Criteria for a difference polynomial sequences are established, and explicit structures of $\mathbb{Z}^{n-1}/\Delta P_n$ and $P_{n-1}/\Delta P_n$ are determined where

$$\Delta P_n = \{\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1} | \text{there exists } f(x) \in \mathbb{Z}[x] \text{ such that } \Delta_F f(i_j) = c_j \text{ for all } 1 \leq j \leq n-1\}.$$

We call the element in ΔP_n a difference polynomial sequence.

In the last part of this dissertation, let D be an integral domain, $I = (i_1, i_2, \ldots, i_n)$ in D^n and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. We answer the following question:

Is there a polynomial $f(x) \in D[x]$ such that

$$\begin{array}{cccc} f(i_1) = a_1^0, & f(i_2) = a_2^0, & \ldots, & f(i_n) = a_n^0, \\ f'(i_1) = a_1^1, & f'(i_2) = a_2^1, & \ldots, & f'(i_n) = a_n^1, \\ \vdots & \vdots & \ddots & \vdots \\ f^{(r_1)}(i_1) = a_1^{r_1}, & f^{(r_2)}(i_2) = a_2^{r_2}, & \ldots, & f^{(r_n)}(i_n) = a_n^{r_n}? \end{array}$$

If the sequence $\mathscr{A}$ satisfies the question above, we call $\mathscr{A}$ a differential polynomial sequence of length n and order $(r_1, \ldots, r_n)$ with respect to I. Denote by $\wp_{n,I}^R$ where $R = (r_1, r_2, \ldots, r_n)$ is the set of all the differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. By using the generalization of Hermite's formula and the Newton form of generalization of Hermite's formula, we will study the differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I.

1.2 Objectives

1. To extend the results of E. F. Cornelius Jr. and P. Schultz [1].
2. To find a necessary and sufficient condition for a sequence in the set of integer to be a difference polynomial sequence, ΔP_n.
3. To find a structure of $\mathbb{Z}^n/\Delta P_n$ and $P_{n-1}/\Delta P_n$.
4. To characterize the differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I.

1.3 Outcomes

1. The criteria for a sequence to be a (D^n, I)-polynomial sequence are established, and explicit structures of $D^n/P_{n,I}$ are determined.
2. A necessary and sufficient condition for a sequence in the set of integer to be a difference polynomial sequence, ΔP_n.
3. A structure of $\mathbb{Z}^n/\Delta P_n$ and $P_{n-1}/\Delta P_n$.
4. The properties of differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I.

1.4 Plan of study

Task	2018								2019				
	5	6	7	8	9	10	11	12	1	2	3	4	5
Study literatures	*	*											
Write up the thesis proposal			*	*									
Present the thesis proposal					*								
Work on the problems						*	*	*	*	*	*	*	
Present the thesis													*

1.5 Outline of the dissertation

This dissertation is organized as follows.

In Chapter 2, we review definitions and basis results that will be used throughout our study.

In Chapter 3, there are three sections. In the first section, let D be an integral domain, $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and $\overline{a} = (a_1, a_2, \ldots, a_n)$ in D^n. Let

$$P_{n,I} = \{\overline{a} \in D^n \mid \text{there exists } f(x) \in D[x] \text{ such that } f(i_j) = a_j,\ for\ all\ 1 \leq j \leq n\}$$

be the set of all (D^n, I)-polynomial sequences. In this part we show that $(P_{n,I}, +)$ is an abelian group. In the second section we extend the results of E. F. Cornelius Jr. and P. Schultz from $\mathbb{Z}$ to be an integral domain D. In the third section we determine the structure of $D^n/P_{n,I}$.

In Chapter 4, let n be a positive integer and

$$\Delta P_n = \{\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1} \mid \text{there exists } f(x) \in \mathbb{Z}[x]$$
$$\text{such that } \Delta_F f(i) = c_i, \ \text{ for all } 1 \leq i \leq n-1\}.$$

Then we characterize ΔP_n and find the structure of $\mathbb{Z}^{n-1}/\Delta P_n$ and $P_{n-1}/\Delta P_n$.

In Chapter 5, let D be an integral domain, $I = (i_1, i_2, \ldots, i_n) \in D^n$ where $i_j \neq i_m$ if $j \neq m$, $R = (r_1, r_2, \ldots, r_n)$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Let

$$\wp_{n,I}^R = \{ \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right) \text{ where } a_1^0, a_1^1,$$
$$\ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n} \in D \mid \text{there exists } f(x) \in D[x]$$
$$\text{such that } f^{(m)}(i_j) = a_j^m, \text{ for all } 1 \leq j \leq n, 0 \leq m \leq r_j\}.$$

In case $r_j = k$ for all j and $I = (1, 2, \ldots, n)$, we get $\wp_{n,I}^{(k)} = \wp_{n,I}^R$. Then we characterize $\wp_{1,c}^{(1)}$, $\wp_{2,I}^{(1)}$, $\wp_{3,I}^{(1)}$ and $\wp_{2,I}^{(2)}$ for $D = \mathbb{Z}$.

In Chapter 6, we summarize the results of this dissertation.

CHAPTER 2

Preliminaries

In this chapter, we will recall some definitions, theorems and examples that will be used throughout our study.

2.1 Lagrange interpolation polynomials

Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $I = (i_1, i_2, \ldots, i_n) \in \mathbb{R}^n$ where $i_j \neq i_{j+1}$ for all $1 \leq j \leq n-1$ and $\mathbb{R}$ is the set of real numbers. By [2, page 33], the Lagrange interpolation polynomial $L_{a,I}(x)$,

$$\begin{aligned} L_{a,I}(x) &= \sum_{j=1}^{n} a_j \prod_{\substack{m=1 \\ m\neq j}}^{n} \frac{x - i_m}{i_j - i_m} \\ &= a_1 \frac{(x-i_2)(x-i_3)(x-i_4)\cdots(x-i_n)}{(i_1-i_2)(i_1-i_3)(i_1-i_4)\cdots(i_1-i_n)} + \\ &\quad a_2 \frac{(x-i_1)(x-i_3)(x-i_4)\cdots(x-i_n)}{(i_2-i_1)(i_2-i_3)(i_2-i_4)\cdots(i_2-i_n)} + \cdots + \\ &\quad a_n \frac{(x-i_1)(x-i_2)(x-i_3)\cdots(x-i_{n-1})}{(i_n-i_1)(i_n-i_2)(i_n-i_3)\cdots(i_n-i_{n-1})}, \end{aligned}$$

is the unique polynomial of degree $\leq n-1$ that passes through n points (i_j, a_j) for all $1 \leq j \leq n$. That is, $L_{a,I}(i_j) = a_j$ for all $1 \leq j \leq n$.
In addition, if $I = (1, 2, 3, \ldots, n)$, then we write $L_a(x)$ for $L_{a,I}$ where

$$L_a(x) = \sum_{j=1}^{n} a_j \prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{x - i}{j - i}.$$

Example 1. Let $\overline{a} = (26, 37, 65)$ and $I = (5, 6, 8)$. Then

$$\begin{aligned} L_{a,I}(x) &= \sum_{j=1}^{3} a_j \prod_{\substack{m=1 \\ m\neq j}}^{3} \frac{x - i_m}{i_j - i_m},\ (i_1 = 5, i_2 = 6, i_3 = 8) \\ &= a_1 \frac{(x-6)(x-8)}{(5-6)(5-8)} + a_2 \frac{(x-5)(x-8)}{(6-5)(6-8)} + a_3 \frac{(x-5)(x-6)}{(8-5)(8-6)} \end{aligned}$$

$$= 26\frac{(x-6)(x-8)}{(5-6)(5-8)} + 37\frac{(x-5)(x-8)}{(6-5)(6-8)} + 65\frac{(x-5)(x-6)}{(8-5)(8-6)}$$
$$= x^2 + 1.$$

We can see that $L_{a,I}(5) = 26$, $L_{a,I}(6) = 37$ and $L_{a,I}(8) = 65$.

Example 2. Let $\bar{b} = (2+i, 4-5i, 10-2i)$ and $I = (2+i, 3, 3+4i)$. Then

$$L_{b,I}(x) = \sum_{j=1}^{3} b_j \prod_{\substack{m=1\\m\neq j}}^{3} \frac{x-i_m}{i_j-i_m}, \qquad (i_1 = 5, i_2 = 6, i_3 = 8)$$
$$= (2+i)\frac{(x-3)(x-3-4i)}{(2+i-3)(2+i-3-4i)} + (4-5i)\frac{(x-2-i)(x-3-4i)}{(3-2-i)(3-3-4i)} +$$
$$(10-2i)\frac{(x-2-i)(x-3)}{(3+4i-2-i)(3-4i-3)}$$
$$= (\frac{-7+41i}{40})x^2 + (\frac{590-139i}{20})x - (\frac{67-53i}{8}).$$

Example 3. Let $\overline{a} = (7, 10, 13)$ and $I = (1, 2, 3)$. Then

$$L_a(x) = \sum_{j=1}^{3} a_j \prod_{\substack{i=1\\i\neq j}}^{3} \frac{x-i}{j-i}$$
$$= \frac{(x-2)(x-3)}{(1-2)(1-3)} \cdot a_1 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \cdot a_2 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \cdot a_3$$
$$= \frac{(x-2)(x-3)}{2} \cdot 7 + \frac{(x-1)(x-3)}{-1} \cdot 10 + \frac{(x-1)(x-2)}{2} \cdot 13$$
$$= 3x + 4.$$

2.2 Newton's interpolation polynomials

Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $I = (i_1, i_2, \ldots, i_n) \in \mathbb{R}^n$ where $i_j \neq i_{j+1}$ for all $1 \leq j \leq n-1$. We define Newton basis polynomials $p_{i_j}(x)$ for $0 \geq j \geq n$ with respect to I as follows

$$p_{i_0}(x) = 1$$
$$p_{i_1}(x) = (x - i_1)$$
$$p_{i_2}(x) = (x - i_1)(x - i_2)$$

$$\vdots$$

$$p_{i_{n-1}}(x) = (x - i_1)(x - i_2)(x - i_3) \cdots (x - i_{n-1}).$$

Thus

$$p_{i_j}(x) = \prod_{m=1}^{j} (x - i_m) = (x - i_1)(x - i_2)(x - i_3) \dots (x - i_j), \qquad j = 1, 2, \dots, n-1.$$

The Newton's interpolation polynomial is defined by

$$N_{a,I}(x) = b_{0,I} + b_{1,I}(x - i_1) + b_{2,I}(x - i_1)(x - i_2) + \cdots + b_{n-1,I}(x - i_1)(x - i_2) \cdots (x - i_{n-1}),$$

where

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{m=1, m \neq j+1}^{k+1} (i_{j+1} - i_m)} \qquad (0 \leq k \leq n-1).$$

The elements

$$1, p_{i_1} := (x - i_1), p_{i_2} := (x - i_1)(x - i_2), \dots, p_{i_{n-1}} := (x - i_1)(x - i_2) \cdots (x - i_n)$$

are referred to as the corresponding Newton basis polynomials [2, page 39-40]. So $N_{a,I}(x)$ is the polynomial of degree $\leq n - 1$ that passes through n points (i_j, a_j) for $1 \leq j \leq n$ and we can see that $N_{a,I}(i_j) = a_j$ for $1 \leq i \leq n$.

Example 4. Let $\overline{a} = (2, 8, 12)$ and $I = (5, 6, 8)$. Then

$$N_{a,I}(x) = \sum_{j=0}^{2} b_{j,I} p_{i_j}(x),$$

where

$$b_{0,I} = \sum_{j=0}^{0} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j}}^{1} (i_{j+1} - i_m)} = a_1 = 2,$$

$$\begin{aligned} b_{1,I} &= \sum_{j=0}^{1} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j}}^{2} (i_{j+1} - i_m)} \\ &= \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{2}{5 - 6} + \frac{8}{6 - 5} = 6, \end{aligned}$$

$$b_{2,I} = \sum_{j=0}^{2} \frac{a_{j+1}}{\prod\limits_{\substack{m=1\\ m\neq j}}^{3} (i_{j+1} - i_m)}$$

$$= \frac{a_1}{(i_1 - i_2)(i_1 - i_3)} + \frac{a_2}{(i_2 - i_1)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)}$$

$$= \frac{2}{(5-6)(5-8)} + \frac{8}{(6-5)(6-8)} + \frac{12}{(8-5)(8-6)} = -\frac{4}{3}.$$

Since $\quad p_{i_0}(x) = 1, p_{i_1}(x) = (x - i_1), p_{i_2}(x) = (x - i_1)(x - i_2),$

we have $N_{a,I}(x) = 2 + 6(x-5) - \frac{4}{3}(x-5)(x-6) = -\frac{4}{3}x^2 + \frac{62}{3}x - 68.$

Example 5. Let $\bar{a} = (-2+i, 2-4i, 8+i, 12-3i)$ and $I = (-2-i, 3+i, 4, 7+2i)$ Then $N_{a,I}(x) = \sum_{j=0}^{3} b_{j,I} p_{i_j}(x)$, where

$$b_{0,I} = \sum_{j=0}^{0} \frac{a_{j+1}}{\prod\limits_{\substack{m=1\\ m\neq j+1}}^{1} (i_{j+1} - i_m)} = a_1 = -2 + i,$$

$$b_{1,I} = \sum_{j=0}^{1} \frac{a_{j+1}}{\prod\limits_{\substack{m=1\\ m\neq j}}^{2} (i_{j+1} - i_m)} = \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{-2+i}{-5-2i} + \frac{2-4i}{5+2i} = \frac{10-33i}{29}$$

$$b_{2,I} = \sum_{j=0}^{2} \frac{a_{j+1}}{\prod\limits_{\substack{m=1\\ m\neq j+1}}^{3} (i_{j+1} - i_m)}$$

$$= \frac{a_1}{(i_1 - i_2)(i_1 - i_3)} + \frac{a_2}{(i_2 - i_1)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)}$$

$$= \frac{-2+i}{((-2-i)-(3+i))((-2-i)-(4))} + \frac{a_2}{((3+i)-(-2-i))((3+i)-(4))} +$$

$$\frac{a+3}{((4)-(-2-i))((4)-(3+i))} = \frac{439+2301i}{2146}$$

$$b_{3,I} = \sum_{j=0}^{3} \frac{a_{j+1}}{\prod\limits_{\substack{m=1\\ m\neq j+1}}^{4} (i_{j+1} - i_m)}$$

$$= \frac{a_1}{(i_1 - i_2)(i_1 - i_3)(i_1 - i_4)} + \frac{a_2}{(i_2 - i_1)(i_2 - i_3)(i_2 - i_4)} +$$

$$\frac{a_3}{(i_3-i_1)(i_3-i_2)(i_3-i_4)} + \frac{a_1}{(i_4-i_1)(i_4-i_2)(i_4-i_3)}$$

$$= \frac{-2+i}{((-2-i)-(3+i))\,((-2-i)-(4))\,((-2-i)-(7+2i))} +$$

$$\frac{2-4i}{((3+i)-(-2-i))\,((3+i)-(4))\,((3+i)-(7+2i))} +$$

$$\frac{8+i}{((4)-(-2-i))\,((4)-(3+i))\,((4)-(7+2i))} +$$

$$\frac{12-3i}{((7+2i)-(2+i))\,((7+2i)-(3+i))\,((7+2i)-(4))}$$

$$= \frac{1116553}{7113990} - \frac{591823i}{2371330}$$

and $p_{i_0}(x) = 1, p_{i_1}(x) = (x+2+i), p_{i_2}(x) = (x+2+i)(x-3-i)$,

$p_{i_3} = (x+2+i)(x-3-i)(x-4)$. We have

$$N_{a,I}(x) = -2+i+\left(\frac{10-33i}{29}\right)(x+2+i)+\left(\frac{439+2301i}{2146}\right)(x+2+i)(x-3-i)+$$

$$\left(\frac{1116553}{7113990} - \frac{591823i}{2371330}\right)(x+2+i)(x-3-i)(x-4)$$

$$= \left(\frac{1116553}{7113990} - \frac{591823i}{2371330}\right)x^3 - \left(\frac{412748}{711399} - \frac{550172i}{237133}\right)x^2 -$$

$$\left(\frac{8996083}{7113990} - \frac{19530341i}{7113990}\right)x + \left(\frac{8747654}{711399} - \frac{6521719i}{711399}\right)$$

In addition, if $I = (1, 2, 3, \ldots, n)$, then we write $N_a(x)$ for $N_{a,I}$, $p_j(x)$ for $p_{i_j}(x)$ and b_j for $b_{j,I}$, for all $j = 0, 1, \ldots, n-1$ where

$$N_a(x) = \sum_{j=0}^{n-1} b_j p_j(x) = b_0 + b_1(x-1) + b_2(x-1)(x-2) + \cdots +$$

$$b_{n-1}(x-1)(x-2)(x-3)\cdots(x-(n-1)).$$

Example 6. Let $\bar{a} = (2, 8, 12, 16)$ and $I = (1, 2, 3, 4)$. Then

$$b_0 = 2,$$

$$b_1 = \sum_{j=0}^{1} \frac{a_{j+1}}{\prod_{\substack{m=1\\ m\neq j+1}}^{2} (j+1-m)} = \frac{a_1}{1-2} + \frac{a_2}{2-1} = -a_1 + a_2 = -2 + 8 = 6,$$

$$b_2 = \sum_{j=0}^{2} \frac{a_{j+1}}{\prod_{\substack{m=1\\ m\neq j+1}}^{3} (j+1-m)} = \frac{a_1}{(1-2)(1-3)} + \frac{a_2}{(2-1)(2-3)} + \frac{a_3}{(3-1)(3-2)}$$

$$= \frac{2}{(-1)(-2)} + \frac{8}{(1)(-1)} + \frac{12}{(2)(1)} = -1,$$

$$b_3 = \sum_{j=0}^{3} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j+1}}^{4} (j+1-m)} = \frac{a_1}{(1-2)(1-3)(1-4)} + \frac{a_2}{(2-1)(2-3)(2-4)}$$

$$+ \frac{a_3}{(3-1)(3-2)(3-4)} + \frac{a_4}{(4-1)(4-2)(4-3)} = \frac{1}{3}.$$

Hence

$$\begin{aligned} N_a(x) &= \sum_{i=0}^{3} b_i p_i(x) = b_0 p_0(x) + b_1 p_1(x) + b_2 p_2(x) + b_3 p_3(x) \\ &= b_0 + b_1(x-1) + b_2(x-1)(x-2) + b_3(x-1)(x-2)(x-3) \\ &= 2 + 6(x-1) - (x-1)(x-2) + \frac{1}{3}(x-1)(x-2)(x-3) \\ &= \frac{1}{3}x^3 - 3x^2 + \frac{38}{3}x - 8. \end{aligned}$$

2.3 Hermite's interpolation formula

Let D be an integral domain and D_Q be the quotient field of D. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1\right), \left(a_2^0, a_2^1\right), \ldots, \left(a_n^0, a_n^1\right) \right) \in \left(D^2\right)^n$$

and $I = (i_1, i_2, \ldots, i_n) \in D^n$ where $i_j \neq i_m$ if $j \neq h$ for all $1 \leq j, m \leq n$. We define polynomials $p_{i_n}(x)$ and $l_{i_j}(x)$ for $1 \leq j \leq n$ with respect to I as follows

$$\begin{aligned} p_{i_n}(x) &= (x-i_1)(x-i_2)\cdots(x-i_n), \\ l_{i_j}(x) &= \frac{p_{i_n}(x)}{(x-i_j)p'_{i_n}(i_j)} = \frac{(x-i_1)\cdots(x-i_{j-1})(x-i_{j+1})\cdots(x-i_n)}{(i_j-i_1)\cdots(i_j-i_{j-1})(i_j-i_{j+1})\cdots(i_j-i_n)}. \end{aligned}$$

By [5], the unique Hermite's formula $H_{\mathscr{A}}(x) \in D_Q[x]$ of degree $< 2n$ such that

$$\begin{aligned} &H_{\mathscr{A}}(i_1) = a_1^0, \quad H_{\mathscr{A}}(i_2) = a_2^0, \quad H_{\mathscr{A}}(i_3) = a_3^0, \quad \ldots, \quad H_{\mathscr{A}}(i_n) = a_n^0, \\ &H'_{\mathscr{A}}(i_1) = a_1^1, \quad H'_{\mathscr{A}}(i_2) = a_2^1, \quad H'_{\mathscr{A}}(i_3) = a_3^1, \quad \ldots, \quad H'_{\mathscr{A}}(i_n) = a_n^1, \end{aligned}$$

where D_Q is the quotient field of D is defined as

$$H_{\mathscr{A}}(x) = \sum_{j=1}^{n} h_{i_j}(x) a_j^0 + \sum_{j=1}^{n} \bar{h}_{i_j}(x) a_j^1$$

where

$$h_{i_j}(x) = \left(1 - \frac{p_n''(i_j)}{p_n'(i_j)}(x - i_j)\right)(l_{i_j}(x))^2$$

and

$$\bar{h}_{i_j}(x) = (x - i_j)(l_{i_j}(x))^2.$$

Example 7. Let $\mathscr{A} = ((a_1^0, a_1^1), (a_2^0, a_2^1)) \in (\mathbb{Z}^2)^2$ and $I = (i_1, i_2) \in \mathbb{Z}^2$. Then we get

$$\begin{aligned}
&p_{i_2}(x) = (x - i_1)(x - i_2),\\
&p_{i_2}'(x) = (x - i_1) + (x - i_2),\\
&p_{i_2}''(x) = 2,\\
&l_{i_1}(x) = \frac{x - i_2}{i_1 - i_2},\\
&l_{i_2}(x) = \frac{x - i_1}{i_2 - i_1},\\
&h_{i_1}(x) = \left(1 - \frac{p_{i_2}''(i_1)}{p_{i_2}'(i_1)}(x - i_1)\right)(l_{i_1}(x))^2 = \left(1 - \frac{2}{i_1 - i_2}(x - i_1)\right)\left(\frac{x - i_2}{i_1 - i_2}\right)^2,\\
&h_{i_2}(x) = \left(1 - \frac{p_{i_2}''(i_2)}{p_{i_2}'(i_2)}(x - i_2)\right)(l_{i_2}(x))^2 = \left(1 - \frac{2}{i_2 - i_1}(x - i_2)\right)\left(\frac{x - i_1}{i_2 - i_1}\right)^2,\\
&\bar{h}_{i_1}(x) = (x - i_1)(l_{i_1}(x))^2 = (x - i_1)\left(\frac{x - i_2}{i_1 - i_2}\right)^2,\\
&\bar{h}_{i_2}(x) = (x - i_2)(l_{i_2}(x))^2 = (x - i_2)\left(\frac{x - i_1}{i_2 - i_1}\right)^2.
\end{aligned}$$

So the unique Hermite's formula $H_{\bar{a}}(x)$ of degree less than 4 such that

$$\begin{aligned}
H_{\mathscr{A}}(i_1) = a_1^0, \quad H_{\mathscr{A}}(i_2) = a_2^0\\
H_{\mathscr{A}}'(i_1) = a_1^1, \quad H_{\mathscr{A}}'(i_2) = a_2^1,
\end{aligned}$$

is given by

$$\begin{aligned}
H_{\mathscr{A}}(x) &= \sum_{j=1}^{2} h_{i_j}(x) a_j^0 + \sum_{j=1}^{2} \bar{h}_{i_j}(x) a_j^1\\
&= \left(1 - \frac{2}{i_1 - i_2}(x - i_1)\right)\left(\frac{x - i_2}{i_1 - i_2}\right)^2 a_1^0 +\\
&\left(1 - \frac{2}{i_2 - i_1}(x - i_2)\right)\left(\frac{x - i_1}{i_2 - i_1}\right)^2 a_2^0 +\\
&(x - i_1)\left(\frac{x - i_2}{i_1 - i_2}\right)^2 a_1^1 + (x - i_2)\left(\frac{x - i_1}{i_2 - i_1}\right)^2 a_2^1.
\end{aligned}$$

In fact if $I = (1,2)$ then we get

$$H_{\mathscr{A}}(x) = a_1^0(1+2(x-1))(x-2)^2 + a_2^0(x-2)^2(x-1)+$$
$$a_1^1(1-2(x-2))(x-1)^2 + a_2^1(x-2)(x-1)^2.$$

Example 8. Let $\mathscr{A} = ((a_1^0, a_1^1), (a_2^0, a_2^1), (a_3^0, a_3^1)) \in (\mathbb{Z}^2)^3$. Then we get

$$\begin{aligned}
p_{i_3}(x) &= (x-i_1)(x-i_2)(x-i_3),\\
p'_{i_3}(x) &= (x-i_2)(x-i_3) + (x-i_1)(x-i_3) + (x-i_1)(x-i_2),\\
p''_{i_3}(x) &= 2(x-i_1) + 2(x-i_2) + 2(x-i_3),\\
l_{i_1}(x) &= \frac{p_{i_n}(x)}{p'_{i_n}(i_1)(x-i_1)} = \frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)},\\
l_{i_2}(x) &= \frac{p_{i_n}(x)}{p'_{i_n}(i_2)(x-i_2)} = \frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)},\\
l_{i_3}(x) &= \frac{p_{i_n}(x)}{p'_{i_n}(i_3)(x-i_3)} = \frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)},\\
h_{i_1}(x) &= \left(1 - \frac{p''_{i_3}(i_1)}{p'_{i_3}(i_1)}(x-i_1)\right)(l_{i_1}(x))^2\\
&= \left(1 - \frac{2(i_1-i_2)+2(i_1-i_3)}{(i_1-i_2)(i_1-i_3)}(x-i_1)\right)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2,\\
h_{i_2}(x) &= \left(1 - \frac{p''_{i_3}(i_2)}{p'_{i_3}(i_2)}(x-i_2)\right)[l_{i_2}(x)]^2\\
&= \left(1 - \frac{2(i_2-i_1)+2(i_2-i_3)}{(i_2-i_1)(i_2-i_3)}(x-i_2)\right)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2,\\
h_{i_3}(x) &= \left(1 - \frac{p''_{i_3}(i_3)}{p'_{i_3}(i_3)}(x-i_3)\right)(l_{i_3}(x))^2\\
&= \left(1 - \frac{2(i_3-i_1)+2(i_3-i_2)}{(i_3-i_1)(i_3-i_2)}(x-i_3)\right)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2,\\
\bar{h}_{i_1}(x) &= (x-i_1)(l_{i_1}(x))^2 = (x-i_1)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2,\\
\bar{h}_{i_2}(x) &= (x-i_2)(l_{i_2}(x))^2 = (x-i_2)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2\\
\bar{h}_{i_3}(x) &= (x-i_3)(l_{i_3}(x))^2 = (x-i_3)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2.
\end{aligned}$$

So the unique Hermite's formula $H_{\mathscr{A}}(x)$ of degree less than 6 such that

$$H_{\mathscr{A}}(i_1) = a_1^0, \quad H_{\mathscr{A}}(i_2) = a_2^0, \quad H_{\mathscr{A}}(i_3) = a_3^0$$
$$H'_{\mathscr{A}}(i_1) = a_1^1, \quad H'_{\mathscr{A}}(i_2) = a_2^1, \quad H'_{\mathscr{A}}(i_3) = a_3^1,$$

is given by

$$\begin{aligned}
H_{\mathscr{A}}(x) &= \sum_{j=1}^{3} h_{i_j}(x) a_j^0 + \sum_{j=1}^{3} \bar{h}_{i_j}(x) a_j^1 \\
&= \left(1 - \frac{2(i_1-i_2)+2(i_1-i_3)}{(i_1-i_2)(i_1-i_3)}(x-i_1)\right)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2 a_1^0 + \\
&\left(1 - \frac{2(i_2-i_1)+2(i_2-i_3)}{(i_2-i_1)(i_2-i_3)}(x-i_2)\right)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2 a_2^0 + \\
&\left(1 - \frac{2(i_3-i_1)+2(i_3-i_2)}{(i_3-i_1)(i_3-i_2)}(x-i_3)\right)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2 a_3^0 + \\
&(x-i_1)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2 a_1^1 + (x-i_2)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2 a_2^1 + \\
&(x-i_3)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2 a_3^1.
\end{aligned}$$

In fact if $I = (1, 2, 3)$ then we get

$$\begin{aligned}
H_{\mathscr{A}}(x) &= a_1^0\left(\frac{1}{4} + \frac{3}{4}(x-1)\right)(x-3)^2(x-2)^2 + \frac{a_1^1}{4}(x-3)^2(x-2)^2(x-1) \\
&\quad + a_2^0(x-3)^2(x-1)^2 + a_2^1(x-3)^2(x-2)(x-1)^2 \\
&\quad + a_3^0\left(\frac{1}{4} - \frac{3}{4}(x-3)\right)(x-2)^2(x-1)^2 + \frac{a_3^1}{4}(x-3)(x-2)^2(x-1)^2 \\
&= (-18a_1^0 + 9a_2^0 + 10a_3^0 - 9a_1^1 - 18a_2^1 - 3a_3^1) \\
&\quad + (57a_1^0 - 24a_2^0 - 33a_3^0 + 24a_1^1 + 57a_2^1 + 10a_3^1)x \\
&\quad + \left(-\frac{127a_1^0}{2} + 22a_2^0 + \frac{83a_3^0}{2} - \frac{97a_1^1}{4} - 68a_2^1 - \frac{51a_3^1}{4}\right)x^2 \\
&\quad + \left(\frac{131a_1^0}{4} - 8a_2^0 - \frac{99a_3^0}{4} + \frac{47a_1^1}{4} + 38a_2^1 + \frac{31a_3^1}{4}\right)x^3 \\
&\quad + \left(-8a_1^0 + a_2^0 + 7a_3^0 - \frac{11a_1^1}{4} - 10a_2^1 - \frac{9a_3^1}{4}\right)x^4 \\
&\quad + \left(\frac{3a_1^0}{4} - \frac{3a_3^0}{4} + \frac{a_1^1}{4} + a_2^1 + \frac{a_3^1}{4}\right)x^5.
\end{aligned}$$

2.4 A generalization of Hermite's formula

Let D be any integral domains and D_Q be the quotient of D. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Let there be given $i_j, r_j, H_{\mathscr{A}}^{(m)}(i_j)$ for all $j = 1, 2, \ldots, n,\ m = 0, 1, 2, \ldots, r_j$. Let $L_{i_j}(x)$ and $g_{i_j}(x)$ be defined by

$$L_{i_j}(x) = (x-i_1)^{r_1+1}(x-i_2)^{r_2+1}\cdots(x-i_{j-1})^{r_{j-1}+1}(x-i_{j+1})^{r_{j+1}+1}\cdots(x-i_n)^{r_n+1},$$

$$g_{i_j}(x) = \left(L_{i_j}(x)\right)^{-1} = \frac{1}{L_{i_j}(x)}.$$

Then the unique polynomial $H_{\mathscr{A}}(x)$ of degree $< n + \sum_{j=1}^{n} r_j$ such that $H_{\mathscr{A}}^{(m)}(i_j) = a_j^m$ for all $j = 1, \ldots, n; m = 0, 1, \ldots, r_j$, or

$$\begin{array}{ccccc}
H_{\mathscr{A}}(i_1) = a_1^0, & H_{\mathscr{A}}(i_2) = a_2^0, & H_{\mathscr{A}}(i_3) = a_3^0, & \ldots, & H_{\mathscr{A}}(i_n) = a_n^0, \\
H'_{\mathscr{A}}(i_1) = a_1^1, & H'_{\mathscr{A}}(i_2) = a_2^1, & H'_{\mathscr{A}}(i_3) = a_3^1, & \ldots, & H'_{\mathscr{A}}(i_n) = a_n^1, \\
H''_{\mathscr{A}}(i_1) = a_1^2, & H''_{\mathscr{A}}(i_2) = a_2^2, & H''_{\mathscr{A}}(i_3) = a_3^2, & \ldots, & H''_{\mathscr{A}}(i_n) = a_n^2, \\
H^{(3)}_{\mathscr{A}}(i_1) = a_1^3, & H^{(3)}_{\mathscr{A}}(i_2) = a_2^3, & H^{(3)}_{\mathscr{A}}(i_3) = a_3^3, & \ldots, & H^{(3)}_{\mathscr{A}}(i_n) = a_n^3, \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H^{(r_1)}_{\mathscr{A}}(i_1) = a_1^{r_1}, & H^{(r_2)}_{\mathscr{A}}(i_2) = a_2^{r_2}, & H^{(r_3)}_{\mathscr{A}}(i_3) = a_3^{r_3}, & \ldots, & H^{(r_n)}_{\mathscr{A}}(i_n) = a_n^{r_n}
\end{array},$$

is given by

$$H_{\mathscr{A}}(x) = \sum_{j=1}^{n}\sum_{m=0}^{r_j} A_{j,m}(x) H_{\mathscr{A}}^{(m)}(i_j),$$

where

$$A_{j,m}(x) = L_{i_j}(x)\frac{(x-i_j)^m}{m!}\sum_{t=0}^{r_j-m}\frac{1}{t!}g_{i_j}^{(t)}(i_j)(x-i_j)^t,$$

is called the generalization of Hermite's formula [6].

Let us see how this applies in the specific case covered by Hermite's formula. Here we have $r_j = 1$ for $j = 1, 2, \ldots, n$. We find

$$L_{i_j}(x) = (x-i_1)^2(x-i_2)^2\ldots(x-i_{j-1})^2(x-i_{j+1})^2\cdots(x-i_n)^2.$$

Then for $m = 1$ we have $A_{j,1}(x) = L_{i_j}(x)(x - i_j)g_{i_j}(i_j)$. Since $g_{i_j}(i_j) = [L_{i_j}(i_j)]^{-1}$, we get

$$A_{j,1}(x) = (x - i_j)\frac{L_{i_j}(x)}{L_{i_j}(i_j)} = (x - i_j)\{l_{i_j}(x)\}^2 = \bar{h}_{i_j}(x).$$

For $m = 0$ we have

$$\begin{aligned}
A_{j,0}(x) &= L_{i_j}(x)[g_{i_j}(i_j) + g'_{i_j}(i_j)(x - i_j)] \\
&= l_{i_j}(x)\left(\frac{1}{l_{i_j}(i_j)} - \frac{l'_{i_j}(i_j)}{[l_{i_j}(i_j)]^2(x - i_j)}\right) \\
&= \left(1 - \frac{l'_{i_j}(i_j)}{l_{i_j}(i_j)}\right)\{l_{i_j}(x)\}^2 \\
&= \left(1 - \frac{p''_{i_n}(i_j)}{p'_{i_j}(i_j)}\right)\{l_{i_j}(x)\}^2 = h_{i_j}(x).
\end{aligned}$$

So

$$H_{\mathscr{A}}(x) = \sum_{j=1}^{n} h_{i_j}(x)H_{\mathscr{A}}(i_j) + \sum_{j=1}^{n} \bar{h}_{i_j}(x)H'_{\mathscr{A}}(i_j).$$

Hence Hermite's formula is seen to be a special case of the generalization of Hermite's formula.

If $r_j = 0$ for $j = 0, 1, \ldots, n$, then the generalization of Hermite's formula is given by the Lagrange interpolation polynomial $L(x)$ or the Newton interpolation polynomial $N(x)$ where $L(i_1) = N(i_1) = a_1^0, L(i_2) = N(i_2) = a_2^0, \ldots,$ $L(i_n) = N(i_n) = a_n^0$.

If $j = 1$, then the generalization of Hermite's formula is given by the Taylor interpolation polynomial $T(x)$ where $T(i_1) = a_1^0, T'(i_1) = a_1^1, T''(i_1) = a_1^2 \ldots,$ $T^{(r_1)}(i_1) = a_1^{r_1}$.

Example 9. Let

$$\begin{array}{lll}
H_{\mathscr{A}}(i_1) = a_1^0 & H_{\mathscr{A}}(i_2) = a_2^0 & H_{\mathscr{A}}(i_3) = a_3^0 \\
H'_{\mathscr{A}}(i_1) = a_1^1 & H'_{\mathscr{A}}(i_2) = a_2^1 & H'_{\mathscr{A}}(i_3) = a_3^1
\end{array}.$$

Since $r_j = 1$ for $j = 1, 2, 3$, we get

$$
\begin{aligned}
L_{i_1}(x) &= (x-i_2)^2(x-i_3)^2, \\
L_{i_2}(x) &= (x-i_1)^2(x-i_3)^2, \\
L_{i_3}(x) &= (x-i_1)^2(x-i_2)^2, \\
g_{i_1}(x) &= \frac{1}{L_{i_1}(x)} = \frac{1}{(x-i_2)^2(x-i_3)^2}, \\
g'_{i_1}(x) &= -\frac{2}{(x-i_2)^3(x-i_3)^2} - \frac{2}{(x-i_2)^2(x-i_3)^3}, \\
g_{i_2}(x) &= \frac{1}{L_{i_2}(x)} = \frac{1}{(x-i_1)^2(x-i_3)^2}, \\
g'_{i_2}(x) &= -\frac{2}{(x-i_1)^3(x-i_3)^2} - \frac{2}{(x-i_1)^2(x-i_3)^3}, \\
g_{i_3}(x) &= \frac{1}{L_{i_3}(x)} = \frac{1}{(x-i_1)^2(x-i_2)^2}, \\
g'_{i_3}(x) &= -\frac{2}{(x-i_1)^3(x-i_2)^2} - \frac{2}{(x-i_1)^2(x-i_2)^3}, \\
A_{1,0}(x) &= L_{i_1}(x)\frac{(x-i_1)^0}{0!}(g_{i_1}(i_1) + g'_{i_1}(i_1)(x-i_1)) = h_{i_1}(x), \\
A_{1,1}(x) &= L_{i_1}(x)\frac{(x-i_1)}{1!}(g_{i_1}(i_1)) = \bar{h}_{i_1}(x), \\
A_{2,0}(x) &= L_{i_2}(x)\frac{(x-i_2)}{0!}(g_{i_2}(i_2) + g'_{i_2}(i_1)(x-i_2)) = h_{i_2}(x), \\
A_{2,1}(x) &= L_{i_2}(x)\frac{(x-i_2)}{1!}(g_{i_2}(i_2)) = \bar{h}_{i_2}(x), \\
A_{3,0}(x) &= L_{i_3}(x)\frac{(x-i_3)}{0!}(g_{i_3}(i_3) + g'_{i_3}(i_3)(x-i_3)) = h_{i_3}(x), \\
A_{3,1}(x) &= L_{i_3}(x)\frac{(x-i_3)}{1!}(g_{i_3}(i_3)) = \bar{h}_{i_3}(x).
\end{aligned}
$$

Then the Hermite's formula is

$$
\begin{aligned}
H_{\mathscr{A}}(x) &= A_{1,0}(x)a_1^0 + A_{2,0}(x)a_2^0 + A_{3,0}(x)a_3^0 + A_{1,1}(x)a_1^1 + A_{2,1}(x)a_2^1 + A_{3,1}(x)a_3^1 \\
&= \left(1 - \frac{2(i_1-i_2)+2(i_1-i_3)}{(i_1-i_2)(i_1-i_3)}(x-i_1)\right)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2 a_1^0 + \\
&\quad \left(1 - \frac{2(i_2-i_1)+2(i_2-i_3)}{(i_2-i_1)(i_2-i_3)}(x-i_2)\right)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2 a_2^0 + \\
&\quad \left(1 - \frac{2(i_3-i_1)+2(i_3-i_2)}{(i_3-i_1)(i_3-i_2)}(x-i_3)\right)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2 a_3^0 +
\end{aligned}
$$

$$(x-i_1)\left(\frac{(x-i_2)(x-i_3)}{(i_1-i_2)(i_1-i_3)}\right)^2 a_1^1+(x-i_2)\left(\frac{(x-i_1)(x-i_3)}{(i_2-i_1)(i_2-i_3)}\right)^2 a_2^1+$$
$$(x-i_3)\left(\frac{(x-i_1)(x-i_2)}{(i_3-i_1)(i_3-i_2)}\right)^2 a_3^1.$$

Example 10. Let $\mathscr{A}=((a_1^0,a_1^1,a_1^2),(a_2^0,a_2^1,a_2^2))\in(\mathbb{Z}^3)^2$ and $I=(i_1,i_2)$ in $\mathbb{Z}^2$. We will find the unique polynomial $H_{\mathscr{A}}(x)$ of degree less than 6 such that

$$H_{\mathscr{A}}(i_1)=a_1^0,\quad H_{\mathscr{A}}(i_2)=a_2^0,$$
$$H'_{\mathscr{A}}(i_1)=a_1^1,\quad H'_{\mathscr{A}}(i_2)=a_2^1,$$
$$H''_{\mathscr{A}}(i_1)=a_1^2,\quad H''_{\mathscr{A}}(i_2)=a_2^2$$

by using the generalization of Hermite's formula. Since $r_j=2$ for $j=1,2$, we get

$$L_{i_1}(x)=(x-i_2)^3,\ L_{i_2}(x)=(x-i_1)^3,$$
$$g_{i_1}(x)=\frac{1}{L_{i_1}(x)}=\frac{1}{(x-i_2)^3},\ g'_{i_1}(x)=-\frac{3}{(x-i_2)^4},\ g''_{i_1}(x)=\frac{12}{(x-i_2)^5},$$
$$g_{i_2}(x)=\frac{1}{L_{i_2}(x)}=\frac{1}{(x-i_1)^3},\ g'_{i_2}(x)=-\frac{3}{(x-i_1)^4},\ g''_{i_2}(x)=\frac{12}{(x-i_1)^5},$$
$$A_{1,0}(x)=L_{i_1}(x)\frac{(x-i_1)^0}{0!}\sum_{t=0}^{2-0}\frac{1}{t!}g_{i_1}^{(t)}(i_1)(x-i_1)^t$$
$$=(x-i_2)^3\left(\frac{1}{(i_1-i_2)^3}-\frac{3}{(i_1-i_2)^4}(x-i_1)+\frac{12}{(i_1-i_2)}(x-i_1)^2\right),$$
$$A_{1,1}(x)=L_{i_1}(x)\frac{(x-i_1)^1}{1!}\sum_{t=0}^{2-1}\frac{1}{t!}g_{i_1}^{(t)}(i_1)(x-i_1)^t$$
$$=(x-i_2)^3(x-i_1)\left(\frac{1}{(i_1-i_2)^3}-\frac{3}{(i_1-i_2)^4}(x-i_1)\right),$$
$$A_{1,2}(x)=L_{i_1}(x)\frac{(x-i_1)^2}{2!}\sum_{t=0}^{2-2}\frac{1}{t!}g_{i_1}^{(t)}(i_1)(x-i_1)^t$$
$$=(x-i_2)^3\frac{(x-i_1)^2}{2}\left(\frac{1}{(i_1-i_2)^3}\right),$$
$$A_{2,0}(x)=L_{i_2}(x)\frac{(x-i_2)^0}{0!}\sum_{t=0}^{2-0}\frac{1}{t!}g_{i_2}^{(t)}(i_2)(x-i_2)^t$$
$$=(x-i_1)^3\left(\frac{1}{(i_2-i_1)^3}-\frac{3}{(i_2-i_1)^4}(x-i_2)+\frac{12}{(i_2-i_1)}(x-i_2)^2\right),$$
$$A_{2,1}(x)=L_{i_2}(x)\frac{(x-i_2)^1}{1!}\sum_{t=0}^{2-1}\frac{1}{t!}g_{i_2}^{(t)}(i_2)(x-i_2)^t$$

$$= (x-i_1)^3(x-i_2)\left(\frac{1}{(i_2-i_1)^3} - \frac{3}{(i_2-i_1)^4}(x-i_2)\right),$$

$$A_{2,2}(x) = L_{i_2}(x)\frac{(x-i_2)^2}{2!}\sum_{t=0}^{2-2}\frac{1}{t!}g_{i_2}^{(t)}(i_2)(x-i_2)^t$$

$$= (x-i_1)^3\frac{(x-i_2)^2}{2}\left(\frac{1}{(i_2-i_1)^3}\right).$$

Thus

$$\begin{aligned} H_{\mathscr{A}}(x) &= \sum_{j=1}^{2}\sum_{m=0}^{2} A_{j,m}(x)H_{\mathscr{A}}^{(m)}(i_j) \\ &= (x-i_2)^3\left(\frac{1}{(i_1-i_2)^3} - \frac{3}{(i_1-i_2)^4}(x-i_1) + \frac{12}{(i_1-i_2)}(x-i_1)^2\right)a_1^0 + \\ &\quad (x-i_1)^3\left(\frac{1}{(i_2-i_1)^3} - \frac{3}{(i_2-i_1)^4}(x-i_2) + \frac{12}{(i_2-i_1)}(x-i_2)^2\right)a_2^0 + \\ &\quad (x-i_2)^3(x-i_1)\left(\frac{1}{(i_1-i_2)^3} - \frac{3}{(i_1-i_2)^4}(x-i_1)\right)a_1^1 + \\ &\quad (x-i_1)^3(x-i_2)\left(\frac{1}{(i_2-i_1)^3} - \frac{3}{(i_2-i_1)^4}(x-i_2)\right)a_2^1 + \\ &\quad (x-i_2)^3\frac{(x-i_1)^2}{2}\left(\frac{1}{(i_1-i_2)^3}\right)a_1^2 + (x-i_1)^3\frac{(x-i_2)^2}{2}\left(\frac{1}{(i_2-i_1)^3}\right)a_2^2. \end{aligned}$$

In fact if $I = (1, 2)$, then we get

$$\begin{aligned} H_{\mathscr{A}}(x) = {} & a_1^0\left(-1 - 3(-1+x) - 6(-1+x)^2\right)(-2+x)^3 \\ & + a_2^0\left(1 - 3(-2+x) + 6(-2+x)^2\right)(-1+x)^3 \\ & + a_1^1\left(-1 - 3(-1+x)\right)(-2+x)^3(-1+x) \\ & + a_2^1\left(1 - 3(-2+x)\right)(-2+x)(-1+x)^3 \\ & - \frac{1}{2}a_1^2(-2+x)^3(-1+x)^2 \\ & + \frac{1}{2}a_2^2(-2+x)^2(-1+x)^3 \\ = {} & (32a_1^0 - 31a_2^0 + 16a_1^1 + 14a_2^1 + 4a_1^2 - 2a_2^2) \\ & + (-120a_1^0 + 120a_2^0 - 64a_1^1 - 55a_2^1 - 14a_1^2 + 8a_2^2)x \\ & + \left(180a_1^0 - 180a_2^0 + 96a_1^1 + 84a_2^1 + 19a_1^2 - \frac{25a_2^2}{2}\right)x^2 \\ & + \left(-130a_1^0 + 130a_2^0 - 68a_1^1 - 62a_2^1 - \frac{25a_1^2}{2} + \frac{19a_2^2}{2}\right)x^3 \end{aligned}$$

$$+\left(45a_1^0 - 45a_2^0 + 23a_1^1 + 22a_2^1 + 4a_1^2 - \frac{7a_2^2}{2}\right)x^4$$
$$+\left(-6a_1^0 + 6a_2^0 - 3a_1^1 - 3a_2^1 - \frac{a_1^2}{2} + \frac{a_2^2}{2}\right)x^5.$$

2.5 Newton form of generalization of Hermite's formula

Let D be an integral domain and D_Q be the quotient field of D. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D and $I = (i_1, i_2, \ldots, i_n) \in D^n$ such that $i_j \neq i_h$ if $j \neq h$. We define polynomials $p_{i_j}^q(x)$ for $0 \leq j \leq n$ and $0 \leq q \leq r_j$ with respect to I as follows

$$\begin{aligned}
&p_{i_0}(x) = 1\\
&p_{i_1}(x) = (x - i_1)\\
&p_{i_1}^2(x) = (x - i_1)^2\\
&\vdots\\
&p_{i_1}^{r_1+1}(x) = (x - i_1)^{r_1+1}\\
&p_{i_2}(x) = (x - i_1)^{r_1+1}(x - i_2)\\
&p_{i_2}^2(x) = (x - i_1)^{r_1+1}(x - i_2)^2\\
&\vdots\\
&p_{i_2}^{r_2+1}(x) = (x - i_1)^{r_1+1}(x - i_2)^{r_2+1}\\
&\vdots\\
&p_{i_n}^{r_n}(x) = (x - i_1)^{r_1+1}(x - i_2)^{r_2+1} \cdots (x - i_{n-1})^{r_{n-1}+1}(x - i_n)^{r_n}.
\end{aligned}$$

Thus

$$p_{i_j}^q(x) = \prod_{h=1}^{j-1}(x - i_h)^{r_h+1}(x - i_j)^q, \qquad 1 \leq j \leq n, 1 \leq q \leq r_j + 1.$$

The Newton form of generalization Hermite's formula [6] corresponding to the points (i_j, a_k^m) $(j = 1, \ldots, n, m = 0, 1, \ldots, r_j)$ is defined as

$$\mathscr{N}_{\mathscr{A}}(x) = [i_1] + [i_1, i_1]p_{i_1}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2]p_{i_1}^{r_1+1}(x) +$$

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2, i_2]p_{i_2}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]p_{i_n}^{r_n}(x)$$

where

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_j, \ldots, i_j}_{r_j+1}] = \sum_{k=1}^{j} \sum_{m=0}^{r_k} \frac{1}{m!} \frac{1}{(r_k - m)!} g_{i_k}^{(r_k - m)}(i_k) a_k^m$$

for $1 \leq j \leq n$ and

$$g_{i_k}(x) = \frac{1}{(x - i_1)^{r_1+1} \cdots (x - i_{k-1})^{r_{k-1}+1} (x - i_{k+1})^{r_{k+1}+1} \cdots (x - i_j)^{r_j+1}}$$

for $1 \leq k \leq j$. The elements

$$1, p_{i_1}(x) := (x - i_1), p_{i_1}^2(x) := (x - i_1)^2, \ldots, p_{i_n}^{r_n}(x) := \prod_{j=1}^{n-1} (x - i_j)^{r_j+1} (x - i_n)^{r_n}$$

are the basis polynomials of the Newton form of the generalization of Hermite's formula. So $\mathscr{N}_{\mathscr{A}}(x)$ is the polynomial of degree $< n + \sum_{j=1}^{n} r_j$ such that

$$\mathscr{N}_{\mathscr{A}}^{(m)}(i_j) = a_j^m$$

for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$. $\mathscr{N}_{\mathscr{A}}^{(m)}(i_j)$ denotes the m^{th} derivative of $\mathscr{N}_{\mathscr{A}}(x)$ at a point i_j.

If $r_j = 0$ for $j = 0, 1, \ldots, n$, then the generalization of Hermite's formula is given by the Newton interpolation polynomial

$$N(x) = b_0 + b_1(x - i_1) + \cdots + b_{n-1}(x - i_1)(x - i_2) \cdots (x - i_{n-1})$$

where

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}^0}{\prod_{m=1, m \neq j+1}^{k+1} (i_{j+1} - i_m)} \qquad (0 \leq k \leq n-1)$$

such that $N(i_1) = a_1^0, N(i_2) = a_2^0, \ldots, N(i_n) = a_n^0$.

If $j = 1$ and $r_1 = k$, then the generalization of Hermite's formula is given by the Taylor interpolation polynomial

$$T(x) = m_0 + m_1(x - i_1) + m_2(x - i_1)^2 + \cdots + m_k(x - i_1)^k$$

where

$$m_j = \frac{a_1^j}{j!} \qquad (0 \leq j \leq k)$$

such that $T(i_1) = a_1^0, T'(i_1) = a_1^1, T''(i_1) = a_1^2 \ldots, T^{(k)}(i_1) = a_1^k$.

Example 11. Let $\mathscr{A} = ((a_1^0, a_1^1), (a_2^0, a_2^1), (a_3^0, a_3^1)) \in (D^2)^3$ and $I = (i_1, i_2, i_3)$ in D^3. Then the Newton form of Hermite's formula $\mathscr{N}_{\mathscr{A}}(x)$ such that

$$\mathscr{N}_{\mathscr{A}}(i_1) = a_1^0, \quad \mathscr{N}_{\mathscr{A}}(i_2) = a_2^0, \quad \mathscr{N}_{\mathscr{A}}(i_3) = a_3^0,$$
$$\mathscr{N}'_{\mathscr{A}}(i_1) = a_1^1, \quad \mathscr{N}'_{\mathscr{A}}(i_2) = a_2^1, \quad \mathscr{N}'_{\mathscr{A}}(i_3) = a_3^1,$$

is

$$\mathscr{N}_{\mathscr{A}}(x) = [i_1] + [i_1, i_1](x - i_1) + [i_1, i_1, i_2](x - i_1)^2 + [i_1, i_1, i_2, i_2](x - i_1)^2(x - i_2) +$$
$$[i_1, i_1, i_2, i_2, i_3](x - i_1)^2(x - i_2)^2 + [i_1, i_1, i_2, i_2, i_3, i_3](x - i_1)^2(x - i_2)^2(x - i_3).$$

All the coefficients can be obtained as follows.
$[i_1] = \mathscr{N}_{\mathscr{A}}(i_1) = a_1^0$.
$[i_1, i_1] = N'_{\mathscr{A}}(i_1) = a_1^1$.
$[i_1, i_1, i_2] = \sum_{j=1}^{2} \sum_{m=0}^{r_j} \frac{1}{m!} \frac{1}{(r_j-m)!} g_{i_j}^{(r_j-m)}(i_j) a_j^m$ where $g_{i_1}(x) = \frac{1}{x-i_2}$, $g_{i_2}(x) = \frac{1}{(x-i_1)^2}$. Thus $[i_1, i_1, i_2] = -\frac{a_1^0}{(i_1-i_2)^2} + \frac{a_1^1}{(i_1-i_2)} + \frac{a_2^0}{(i_2-i_1)^2}$.
$[i_1, i_1, i_2, i_2] = \sum_{j=1}^{2} \sum_{m=0}^{r_j} \frac{1}{m!} \frac{1}{(r_j-m)!} g_{i_j}^{(r_j-m)}(i_j) a_j^m$ where $g_{i_1}(x) = \frac{1}{(x-i_2)^2}$, $g_{i_2}(x) = \frac{1}{(x-i_1)^2}$. Thus $[i_1, i_1, i_2, i_2] = -\frac{2a_1^0}{(i_1-i_2)^3} + \frac{a_1^1}{(i_1-i_2)^2} - \frac{2a_2^0}{(i_2-i_1)^3} + \frac{a_2^1}{(i_2-i_1)^2}$.
$[i_1, i_1, i_2, i_2, i_3] = \sum_{j=1}^{3} \sum_{m=0}^{r_j} \frac{1}{m!} \frac{1}{(r_j-m)!} g_{i_j}^{(r_j-m)}(i_j) a_j^m$ where $g_{i_1}(x) = \frac{1}{(x-i_2)^2(x-i_3)}$, $g_{i_2}(x) = \frac{1}{(x-i_1)^2(x-i_3)}$, $g_{i_3}(x) = \frac{1}{(x-i_1)^2(x-i_2)^2}$. Thus

$$[i_1, i_1, i_2, i_2, i_3] = -\frac{2a_1^0}{(i_1 - i_2)^3(i_1 - i_3)} - \frac{a_1^0}{(i_1 - i_2)^2(i_1 - i_3)^2} + \frac{a_1^1}{(i_1 - i_2)^2(i_1 - i_3)} -$$
$$\frac{2a_2^0}{(i_2 - i_1)^3(i_2 - i_3)} - \frac{a_2^0}{(i_2 - i_1)^2(i_2 - i_3)^2} + \frac{a_2^1}{(i_2 - i_1)^2(i_2 - i_3)} +$$
$$\frac{a_3^0}{(i_3 - i_1)^2(i_3 - i_2)^2}$$

$[i_1, i_1, i_2, i_2, i_3, i_3] = \sum_{j=1}^{3} \sum_{m=0}^{r_j} \frac{1}{m!} \frac{1}{(r_j - m)!} g_{i_j}^{(r_j - m)}(i_j) a_j^m$ where $g_{i_1}(x) = \frac{1}{(x-i_2)^2(x-i_3)^2}$, $g_{i_2}(x) = \frac{1}{(x-i_1)^2(x-i_3)^2}$, $g_{i_3}(x) = \frac{1}{(x-i_1)^2(x-i_2)^2}$. Thus

$$\begin{aligned}
[i_1, i_1, i_2, i_2, i_3, i_3] = & -\frac{2a_1^0}{(i_1 - i_2)^3(i_1 - i_3)^2} - \frac{2a_1^0}{(i_1 - i_2)^2(i_1 - i_3)^3} + \\
& -\frac{a_1^1}{(i_1 - i_2)^2(i_1 - i_3)^2} \frac{2a_2^0}{(i_2 - i_1)^3(i_2 - i_3)^2} - \frac{2a_2^0}{(i_2 - i_1)^2(i_2 - i_3)^3} + \\
& \frac{a_2^1}{(i_2 - i_1)^2(i_2 - i_3)^2} - \frac{2a_3^0}{(i_3 - i_1)^3(i_3 - i_2)^2} - \frac{2a_3^0}{(i_3 - i_1)^2(i_3 - i_2)^3} + \\
& \frac{a_3^1}{(i_3 - i_1)^2(i_3 - i_2)^2}.
\end{aligned}$$

In fact if $I = (1, 2, 3)$, then we get

$$\begin{aligned}
[1] &= a_1^0, \\
[1, 1] &= a_1^1, \\
[1, 1, 2] &= -\frac{a_1^0}{(-1)^2} + \frac{a_1^1}{(-1)} + \frac{a_2^0}{(1)^2} = -a_1^0 - a_1^1 + a_2^0, \\
[1, 1, 2, 2] &= -\frac{2a_1^0}{(-1)^3} + \frac{a_1^1}{(-1)^2} - \frac{2a_2^0}{(1)^3} + \frac{a_2^1}{(1)^2} = 2a_1^0 + a_1^1 - 2a_2^0 + a_2^1, \\
[1, 1, 2, 2, 3] &= -\frac{2a_1^0}{(-1)^3(-2)} - \frac{a_1^0}{(-1)^2(-2)^2} + \frac{a_1^1}{(-1)^2(-2)} - \\
& \frac{2a_2^0}{(1)^3(-1)} - \frac{a_2^0}{(1)^2(-1)^2} + \frac{a_2^1}{(1)^2(-1)} + \frac{a_3^0}{(2)^2(1)^2} \\
&= -\frac{5a_1^0}{4} - \frac{a_1^1}{2} + a_2^0 - a_2^1 + \frac{a_3^0}{4}, \\
[1, 1, 2, 2, 3, 3] &= -\frac{2a_1^0}{(-1)^3(-2)^2} - \frac{2a_1^0}{(-1)^2(-2)^3} + \frac{a_1^1}{(-1)^2(-2)^2} - \frac{2a_2^0}{(1)^3(-1)^2} - \\
& \frac{2a_2^0}{(1)^2(-1)^3} + \frac{a_2^1}{(1)^2(-1)^2} - \frac{2a_3^0}{(2)^3(1)^2} - \frac{2a_3^0}{(2)^2(1)^3} + \frac{a_3^1}{(2)^2(1)^2} \\
&= \frac{3a_1^0}{4} + \frac{a_1^1}{4} + a_2^1 - \frac{3a_3^0}{4} + \frac{a_3^1}{4}.
\end{aligned}$$

2.6 Background in Algebra

In this section, we will review definitions and theorem about group that we will use later.

Definition 2.1. [4, page 16-21] A group is an ordered pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G satisfying the following axioms:

1. $(a \star b) \star c = a \star (b \star c)$, for all $a, b, c \in G$, i.e., $\star$ is associative,
2. there exists an element e in G, called an identity of G, such that for all $a \in G$ we have $a \star e = e \star a = a$,
3. for each $a \in G$ there is an element a^{-1} of G, called an inverse of a, such that $a \star a^{-1} = a^{-1} = e$.

Note: The group G is called abelian (or commutative) if $a \star b = b \star a$ for all $a, b \in G$.

Definition 2.2. [4]**(Subgroup)**
Let G be a group. The subset H of G is a subgroup of G if H is nonempty and H is closed under products and inverses (i.e., $x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$). If H is a subgroup of G we shall write $H \leq G$.

Definition 2.3. [4]**(Normal Subgroup)**
Let N be subgroup of the group G. Then N is called a normal subgroup of G if $gNg^{-1} = N$ for all $g \in G$ and we write $N \trianglelefteq G$.

Now we review the Fundamental Theorem of Finite Generated Abelian Group and example of it as follow:

Definition 2.4. [4, page 158] Let $A = \{a_1, a_2, \ldots, a_n\}$. Define

$$\langle A \rangle = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} | a_i \in A, \alpha_i \in \mathbb{Z}\}.$$

Then we have

1. A group G is *finitely generated* if there is a finite set A of G such that $G = \langle A \rangle$.
2. For each $r \in \mathbb{Z}$ with $r \leq 0$, let $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the direct product of r copies of the group $\mathbb{Z}$, where $\mathbb{Z}^0 = 1$. The group $\mathbb{Z}^r$ is called *the free abelian group of rank* r.

Note that any finite group G is, a fortiori, finitely generated: simply take $A = G$ as a set of generators. Also, $\mathbb{Z}^r$ is finitely generated by $e_1, e_2, \ldots, e_r$, where e_i is the n- tuple with 1 in position i and zeros elsewhere. We can state the fundamental classification theorem for (finitely generated) abelian groups.

Example 12. Let $G = \{(m, n)|m, n \in \mathbb{Z}\}$. Then $A = \{(1, 0), (0, 1)\}$ is a basis of G. Denote $e_1 = (1, 0), e_2 = (0, 1)$. Then all $(m, n) \in G$, we can write as $(m, n) = me_1 + ne_2$. Thus G is a free abelian group.
If G is a finite generated, then the cardinal of G is finite $|G| = n$ and $G \cong \mathbb{Z}^n$.

Theorem 2.1. [4, page 158] (***Fundamental Theorem of Finitely Generated Free Abelian Groups***) *Let G be a finitely generated abelian group. Then*

(1) $G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \mathbb{Z}_{n_s}$, *for some integers* $r, n_1, n_2, \ldots, n_s$ *satisfying the following conditions:*

(a) $r > 0$ *and* $n_j \leq 2$ *for all* j, *and*

(b) $n_{i+1}|n_i$ *for* $1 \leq i \leq s - 1$

(2) the expression in (1) *is unique: if* $G \cong \mathbb{Z}^t \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_u}$, *where* t *and* $m_1, m_2, \ldots, m_u$ *satisfy* (a) *and* (b) *(i.e.,* $t \geq 0, m_j \geq 2$ *for all* j *and* $m_{i+1}|m_i$ *for* $1 \leq i \leq u - 1$*), then* $t = r, u = s$ *and* $m_i = n_i$.

We now give a definition of a $\mathbb{Z}$-basis [7].
If G is finitely generated as a $\mathbb{Z}-$module, so that there exist $g_1, g_2, g_3, \ldots, g_n \in G$ such that every $g \in G$ is a sum

$$g = m_1 g_1 + \cdots + m_n g_n, \ (m_i \in \mathbb{Z})$$

then G is called a finitely generated abelian group.

Generalizing the notation of linear independence in a vector space, we say that elements $g_1, \ldots, g_n$ in an abelian group G are linearly independent over $\mathbb{Z}$ if any equation

$$m_1 g_1 + \cdots + m_n g_n = 0$$

with $m_1, \ldots, m_n \in \mathbb{Z}$ implies $m_1 = \cdots = m_n = 0$. A linearly independent set which generates G is called a $\mathbb{Z}-$basis. If $\{g_1, \ldots, g_n\}$ is a basis, then every $g \in G$ has a unique representation:

$$g = m_1 g_1 + \cdots + m_n g_n \; (m_i = \mathbb{Z})$$

because an alternative expression

$$g = k_1 g_1 + \cdots + k_n g_n \; (k_i \in \mathbb{Z})$$

implies

$$(m_1 - k_1) g_1 + \cdots + (m_n - k_n) g_n = 0$$

and linearly independence implies $m_i = k_i, (1 \leq i \leq n)$.

Definition 2.5. [7]**(Unimodular)** A square matrix over $\mathbb{Z}$ with determinant ± 1 is said to be unimodular.

Theorem 2.2. [7] *Let G be a free abelian group of rank n with basis $\{x_1, \ldots, x_n\}$. Suppose (a_{ij}) is an $n \times n$ matrix with integer entries. Then the elements*

$$y_i = \sum_j a_{ij} x_j$$

form a basis of G if and only if (a_{ij}) is unimodular.

Theorem 2.3. [4]**(Fundamental Theorem of Finitely Generated Abelian Groups)** *Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form*

1). $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}} \times \mathbb{Z} \times \ldots \mathbb{Z}$, where the p_i are primes, not necessarily distinct, and also in the form

2). $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, where m_i divides m_{i+1}.

*In both cases the direct product is unique up to order of the factors, i.e., the number (**betti number of** G) of factors of $\mathbb{Z}$ is unique, the torsion coefficients m_i of G are unique, and the prime powers $(p_i)^{r_i}$ are unique.*
*By [1], this expression is called the **Smith Normal Form of** G. The sequence $(m_1, m_2, \ldots, m_r)$ is called the **Smith Invariant of the group** G and the order of G is $|G| = m_1 m_2 \cdots m_r$. Moreover, m_1 is the exponent of G; that is , the least positive integer n which $na = 0$ for all $a \in G$.*

Theorem 2.4. [4]**(The first fundamental homomorphism theorem)**
Let ϕ be a homomorphism of a group G into a group G' with kernel K. Then $\phi(G)$ is a group, and there is a canonical (natural) isomorphism of $\phi(G)$ with G/K.

Theorem 2.5. [4]
Let B_1 be a normal subgroup of the group of the group A_1 and let B_2 be a normal subgroup of the group of the group A_2. Then $(B_1 \times B_2) \trianglelefteq (A_1 \times A_2)$ and $\frac{A_1 \times A_2}{B_1 \times B_2} \cong \frac{A_1}{B_1} \times \frac{A_2}{B_2}$.

It is not hard to see that if B_i is a normal subgroup of A_i for all $1 \leq i \leq n$ then

$$\frac{A_1 \times A_2 \times \cdots \times A_n}{B_1 \times B_2 \times \cdots \times B_n} \cong \frac{A_1}{B_1} \times \frac{A_2}{B_2} \times \cdots \times \frac{A_n}{B_n}.$$

We next recall the division algorithm which we use in many parts of our work.

Theorem 2.6. [4]**(Division Algorithm)** *Let F be a field and $F[x]$ be ring of polynomials with coefficients in F. Let*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$$

be two elements of $F[x]$, with a_n and b_m both nonzero elements of F and $m > 0$. Then there are unique polynomial $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$, with the degree of $r(x)$ less than m =degree $g(x)$.

CHAPTER 3

Polynomial Sequences over Integral Domain

Let D be an integral domain. For sequences $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $I = (i_1, i_2, \ldots, i_n)$ in D^n with distinct i_j, call $\bar{a}$ a (D^n, I)-polynomial sequence if there exists $f(x)$ in $D[x]$ such that $f(i_j) = a_j$ $(j = 1, \ldots, n)$. Criteria for a sequence to be a (D^n, I)-polynomial sequence are established, and explicit structures of $D^n/P_{n,I}$ are determined.

3.1 Introduction

For a fixed $n \in \mathbb{N}$, by a polynomial sequence (of length n), we mean a sequence $\bar{a} := (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$ for which there exists $f(x) \in \mathbb{Z}[x]$ such that $f(i) = a_i$ for all $i = 1, 2, \ldots, n$; we refer to $f(x)$ as a polynomial which generates the sequence $\bar{a}$. Denote by P_n the set of all polynomial sequences. E. F. Cornelius Jr. and P. Schultz [1] characterized P_n using Lagrange and (implicitly) Newton interpolation polynomials, and determined the structure of $\mathbb{Z}^n/P_n$.

The main objectives in this chapter are first to extend the characterization of E. F. Cornelius Jr. and P. Schultz from $\mathbb{Z}$ to an integral domain D, and second, to determine their corresponding structure.

Definition 3.1. Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and $\overline{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Let

$$P_{n,I} = \{\overline{a} \in D^n \mid \text{there exists } f(x) \in D[x] \text{ such that } f(i_j) = a_j, \text{ for all } 1 \leq j \leq n\}$$

be the set of all (D^n, I)-polynomial sequences. We call $\overline{a}$ an element in $P_{n,I}$, a polynomial sequence over D with respect to I or a (D^n, I)-polynomial sequence.

If $I = (1, 2, 3, \ldots, n)$, then we write P_n for $P_{n,I}$ and call an element in P_n, a polynomial sequence.

The set $P_{n,I}$ is a group under addition. We will show this as follows:
For any $\overline{a} = (a_1, a_2, \ldots, a_n), \overline{b} = (b_1, b_2, \ldots, b_n) \in D^n$ we define

$$\overline{a} + \overline{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

It is easy to see that $\overline{a} + \overline{b} = \overline{b} + \overline{a}$.

Theorem 3.1. *$P_{n,I}$ is a group under addition.*

Proof. Let $\overline{a}, \overline{b}, \overline{c} \in P_{n,I}$ where

$$\overline{a} = (a_1, a_2, \ldots, a_n),$$
$$\overline{b} = (b_1, b_2, \ldots, b_n),$$
$$\overline{c} = (c_1, c_2, \ldots, c_n)$$

such that there exist $f(x), g(x), h(x) \in D[x]$ such that

$$f(i_j) = a_j, g(i_j) = b_j, h(i_j) = c_j$$

for all $1 \leq j \leq n$. We next show that $P_{n,I}$ is a group under addition.

1. For all $\overline{a}, \overline{b} \in P_{n,I}$, we get

$$\begin{aligned} \overline{a} + \overline{b} &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\ &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\ &= (f(i_1) + g(i_1), f(i_2) + g(i_2), \ldots, f(i_n) + g(i_n)) \\ &= ((f + g)(i_1), (f + g)(i_2), \ldots, (f + g)(i_n)) \end{aligned}$$

   Since $f(x), g(x) \in D[x]$, $(f + g)(x) \in D[x]$.
   Hence $\overline{a} + \overline{b} \in P_{n,I}$. Thus $(P_{n,I}, +)$ is closed under addition.

2. Since $\overline{0} = (0, 0, \ldots, 0) = (f(i_1), f(i_2), \ldots, f(i_n))$, there exists $f(x) = 0$ in $D[x]$ such that $f(i_j) = 0$ for all $1 \leq j \leq n$. so $\overline{0} \in P_{n,I}$. Moreover

$$\overline{a} + \overline{0} = (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0)$$

$$= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) = (a_1, a_2, \ldots, a_n)$$
$$= (0 + a_1, 0 + a_2, \ldots, 0 + a_n) = \overline{0} + \overline{a}.$$

So $\overline{0}$ is the identity.

3. For all $\overline{a} \in P_{n,I}$, there exists $-\overline{a} = (-a_1, -a_2, \ldots, -a_n) \in P_{n,I}$, a polynomial sequence generated by $-f(x)$ such that

$$\overline{a} + (-\overline{a}) = \overline{0} = -\overline{a} + \overline{a}.$$

Thus $-\overline{a}$ is the inverse of $\overline{a}$.

4. It is easy to see that the associative law holds.

Therefore $P_{n,I}$ is a group under addition. □

Lemma 3.2. *If $\overline{a} \in P_{n,I}$ then $c\overline{a} \in P_{n,I}$ for any $c \in D$.*

Proof. Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in P_{n,I}$. Then there exists $f(x) \in D[x]$ such that $f(i_j) = a_j$ for all $1 \leq j \leq n$. Then

$$c \cdot \overline{a} = (c \cdot a_1, c \cdot a_2, \ldots, c \cdot a_n)$$
$$= (c \cdot f(i_1), c \cdot f(i_2), \ldots, c \cdot f(i_n))$$

So $g(x) = c \cdot f(x) \in D[x]$ generates $c \cdot \overline{a}$. Thus $c \cdot \overline{a} \in P_{n,I}$. □

Now define a multiplication on a set $P_{n,I}$ as follows:

$$\overline{a} \cdot \overline{b} = (a_1 b_1, a_2 b_2, a_3 b_3, \ldots, a_n b_n).$$

Lemma 3.3. *If $\overline{a}, \overline{b} \in P_{n,I}$ then $\overline{a} \cdot \overline{b} \in P_{n,I}$.*

Proof. Let $\overline{a}, \overline{b} \in P_{n,I}$ and write $\overline{a} = (a_1, a_2, \ldots, a_n),\ \overline{b} = (b_1, b_2, \ldots, b_n)$.
Then there exist $f(x), g(x) \in D[x]$ such that $f(i_j) = a_j \text{ and } g(i_j) = b_j$ for all $1 \leq j \leq n$. Then

$$\overline{a} \cdot \overline{b} = (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n)$$

$$= (f(i_1) \cdot g(i_1), f(i_2) \cdot g(i_2), \ldots, f(i_n) \cdot g(i_n)).$$

So $h(x) = f(x) \cdot g(x) \in D[x]$ generates $\overline{a} \cdot \overline{b}$.

Thus $\overline{a} \cdot \overline{b} \in P_{n,I}$. □

Theorem 3.4. $(P_{n,I}, +, \cdot)$ *is a commutative ring with identity* $\overline{1}$.

Proof. We will show that $(P_{n,I}, +, \cdot)$ is a commutative ring with identity $\overline{1}$.

1. By Theorem 3.1 and the fact that $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ for any $\bar{a}, \bar{b} \in P_{n,I}$, we obtain that $(P_{n,I}, +)$ is an abelian group.

2. Let $\overline{a}, \overline{b}, \overline{c} \in P_{n,I}$. Then

$$\begin{aligned}
(\overline{a} \cdot \overline{b}) \cdot \overline{c} &= (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n) \cdot (c_1, c_2, \ldots, c_n) \\
&= ((a_1 \cdot b_1) \cdot c_1, (a_2 \cdot b_2) \cdot c_2, \ldots, (a_n \cdot b_n) \cdot c_n) \\
&= (a_1 \cdot (b_1 \cdot c_1), a_2 \cdot (b_2 \cdot c_2), \ldots, a_n \cdot (b_n \cdot c_n)) \\
&= (a_1, a_2, \ldots, a_n) \cdots (b_1 \cdot c_1, b_2 \cdot c_2, \ldots, b_n \cdot c_n) \\
&= \overline{a} \cdot (\overline{b} \cdot \overline{c}).
\end{aligned}$$

Hence the associative law for multiplication holds.

3. Let $\overline{a}, \overline{b}, \overline{c} \in P_{n,I}$. Then

$$\begin{aligned}
(\overline{a} + \overline{b}) \cdot \overline{c} &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \cdot (c_1, c_2, \ldots, c_n) \\
&= ((a_1 + b_1) \cdot c_1, (a_2 + b_2) \cdot c_2, \ldots, (a_n + b_n) \cdot c_n) \\
&= (a_1 \cdot c_1 + b_1 \cdot c_1, a_2 \cdot c_2 + b_2 \cdot c_2, \ldots, a_n \cdot c_n + b_n \cdot c_n) \\
&= (a_1 \cdot c_1, a_2 \cdot c_2, \ldots, a_n \cdot c_n) + (b_1 \cdot c_1, b_2 \cdot c_2, \ldots, b_n \cdot c_n) \\
&= \overline{a} \cdot \overline{c} + \overline{b} \cdot \overline{c}. \\
\overline{a} \cdot (\overline{b} + \overline{c}) &= (a_1, a_2, \ldots, a_n) \cdot (b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n) \\
&= (a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2), \ldots, a_n \cdot (b_n + c_n)) \\
&= (a_1 \cdot b_1 + a_1 \cdot c_1, a_2 \cdot b_2 + a_2 \cdot c_2, \ldots, a_n \cdot b_n + a_n \cdot c_n) \\
&= (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n) + (a_1 \cdot c_1, a_2 \cdot c_2, \ldots, a_n \cdot c_n) \\
&= \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c}.
\end{aligned}$$

4. Let $\overline{a}, \overline{b} \in P_{n,I}$. Then

$$\begin{aligned}\overline{a} \cdot \overline{b} &= (a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) \\ &= (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n) \\ &= (b_1 \cdot a_1, b_2 \cdot a_2, \ldots, b_n \cdot a_n) \\ &= (b_1, b_2, \ldots, b_n) \cdot (a_1, a_2, \ldots, a_n) \\ &= \overline{b} \cdot \overline{a}.\end{aligned}$$

   Thus commutative law for multiplication holds.

5. Let $\overline{a} \in P_{n,I}$. Then there exists $\overline{1} = (1, 1, \ldots, 1) \in P_{n,I}$ such that

$$\begin{aligned}\overline{a} \cdot \overline{1} &= (a_1, a_2, \ldots, a_n) \cdot (1, 1, \ldots, 1) \\ &= (a_1, a_2, \ldots, a_n) = \overline{a} \\ &= (1, 1, \ldots, 1) \cdot (a_1, a_2, \ldots, a_n) \\ &= \overline{1} \cdot \overline{a}.\end{aligned}$$

   Thus $\overline{1}$ is an identity of $(P_{n,I}, \cdot)$.

Therefore $(P_{n,I}, +, \cdot)$ is a commutative ring with identity. □

Note that for $n \geq 2$, we see that for all $\overline{a}$ in D^n with respect to $I = (i_1, i_2, \ldots, i_n)$ in D^n where i_j's are all distinct. Choose $\bar{a}_1 = \Big(\prod_{m=2}^{n} (i_1 - i_m), 0, 0, \ldots, 0 \Big)$ and $\bar{a}_2 = \Big(0, \prod_{\substack{m=1 \\ m \neq 2}}^{n} (i_2 - i_m), 0, \ldots, 0\Big)$ be the integral sequences of length n. Then there exist the polynomials

$$f(x) = (x - i_2)(x - i_3) \cdots (x - i_n) \text{ and } g(x) = (x - i_1)(x - i_3) \cdots (x - i_n)$$

in $D[x]$ that generate the sequences $\bar{a}_1$ and $\bar{a}_2$ respectively. So $\bar{a}_1, \bar{a}_2 \in P_{n,I}$ and $\bar{a}_1 \cdot \bar{a}_2 = \bar{0}$. Hence $(P_{n,I}, +, \cdot)$ has zero divisors for all n and I.

3.2 Properties of the polynomial sequences

For a fixed sequence I as above, and a sequence $\bar{a} := (a_1, \ldots, a_n) \in D^n$, the Lagrange interpolation polynomial, [2, page 33], which interpolates the points (i_j, a_j) for all $1 \leq j \leq n$, is defined by

$$L_{a,I}(x) := \sum_{j=1}^{n} a_j \prod_{m=1, m\neq j}^{n} \frac{x - i_m}{i_j - i_m} \in D_Q[x] \quad (D_Q \text{ the quotient field of } D)$$

and satisfies

$$L_{a,I}(i_j) = a_j \ \ (1 \leq j \leq n).$$

Lemma 3.5. *Let $r(x), s(x) \in D_Q[x]_n$ where $r(x)$ and $s(x)$ agree at n distinct points. Then $r(x) = s(x)$.*

Proof. Let $g(x) = r(x) - s(x)$. Let $i_1, i_2, \ldots, i_n$ be all n distinct points such that $r(i_j) = s(i_j)$ for $1 \leq j \leq n$. Then for any $1 \leq j \leq n$

$$g(i_j) = r(i_j) - s(i_j) = 0.$$

Hence

$$\prod_{j=1}^{n-1}(x - i_j) \mid g(x).$$

Since $g(x)$ is a polynomial of degree less that n, $g(x) = 0$.
Therefore $r(x) = s(x)$. □

Theorem 3.6. *Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and $\bar{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Then $\bar{a}$ is a (D^n, I)-polynomial sequence if and only if $L_{a,I}(x) \in D[x]_n$, the set of all polynomials in $D[x]$ of degree $< n$.*
Furthermore, $L_{a,I}(x)$ is the unique polynomial of degree $< n$ in $D_Q[x]$ that generates $\bar{a}$.

Proof. If $a \in P_{n,I}$, then there is $f(x) \in D[x]$ such that $f(i_j) = a_j$ $(1 \leq j \leq n)$. Let $p(x) := (x - i_1) \cdots (x - i_n) \in D[x]$, $\deg p(x) = n$. Since $p(x)$ is monic, by the division algorithm, $f(x) = q(x)p(x) + r(x)$, where $q, r \in D[x]$ with $\deg r < n$. Evaluating at the points i_j $(1 \leq j \leq n)$, we see that $r(x)$ generates the sequence

$\bar{a}$ which shows that both $r(x)$ and $L_{a,I}(x)$ are polynomials in $D_Q[x]$ of degree less than n which agree at n distinct points, and so both must be identical. The remaining assertions are trivial. □

Taking $I = (1, 2, \ldots, n), D = \mathbb{Z}$ in Theorem 3.6 , We recover [1, Theorem 2.1].

Corollary 3.7. [1, Theorem 2.1] *Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. Then $\bar{a} \in P_n$ if and only if $L_a(x) \in \mathbb{Z}[x]_n$. Furthermore, $L_a(x)$ is the unique polynomial of degree $< n$ with real coefficients that generates $\bar{a}$.*

Proof. Let $D = \mathbb{Z}$ and take $I = (1, 2, 3, \ldots, n)$. Then the corollary follows immediately from Theorem 3.6. □

Given a set of n points (i_k, a_k) $(k = 1, \ldots, n)$, with distinct i_k, and a_k being in D, the Newton interpolation polynomial corresponding to the points (i_k, a_k) for all $k = 1, \ldots, n$ is defined as

$$\begin{aligned} N_{a,I}(x) = & b_{0,I} + b_{1,I}(x - i_1) + b_{2,I}(x - i_1)(x - i_2) + \cdots + \\ & b_{n-1,I}(x - i_1)(x - i_2) \cdots (x - i_{n-1}) \in D_Q[x], \end{aligned}$$

where

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j+1}}^{k+1} (i_{j+1} - i_m)} \in D_Q \qquad (0 \leq k \leq n - 1).$$

The elements

$$1, p_{i_1} := (x - i_1), p_{i_2} := (x - i_1)(x - i_2), \ldots, p_{i_{n-1}} := (x - i_1)(x - i_2) \cdots (x - i_n)$$

are referred to as the corresponding Newton basis polynomials [2, page 39-40].

Theorem 3.8. *Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Let*

$$N_{a,I}(x) = b_{0,I} p_{i_0}(x) + b_{1,I} p_{i_1}(x) + \cdots + b_{n-1,I} p_{i_{n-1}}(x) \in D_Q[x]$$

where

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j+1}}^{k+1} (i_{j+1} - i_m)} \in D_Q, \qquad (k = 0, 1, \ldots, n - 1).$$

Then $N_{a,I}(x) = L_{a,I}(x)$.

Proof. By Theorem 3.6, $L_{a,I}(x)$ is the unique polynomial with coefficients in D_Q of degree $< n$ generating $\overline{a}$. Since $N_{a,I}(i_j) = a_j = L_{a,I}(i_j)$ for $1 \le j \le n-1$ and $N_{a,I}(x)$ is the polynomial of degree $< n$, they are identical. □

Taking $D = \mathbb{Z}, I = (1, 2, \dots, n)$ in Theorem 3.8, then we have the following result.

Corollary 3.9. [1, Lemma 2.2] *Let* $\overline{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. *Let*

$$N_a(x) = b_0p_0(x) + b_1p_1(x) + b_2p_2(x) + \cdots + b_{n-1}p_{n-1}(x)$$

where

$$b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j}}{j!(k-j)!} a_{j+1}$$

for $k = 0, \dots, n-1$. *Then* $N_a(x) = L_a(x)$.

Proof. Take $D = \mathbb{Z}, I = (1, 2, \dots, n)$. Then the result follows from Theorem 3.8. □

Remark Let $L_{a,I}(x) = c_{n-1,I}x^{n-1} + c_{n-2,I}x^{n-2} + \cdots + c_{1,I}x + c_{0,I} \in D[x]_n$. Then by Theorem 3.8, $L_{a,I}(x) = N_{a,I}(x)$. Since $N_{a,I}(x) = b_{n-1,I}p_{i_{n-1}}(x) + b_{n-2,I}p_{i_{n-2}}(x) + \cdots + b_{1,I}p_{i_1}(x) + b_{0,I}$, $b_{n-1,I}$ is the coefficient of x^{n-1}. So $b_{n-1,I} = c_{n-1,I}$. We see that $b_{n-2,I}$, is the coefficient of x^{n-2} in $N_{a,I}(x) - b_{n-1,I}p_{i_{n-1}}(x)$ which is an element in integral domain D. So $b_{i,I} \in D$ for all $i = 0, 1, \dots, n-1$, by the same reasoning. Therefore $L_{a,I}(x) \in D[x]$ if and only if $N_{a,I}(x) \in D[x]$.

Corollary 3.10. *Let* $f(x) \in D[x]_n$. *Then there are unique element in* D $b_{0,I}, b_{1,I}, \dots, b_{n-1,I}$ *such that* $f(x) = b_{0,I}p_{i_0}(x) + b_{1,I}p_{i_1}(x) + \cdots + b_{n-1,I}p_{i_{n-1}}(x)$.

Proof. Let $\overline{a}$ be the sequence $(f(i_1), f(i_2), \dots, f(i_n))$ in D^n. Then $f(x)$ and $N_{a,I}(x)$ are both polynomials of degree less than n which agree at n points. Hence they are identical. □

Taking $D = \mathbb{Z}, I = (i_1, i_2, \ldots, i_n)$ from the above corollary we get the following result.

Corollary 3.11. [1, Corollary 2.3] *Let $f(x) \in \mathbb{Z}[x]_n$. Then there are unique integers $b_0, b_1, \ldots, b_{n-1}$ such that $f(x) = b_0 p_0(x) + b_1 p_1(x) + \cdots + b_{n-1} p_{n-1}(x)$.*

Proof. Take $D = \mathbb{Z}, I = (1, 2, 3, \ldots, n)$ and let $b_i = b_{i,I}$ for all $0 \le i \le n-1$. Then the corollary follows by Corollary 3.10. □

Corollary 3.12. *Let $\bar{a} \in D^n$. Then $\bar{a} \in P_{n,I}$ if and only if*

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{\substack{m=1 \\ m \neq j+1}}^{k+1} (i_{j+1} - i_m)} \qquad (k = 0, 1, \ldots, n-1)$$

is an element in D.

Proof. The result follows immediately from Theorem 3.6 and Theorem 3.8. So $N_{a,I}(x)$ has coefficients in D if and only if $L_{a,I}(x)$ does. □

Taking $D = \mathbb{Z}, I = (1, 2, \ldots, n)$ in Corollary 3.12, then we have the following result.

Corollary 3.13. [1, Corollary 2.4] *Let $\bar{a} \in \mathbb{Z}^n$. Then $\bar{a}$ is a polynomial sequence if and only if for all $k = 0, 1, \ldots, n-1$ then*

$$b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j} a_{j+1}}{j!(k-j)!}$$

is an integer.

Proof. Take $D = \mathbb{Z}, I = (1, 2, \ldots, n)$. Then the result follows from Corollary 3.12. □

It is of interest to investigate the above results for small values of n, which we do now.

Lemma 3.14. *For any $I = (i_1) \in \mathbb{Z}$, we have $P_{1,I} = \mathbb{Z}$.*

Proof. For any $a \in \mathbb{Z}$ there exists $f(x) = a$ such that $f(i_1) = a$. Thus $P_{1,I} = \mathbb{Z}$ as desired. □

Lemma 3.15. *For any* $\overline{a} = (a_1, a_2), I = (i_1, i_2) \in \mathbb{Z}^2$ *where* $i_1 < i_2$*, we have*

$$\overline{a} \in P_{2,I} \text{ if and only if } a_1 \equiv a_2 \mod (i_1 - i_2).$$

In fact if $I = (1, 2)$ *then* $P_2 = \mathbb{Z}^2$.

Proof. Let $\overline{a} = (a_1, a_2) \in \mathbb{Z}^2$. By Corollary 3.12 $\overline{a} \in P_{2,I}$ if and only if

$$\begin{aligned} &b_{0,I} = a_1 \text{ and} \\ &b_{1,I} = \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{a_1 - a_2}{i_1 - i_2} \text{ are integers.} \end{aligned}$$

Hence $\overline{a} \in P_{2,I}$ if and only if $a_1 \equiv a_2 \mod (i_1 - i_2)$.
In fact if $I = (1, 2)$, then $i_1 - i_2 = 1$. Thus $P_2 = \mathbb{Z}^2$. □

Lemma 3.16. *For any* $\overline{a} = (a_1, a_2, a_3), I = (i_1, i_2, i_3) \in \mathbb{Z}^3$ *where* $i_1 < i_2 < i_3$*, we have*

$$\overline{a} \in P_{3,I} \text{ if and only if } \frac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)} \text{ and } m = \frac{a_1 - a_2}{i_1 - i_2} \text{ are integers.}$$

In fact if $I = (1, 2, 3)$ *then* $P_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 \equiv a_3 \mod 2\}$.

Proof. Let $\overline{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ then

$$\begin{aligned} b_{0,I} &= a_1, \\ b_{1,I} &= \frac{a_1}{i_1 - i_2} + \frac{a_2}{i_2 - i_1} = \frac{a_1 - a_2}{i_1 - i_2}, \\ b_{3,I} &= \frac{a_1}{(i_1 - i_2)(i_1 - i_3)} + \frac{a_2}{(i_2 - i_1)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} \\ &= \frac{a_1(i_2 - i_3) - a_2(i_1 - i_3)}{(i_1 - i_2)(i_1 - i_3)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} \\ &= \frac{i_2 a_1 - i_3 a_1 - i_1 a_2 + i_3 a_2}{(i_1 - i_2)(i_1 - i_3)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} \\ &= \frac{-i_3(a_1 - a_2) + (i_2 a_1 - i_1 a_2)}{(i_1 - i_2)(i_1 - i_3)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} \\ &= \frac{-i_3 m}{(i_1 - i_3)(i_2 - i_3)} + \frac{i_2(a_1 - a_2) - a_2(i_1 - i_2)}{(i_1 - i_2)(i_1 - i_3)(i_2 - i_3)} + \frac{a_3}{(i_3 - i_1)(i_3 - i_2)} \end{aligned}$$

$$= \frac{a_3 - i_3 m}{(i_1 - i_3)(i_2 - i_3)} + \frac{i_2 m - a_2}{(i_1 - i_3)(i_2 - i_3)}, \qquad \left(m = \frac{a_1 - a_2}{i_1 - i_2}\right)$$
$$= \frac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)}.$$

By Corollary 3.12, $\bar{a} \in P_{3,I}$ if and only if $m = \frac{a_1 - a_2}{i_1 - i_2} \in \mathbb{Z}$ and $\frac{(a_3-a_2)+m(i_2-i_3)}{(i_1-i_3)(i_2-i_3)} \in \mathbb{Z}$ as desired.

In fact if $I = (1, 2, 3)$, then $m = \frac{a_1 - a_2}{1-2} = a_2 - a_1$ is an integer. Hence $\frac{(a_3-a_2)+m(2-3)}{(1-3)(2-3)} = \frac{(a_3-a_2)+(a_2-a_1)(-1)}{2} = \frac{a_3-a_1}{2} - a_2$ is an integer if and only if $2 | a_3 - a_1$. Thus, $\overline{a} \in \mathbb{Z}^3$ is a polynomial sequence of length 3 if and only if a_1 and a_3 are of the same parity. □

Example 13. By the lemma above, it is easy to see that the sequences $(2, 5, 12)$ and $(3, 9, 17)$ are in P_3 while the sequences $(0, 1, 3)$ and $(3, 8, 12)$ are not.

The next result shows how to turn a sequence into a (D^n, I)-polynomial sequence.

Theorem 3.17. *Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ with distinct i_j, let $\bar{a} = (a_1, a_2, \ldots, a_n)$ in D^n and let*

$$M = \prod_{j=0}^{n-1} M_j \quad \textit{where} \quad M_j = \prod_{\substack{m=1,\\ m \neq j+1}}^{n} (i_{j+1} - i_m) \textit{ for all } j = 0, 1, 2, \ldots, n-1.$$

Then $M\bar{a} = (Ma_1, Ma_2, \ldots, Ma_n) \in P_{n,I}$.

Moreover, if the integral domain D is a unique factorization domain, then

$$M'\bar{a} = (M'a_1, M'a_2, \ldots, M'a_n) \in P_{n,I} \textit{ where } M' = \operatorname{lcm}\{M_j\}_{j=0}^{n-1}$$

and M' is the minimal element in D for which this is true for every sequence of length n.

Proof. Let $M = \prod_{j=0}^{n-1} M_j$ where $M_j = \prod_{\substack{m=1\\ m \neq j+1}}^{n} (i_{j+1} - i_m), j = 0, 1, 2, \ldots, n-1$.

Since

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{\substack{m \neq j+1\\ m=1}}^{k+1} (i_{j+1} - i_m)} = \sum_{j=0}^{k} \frac{\frac{a_{j+1}}{M_j}}{\prod_{\substack{m=k+2\\ m \neq j+1}}^{n} (i_{j+1} - i_m)} \text{ for all } 0 \leq k \leq n-1,$$

we see that $Mb_{k,I}$ is an integer. Therefore $M\bar{a}$ is a (D^n, I)-polynomial sequence.

In fact if the integral domain D is a unique factorization domain, then letting $M_j = \prod_{\substack{m=1 \\ m\neq j+1}}^{n} (i_{j+1} - i_m)$ and $M' = \operatorname{lcm}\{M_j\}_{j=0}^{n-1}$ for $j = 0, 1, 2, \ldots, n-1$. Thus

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod_{\substack{m\neq j+1 \\ m=1}}^{k+1} (i_{j+1} - i_m)} = \sum_{j=0}^{k} \frac{\frac{a_{j+1}}{M_j}}{\prod_{\substack{m=k+2 \\ m\neq j+1}}^{n} (i_{j+1} - i_m)} \text{ for all } 0 \leq k \leq n-1.$$

It is easy to see that $M'b_{k,I}$ is an integer.

To see that M' is the minimal element with the stated property, consider the sequences:

Sequence $\overline{a}$	$b_{0,I}$	$b_{1,I}$	$b_{2,I}$	$\ldots$	$b_{n-1,I}$
$\overline{a}_1 = (1, 0, 0, \ldots, 0)$	1	$\frac{1}{i_1-i_2}$	$\frac{1}{(i_1-i_2)(i_1-i_3)}$	$\ldots$	$\frac{1}{(i_1-i_2)(i_1-i_3)\cdots(i_1-i_n)}$
$\overline{a}_2 = (0, 1, 0, \ldots, 0)$	0	$\frac{1}{i_2-i_1}$	$\frac{1}{(i_2-i_1)(i_2-i_3)}$	$\ldots$	$\frac{1}{(i_2-i_1)(i_2-i_3)\cdots(i_2-i_n)}$
$\overline{a}_3 = (0, 0, 1, \ldots, 0)$	0	0	$\frac{1}{(i_3-i_1)(i_3-i_2)}$	$\ldots$	$\frac{1}{(i_3-i_1)(i_3-i_2)(i_3-i_4)\cdots(i_3-i_n)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{a}_n = (0, 0, 0, \ldots, 1)$	0	0	0	$\ldots$	$\frac{1}{(i_n-i_1)(i_n-i_2)\cdots(i_n-i_{n-1})}$

For each $\overline{a}_i, (1 \leq i \leq n)$, we can see that $M_{i-1}\overline{a}_i \in P_{n,I}$ and for any element $L \in D$ such that $L\overline{a}_i \in P_{n,I}$, we have $M_{i-1}|L$ for all $1 \leq i \leq n$. Therefore by the definition of M', we have $M'|L$, showing that M' is the minimal element such that $M'\overline{a} \in P_{n,I}$. $\square$

Before proceeding, let us work out some examples.

Example 14. Let $D = \mathbb{Z}$, $I = (5, 6, 8)$ and $\overline{a} = (2, 8, 12)$. From Example 4, we see that

$$N_{a,I}(x) = -\frac{4}{3}x^2 + \frac{62}{3}x - 68 \notin \mathbb{Z}[x].$$

So $\bar{a} \notin P_{3,I}$. Since

$$M_0 = \prod_{\substack{m=1 \\ m\neq 1}}^{3} (i_1 - i_m) = (i_1 - i_2)(i_1 - i_3) = (5-6)(5-8) = 3,$$

$$M_1 = \prod_{\substack{m=1 \\ m \neq 2}}^{3} (i_2 - i_m) = (i_2 - i_1)(i_2 - i_3) = (6-5)(6-8) = -2,$$

$$M_2 = \prod_{\substack{m=1 \\ m \neq 3}}^{3} (i_3 - i_m) = (i_3 - i_1)(i_3 - i_2) = (8-5)(8-6) = 6,$$

$M' = \mathrm{lcm}(M_0, M_1, M_2) = \mathrm{lcm}(3, -2, 6) = 6$. So $M'\overline{a} = (12, 48, 72)$. It is now easy to see that $M'\overline{a} = (12, 48, 72)$ is a polynomial sequence generated by

$$-8x^2 + 24x - 408$$

with respect to $I = (5, 6, 8)$ in $\mathbb{Z}$.

Example 15. Let $D = \mathbb{Z}$, $I = (5, 6, 8)$ and $\overline{a}_1 = (1, 0, 0)$. We get $b_{0,I} = 1$, $b_{1,I} = -1, b_{2,I} = \frac{1}{3}$. So

$$N_{a_1,I}(x) = -\frac{1}{3}x^2 - \frac{14}{3}x - 16 \notin \mathbb{Z}[x].$$

By Example 14, we have $M' = \mathrm{lcm}(M_0, M_1, M_2) = \mathrm{lcm}(3, -2, 6) = 6$ and $M'\overline{a}_1 = (6, 0, 0)$. It is easy to see that $M'\overline{a}_1 = (6, 0, 0)$ is a polynomial sequence generated by $2x^2 - 28x + 96$ with respect to $I = (5, 6, 8)$ in $\mathbb{Z}$.

Example 16. Let $D = \mathbb{Z}$, $I = (5, 6, 8)$ and $\overline{a}_2 = (0, 1, 0)$. Similarly to Example 15, we see that $N_{a_2,I}(x) = \frac{1}{2}x^2 - \frac{9}{2}x - 10 \notin \mathbb{Z}[x]$ and $M' = 6$. Then $M'\overline{a}_2 = (0, 6, 0)$ and $N_{M'a_2,I}(x) = -3x^2 + 39x - 120 \in \mathbb{Z}[x]$. Thus $6\overline{a}_2$ is a $(\mathbb{Z}^3, I)$-polynomial sequence.

Example 17. Let $D = \mathbb{Z}$, $I = (5, 6, 8)$ and $\overline{a}_3 = (0, 0, 1)$. Similarly to Example 15, we see that $N_{a_3,I}(x) = \frac{1}{6}x^2 - \frac{11}{6}x + 5 \notin \mathbb{Z}[x]$ and $M' = 6$. Then $M'\overline{a}_3 = (0, 0, 6)$ and $N_{M'a_3,I}(x) = x^2 - 11x + 30 \in \mathbb{Z}[x]$. Thus $6\overline{a}_3$ is a $(\mathbb{Z}^3, I)$-polynomial sequence.

Therefore, for $D = \mathbb{Z}$ and $I = (5, 6, 8)$ we can multiply the sequences in Example 15, 16 and 17 with $3, 2$ and 6 respectively to make the sequences in these examples the $(\mathbb{Z}^3, I)$-polynomial sequences. By Theorem 3.17 the best integer M' that is true for all the sequences of length 3 with respect to $I = (5, 6, 8)$ in $\mathbb{Z}^3$ is $M' = 6$.

Example 18. Let $D = \mathbb{Z}[i]$, $I = (i, 3i, 2+i, 4) \in \mathbb{Z}[i]^4$ and $\overline{a} = (2i, 8i, 4+2i, 10)$ in $\mathbb{Z}[i]^4$. we see that

$$\begin{aligned} N_a(x) &= 2i + 3(x-i) + \frac{-1-i}{4}(x-i)(x-3i) + \frac{33+191i}{1700}(x-i)(x-3i)(x-2-i) \\ &= \frac{33+199i}{1700}x^3 + \frac{116-243i}{425}x^2 + \frac{1641+627i}{1700}x + \frac{9}{17} + \frac{41i}{85} \notin \mathbb{Z}[i][x]. \end{aligned}$$

Since

$$\begin{aligned} M_0 &= \prod_{\substack{m=1 \\ m\neq 1}}^{4} (i_1 - i_m) = (i-3i)(i-2-i)(i-4) = -4-16i, \\ M_1 &= \prod_{\substack{m=1 \\ m\neq 2}}^{4} (i_2 - i_m) = (3i-i)(3i-2-i)(3i-4) = 28+4i, \\ M_2 &= \prod_{\substack{m=1 \\ m\neq 3}}^{4} (i_3 - i_m) = (2+i-i)(2+i-3i)(2+i-4) = -4+12i, \\ M_3 &= \prod_{\substack{m=1 \\ m\neq 4}}^{4} (i_4 - i_m) = (4-i)(4-3i)(4-2-i) = 10-45i, \\ M' &= \operatorname{lcm}(M_1, M_2, M_3, M_4) \\ &= \operatorname{lcm}(-4-16i, 28+4i, -4+12i, 10-45i) = -140-220i. \end{aligned}$$

So $M'\overline{a} = (-140-220i)\cdot(2i, 8i, 4+2i, 10) = (440-280i, 1760-1120i, -120-1160i, -1400-2200i)$. Hence

$$\begin{aligned} N_{M'\overline{a}}(x) = &(440-280i) + (-420-660i)(x-i) + \\ &(-120+90i)(x-i)(x-3i) + (22-20i)(x-i)(x-3i)(x-2-i) \\ = &(22-20i)x^3 - (164-20i)x^2 - (54+264i)x + (32-184i) \in \mathbb{Z}[i][x]. \end{aligned}$$

Thus $M'\overline{a}$ is a $(\mathbb{Z}[i]^4, I)$-polynomial sequence.

To see $M' = -140-220i$ is the best number that $M'\overline{a} \in P_{4,I}$ for all the sequences of length 4 in $\mathbb{Z}[i]^4$ for $I = (1, 3i, 2+1, 4)$, let see some these examples:

Example 19. Let $D = \mathbb{Z}[i]$, $I = (i, 3i, 2+i, 4) \in \mathbb{Z}[i]^4$ and $\bar{a}_1 = (i, 0, 0, 0)$. We see that

$$\begin{aligned} N_{a_1,I}(x) &= i - \frac{1}{2}(x-i) + \frac{1}{4}(x-i)(x-3i) + \frac{-4-i}{68}(x-i)(x-3i)(x-2-i) \\ &= \frac{-4-i}{68}x^3 + \frac{10+11i}{34}x^2 + \frac{2-93i}{68}x - \frac{18+21i}{17} \notin \mathbb{Z}[x]. \end{aligned}$$

By Example 18, we have $M' = \operatorname{lcm}(M_0, M_1, M_2, M-3) = \operatorname{lcm}(-4-16i, 28+4i, -4+12i, 10-45i) = -140-220i$ and $M'\bar{a}_1 = (220-140i, 0, 0, 0)$. It is easy to see that $M'\bar{a}_1 = (220-140i, 0, 0, 0)$ is a $(\mathbb{Z}[i]^4, I)$-polynomial sequence generated by

$$(5+15i)x^3 + (30-110i)x^2 - (305-185i)x + (420+60i) \in \mathbb{Z}[i][x]$$

with respect to $I = (i, 3i, 2+i, 4)$ in $\mathbb{Z}[i]^4$.

Example 20. Let $D = \mathbb{Z}[i]$, $I = (i, 3i, 2+i, 4) \in \mathbb{Z}[i]^4$ and $\bar{a}_2 = (0, i, 0, 0)$. We see that

$$\begin{aligned} N_{a_2,I}(x) &= \frac{1}{2}(x-i) + \frac{-1-i}{8}(x-i)(x-3i) + \frac{1+7i}{68}(x-i)(x-3i)(x-2-i) \\ &= \frac{1+7i}{200}x^3 + \frac{2-11i}{50}x^2 - \frac{63-59i}{200}x + \frac{3+i}{10} \notin \mathbb{Z}[x]. \end{aligned}$$

By Example 18, we have $M' = \operatorname{lcm}(M_0, M_1, M_2, M_3) = \operatorname{lcm}(-4-16i, 28+4i, -4+12i, 10-45i) = -140-220i$ and $M'\bar{a}_2 = (0, 220-140i, 0, 0)$. It is easy to see that $M'\bar{a}_2 = (0, 220-140i, 0, 0)$ is a $(\mathbb{Z}[i]^4, I)$-polynomial sequence generated by

$$(7-6i)x^3 - (54-22i)x^2 + (309+28i)x - (20+80i) \in \mathbb{Z}[i][x]$$

with respect to $I = (i, 3i, 2+i, 4)$ in $\mathbb{Z}[i]^4$.

Example 21. Let $D = \mathbb{Z}[i]$, $I = (i, 3i, 2+i, 4) \in \mathbb{Z}[i]^4$ and $\bar{a}_3 = (0, 0, i, 0)$. We see that

$$\begin{aligned} N_{a_3,I}(x) &= \frac{-1+i}{8}(x-i)(x-3i) + \frac{3-i}{40}(x-i)(x-3i)(x-2-i) \\ &= \frac{3-i}{4}x^3 - \frac{2-i}{5}x^2 + \frac{7+51i}{40}x + \frac{9-3i}{10} \notin \mathbb{Z}[x]. \end{aligned}$$

By Example 18, we have $M' = \operatorname{lcm}(M_0, M_1, M_2, M_3) = \operatorname{lcm}(-4-16i, 28+4i, -4+12i, 10-45i) = -140-220i$ and $M'\bar{a}_3 = (0, 0, 220-140i, 0)$. It is easy to see that $M'\bar{a}_1 = (0, 0, 220-140i, 0)$ is a $(\mathbb{Z}[i]^4, I)$-polynomial sequence generated by

$$(-16-13i)x^3 + (12+116i)x^2 + (256-217i)x - (192+156i) \in \mathbb{Z}[i][x]$$

with respect to $I = (i, 3i, 2+i, 4)$ in $\mathbb{Z}[i]^4$.

Example 22. Let $D = \mathbb{Z}[i]$, $I = (i, 3i, 2+i, 4) \in \mathbb{Z}[i]^4$ and $\overline{a}_4 = (0, 0, 0, i)$. We see that

$$\begin{aligned} N_{a_4,I}(x) &= \frac{-9+2i}{425}(x-i)(x-3i)(x-2-i) \\ &= \frac{-9+2i}{425}x^3 + \frac{28+41i}{425}x^2 + \frac{47-86i}{425}x + \frac{-12-3i}{85} \notin \mathbb{Z}[x]. \end{aligned}$$

By Example 18, we have $M' = \operatorname{lcm}(M_0, M_1, M_2, M_3) = \operatorname{lcm}(-4-16i, 28+4i, -4+12i, 10-45i) = -140-220i$ and $M'\bar{a}_4 = (0, 0, 0, 220-140i)$. It is easy to see that $M'\bar{a}_4 = (0, 0, 0i, 220-140i)$ is a $(\mathbb{Z}[i]^4, I)$-polynomial sequence generated by

$$(4+4i)x^3 + (12-28i)x^2 - (60+4i)x + (12+36i) \in \mathbb{Z}[i][x]$$

with respect to $I = (i, 3i, 2+i, 4)$ in $\mathbb{Z}[i]^4$.

Therefore, for $D = \mathbb{Z}[i]$ and $I = (i, 3i, 2+i, 4)$ we can multiply the sequences in Example 19, 20,21 and 22 with $-4-16i, 28+4i, -4+12i$ and $10-45i$ respectively to make the sequences in these examples the $(\mathbb{Z}[i]^4, I)$-polynomial sequences. By Theorem 3.17 the best integer M' that is true for all the sequences of length 4 with respect to $I = (i, 3i, 2+i, 4)$ in $\mathbb{Z}[i]^4$ is $M' = -140-220i$.

If $D = \mathbb{Z}$ and $I = (1, 2, \ldots, n)$, then we have the following result which is [1, Theorem 2.5].

Corollary 3.18. *Let* $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. *Then*

$$(n-1)!a = \big((n-1)!a_1, (n-1)!a_2, \ldots, (n-1)!a_n\big) \in P_{n,I}.$$

Moreover, $(n-1)!$ *is the least positive integer for which is true every sequence of length* n.

Proof. Take $I = (1, 2, 3, \ldots, n)$. Using the same notation as in Theorem 3.17, we first compute M_j for $0 \leq j \leq n-1$, we have $M_j = \prod_{m=1, m\neq j+1}^{n}(j+1-m)$. Thus we can see that

$$M_0 = \prod_{\substack{m=1\\ m\neq 1}}^{n}(1-m) = (1-2)(1-3)\cdots(1-n) = (-1)^{n-1}(1-1)!(n-1)!$$

$$M_1 = \prod_{\substack{m=1\\ m\neq 2}}^{n}(2-m) = (2-1)(2-3)\cdots(2-n) = (-1)^{n-2}(2-1)!(n-2)!$$

$$M_2 = \prod_{\substack{m=1\\ m\neq 3}}^{n}(3-m) = (3-1)(3-2)\cdots(3-n) = (-1)^{n-3}(3-1)!(n-3)!$$

$$\vdots$$

$$M_j = \prod_{\substack{m=1\\ m\neq j+1}}^{n}(j+1-m) = (-1)^{n-j-1}(j)!(n-j-1)!$$

$$\vdots$$

$$M_{n-1} = \prod_{\substack{m=1\\ m\neq n}}^{n}(n-m) = (n-1)!.$$

Then M_j is the divisor of $(n-1)!$ for all $0 \leq j \leq n-1$ and $M_{n-1} = (n-1)!$. Hence $M = \text{lcm}(M_1, M_2, \ldots, M_n) = (n-1)!$. □

Example 23. Let $D = \mathbb{Z}$, $I = (1, 2, 3, 4)$ and $\overline{a} = (2, 8, 12, 16)$. From Example 6, we see that

$$N_a(x) = \frac{1}{3}x^3 - 3x^2 + \frac{38}{3}x - 8 \notin \mathbb{Z}[x].$$

Since

$$M_0 = \prod_{\substack{m=1\\ m\neq 1}}^{4}(1-m) = (1-2)(1-3)(1-4) = -6,$$

$$M_1 = \prod_{\substack{m=1\\ m\neq 2}}^{4}(2-m) = (2-1)(2-3)(2-4) = 2,$$

$$M_2 = \prod_{\substack{m=1\\ m\neq 3}}^{4}(3-m) = (3-1)(3-2)(3-4) = -2,$$

$$M_3 = \prod_{\substack{m=1 \\ m\neq 4}}^{4} (4-m) = (4-1)(4-2)(4-3) = 6,$$

$M' = \text{lcm}(M_1, M_2, M_3, M_4) = \text{lcm}(-6, 2, -2, 6) = 6 = (4-1)!$.

So $M'\overline{a} = 6(2, 8, 12, 16) = (12, 48, 72, 96)$.

$$\begin{aligned} \text{Hence } N_{3!\overline{a}}(x) &= b_0 + b_1 p_1(x) + b_2 p_2(x) \\ &= 12 + 36(x-1) - 6(x-1)(x-2) + 2(x-1)(x-2)(x-3) \\ &= 2x^3 - 18x^2 + 76x - 48 \in \mathbb{Z}[x]. \end{aligned}$$

Thus $3!\overline{a}$ is a polynomial sequence.

Example 24. Let $D = \mathbb{Z}$, $I = (1, 2, 3, 4)$ and $\bar{a}_1 = (1, 0, 0, 0)$. We see that b_i for $i = 0, 1, 2, 3$ with respect to $\bar{a}_1$ and I are $b_0 = 1, b_1 = -1, b_2 = \frac{1}{2}, b_3 = -\frac{1}{6}$. Then $N_{\bar{a}_1}(x) = -\frac{1}{6}x^3 + \frac{3}{2}x^2 - \frac{13}{3}x + 4 \notin \mathbb{Z}[x]$. Since $I = (1, 2, 3, 4)$, $M' = (4-1)! = 3!$ and $M'\bar{a}_1 = (6, 0, 0, 0)$. So The Newton's polynomial that generate $M'\bar{a}_1$ is

$$N_{3!a_1}(x) = -x^3 + 9x^2 - 26x + 24 \in \mathbb{Z}[x].$$

Hence $3!\bar{a}_1$ is a polynomial sequence.

Example 25. Let $D = \mathbb{Z}$, $I = (1, 2, 3, 4)$ and $\bar{a}_2 = (0, 1, 0, 0)$. Similarly to Example 24, we see that the Newton's polynomial that generate $\bar{a}_2$ is $N_{a_2}(x) = \frac{1}{2}x^3 - 4x^2 + \frac{19}{2}x - 6 \notin \mathbb{Z}[x]$ and $M' = 3!$. So $M'\bar{a}_2 = (0, 6, 0, 0)$ and the Newton's polynomial that generate $M'\bar{a}_2$ is $N_{3!a_2}(x) = 3x^3 - 24x^2 + 57x - 36 \in \mathbb{Z}[x]$. Thus $3!\bar{a}_2$ is a polynomial sequence.

Example 26. Let $D = \mathbb{Z}$, $I = (1, 2, 3, 4)$ and $\bar{a}_3 = (0, 0, 1, 0)$. Similarly to Example 24, we see that the Newton's polynomial that generate $\bar{a}_3$ is $N_{a_3}(x) = -\frac{1}{2}x^3 - \frac{7}{2}x^2 - 7x + 4 \notin \mathbb{Z}[x]$ and $M' = 3!$. So $M'\bar{a}_3 = (0, 0, 6, 0)$ and the Newton's polynomial that generate $M'\bar{a}_2$ is $N_{3!a_3}(x) = -x^3 + 21x^2 - 42x - 24 \in \mathbb{Z}[x]$. Thus $3!\bar{a}_3$ is a polynomial sequence.

Example 27. Let $D = \mathbb{Z}$, $I = (1, 2, 3, 4)$ and $\bar{a}_4 = (0, 0, 0, 1)$. Similarly to Example 24, we see that the Newton's polynomial that generate $\bar{a}_4$ is $N_{a_4}(x) = -\frac{1}{6}x^3 - x^2 +$

$\frac{11}{6}x - 1 \notin \mathbb{Z}[x]$ and $M' = 3!$. So $M'\bar{a}_4 = (0, 0, 0, 6)$ and the Newton's polynomial that generate $M'\bar{a}_4$ is $N_{3!a_4}(x) = x^3+6x^2-11x-6 \in \mathbb{Z}[x]$. Thus $3!\bar{a}_4$ is a polynomial sequence.

Therefore, for $D = \mathbb{Z}$ and $I = (1, 2, 3, 4)$ we can multiply the sequences in Example 24, 25, 26 and 27 by $6, 2, 2$ and 6 respectively to make the sequences in these examples the polynomial sequences. By Theorem 3.18 the best integer M' that is true for all the sequences of length 4 with respect to $I = (1, 2, 3, 4)$ in $\mathbb{Z}^4$ is $M' = (4 - 1)!$.

3.3 Structure of $P_{n,I}$

In this section, we will show that $P_{n,I}$ is a rank n subgroup of the free abelian group D^n.

We first show that for any $I \in D^n$, we have $P_{n,I} \cong D[x]_n$ as a group where $D[x]_n$ is the set of polynomial in $D[x]$ of degree less than n.
Let $v : D[x] \longrightarrow D^n$ be defined by $v(f(x)) = (f(i_1), f(i_2), \ldots, f(i_n))$. It easy to see that v is an additive homomorphism with image $P_{n,I}$.

Theorem 3.19. *The group $P_{n,I}$ is isomorphic to $D[x]_n$.*

Proof. We will show that $D[x]_n \cong P_{n,I}$.
Define $v : D[x]_n \to P_{n,I}$ by $v(f(x)) = (f(i_1), f(i_2), \ldots, f(i_n))$.
Let $f_1, f_2 \in D[x]_n$. Then

$$\begin{aligned} v((f_1 + f_2)(x)) &= ((f_1 + f_2)(i_1), (f_1 + f_2)(i_2), \ldots, (f_1 + f_2)(i_n)) \\ &= (f_1(i_1) + f_2(i_1), f_1(i_2) + f_2(i_2), \ldots, f_1(i_n) + f_2(i_n)) \\ &= v(f_1(x)) + v(f_2(x)). \end{aligned}$$

Thus v is an additive homomorphism.
We next show that v restricted to $D[x]_n$ is an isomorphism from $D[x]_n$ to $P_{n,I}$.
Let $\overline{a} = (a_1, a_2, \ldots, a_n) \in P_{n,I}$. Then there exists $f(x) \in D[x]$ such that

$f(i_1) = a_1, f(i_2) = a_2, \ldots, f(i_n) = a_n$. Again as in Theorem 3.6,

$$f(x) = q(x)p(x) + r(x)$$

where $p(x) = (x - i_1) \cdots (x - i_n)$, $q, r \in D[x]$ with $r = 0$ or $\deg r < n$. Evaluating at the points i_j $(1 \le j \le n)$, we see that $r(x)$ generates the sequence $\bar{a}$. So v is onto. Let $f, g \in D[x]_n$. Suppose $v(f(x)) = v(g(x))$.
Then $(f(i_1), f(i_2), \ldots, f(i_n)) = (g(i_1), g(i_2), \ldots, g(i_n))$. So $f(i_k) = g(i_k)$ for all $1 \le k \le n$. Since both $\deg(f)$ and $\deg(g)$ are less than n, and the polynomials f, g agree at n distinct points, they are identical, i.e., v is one-to-one.
Therefore v is an isomorphism of $D[x]_n$ onto $P_{n,I}$. □

We next consider the structure of $\mathbb{Z}^n/P_{n,I}$. For $I = (1, 2, \ldots, n) \in \mathbb{Z}^n$, it was shown in [1, Theorem 3.2] that

$$\mathbb{Z}^n/P_n = \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}.$$

For $n = 1, 2$, $\mathbb{Z}^n/P_n$ isomorphic to $\mathbb{Z}/0!\mathbb{Z}$ and $\mathbb{Z}/1!\mathbb{Z}$ respectively. We use the technique similar to that in [1] to generalize the above result to $D^n/P_{n,I}$.

Theorem 3.20. *For $n \ge 2$, let $I = (i_1, i_2, \ldots, i_n) \in D^n$. If*

$$\prod_{m=1}^{k-1}(i_j - i_m) / \prod_{m=1}^{k-1}(i_k - i_m) \in D \quad (1 < k < j \le n),$$

then

$$D^n/P_{n,I} \cong D/(i_2-i_1)D \oplus D/(i_3-i_1)(i_3-i_2)D \oplus \cdots \oplus D/(i_n-i_1)\cdots(i_n-i_{n-1})D.$$

Proof. For $j, k \in \{1, 2, \ldots, n\}$, let $A_n = (a_{jk})$ be the matrix $n \times n$ where

$$a_{jk} = \begin{cases} \prod_{m=1}^{k-1}(i_j - i_m) / \prod_{m=1}^{k-1}(i_k - i_m) & \text{if } j \ge k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k, \end{cases}$$

so that

$$A_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \frac{i_3-i_1}{i_2-i_1} & 1 & 0 & \dots & 0 \\ 1 & \frac{i_4-i_1}{i_2-i_1} & \frac{(i_4-i_1)(i_4-i_2)}{(i_3-i_1)(i_3-i_2)} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{i_n-i_1}{i_2-i_1} & \frac{(i_n-i_1)(i_n-i_2)}{(i_3-i_1)(i_3-i_2)} & \frac{(i_n-i_1)(i_n-i_2)(i_n-i_3)}{(i_4-i_1)(i_4-i_2)(i_4-i_3)} & \dots & 1 \end{bmatrix}.$$

Let $e_I(j-1)$ be the j^{th} column of A_n $(j = 1, 2, \dots, n)$. Since $\det A_n = 1$ and

$$a_{jk} = \prod_{m=1}^{k-1}(i_j - i_m) / \prod_{m=1}^{k-1}(i_k - i_m) \in D \qquad (1 < k < j),$$

the matrix A_n is an unimodular [7, Lemma 1.15]. In this case, we see that $\{e_I(j-1), j = 1, 2, \dots, n\}$ forms a D-basis for D^n.

Now let $C_n = (c_{jk})$ where

$$c_{jk} = \begin{cases} (i_j - i_1)(i_j - i_2)\cdots(i_j - i_{k-1}) & \text{if } 1 < k \leq j \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k, \text{ i.e,} \end{cases}$$

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & i_2 - i_1 & 0 & \dots & 0 \\ 1 & i_3 - i_1 & (i_3 - i_1)(i_3 - i_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & i_n - i_1 & (i_n - i_1)(i_n - i_2) & \dots & (i_n - i_1)\dots(i_n - i_{n-1}) \end{bmatrix}$$

Next, let D_n be the diagonal matrix whose j^{th} diagonal entries are

$$d_{j,I} = (i_j - i_1)(i_j - i_2)\cdots(i_j - i_{j-1}) \quad (j = 1, 2, \dots, n) \text{ i.e.,}$$

$$D_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & i_2 - i_1 & 0 & \dots & 0 \\ 0 & 0 & (i_3 - i_1)(i_3 - i_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (i_n - i_1)(i_n - i_2)\dots(i_n - i_1) \end{bmatrix}$$

It is easy to see that $C_n = A_n D_n$ i.e.,

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & i_2 - i_1 & 0 & \dots & 0 \\ 1 & i_3 - i_1 & (i_3 - i_1)(i_3 - i_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & i_n - i_1 & (i_n - i_1)(i_n - i_2) & \dots & (i_n - i_1)\dots(i_n - i_{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ 1 & \frac{i_3 - i_1}{i_2 - i_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{i_n - i_1}{i_2 - i_1} & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & i_2 - i_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (i_n - i_1)\dots(i_n - i_{n-1}) \end{bmatrix}.$$

Since $\{1, p_{i_1}(x), \dots, p_{i_{n-1}}(x)\}$ forms a D-basis for $D[x]_n$, by Theorem 3.19, the map $v : D[x]_n \longrightarrow P_{n,I}$ is an isomorphism. So the images

$$\{v(p_{i_0}(x)), v(p_{i_1}(x)), \dots, v(p_{i_{n-1}}(x))\}$$

forms a $D-$basis for $P_{n,I}$. From

$$v(p_{i_0}(x)) = \begin{bmatrix} p_{i_0}(i_1) \\ p_{i_0}(i_2) \\ p_{i_0}(i_3) \\ \vdots \\ p_{i_0}(i_n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$v(p_{i_1}(x)) = \begin{bmatrix} p_{i_1}(i_1) \\ p_{i_1}(i_2) \\ p_{i_1}(i_3) \\ \vdots \\ p_{i_1}(i_n) \end{bmatrix} = \begin{bmatrix} 0 \\ i_2 - i_1 \\ i_3 - i_1 \\ \vdots \\ i_n - i_1 \end{bmatrix},$$

$$v(p_{i_2}(x)) = \begin{bmatrix} p_{i_2}(i_1) \\ p_{i_2}(i_2) \\ p_{i_2}(i_3) \\ \vdots \\ p_{i_2}(i_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (i_3-i_1)(i_3-i_2) \\ \vdots \\ (i_n-i_1)(i_n-i_2) \end{bmatrix}$$

$$\vdots$$

$$v(p_{i_{n-1}}(x)) = \begin{bmatrix} p_{i_{n-1}}(i_1) \\ p_{i_{n-1}}(i_2) \\ p_{i_{n-1}}(i_3) \\ \vdots \\ p_{i_{n-1}}(i_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ (i_n-i_1)(i_n-i_2)\dots(i_n-i_{n-1}) \end{bmatrix},$$

we see that $v(p_{i_{j-1}}(x))$ is the j^{th} column of C_n $(j = 1, 2, \dots, n)$. Since $C_n = A_n D_n$, i.e.,

$$\begin{bmatrix} v(p_{i_0(x)}) & v(p_{i_1(x)}) & \dots & v(p_{i_{n-1}(x)}) \end{bmatrix} =$$

$$\begin{bmatrix} e_I(0) & e_I(1) & \dots & e_I(n-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & i_2-i_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (i_n-i_1)\dots(i_n-i_{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot e_I(0) & (i_2-i_1)e_I(1) & \dots & [(i_n-i_1)\dots(i_n-i_{n-1})]e_I(n-1) \end{bmatrix},$$

we have

$$v(p_{i_{j-1}}(x)) = (i_j - i_1)\cdots(i_j - i_{j-1})e_I(j-1) = \prod_{m=1}^{j-1}(i_j - i_m)e_I(j-1) \ (j = 1, 2, \dots, n).$$

Thus,

$$D^n/P_{n,I} = \frac{\langle e_I(0)\rangle \oplus \langle e_I(1)\rangle \oplus \langle e_I(2)\rangle \oplus \cdots \oplus \langle e_I(n-1)\rangle}{\langle e_I(0)\rangle \oplus \prod_{m=1}^{1}(i_2-i_m)\langle e_I(1)\rangle \oplus \cdots \oplus \prod_{m=1}^{n-1}(i_n-i_m)\langle e_I(n-1)\rangle}$$

$$= \frac{\langle e_I(0)\rangle}{\langle e_I(0)\rangle} \oplus \frac{\langle e_I(1)\rangle}{\prod_{m=1}^{1}(i_2-i_m)\langle e_I(1)\rangle} \oplus \cdots \oplus \frac{\langle e_I(n-1)\rangle}{\prod_{m=1}^{n-1}(i_n-i_m)\langle e_I(n-1)\rangle}$$

$$= \frac{\big\langle e_I(1)\big\rangle}{\prod_{m=1}^{1}(i_2 - i_m)\big\langle e_I(1)\big\rangle} \oplus \cdots \oplus \frac{\big\langle e_I(n-1)\big\rangle}{\prod_{m=1}^{n-1}(i_n - i_m)\big\langle e_I(n-1)\big\rangle}$$

$$\cong D\big/(i_2 - i_1)D \oplus D\big/\prod_{m=1}^{2}(i_3 - i_m)D \oplus \cdots \oplus D\big/\prod_{m=1}^{n-1}(i_n - i_m)D$$

$$= D/(i_2 - i_1)D \oplus D/(i_3 - i_1)(i_3 - i_2)D \oplus \cdots \oplus D\big/\prod_{m=1}^{n-1}(i_n - i_m)D.$$

□

By Theorem 3.20, for $1 \leq j \leq n$, if $a_{jk} = \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) \in D$ for all $1 < k \leq j$, choosing $k = j - 1$, we get

$$a_{j,j-1} = \prod_{m=1}^{j-2}(i_j - i_m)/\prod_{m=1}^{j-2}(i_{j-1} - i_m) \in D \quad (j = 0, 1, \ldots, n-1).$$

Thus,

$$d_{j,I} = \prod_{m=1}^{j-1}(i_j - i_m) = a_{j,j-1} \cdot (i_j - i_{j-1}) \cdot d_{j-1,I},$$

i.e., $d_{j-1,I}$ is the factor of $d_{j,I}$ $(j = 1, 2, \ldots, n)$, yielding

Corollary 3.21. *With the set up above, $D^n/P_{n,I}$ is a finite abelian group of the form*

$$D/d_{n-1,I}D \oplus \cdots \oplus D/d_{2,I}D \oplus D/d_{1,I}D$$

where $d_{1,I} \mid d_{2,I} \mid \cdots \mid d_{n-1,I}$.

If we take $D = \mathbb{Z}$ and $I = (1, 2, \ldots, n)$, we deduce the following

Corollary 3.22. [1, Corollary 3.3] *If $I = (1, 2, \ldots, n)$ $(n \geq 3)$, then $\mathbb{Z}^n/P_n$ is a finite abelian group with Smith normal form*

$$\mathbb{Z}/(n-1)!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z}$$

and Smith invariant $((n-1)!, \ldots, 3!, 2!)$. Moreover, $|\mathbb{Z}^n/P_n| = \prod_{i=1}^{n-1} i!$.

We pause to look at one simple example.

Example 28. Let $D = \mathbb{Z}[i]$ and $I = (2+i, 3+4i, 2+11i)$. Since

$$a_{3,2} = \frac{i_3 - i_1}{i_2 - i_1} = \frac{(2+11i)-(2+i)}{(3+4i)-(2+i)} = 3+i \in \mathbb{Z}[i],$$

all the elements a_{jk} of the matrix A_3 are in $\mathbb{Z}[i]$. By Theorem 3.20 we get

$$\mathbb{Z}[i]^3/P_{3,I} \cong \frac{\mathbb{Z}[i]}{(1+3i)\mathbb{Z}[i]} \oplus \frac{\mathbb{Z}[i]}{(10i)(-1+7i)\mathbb{Z}[i]} = \frac{\mathbb{Z}[i]}{(1+3i)\mathbb{Z}[i]} \oplus \frac{\mathbb{Z}[i]}{(-70-10i)\mathbb{Z}[i]}.$$

The quotient condition in Theorem 3.20 simplifies for some particular sets I as witnessed in the next corollary.

Corollary 3.23. *Let a, q be elements in D and $n \geq 2$. If $i_k = aq^k$ $(1 \leq k \leq n)$, then*

$$D^n/P_{n,I} \cong D/aq(q-1)D \oplus D/a^2q^{1+2}(q^2-1)(q-1)D \oplus \cdots \oplus D/a^{n-1}q^{\sum_{i=1}^{n-1} i}\prod_{i=1}^{n-1}(q^i-1)D.$$

Proof. Since $i_k = aq^k$, $i_{k+1} - i_k = aq^k(q-1)$ $(1 \leq k \leq n-1)$,
we have $i_j - i_k = aq^j - aq^k = aq^k(q^{j-k}-1)$ $(j > k)$. By the proof of Theorem 3.20, we get $A_n = (a_{jk})$ where

$$a_{jk} = \begin{cases} \dfrac{\prod_{m=1}^{k-1}(i_j - i_m)}{\prod_{m=1}^{k-1}(i_k - i_m)} = \prod_{m=1}^{k-1}(q^{j-m}-1)/\prod_{m=1}^{k-1}(q^m-1) & \text{if } j \geq k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k. \end{cases}$$

For $1 \leq k \leq j \leq n$, since $\prod_{m=1}^{k-1}(q^{j-m}-1)/\prod_{m=1}^{k-1}(q^m-1)$ is a q-binomial coefficient, it is in D, and by Theorem 3.20 we have

$$D^n/P_{n,I} \cong D/aq(q-1)D \oplus D/a^2q^{2+1}(q^2-1)(q-1)D \oplus \cdots \oplus D/a^{n-1}q^{\sum_{i=1}^{n-1} i}\prod_{i=1}^{n-1}(q^i-1)D.$$

□

Corollary 3.24. *For $n \geq 2$,, if $i_{k+1} - i_k$ is a constant $c \in D$ for all $1 \leq k \leq n-1$, then*

$$D^n/P_{n,I} \cong D/c \cdot D \oplus D/2!c^2D \oplus D/3!c^3D \oplus \cdots \oplus D/(n-1)!c^{n-1}D.$$

Proof. Since $i_{k+1} - i_k = c\,(1 \le k \le n-1)$, we have

$$i_j - i_k = (i_j - i_{j-1}) + (i_{j-1} - i_{j-2}) + \cdots + (i_{k+1} - i_k) = (j-k)c \;\; (j > k).$$

By the proof of Theorem 3.20, we get

$$A_n = (a_{jk}), \;\; a_{jk} = \begin{cases} \prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) = \binom{j-1}{k-1} & \text{if } j \ge k > 1 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } j < k. \end{cases}$$

Thus, $a_{jk} \in D$, and by Theorem 3.20, it is easy to see that

$$D^n/P_{n,I} \cong D/cD \oplus D\big/2!c^2 D \oplus \cdots \oplus D\big/(n-1)!c^{n-1}D.$$

□

Taking $D = \mathbb{Z},\ I = (1, 2, \ldots, n),\ c = 1$ in Corollary 3.24 yields the following result.

Corollary 3.25. [1, Theorem 3.2] For $n \ge 3$, and $I = \{1, 2, \ldots, n\}$ then

$$\mathbb{Z}^n/P_n \cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}.$$

Proof. Since $I = (1, 2, \ldots, n)$, $i_k - i_{k-1} = 1$ for all $k = 2, 3, \ldots, n$. By Lemma 3.24 we get

$$\begin{aligned} \mathbb{Z}^n/P_{n,I} &\cong \mathbb{Z}/1 \cdot \mathbb{Z} \oplus \mathbb{Z}/2!1^2\mathbb{Z} \oplus \mathbb{Z}/3!1^3\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!1^{n-1}\mathbb{Z} \\ &= \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z} \\ &\cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}. \end{aligned}$$

□

CHAPTER 4

Difference Polynomial Sequences

In this chapter, we study the difference of the polynomial and the difference of the sequence over $\mathbb{Z}$. For sequence $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$, call $\bar{c}$ a difference polynomial sequence of length n if there exists $f(x) \in \mathbb{Z}[x]$ such that $\Delta_F f(i) = c_i$ for all $1 \leq i \leq n-1$. Denote by ΔP_n the set of all difference polynomial sequences of length $n-1$. Criteria for a difference polynomial sequences are established and explicit structures of $\mathbb{Z}^{n-1}/\Delta P_n$ and $P_{n-1}/\Delta P_n$ are determined.

4.1 Introduction

In this section, we first introduce definitions that we will use throughout this chapter.

Definition 4.1. Let $f(x) \in \mathbb{Z}[x]$. Then we define

$$\Delta_F f(x) = f(x+1) - f(x).$$

We next find $\Delta_F(p_i(x))$ where $p_0(x) = 1$ and $p_i(x) = (x-1)(x-2)\cdots(x-i)$ for any $i \geq 1$.

Lemma 4.1. *With the above notation, for any $f(x), g(x) \in \mathbb{Z}[x]$*

1. $\Delta_F(p_0(x)) = 0$ *and* $\Delta_F(p_i(x)) = ip_{i-1}(x)$ *for* $i \geq 1$.
2. $\Delta_F(cf(x)) = c\Delta_F f(x)$ *for any constant c.*
3. $\Delta_F(f+g) = \Delta_F f + \Delta_F g$.

Proof. 1. We first compute $\Delta_F\left(p_0(x)\right)$ where $p_0(x) = 1$.
Thus $\Delta_F(p_0(x)) = p_0(x+1) - p_0(x) = 1 - 1 = 0$.
For $i \geq 1$, we have

$$\Delta_F(p_i(x)) = (x)(x-1)\cdots(x+1-i) - (x-1)(x-2)\cdots(x-i)$$

$$= (x-1)(x-2)\cdots(x+1-i)(x-(x-i))$$
$$= i(x-1)\cdots(x+1-i) = ip_{i-1}(x).$$

2. For any $c \in \mathbb{Z}$, $\Delta_F cf(x) = cf(x+1) - cf(x) = c(f(x+1) - f(x)) = c\Delta_F f(x)$.

3. Let $f(x), g(x) \in \mathbb{Z}[x]$. Then

$$\begin{aligned}\Delta_F(f+g) &= \big(f+g\big)(x+1) - \big(f+g\big)(x)\\ &= f(x+1) + g(x+1) - f(x) - g(x)\\ &= f(x+1) - f(x) + g(x+1) - g(x)\\ &= \Delta_F f + \Delta_F g.\end{aligned}$$

□

Lemma 4.2. *For any* $\bar{a} \in P_n$ *and* $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$, *then*

$$\Delta_F N_{\bar{a}}(x) = \sum_{i=0}^{n-2} (i+1) b_{i+1} p_i(x).$$

Proof. Let $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$. Then

$$\begin{aligned}\Delta_F N_{\bar{a}}(x) &= \Delta_F\Big(\sum_{i=0}^{n-1} b_i p_i(x)\Big)\\ &= \sum_{i=0}^{n-1} \Delta_F(b_i p_i(x)) \quad \text{(By Lemma 4.1)}\\ &= \Delta_F(b_0 p_0(x)) + \sum_{i=1}^{n-1} \Delta_F(b_i p_i(x))\\ &= \sum_{i=1}^{n-1} b_i i p_{i-1}(x) \quad \text{(By Lemma 4.1)}\\ &= \sum_{i=0}^{n-2} (i+1) b_{i+1} p_i(x).\end{aligned}$$

□

We next define a difference sequence. For any $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, we define $\Delta\bar{a} = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}) \in \mathbb{Z}^{n-1}$.

Example 29. Let $\bar{a} = (3, 7, 4, 8, 15) \in \mathbb{Z}^5$. Then

$$\Delta\bar{a} = (7-3, 4-7, 8-4, 15-8) = (4, -3, 4, 7) \in \mathbb{Z}^4.$$

Lemma 4.3. *If* $\bar{a} \in P_n$ *then* $\Delta\bar{a} \in P_{n-1}$.

Proof. Let $\bar{a} \in P_n$. Then $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$ for some $b_i \in \mathbb{Z}$ and $p_0(x) = 1$. Thus

$$\Delta_F N_{\bar{a}}(x) = \sum_{i=1}^{n-1} i b_i p_{i-1}(x).$$

Since $ib_i \in \mathbb{Z}$ and $\Delta_F N_{\bar{a}}(x)$ generates $\Delta\bar{a}$, $\Delta\bar{a} \in P_{n-1}$. □

Remark: The converse is not true as we can see from the following example:

Example 30. Let $\Delta\bar{a} = (1, 2)$. Thus $\bar{a} = (n, n+1, n+3)$ for some n. It is easy to see that $\Delta\bar{a} \in P_2$ but by Lemma 3.16, $\bar{a} \notin P_3$.

Definition 4.2. Let n be a positive integer. Let $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$. We define

$$\Delta P_n = \{\bar{c} \in \mathbb{Z}^{n-1} \mid \text{there exists } f(x) \in \mathbb{Z}[x] \text{ such that } \Delta_F f(i) = c_i, \\ \text{for all } 1 \leq i \leq n-1\}.$$

Example 31. Let $\bar{c} = (14, 40, 78) \in \mathbb{Z}^3$. Since there exists $f(x) = 2x^3 + x^2 - 3x + r$ in $\mathbb{Z}[x]$ where $r \in \mathbb{Z}$ such that $\Delta_F f(x) = 6x^2 + 8x \in \mathbb{Z}[x]$ generates $\bar{c}$, we get $\bar{c} \in \Delta P_4$.

Example 32. Let $\bar{a} = (1, 0, 0) \in \mathbb{Z}^3$.
Suppose there exists $f(x) \in \mathbb{Z}[x]$ such that $\Delta_F f(1) = 1, \Delta_F f(2) = 0, \Delta_F f(3) = 0$.
So $\Delta_F f(x) = (x-2)(x-3)g(x)$ where $g(x) \in \mathbb{Z}[x]$.
Then $\Delta_F f(1) = 2g(1)$. This is a contradiction because $\Delta_F f(1)$ is always even. Thus $\bar{a}$ is not a sequence in ΔP_4.

Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$. Then

$$\begin{aligned}
\Delta_F f(x) &= f(x+1) - f(x) \\
&= a_m(x+1)^m + a_{m-1}(x+1)^{m-1} + a_{m-2}(x+1)^{m-2} + \cdots + a_2(x+1)^2 + \\
&\quad a_1(x+1) + a_0 - \big(a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0\big) \\
&= a_m \sum_{i=0}^{m} \binom{m}{i} x^{m-i} + a_{m-1} \sum_{i=0}^{m-1} \binom{m-1}{i} x^{m-1-i} + \cdots + a_2 \sum_{i=0}^{2} \binom{2}{i} x^{2-i} + \\
&\quad a_0 - \big(a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0\big) \\
&= \sum_{i=m}^{m} a_i \binom{i}{i-m+1} x^{m-1} + \sum_{i=m-1}^{m} a_i \binom{i}{i-m+2} x^{m-2} + \cdots + \\
&\quad \sum_{i=3}^{m} a_i \binom{i}{i-2} x^2 + \sum_{i=2}^{m} a_i \binom{i}{i-1} x + \sum_{i=1}^{m} a_i \binom{i}{i} \in \mathbb{Z}[x].
\end{aligned}$$

Thus if $f(x) \in \mathbb{Z}[x]_{m+1}$ then $\Delta_F f(x) \in \mathbb{Z}[x]_m$.

Definition 4.3. For any positive integer n, let

$\Delta\mathbb{Z}[x]_n = \{f(x) \in \mathbb{Z}[x]_{n-1} |$ there exists $g(x) \in \mathbb{Z}[x]_n$ such that $\Delta_F(g(x)) = f(x)\}$.

Example 33. Let $g(x) = 2x + 1$ be a polynomial in $\mathbb{Z}[x]_2$.
Choose $f(x) = x^2 + m \in \mathbb{Z}[x]_3$ where $m \in \mathbb{Z}$.
So $\Delta_F f(x) = f(x+1) - f(x) = 2x + 1 = g(x)$.
Hence $g(x) \in \Delta\mathbb{Z}[x]_3$.

Example 34. Let $f(x) = 3x + 1$ be a polynomial in $\mathbb{Z}[x]_2$.
Suppose there exists $g(x) = ax^2 + bx + c \in \mathbb{Z}[x]_3$ such that $\Delta_F(g(x)) = 3x + 1$. Then

$$\begin{aligned}
\Delta_F(g(x)) &= g(x+1) - g(x) = a(x+1)^2 + b(x+1) + c - ax^2 - bx - c \\
&= 2ax + a + b = 3x + 1.
\end{aligned}$$

So $2a = 3$ and $a + b = 1$. Hence $a = 3/2$ which is not an integer. This is a contradiction.
Thus $f(x) \notin \Delta\mathbb{Z}[x]_3$.

4.2 Structure of ΔP_n

Theorem 4.4. $\Delta\mathbb{Z}[x]_n$ *is an abelian group under addition.*

Proof. Let $f(x), g(x), h(x) \in \Delta\mathbb{Z}_n$. Then there exist $F(x), G(x), H(x) \in \mathbb{Z}[x]_n$ such that

$$\Delta_F F(x) = F(x+1) - F(x) = f(x),$$

$$\Delta_F G(x) = G(x+1) - G(x) = g(x),$$

$$\Delta_F H(x) = H(x+1) - H(x) = h(x).$$

We next show that $\Delta\mathbb{Z}[x]_n$ is an abelian group under addition.

1. For all $f(x), g(x) \in \Delta\mathbb{Z}[x]_n$, we get

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= F(x+1) - F(x) + G(x+1) - G(x) \\ &= (F+G)(x+1) - (F+G)(x) \\ &= \Delta_F (F+G)(x).\end{aligned}$$

   Since $(\mathbb{Z}[x]_n, +)$ is a group and $F(x), G(x) \in \mathbb{Z}[x]_n$, $(F+G)(x) \in \mathbb{Z}[x]_n$. Hence $(f+g)(x) \in \Delta\mathbb{Z}[x]_n$. Thus $\Delta\mathbb{Z}[x]_n$ is closed under addition.

2. Since there exists $p(x) = m \in \mathbb{Z}[x]_n$ such that $\Delta_F p(x) = p(x+1) - p(x) = m - m = 0$ for all $m \in \mathbb{Z}$, $0 \in \mathbb{Z}[x]_n$. Moreover

$$f(x) + 0 = f(x) = 0 + f(x).$$

   So 0 is the identity.

3. For all $f(x) \in \Delta\mathbb{Z}[x]_n$ there exists $F(x) \in \mathbb{Z}[x]_n$ such that $f(x) = \Delta_F F(x)$. Thus $-f(x) \in \Delta\mathbb{Z}[x]_n$ and

$$f(x) + (-f(x)) = 0 = (-f(x)) + f(x).$$

   Therefore $-f(x)$ is the inverse of $f(x)$.

4. It is easy to see that $f(x)+g(x)=g(x)+f(x)$ for all $f,g\in\Delta\mathbb{Z}[x]_n$ and the associative law holds.

Therefore $\Delta\mathbb{Z}[x]_n$ is an abelian group under addition. □

Theorem 4.5. *ΔP_n is an abelian group under addition.*

Proof. Let $\overline{a},\overline{b},\overline{c}\in\Delta P_n$ where

$$\overline{a}=(a_1,a_2,\ldots,a_{n-1}),$$
$$\overline{b}=(b_1,b_2,\ldots,b_{n-1}),$$
$$\overline{c}=(c_1,c_2,\ldots,c_{n-1})$$

such that there exist $f(x),g(x),h(x)\in\Delta\mathbb{Z}[x]_n$ such that

$$f(i)=a_i, g(i)=b_i, h(i)=c_i$$

for $1\le i\le n-1$. We next show that ΔP_n is a group under addition.

1. For all $\overline{a},\overline{b}\in\Delta P_n$, we get

$$\begin{aligned}\overline{a}+\overline{b}&=(a_1,a_2,\ldots,a_{n-1})+(b_1,b_2,\ldots,b_{n-1})\\&=(a_1+b_1,a_2+b_2,\ldots,a_{n-1}+b_{n-1})\\&=(f(1)+g(1),f(2)+g(2),\ldots,f(n-1)+g(n-1))\\&=((f+g)(1),(f+g)(2),\ldots,(f+g)(n-1))\end{aligned}$$

   Since $f(x),g(x)\in\Delta\mathbb{Z}[x]_n$, $(f+g)(x)\in\Delta\mathbb{Z}[x]_n$.
   Hence $\overline{a}+\overline{b}\in\Delta P_n$. Thus $(\Delta P_n,+)$ is closed under addition.

2. Since $\overline{0}=(0,0,\ldots,0)=(f(1),f(2),\ldots,f(n-1))$, there exists $f(x)=0$ in $\Delta\mathbb{Z}[x]_n$ such that $f(i)=0$ for all $1\le i\le n-1$. So $\overline{0}\in\Delta P_n$. Moreover

$$\begin{aligned}\overline{a}+\overline{0}&=(a_1,a_2,\ldots,a_{n-1})+(0,0,\ldots,0)\\&=(a_1+0,a_2+0,\ldots,a_{n-1}+0)=(a_1,a_2,\ldots,a_{n-1})\\&=(0+a_1,0+a_2,\ldots,0+a_{n-1})=\overline{0}+\overline{a}.\end{aligned}$$

   So $\overline{0}$ is the identity.

3. Let $\overline{a} \in \Delta P_n$. There exists $f(x) \in \Delta\mathbb{Z}[x]_n$ such that $f(i) = a_i$ for all $1 \leq i \leq n-1$. Thus there exists $-\overline{a} = (-a_1, -a_2, \ldots, -a_{n-1}) \in \Delta P_n$ such that

$$\overline{a} + (-\overline{a}) = \overline{0} = -\overline{a} + \overline{a}.$$

Therefore $-\overline{a}$ is the inverse of $\overline{a}$.

4. It is easy to see that $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ for all $\bar{a}, \bar{b} \in \Delta P_N$ and the associative law holds.

Therefore ΔP_n is an abelian group under addition. □

Theorem 4.6. *The group $\Delta\mathbb{Z}[x]_n$ is isomorphic to ΔP_n.*

Proof. We will show that $\Delta\mathbb{Z}[x]_n \cong \Delta P_n$.
Define a map $v : \Delta\mathbb{Z}[x]_n \to \Delta P_n$ by $v(f(x)) = (f(1), f(2), \ldots, f(n-1))$.
Let $f, g \in \Delta\mathbb{Z}[x]_n$. Then

$$\begin{aligned} v(f+g) &= \big((f+g)(1), (f+g)(2), \ldots, (f+g)(n-1)\big) \\ &= \big(f(1), f(2), \ldots, f(n-1)\big) + \big(g(1), g(2), \ldots, g(n-1)\big) \\ &= v(f) + v(g). \end{aligned}$$

Thus v is an additive homomorphism.
We next show that v restricted to $\Delta\mathbb{Z}[x]_n$ is an isomorphism from $\Delta\mathbb{Z}[x]_n$ to ΔP_n.
Let $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \Delta P_n$. Then there exists $f(x) \in \mathbb{Z}[x]$ such that $\Delta_F f(i) = c_i$ for all $1 \leq i \leq n-1$.
Let $p_n(x) := (x-1)(x-2)\cdots(x-n) \in \mathbb{Z}[x]$, $\deg p_n(x) = n$. Since $p_n(x)$ is monic, by division algorithm, $f(x) = q(x)p_n(x) + r(x)$, where $q, r \in \mathbb{Z}[x]$ with $\deg r < n$. Then we get

$$\Delta_F f(x) = f(x+1) - f(x) = q(x+1)p_n(x+1) + r(x+1) - q(x)p_n(x) - r(x).$$

Since $p_n(x+1) = xp_{n-1}(x)$,

$$\Delta_F f(x) = p_{n-1}(x)\left[xq(x+1) - (x-n)q(x)\right] + \Delta_F r(x)$$

with $\deg \Delta_F r(x) < n-1$. Thus $\Delta_F r(x) \in \Delta\mathbb{Z}[x]_n$.

Evaluating at the points i for all $1 \le i \le n-1$, we see that $\Delta_F r(x)$ generates the sequence $\bar{c}$. Hence there exists $r(x) \in \mathbb{Z}[x]_n$ such that $\Delta_F r(x)$ generates $\bar{c}$. Therefore v is onto.

Let $f, g \in \Delta\mathbb{Z}[x]_n$. Suppose $v(f(x)) = v(g(x))$. Then $\big(f(1), f(2), \ldots,$ $f(n-1)\big) = \big(g(1), g(2), \ldots, g(n-1)\big)$. So $f(i) = g(i)$ for all $1 \le i \le n-1$. Since $\deg(f)$ and $\deg(g)$ are less than $n-1$, and the polynomials f, g agree at $n-1$ distinct points, they are identical, i.e., v is one-to-one.

Therefore v is an isomorphism. □

Theorem 4.7. *For any positive integer* $n \ge 2$*,*

$$\mathbb{Z}^{n-1}/\Delta P_n \cong \mathbb{Z}/1!\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}.$$

Proof. We first observe that $\{1, 2p_1(x), 3p_2(x), \ldots, (n-1)p_{n-2}(x)\}$ is a $\mathbb{Z}$-basis for $\Delta\mathbb{Z}[x]_n$. Since the map $v : \Delta\mathbb{Z}[x]_n \to \Delta P_n$ is an isomorphism, $\{v\big((j+1)p_j(x)\big) \mid j = 0, 1, \ldots, n-2\}$ is a $\mathbb{Z}$-basis for ΔP_n.

For $i, j \in \{1, 2, \ldots, n-1\}$, let $C^*_{n-1} = (c^*_{i,j})$ where

$$c^*_{i,j} = \begin{cases} j(i-1)_{j-1} = j(i-1)(i-2)\cdots(i-j) & \text{if } 2 \le j \le i \le n-1 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } i < j, \text{ i.e.,} \end{cases}$$

$$C^*_{n-1} = \begin{bmatrix} 1! & 0 & 0 & 0 & \ldots & 0 \\ 1 & 2! & 0 & 0 & \ldots & 0 \\ 1 & 2\cdot(2)_1 & 3! & 0 & \ldots & 0 \\ 1 & 2\cdot(3)_1 & 3\cdot(3)_2 & 4! & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cdot(n-2)_1 & 3\cdot(n-2)_2 & 4\cdot(n-2)_3 & \ldots & (n-1)! \end{bmatrix}.$$

We see that $v((j)p_{j-1}(x))$ is the jth column of C^*_{n-1} for $j = 1, 2, \ldots, n-1$.

Now let $A^*_{n-1} = (a^*_{i,j})$ where

$$a^*_{i,j} = \begin{cases} \binom{i-1}{j-1} & \text{if } 2 \leq j \leq i \leq n-1 \\ 1 & \text{if } j = 1 \\ 0 & \text{if } i < j. \end{cases}$$

Thus

$$A^*_{n-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \binom{2}{1} & 1 & 0 & \dots & 0 \\ 1 & \binom{3}{1} & \binom{3}{2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & \binom{n-2}{3} & \dots & 1 \end{bmatrix}.$$

Let $e(j-1)$ be the j^{th} column of A^*_{n-1} for all $j = 1, 2, \dots, n-1$. Since

$$a^*_{i,j} = \binom{i-1}{j-1} \in \mathbb{Z} \qquad (2 \leq j \leq i),$$

and $\det A^*_{n-1} = 1$, A^*_n is an unimodular by Theorem 2.2. In this case, we see that $\{e(j-1), j = 1, 2, \dots, n-1\}$ forms a $\mathbb{Z}-$basis for $\mathbb{Z}^{n-1}$.

Next, let D^*_{n-1} be the diagonal matrix whose j^{th} diagonal entry is $j!$ for all $j = 1, 2, \dots, n-1$, i.e.,

$$D^*_{n-1} = \begin{bmatrix} 1! & 0 & 0 & \dots & 0 \\ 0 & 2! & 0 & \dots & 0 \\ 0 & 0 & 3! & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (n-1)! \end{bmatrix}.$$

It is not hard to see that $C^*_{n-1} = A^*_{n-1}D^*_{n-1}$ i.e.,

$$\begin{bmatrix} 1! & 0 & 0 & 0 & \dots & 0 \\ 1 & 2! & 0 & 0 & \dots & 0 \\ 1 & 2\cdot(2)_1 & 3! & 0 & \dots & 0 \\ 1 & 2\cdot(3)_1 & 3\cdot(3)_2 & 4! & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cdot(n-2)_1 & 3\cdot(n-2)_2 & 4\cdot(n-2)_3 & \dots & (n-1)! \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & \binom{2}{1} & 1 & 0 & \dots & 0 \\ 1 & \binom{3}{1} & \binom{3}{2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & \binom{n-2}{3} & \dots & 1 \end{bmatrix} \begin{bmatrix} 1! & 0 & 0 & \dots & 0 \\ 0 & 2! & 0 & \dots & 0 \\ 0 & 0 & 3! & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (n-1)! \end{bmatrix}.$$

We have $v(jp_{j-1}(x)) = j!e(j-1)$ for all $j = 1, 2, \dots, n-1$.

So $\{j!e(j-1) : j = 1, 2, \dots, n-1\}$ forms a basis for ΔP_n. Then it is easy to see that

$$\begin{aligned} \mathbb{Z}^{n-1}/\Delta P_n &= \frac{\langle e(0)\rangle \oplus \langle e(1)\rangle \oplus \langle e(2)\rangle \oplus \dots \oplus \langle e(n-2)\rangle}{1!\langle e(0)\rangle \oplus 2!\langle e(1)\rangle \oplus 3!\langle e(2)\rangle \oplus \dots \oplus (n-1)!\langle e(n-2)\rangle} \\ &= \frac{\langle e(0)\rangle}{\langle e(0)\rangle} \oplus \frac{\langle e(1)\rangle}{2!\langle e(1)\rangle} \oplus \frac{\langle e(2)\rangle}{3!\langle e(2)\rangle} \oplus \dots \oplus \frac{\langle e(n-2)\rangle}{(n-1)!\langle e(n-2)\rangle} \\ &\cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \mathbb{Z}/4!\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}. \end{aligned}$$

□

Theorem 4.8. *For $n \geq 2$, $P_{n-1}/\Delta P_n \cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/(n-1)\mathbb{Z}$.*

Proof. Since $\{(j-1)!e(j-1) \mid j = 1, \dots, n-1\}$ forms a $\mathbb{Z}$-basis for P_{n-1} and $\{j!e(j-1) \mid j = 1, 2, \dots, n-1\}$ forms a $\mathbb{Z}$-basis for ΔP_n, we have

$$\begin{aligned} \frac{P_{n-1}}{\Delta P_n} &= \frac{0!\langle e(0)\rangle \oplus 1!\langle e(1)\rangle \oplus 2!\langle e(2)\rangle \oplus \dots \oplus (n-2)!\langle e(n-2)\rangle}{1!\langle e(0)\rangle \oplus 2!\langle e(1)\rangle \oplus 3!\langle e(2)\rangle \oplus \dots \oplus (n-1)!\langle e(n-2)\rangle} \\ &= \frac{\langle e(0)\rangle}{\langle e(0)\rangle} \oplus \frac{\langle e(1)\rangle}{2\langle e(1)\rangle} \oplus \frac{\langle e(2)\rangle}{3\langle e(2)\rangle} \oplus \dots \oplus \frac{\langle e(n-2)\rangle}{(n-1)\langle e(n-2)\rangle} \end{aligned}$$

$$\cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)\mathbb{Z}.$$

□

We next find a necessary and sufficient condition to have $\bar{c} \in \Delta P_n$.

Theorem 4.9. *Let* $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$ *with* $N_{\bar{c}}(x) = \sum_{i=0}^{n-2} d_i p_i(x)$ *for some* $d_i \in \mathbb{Q}$. *Then* $\bar{c} \in \Delta P_n$ *if and only if* $(i+1)|d_i$ *for all* $0 \leq i \leq n-2$.

Proof. ($\Rightarrow$) Let $\bar{c} \in \Delta P_n$. There exists $\bar{a} \in P_n$ such that $\Delta \bar{a} = \bar{c}$.
Let $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$. Since $\Delta_F N_{\bar{a}}(i) = N_{\bar{a}}(i+1) - N_{\bar{a}}(i) = a_{i+1} - a_i = c_i$ for all $1 \leq i \leq n-1$, $\Delta_F N_{\bar{a}}(x)$ generates $\bar{c}$. Since $N_{\bar{c}}(x)$ and $\Delta_F N_{\bar{a}}(x)$ are polynomials whose degree less than $n-1$ and they agree on $n-1$ points, $N_{\bar{c}}(x) = \Delta_F N_{\bar{a}}(x)$. So

$$\begin{aligned} \sum_{i=0}^{n-2} d_i p_i(x) &= \sum_{i=1}^{n-1} b_i i p_{i-1}(x) \\ &= \sum_{i=0}^{n-2} b_{i+1}(i+1) p_i(x). \end{aligned}$$

Since $\{p_0(x), p_1(x), \ldots, p_{n-2}(x)\}$ is linearly independent,

$$d_i = (i+1) b_{i+1}, \quad 0 \leq i \leq n-2.$$

Thus $b_{i+1} = \frac{d_i}{i+1} \in \mathbb{Z}$ for all $i = 0, 1, \ldots, n-2$ because $\bar{a} \in P_n$.
Therefore $(i+1)|d_i$ for all $0 \leq i \leq n-2$.

($\Leftarrow$) Let $a_1 = 1$ and $a_{i+1} = c_i + a_i$ for all $1 \leq i \leq n-1$. Similarly as in the other direction, let $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$. We have $d_i = b_{i+1}(i+1)$. Thus $b_{i+1} = \frac{d_i}{i+1} \in \mathbb{Z}$.
Therefore $\bar{a} \in P_n$.
$\therefore \bar{c} \in \Delta P_n$. □

Theorem 4.10. *Suppose* $\bar{c}_1 \in \mathbb{Z}^{n-1}$. *Let* $\bar{c}_2 = \Delta \bar{c}_1, \bar{c}_3 = \Delta \bar{c}_2 = \Delta^2 \bar{c}_1, \ldots,$ $\bar{c}_k = \Delta \bar{c}_{k-1} = \Delta^{k-1} \bar{c}_1$ *for* $1 \leq k \leq n-2$. *There exists* $\bar{a} \in P_n$ *such that* $\Delta^j \bar{a} = \bar{c}_j$ *for* $1 \leq j \leq k$ *if and only if*

1. $(i+1)(i+2)\cdots(i+k)|b_i^{(k)}$ *where*

$$N_{\bar{c}_k}(x)=\sum_{i=0}^{n-k-1} b_i^{(k)}p_i(x)$$

for all $0\le i\le n-k-1$ *and*

2. $j!|a_1^{(j)}$ *where* $\Delta^j\bar{a}=(a_1^{(j)},a_2^{(j)},\ldots,a_{n-j}^{(j)})$ *for all* $1\le j\le k$.

Proof. $(\Rightarrow)$ Let $\bar{a}\in P_n$ such that

$$\begin{aligned}\Delta\bar{a}&=(a_1^{(1)},a_2^{(1)},\ldots,a_{n-1}^{(1)})=\bar{c}_1,\\ \Delta^2\bar{a}&=(a_1^{(2)},a_2^{(2)},\ldots,a_{n-2}^{(2)})=\bar{c}_2,\\ \Delta^3\bar{a}&=(a_1^{(3)},a_2^{(3)},\ldots,a_{n-3}^{(3)})=\bar{c}_3,\\ &\vdots\\ \Delta^k\bar{a}&=(a_1^{(k)},a_2^{(k)},\ldots,a_{n-k}^{(k)})=\bar{c}_k.\end{aligned}$$

We will prove by induction. The statement is true for $k=1$ by Theorem 4.9.
Assume the statement holds for $k-1$. We must show that it is true for k.
Suppose $\Delta^j\bar{a}=\bar{c}_j$ for $1\le j\le k$.
Consider $N_{\bar{c}_{k-1}}(x)=\sum_{i=0}^{n-k} b_i^{(k-1)}p_i(x)$.
By assumption, $(i+1)(i+2)\cdots(i+k-1)|b_i^{(k-1)}$ and $j!|a_1^{(j)}$ for all $0\le i\le n-k$, $1\le j\le k-1$. Since $\Delta N_{\bar{c}_{k-1}}(x)=\sum_{i=0}^{n-k-1}(i+1)b_{i+1}^{(k-1)}p_i(x)=N_{\bar{c}_k}(x)=\sum_{i=0}^{n-k-1}b_i^{(k)}p_i(x)$,

$$b_i^{(k)}=(i+1)b_{i+1}^{(k-1)}$$

for all $0\le i\le n-k-1$.
Since $(i+1)(i+2)\cdots(i+k-1)|b_i^{(k-1)}$ for all $0\le i\le n-k$, we have $(i+2)(i+3)\cdots(i+k)|b_{i+1}^{(k-1)}$ for all $0\le i\le n-k-1$. Hence

$$(i+1)(i+2)\cdots(i+k)|b_i^{(k)}$$

for $0\le i\le n-k-1$.
We next consider $a_1^{(k)}$. Since $a_1^{(k)}=b_0^{(k)}$, $k!|a_1^{(k)}$ as desired.

($\Leftarrow$) We will show by induction on k. If $k = 1$, then the statement holds by Theorem 4.9.

Assume that the statement holds for all $k-1$. Suppose $(i+1)(i+2)\cdots(i+k)|b_i^{(k)}$ for $0 \le i \le n-k-1$ and $j!|a_1^{(j)}$ for all $1 \le j \le k$.

Let $N_{\bar{c}_{k-1}}(x) = \sum_{i=0}^{n-k} b_i^{(k-1)} p_i(x)$. Since $\Delta N_{\bar{c}_{k-1}}(x) = N_{\bar{c}_k}(x)$, $(i+1)b_{i+1}^{(k-1)} = b_i^{(k)}$ for $1 \le i \le n-k-1$.

Thus $b_{i+1}^{(k-1)} = \frac{b_i^{(k)}}{i+1}$. By assumption $(i+2)(i+3)\cdots(i+k)|b_{i+1}^{(k-1)}$ for $0 \le i \le n-k-1$. Hence $(i+1)(i+2)\cdots(i+k-1)|b_i^{(k-1)}$ for $0 \le i \le n-k$. Now for $i = 0$, $b_0^{(k-1)} = a_1^{(k-1)}$. Since $(k-1)!|a_1^{(k-1)}$, $(k-1)!|b_0^{(k-1)}$.

Then by an inductive hypotheses, there exists $\bar{a} \in P_n$ such that $\Delta^i \bar{a} = \bar{c}_i$ for all $1 \le i \le k-1$. $\square$

Example 35. Let $\bar{a} = (2, 17, 82, 257, 626) \in \mathbb{Z}^5$. Since $N_{\bar{a}}(x) = 2 + 15(x-1) + 28(x-1)(x-2) + 10(x-1)(x-2)(x-3) + (x-1)(x-2)(x-3)(x-4) = x^4 + 1 \in \mathbb{Z}[x]$, $\bar{a} \in P_5$. Then $\Delta\bar{a} = (15, 65, 175, 369) \in P_4$, $\Delta^2\bar{a} = (50, 110, 194) \in P_3$. We see that $N_{\Delta^2\bar{a}}(x) = 50 + 60(x-1) + 12(x-1)(x-2)$ generates the sequence $\Delta^2\bar{a} = (50, 110, 194) \in P_3$ and $2|50, 6|60, 12|12$. So $(i+1)(i+2)|b_i$ for $i = 0, 1, 2$ where $b_0 = 50, b_1 = 60$ and $b_2 = 12$.

Example 36. Let $\bar{a} = (3, 66, 731, 4098, 15627, 46658, 117651)$. Since the polynomial $N_{\bar{a}}(x) = 3 + 63(x-1) + 301(x-1)(x-2) + 350(x-1)(x-2)(x-3) + 140(x-1)(x-2)(x-3)(x-4) + 21(x-1)(x-2)(x-3)(x-4)(x-5) + (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = x^6 + 2$ generates $\bar{a}$, $\bar{a} \in P_7$. Then

$\Delta\bar{a} = (\underline{63}, 665, 3367, 11529, 31031, 70993) \in P_6$ since

$$\Delta_F N_{\bar{a}}(x) = 63 + 602(x-1) + 1050(x-1)(x-2) + 560(x-1)(x-2)(x-3) + 105(x-1)(x-2)(x-3)(x-4) + 6(x-1)(x-2)(x-3)(x-4)(x-5),$$

$\Delta^2\bar{a} = (\underline{602}, 2702, 8162, 19502, 39962) \in P_5$ since

$$\Delta_F N_{\Delta\bar{a}}(x) = 602 + 2100(x-1) + 1680(x-1)(x-2) + 420(x-1)(x-2)(x-3) + 30(x-1)(x-2)(x-3)(x-4),$$

$\Delta^3\bar{a} = (\underline{2100}, 5460, 11340, 20460) \in P_4$ since

$\Delta_F N_{\Delta^2\bar{a}}(x) = 2100 + 3360(x-1) + 1260(x-1)(x-2)+$

$120(x-1)(x-2)(x-3),$

$\Delta^4\bar{a} = (\underline{3360}, 5880, 9120) \in P_3$ since

$\Delta_F N_{\Delta^3\bar{a}}(x) = 3360 + 2520(x-1) + 360(x-1)(x-2).$

We see that $N_{\Delta^4\bar{a}}(x) = \sum_{i=0}^{2} b_i p_i(x) = 3360 + 2520(x-1) + 360(x-1)(x-2)$ generates $\Delta^4\bar{a}$ where $(i+1)(i+2)(i+3)(i+4)|b_i$ for all $i = 0, 1, 2$ and $j!|a_1^{(j)}$ where $a_1^{(j)}$ is the first term of $\Delta^j\bar{a}$ for all $j = 1, 2, 3, 4$.

Example 37. Let $\bar{c} = (2, 3, 10) \in \mathbb{Z}^3$. Since $N_{\bar{c}}(x) = 2 + (x-1) + 3(x-1)(x-2)$ in $\mathbb{Z}[x]$, $\bar{c} \in P_3$. Assume that $\bar{c} \in \Delta P_4$. Then there exists

$$N(x) = a_0 + a_1(x-1) + a_2(x-1)(x-2) + a_3(x-1)(x-2)(x-3) \in \mathbb{Z}[x]_4$$

such that $\Delta_F N(x) = N_{\bar{c}}(x)$.
Thus

$$\Delta_F N(x) = a_1 + 2a_2(x-1) + 3a_3(x-1)(x-2) = 2 + (x-1) + 3(x-1)(x-2).$$

So $a_2 = \frac{1}{2} \notin \mathbb{Z}$. This is a contradiction.
Therefore $\bar{c} \notin \Delta P_3$.

In fact, if we let

$$N_{\bar{c}}(x) = N_{\bar{c}}(x) = b_0 + b_1(x-1) + b_2(x-1)(x-2) = 2 + (x-1) + 3(x-1)(x-2).$$

Then $b_0 = 2, b_1 = 1$ and $b_2 = 3$. By Theorem 4.10 it is easy to see that $\bar{c} \notin \Delta P_4$ because $2 \nmid b_1$.

CHAPTER 5

Differential Polynomial Sequences

Let D be an integral domain. For $I = (i_1, i_2, \ldots, i_n) \in D^n$ with $i_j \neq i_k$ if $j \neq k$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. We call $\mathscr{A}$ a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I if there exists $f(x) \in D[x]$ such that

$$\begin{array}{cccc} f(i_1) = a_1^0, & f(i_2) = a_2^0, & \ldots, & f(i_n) = a_n^0, \\ f'(i_1) = a_1^1, & f'(i_2) = a_2^1, & \ldots, & f'(i_n) = a_n^1, \\ \vdots & \vdots & \ddots & \vdots \\ f^{(r_1)}(i_1) = a_1^{r_1}, & f^{(r_2)}(i_2) = a_2^{r_2}, & \ldots, & f^{(r_n)}(i_n) = a_n^{r_n}. \end{array}$$

Criteria for a sequence to be a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I is established.

5.1 Introduction

For a fixed $n \in \mathbb{N}$, let D be an integral domain, $I = (i_1, i_2, \ldots, i_n) \in D^n$, $R = (r_1, r_2, \ldots, r_n)$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D for which there exists $f(x) \in D[x]$ such that $f^{(m)}(i_j) = a_j^m$ for all $j = 1, 2, \ldots, n$ and $m = 0, 1, \ldots, r_j$; where $f^{(m)}(i_j) = a_j^m$ denotes the m^{th} derivative of $f(x)$ evaluated at the point i_j, be a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. Denote by $\wp_{n,I}^R$ the set of all differential polynomial sequences of length n and order $(r_1, \ldots, r_n)$ with respect to I.

In this chapter we will first characterize $\wp_{n,I}^{R}$ over D using the generalization of Hermite's interpolation formula.

Definition 5.1. Let D be an integral domain, $R = (r_1, r_2, \ldots, r_n)$, $I = (i_1, i_2, \ldots, i_n)$ in D^n where $i_j \neq i_m$ if $j \neq m$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Let

$$\begin{aligned} \wp_{n,I}^{R} = \{\mathscr{A} = &\left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right) \\ &\text{where } a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n} \in D \mid \text{there exists} \\ &f(x) \in D[x] \text{ such that } f^{(m)}(i_j) = a_j^m, \text{ for all } 1 \leq j \leq n, 0 \leq m \leq r_j\} \end{aligned}$$

be the set of all differential polynomial sequences of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. We call $\mathscr{A}$ an element in $\wp_{n,I}^{R}$, a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I.

If $r_j = k$ for all $j = 1, 2, \ldots, n$, then we let $\wp_{n,I}^{(k)} = \wp_{n,I}^{R}$. We call $\mathscr{A}$ an element in $\wp_{n,I}^{(k)}$, a differential polynomial sequence of length n and order k with respect to I. We next define some definition as follows:

1. If $D = \mathbb{Z}$ and $I = (1, 2, \ldots, n)$, then we let $\wp_{n}^{(k)} = \wp_{n,I}^{(k)}$. We call $\mathscr{A}$ an element in $\wp_{n}^{(k)}$, a differential polynomial sequence of length n and order k.

2. If $I = (c)$ where $c \in D$, then we let $\wp_{1,c}^{(k)} = \wp_{n,I}^{(k)}$. We call $\mathscr{A}$ an element in $\wp_{1,c}^{(k)}$, a differential polynomial sequence of length 1 and order k with respect to c.

Example 38. Let $D = \mathbb{Z}$, $I = (1, 4, 5)$ and $\mathscr{A} = ((4, 8), (1036, 1283), (3140, 3128))$. We see that there exists $f(x) = x^5 + 6x^3 \in \mathbb{Z}[x]$ such that

$$f(1) = 4, f(4) = 1036, f(5) = 3140,$$

$$f'(1) = 8, f'(4) = 1283, f'(5) = 3128.$$

So $\mathscr{A} \in \wp_{3,I}^{(1)}$.

Example 39. Let $D = \mathbb{Z}$, $I = (1, 2, 3)$ and $\mathscr{A} = ((4, 8), (38, 83), (252, 408))$. We see that there exists $f(x) = x^5 + 6x^3 \in \mathbb{Z}[x]$ such that

$$f(1) = 4, f(2) = 38, f(3) = 252,$$
$$f'(1) = 8, f'(2) = 83, f'(3) = 408.$$

So $\mathscr{A} \in \wp_3^{(1)}$.

Example 40. Let $D = \mathbb{Z}$, $c = 2$ and $\mathscr{A} = ((38, 83, 160, 240, 240, 120))$. We see that there exists $f(x) = x^5 + 6x^3$ such that

$$f(2) = 38, f'(2) = 83, f''(2) = 160, f^{(3)}(2) = 240, f^{(4)}(2) = 240, f^{(5)}(2) = 120.$$

So $\mathscr{A} \in \wp_{1,2}^{(5)}$.

Throughout let D be an integral domain, D_Q be a quotient field of D, $I = (i_1, i_2, \dots, i_n)$ in D^n, $R = (r_1, r_2, \dots, r_n)$ and n be a positive integer. The set $\wp_{n,I}^R$ is a group under addition. We will show this as follows:
For any

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \dots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \dots, a_2^{r_2}\right), \dots, \left(a_n^0, a_n^1, \dots, a_n^{r_n}\right)\right)$$

and

$$\mathscr{B} = \left(\left(b_1^0, b_1^1, \dots, b_1^{r_1}\right), \left(b_2^0, b_2^1, \dots, b_2^{r_2}\right), \dots, \left(b_n^0, b_n^1, \dots, b_n^{r_n}\right)\right)$$

where $a_1^0, \dots, a_1^{r_1}, \dots, a_n^0, \dots, a_n^{r_n}, b_1^0, \dots, b_1^{r_1}, \dots, b_n^0, \dots, b_n^{r_n}$ are elements in D. we define

$$\mathscr{A} + \mathscr{B} = \left(\left(a_1^0 + b_1^0, \dots, a_1^{r_1} + b_1^{r_1}\right), \dots, \left(a_n^0 + b_n^0, \dots, a_n^{r_n} + b_n^{r_n}\right)\right)$$

where $a_1^0 + b_1^0, \dots, a_1^{r_1} + b_1^{r_1}, \dots, a_n^0 + b_n^0, \dots, a_n^{r_n} + b_n^{r_n}$ are elements in D. It is easy to see that $\mathscr{A} + \mathscr{B} = \mathscr{B} + \mathscr{A}$.

Theorem 5.1. *$\wp_{n,I}^R$ is an abelian group under addition.*

Proof. Let $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \wp_{n,I}^R$ where

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \dots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \dots, a_2^{r_2}\right), \dots, \left(a_n^0, a_n^1, \dots, a_n^{r_n}\right)\right),$$

$$\mathscr{B} = \left(\left(b_1^0, b_1^1, \ldots, b_1^{r_1}\right), \left(b_2^0, b_2^1, \ldots, b_2^{r_2}\right), \ldots, \left(b_n^0, b_n^1, \ldots, b_n^{r_n}\right)\right),$$
$$\mathscr{C} = \left(\left(c_1^0, c_1^1, \ldots, c_1^{r_1}\right), \left(c_2^0, c_2^1, \ldots, c_2^{r_2}\right), \ldots, \left(c_n^0, c_n^1, \ldots, c_n^{r_n}\right)\right).$$

Then there exist $f(x), g(x), h(x) \in D[x]$ such that

$$f^{(m)}(i_j) = a_j^m, g^{(m)}(i_j) = b_j^m, h^{(m)}(i_j) = c_j^m$$

for $1 \leq j \leq n$ and $0 \leq m \leq r_j$. We next show that $\wp_{n,I}^R$ is a group under addition.

1. For all $\mathscr{A}, \mathscr{B} \in \wp_{n,I}^R$, we have

   $$\mathscr{A} + \mathscr{B} = \left(\left(a_1^0 + b_1^0, \ldots, a_1^{r_1} + b_1^{r_1}\right), \ldots, \left(a_n^0 + b_n^0, \ldots, a_n^{r_n} + b_n^{r_n}\right)\right)$$
   $$= \left(\left((f+g)(i_1), \ldots, (f+g)^{(r_1)}(i_1)\right), \ldots, \left((f+g)(i_n), \ldots, (f+g)^{(r_n)}(i_n)\right)\right).$$

   Since $f(x), g(x) \in D[x]$, $(f+g)(x) \in D[x]$.
   Hence $\mathscr{A} + \mathscr{B} \in \wp_{n,I}^R$.
   Thus $(\wp_{n,I}^R, +)$ is closed under addition.

2. Let $\bar{0} = ((0, 0, \ldots, 0), (0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0))$. There exists $f(x) = 0$ in $D[x]$ such that $f^{(m)}(i_j) = 0$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$. So $\overline{0} \in \wp_{n,I}^R$. Moreover

   $$\mathscr{A} + \bar{0} = \mathscr{A} = \bar{0} + \mathscr{A}.$$

   So $\overline{0}$ is the identity.

3. For all $\mathscr{A} \in \wp_{n,I}^R$, there exists

   $$-\mathscr{A} = \left(\left(-a_1^0, -a_1^1, \ldots, -a_1^{r_1}\right), \ldots, \left(-a_n^0, -a_n^1, \ldots, -a_n^{r_n}\right)\right)$$

   a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ generated by $-f(x)$ such that

   $$\mathscr{A} + (-\mathscr{A}) = \overline{0} = -\mathscr{A} + \mathscr{A}.$$

   Thus $-\mathscr{A}$ is the inverse of $\mathscr{A}$.

4. It is easy to see that the associative law holds.

Therefore $\wp^R_{n,I}$ is an abelian group under addition. $\square$

Lemma 5.2. *If $\mathscr{A} \in \wp^R$ then $c \cdot \mathscr{A} \in \wp^R_{n,I}$ for any $c \in D$.*

Proof. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}\right), \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Then there exists $f(x) \in D[x]$ such that $f^{(m)}(i_j) = a_j^m$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$. Then

$$\begin{aligned}
c \cdot \mathscr{A} &= c \cdot \left(\left(a_1^0, \ldots, a_1^{r_1}\right), \left(a_2^0, \ldots, a_2^{r_2}\right), \ldots, \left(a_n^0, \ldots, a_n^{r_n}\right)\right) \\
&= \left(\left(c \cdot a_1^0, \ldots, c \cdot a_1^{r_1}\right), \left(c \cdot a_2^0, \ldots, c \cdot a_2^{r_2}\right), \ldots, \left(c \cdot a_n^0, \ldots, c \cdot a_n^{r_n}\right)\right) \\
&= \left((cf(i_1), \ldots, cf^{(r_1)}(i_1)), (cf(i_2), \ldots, cf^{(r_2)}(2)), \ldots, (cf(i_n), \ldots, cf^{(r_n)}(i_n))\right)
\end{aligned}$$

So $g(x) = c \cdot f(x) \in D[x]$ generates $c \cdot \mathscr{A}$.
Thus $c \cdot \mathscr{A} \in \wp^R_{n,I}$. $\square$

5.2 Properties of the differential polynomial sequences

For a fixed sequence I as above and a sequence

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D, there is the generalization of Hermite's formula $H_{\mathscr{A}}(x)$ such that

$$H_{\mathscr{A}}^{(m)}(i_j) = a_j^m \;\; (1 \leq j \leq n, 0 \leq m \leq r_j).$$

Lemma 5.3. *Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ and*

$$p(x) := (x - i_1)^{r_1+1}(x - i_2)^{r_2+1} \cdots (x - i_n)^{r_n+1}$$

with degree $n + \sum_{j=1}^{n} r_j$. Then $p^{(m)}(i_j) = 0$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$.

Proof. Let $I = (i_1, i_2, \ldots, i_n) \in D^n$,

$$p(x) := (x - i_1)^{r_1+1}(x - i_2)^{r_2+1} \cdots (x - i_n)^{r_n+1} \in D[x],$$

and $B_j(x) = (x - i_1)^{r_1+1} \cdots (x - i_{j-1})^{r_{j-1}+1}(x - i_{j+1})^{r_{j+1}+1} \cdots (x - i_n)^{r_n+1}$ for all $1 \le j \le n$. So $p(x) = (x - i_j)^{r_j+1} B_j(x)$ for all $1 \le j \le n$. By Leibniz's Lemma we have

$$\begin{aligned} p^{(m)}(x) &= \sum_{t=0}^{m} \binom{m}{t} \left((x - i_j)^{r_j+1}\right)^{(t)} B_j^{(m-t)}(x) \\ &= (x - i_j)^{r_j+1} B_j^{(m)}(x) + \sum_{t=1}^{m} \binom{m}{t} (r_j + 1)_{t-1} (x - i_j)^{r_j - t + 1} B_j^{(m-t)}(x). \end{aligned}$$

It is easy to see that $p^{(m)}(i_j) = 0$ for all $1 \le j \le n$ and $0 \le m \le r_j$.
Thus $p^{(m)}(i_j) = 0$ for all $1 \le j \le n$ and $0 \le m \le r_j$. □

Theorem 5.4. *Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and*

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Then $\mathscr{A}$ is a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I if and only if $H_{\mathscr{A}}(x)$ is a polynomial in $D[x]$ of degree less than $n + \sum_{j=1}^{n} r_j$. Furthermore, $H_{\mathscr{A}}(x)$ is the unique polynomial of degree $< n + \sum_{j=1}^{n} r_j$ in $D_Q[x]$ where D_Q is the quotient of D, such that

$$H_{\mathscr{A}}^{(m)}(i_j) = a_j^m \quad (1 \le j \le n, 0 \le m \le r_j).$$

Proof. If $\mathscr{A} \in \wp_{n,I}^{R}$, then there exists $f(x)$ in $D[x]$ such that $f^{(m)}(i_j) = a_j^m$ for all $1 \le j \le n, 0 \le m \le r_j$. Let $p(x) := (x - i_1)^{r_1+1}(x - i_2)^{r_2+1} \cdots (x - i_n)^{r_n+1}$ be a polynomial in $D[x]$. Thus $\deg p(x) = n + \sum_{j=1}^{n} r_j$. Since $p(x)$ is monic, by the division algorithm, $f(x) = q(x)p(x) + r(x)$, where $q, r \in D[x]$ with
$\deg r < n + \sum_{j=1}^{n} r_j$. Then

$$\begin{aligned} f'(x) &= q'(x)p(x) + q(x)p'(x) + r'(x) \\ f''(x) &= q''(x)p(x) + 2q'(x)p'(x) + q(x)p''(x) + r''(x) \end{aligned}$$

$$\vdots$$

$$f^{(r_j)}(x) = \sum_{m=0}^{r_j} \binom{r_j}{m} p^{(m)}(x) q^{(r_j-m)}(x) + r^{(r_j)}(x).$$

From Lemma 5.3, we have $p^{(m)}(i_j) = 0$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$. Evaluating at the points i_j $(1 \leq j \leq n)$, we see that $r^{(m)}(i_j) = a_j^m$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$. Since $H_{\mathscr{A}}(x)$ is the unique polynomial in $D_Q[x]$ of degree less than $n + \sum_{j=1}^{n} r_j$ such that $H_{\mathscr{A}}^{(m)}(i_j) = a_j^m$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$, $r(x) = H_{\mathscr{A}}(x)$. The remaining assertions are trivial. □

Given a set of $n + \sum_{j=1}^{n} r_j$ points (i_j, a_j^m) $(j = 1, \ldots, n, m = 0, 1, \ldots, r_j)$, with distinct i_j, and a_j^m being in D, the Newton form of generalization of Hermite's formula corresponding to the points (i_j, a_j^m) $(j = 1, \ldots, n, m = 0, 1, \ldots, r_j)$ is defined as

$$\mathscr{N}_{\mathscr{A}}(x) = [i_1] + [i_1, i_1]p_{i_1}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2]p_{i_1}^{r_1+1}(x) +$$

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2, i_2]p_{i_2}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]p_{i_n}^{r_n}(x)$$

where

$$p_{i_j}^q(x) = \prod_{h=1}^{j-1}(x - i_h)^{r_h+1}(x - i_j)^q, \qquad 1 \leq j \leq n, 1 \leq q \leq r_j + 1$$

and

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_j, \ldots, i_j}_{r_j+1}] = \sum_{k=1}^{j} \sum_{m=0}^{r_k} \frac{1}{m!} \frac{1}{(r_k - m)!} g_{i_k}^{(r_k-m)}(i_k) a_k^m$$

for $1 \leq j \leq n$ and

$$g_{i_k}(x) = \frac{1}{(x - i_1)^{r_1+1} \cdots (x - i_{k-1})^{r_{k-1}+1}(x - i_{k+1})^{r_{k+1}+1} \cdots (x - i_j)^{r_j+1}}$$

for $1 \leq k \leq j$. The elements $1, p_{i_1}(x), \ldots, p_{i_n}^{r_n}(x)$ are referred to as the basis polynomial of the corresponding the Newton form of generalization of Hermite's formula.

Theorem 5.5. *Let*

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ *are elements in* D*. Let*

$$\mathscr{N}_{\mathscr{A}}(x) = [i_1] + [i_1, i_1]p_{i_1}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2]p_{i_1}^{r_1+1}(x) +$$

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2, i_2]p_{i_2}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]p_{i_n}^{r_n}(x)$$

be a polynomial in $D_Q[x]$ *where*

$$p_{i_j}^q(x) = \prod_{h=1}^{j-1}(x - i_h)^{r_h+1}(x - i_j)^q, \qquad 1 \le j \le n, 1 \le q \le r_j + 1$$

and

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_j, \ldots, i_j}_{r_j+1}] = \sum_{k=1}^{j}\sum_{m=0}^{r_k}\frac{1}{m!}\frac{1}{(r_k - m)!}g_{i_k}^{(r_k-m)}(i_k)a_k^m \in D_Q$$

for $1 \le j \le n$ *and*

$$g_{i_k}(x) = \frac{1}{(x - i_1)^{r_1+1}\cdots(x - i_{k-1})^{r_{k-1}+1}(x - i_{k+1})^{r_{k+1}+1}\cdots(x - i_j)^{r_j+1}}$$

for $1 \le k \le j$*. Then* $\mathscr{N}_{\mathscr{A}}(x) = H_{\mathscr{A}}(x)$*.*

Proof. The generalization of Hermite's formula $H_{\mathscr{A}}(x)$of $\mathscr{A}$ is the unique polynomial with coefficients in D_Q of degree $< n + \sum_{j=1}^{n} r_n$ such that $H_{\mathscr{A}}^{(m)}(i_j) = a_j^m$ for all $1 \le j \le n$ and $0 \le m \le r_j$. Since $\mathscr{N}_{\mathscr{A}}^{(m)}(i_j) = a_j^m = H_{\mathscr{A}}^{(m)}(i_j)$ for all $1 \le j \le n, 0 \le m \le r_j$ and $N_{\mathscr{A}}(x)$ is the polynomial of degree $< n + \sum_{j=1}^{n} r_j$, they are identical. □

Remark Let $H_{\mathscr{A}}(x) = c_M x^M + c_{M-1}x^{M-1} + \cdots + c_1 x + c_0$ where $M = n - 1 + \sum_{j=1}^{n} r_j$ be the polynomial in $D[x]$ with degree $< n + \sum_{j=1}^{n} r_j$. Since $\mathscr{N}_{\mathscr{A}}(x) = H_{\mathscr{A}}(x)$ and $[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]$ is the coefficient of x^M,

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}] = c_M \in D.$$

We see that $[\underbrace{i_1,\ldots,i_1}_{r_1+1},\ldots,\underbrace{i_n,\ldots,i_n}_{r_n}]$ is the coefficient of x^{M-1} in polynomial

$$\mathscr{N}_{\mathscr{A}}(x) - [\underbrace{i_1,\ldots,i_1}_{r_1+1},\ldots,\underbrace{i_n,\ldots,i_n}_{r_n+1}](x-i_1)^{r_1+1}\cdots(x-i_n)^{r_n}$$

which is an element in D. Similarly by the same reasoning,

$$[i_1], [i_1, i_1], \ldots, [\underbrace{i_1,\ldots,i_1}_{r_1+1},\ldots,\underbrace{i_n,\ldots,i_n}_{r_n-1}]$$

are elements in D. Therefore $H_{\mathscr{A}}(x) \in D[x]$ if and only if $\mathscr{N}_{\mathscr{A}}(x) \in D[x]$.

Example 41. Let $D = \mathbb{Z}$ and $c \in \mathbb{Z}$. We now compute $\wp_c$ and $\wp_c^{(1)}$.

1. For any $c \in \mathbb{Z}$, $\wp_c = \mathbb{Z}$ since $f(x) = a_1$ and $f(c) = a_1$.
2. $\wp_c^{(1)} = \mathbb{Z} \times \mathbb{Z}$ for any $c \in \mathbb{Z}$ since $f(x) = a_1^0 x + (a_1^0 - a_1^1 c)$ gives $f(c) = a_1^0$ and $f'(c) = a_1^1$ for any $a_1^0, a_1^1 \in \mathbb{Z}$.

Example 42. Let $D = \mathbb{Z}$, $\mathscr{A} = ((18, 16, 6)) \in \mathbb{Z}^3$ and $c = 2$.
Let $f(x) = 3x^2 + 4x - 2$. Then we see that $f(2) = 18, f'(2) = 16$ and $f''(2) = 6$. So the sequence $\mathscr{A}$ is a difference polynomial sequence of length 1 and order 2 with respect to 2.

Example 43. Let $D = \mathbb{Z}$ and $c \in D$. The sequence $((0, 0, 0, 1)) \notin \wp_c^{(3)}$ for any integer c. Suppose that the sequence $((0, 0, 0, 1)) \in \wp_c^{(3)}$. Then there exists a polynomial $f(x) \in \mathbb{Z}[x]$ and a constant $c \in \mathbb{Z}$ such that $f(c) = 0, f'(c) = 0, f''(c) = 0$, $f^{(3)}(c) = 1$. Since $f(c) = 0, f'(c) = 0, f''(c) = 0$, we get that $(x - c)^3 | f(x)$.
Let $f(x) = (x - c)^3 g(x)$ where $g(x) \in \mathbb{Z}[x]$. Then

$$\begin{aligned}
f'(x) &= 3(x-c)^2 g(x) + (x-c)^3 g'(x),\\
f''(x) &= 6(x-c)g(x) + 6(x-c)^2 g'(x) + (x-c)^3 g''(x),\\
f^{(3)}(x) &= 6g(x) + 18(x-c)g'(x) + 9(x-c)^2 g''(x) + (x-c)^3 g^{(3)}(x).
\end{aligned}$$

We see that $f^{(3)}(c) = 6g(c) = 1$. So $g(c) = \frac{1}{6} \notin \mathbb{Z}$. This is a contradiction.

Theorem 5.6. *Let*

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ *are elements in* D. *Then* $\mathscr{A} \in \wp_{n,I}^R$ *if and only if*

$$[i_1], [i_1, i_1], \ldots, [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]$$

are elements in D.

Proof. The result follows immediately from Theorem 5.4 and Theorem 5.5. So $\mathscr{N}_{\mathscr{A}}(x)$ has coefficients in D if and only if $H_{\mathscr{A}}(x)$ does. □

Next we will characterize the differential polynomial sequence of length 1 and order k with respect to c for any constant $c \in D$.

Corollary 5.7. *Let* $\mathscr{A} = ((a^0, a^1, \ldots, a^k)) \in D^{k+1}$ *and* $c \in D$. *Then there exists*

$$\mathscr{T}(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots + b_k(x-c)^k \in D_Q[x]_{k+1}$$

where

$$b_j = \frac{a^j}{j!} \in D$$

for all $j = 0, 1, 2, \ldots, k$ *such that* $\mathscr{T}^{(j)}(c) = a^j$ *for all* $j = 0, 1, 2, \ldots, k$.
Furthermore $\mathscr{A} \in \wp_c^{(k)}$ *if and only if* $b_j \in D$ *for all* $j = 0, 1, \ldots, k$.

Proof. The result follows immediately from Theorem 5.6. □

Example 44. Let $\mathscr{A} = ((1, 3, 5, 7)) \in \mathbb{Z}^4$ and $c \in \mathbb{Z}$. Then we get

$$\begin{aligned} b_0 &= 1 - 3c + \frac{5c^2}{2} - \frac{7c^3}{6} \\ b_1 &= 3 - 5c + \frac{7c^2}{2} \\ b_2 &= \frac{5}{2} - \frac{7c}{2} \\ b_3 &= \frac{7}{6}. \end{aligned}$$

So the polynomial $f(x)$ such that $f(c) = 1, f'(c) = 3, f''(c) = 5$ and $f^{(3)}(c) = 7$ is $f(x) = \frac{7}{6}x^3 + (\frac{5}{2} - \frac{7c}{2})x^2 + (3 - 5c + \frac{7c^2}{2})x + (1 - 3c + \frac{5c^2}{2} - \frac{7c^3}{6})$.

Next we will show how to turn a sequence $\mathscr{A} \in (D^{k+1})$ to be a sequence in $\wp_{1,c}^{(k)}$.

Corollary 5.8. *If* $\mathscr{A} = ((a^0, a^1, \ldots, a^k)) \in D^{k+1}$ *then* $k!\mathscr{A} \in \wp_{1,c}^{(k)}$.
Moreover, $k!$ *is the least positive integer for which this is true for every sequence of length* 1 *and order* k *with respect to c.*

Proof. Denote the sequence $k!\mathscr{A}$ by $\big((c^0, c^1, \ldots, c^k)\big)$ and let $b'_0, b'_1, \ldots, b'_k$ be the corresponding sequence of coefficients defined in Corollary 5.7.
Note that $j!|k!$ for all $0 \le j \le k$. Hence each b'_j is an element in D. So $k!\mathscr{A} \in \wp_{1,c}^{(k)}$.
To see that this result is the best possible, consider $\mathscr{A} = ((0, 0, \ldots, 0, 1))$. Then for all $j = 0, 1, \ldots, k$, $b_j = \frac{a^j}{j!}$ and in particular, $b_k = \frac{1}{k!}$.
Hence the least positive integer m such that $m\mathscr{A}$ is a differential polynomial sequence of length 1 and order k is $m = k!$. □

Example 45. Let $D = \mathbb{Z}[i]$, $\mathscr{A} = ((2i, 3i, 17+i, 1)) \in \mathbb{Z}[i]^4$ and $c = 7+i \in \mathbb{Z}[i]$. By Corollary 5.7, $\mathscr{A} \notin \wp_{1,c}^{(3)}$. We see that the sequence $3!\mathscr{A} = ((12i, 36i, 204+12i, 12))$ in $(\mathbb{Z}^4)$. There exists $g(x) = 12i+36i(x-7-i)+(102+6i)(x-7-i)^2+(x-7-i)^3 \in \mathbb{Z}[i][x]_4$ such that $g(7+i) = 12i, g'(7+i) = 36i$,
$g''(7+i) = 204+12i, g^{(3)}(7+i) = 12$. Hence $3!\mathscr{A} \in \wp_{1,c}^{(3)}$.

Example 46. Let $D = \mathbb{Z}$, $\mathscr{A} = ((2, 3, 17, 21)) \in \mathbb{Z}^4$ and $c = 7 \in \mathbb{Z}$. By Corollary 5.7, $\mathscr{A} \notin \wp_{1,c}^{(3)}$. We see that the sequence $2\mathscr{A} = ((4, 6, 34, 42)) \in (\mathbb{Z}^4)$. There exists $g(x) = -1606 + 797x - 130x^2 + 7x^3 \in \mathbb{Z}[x]_4$ such that $g(7) = 4, g'(7) = 6$,
$g''(7) = 34, g^{(3)}(7) = 42$. Hence $2\mathscr{A} \in \wp_{1,c}^{(3)}$.

Example 47. Let $D = \mathbb{Z}$, $\mathscr{A} = ((1, 0, 0, 0)) \in \mathbb{Z}^4$ and $c \in \mathbb{Z}$. We see that there exists $f(x) = 1$ such that $f(c) = 1, f'(c) = 0, f''(c) = 0$ and $f^{(3)}(c) = 0$. So $\mathscr{A} \in \wp_{1,c}^{(3)}$.

Example 48. Let $D = \mathbb{Z}$, $\mathscr{A} = ((0, 1, 0, 0)) \in \mathbb{Z}^4$ and $c \in \mathbb{Z}$. We see that there exists $f(x) = -c + x$ such that $f(c) = 0, f'(c) = 1, f''(c) = 0$ and $f^{(3)}(c) = 0$. So $\mathscr{A} \in \wp_{1,c}^{(3)}$.

Example 49. Let $D = \mathbb{Z}$, $\mathscr{A} = ((0, 0, 1, 0)) \in \mathbb{Z}^4$ and $c \in \mathbb{Z}$. There exists $f(x) = \frac{c^2}{2} - cx + \frac{x^2}{2}$ such that $f(c) = 0, f'(c) = 1, f''(c) = 0, f^{(3)}(c) = 0$. So $\mathscr{A} \notin \wp_{1,c}^{(3)}$.

We see that $g(x) = 2f(x) \in \mathbb{Z}[x]$ and $g(c) = 0, g'(c) = 0, g''(c) = 2$ and $g_2^{(3)}(c) = 0$. So $g(x)$ generates $2\mathscr{A}$. Hence $2\mathscr{A} \in \wp_{1,c}^{(3)}$.

To see that 3! is the best integer such that $3!\mathscr{A} \in \wp_{1,c}^{(3)}$, we can see from the following example.

Example 50. Let $D = \mathbb{Z}$ and $\mathscr{A} = ((0,0,0,1)) \in \mathbb{Z}^4$ and $c \in \mathbb{Z}$. Then there exists $f(x) = -\frac{c^3}{6} + \frac{c^2}{2}x - \frac{c}{2}x^2 + \frac{x^3}{6}$ such that $f(c) = 0, f'(c) = 0, f''(c) = 0, f^{(3)}(c) = 1$. Since $f(x) \notin \mathbb{Z}[x]$, $\mathscr{A} \notin \wp_{1,c}^{(3)}$. We see that $g(x) = 3!f(x) = -c^3 + 3c^2x - 3cx^2 + x^3$ and $g(c) = 0, g'(c) = 0, g''(c) = 0, g^{(3)}(c) = 6$. Hence $3!\mathscr{A} \in \wp_{1,c}^{(3)}$.

For $c \in \mathbb{Z}$ and $k = 3$ we can multiply the sequences in Example 47, 48 and 49 by $1, 1$ and 2 respectively to make the sequences in these examples to be the differential polynomial sequences of length 1 and order 3 with respect to c. By Corollary 5.8 the best integer m that is true for all the sequences in $\mathbb{Z}^4$ with respect to $c \in \mathbb{Z}$ is $m = 3!$ which we can see by Example 50 that we need 3! to make the sequence in Example 50 to be the differential polynomial sequence of length 1 and order 3 with respect to c.

Corollary 5.9. *Let $D = \mathbb{Z}$, $I = (1,2)$ and $\mathscr{A} = ((a_1^0, a_1^1), ((a_2^0, a_2^1)) \in (\mathbb{Z}^2)^2$. Then $\mathscr{A} \in \wp_2^{(1)}$.*

Proof. By Theorem 5.6, $A \in \wp_2^{(1)}$ if and only if

$$[1] = a_1^0, [1,1] = a_1^1, [1,1,2] = a_2^0 - a_1^0 - a^1 \text{ and } [1,1,2,2] = 2a_1^0 + a_1^1 - 2a_2^0 + a_2^1$$

are integers. □

Example 51. Let $\mathscr{A} = ((3,13),(22,9)) \in (\mathbb{Z}^2)^2$. By Corollary 5.9 we see that $\mathscr{A} \in \wp_2^{(1)}$. In fact, the unique Hermite's polynomial formula of $\mathscr{A}$ with degree less than 4 such that

$$H_{\mathscr{A}}(1) = 3, \quad H_{\mathscr{A}}(2) = 22$$
$$H'_{\mathscr{A}}(1) = 13, \quad H'_{\mathscr{A}}(2) = 9,$$

is

$$H_{\mathscr{A}}(x) = -53 + 101x - 56x^2 + 11x^3 \in \mathbb{Z}[x].$$

Corollary 5.10. *Let* $D = \mathbb{Z}$*,* $I = (1, 2, 3)$ *and* $\mathscr{A} = \big((a_1^0, a_1^1), (a_2^0, a_2^1), (a_3^0, a_3^1)\big)$ *in* $(\mathbb{Z}^2)^3$*. Then* $\mathscr{A} \in \wp_3^{(1)}$ *if and only if either one of these holds:*

1. $a_1^1 \equiv a_3^1 \equiv 1, 3 \mod 4$ *and* $a_1^0 \equiv a_3^0 + 2 \mod 4$ *or*

2. $a_1^1 \equiv a_3^1 \equiv 0, 2 \mod 4$ *and* $a_1^0 \equiv a_3^0 \mod 4$.

Proof. By Theorem 5.6, $A \in \wp_3^{(1)}$ if and only if

$$\begin{aligned}
&[1] = a_1^0, [1, 1] = a_1^1, [1, 1, 2] = -a_1^0 - a_1^1 + a_2^0,\\
&[1, 1, 2, 2] = 2a_1^0 + a_1^1 - 2a_2^0 + a_2^1,\\
&[1, 1, 2, 2, 3] = \left(-\frac{5a_1^0}{4} - \frac{a_1^1}{2} + a_2^0 - a_2^1 + \frac{a_3^0}{4}\right),\\
&[1, 1, 2, 2, 3, 3] = \left(\frac{3a_1^0}{4} + \frac{a_1^1}{4} + a_2^1 - \frac{3a_3^0}{4} + \frac{a_3^1}{4}\right)
\end{aligned}$$

are integers. Those rational numbers are integers if and only if $n_1 = -\frac{a_1^0}{4} - \frac{a_1^1}{2} + \frac{a_3^0}{4}$ and $n_2 = \frac{3a_1^0}{4} + \frac{a_1^1}{4} - \frac{3a_3^0}{4} + \frac{a_3^1}{4}$ are integers. So $n_2 - n_1 = a_1^0 + \frac{3a_1^1}{4} - a_3^0 + \frac{a_3^1}{4}$ is integer. This implies that $a_1^1 \equiv a_3^1 \mod 4$. If $a_1^1 \equiv 0, 2 \mod 4$ then $a_1^0 \equiv a_3^0 \mod 4$ and if $a_1^1 \equiv 1, 3 \mod 4$ then $a_1^0 \equiv a_3^0 + 2 \mod 4$.

For the converse it is easy to verify that if $a_1^0, a_1^1, a_3^0, a_3^1$ satisfy the conditions in the theorem then $[1], [1, 1], [1, 1, 2], [1, 1, 2, 2], [1, 1, 2, 2, 3]$ and $[1, 1, 2, 2, 3, 3]$ are integers. □

Example 52. Let $\mathscr{A} = \big((4, 4), (3, 3), (12, 12)\big) \in (\mathbb{Z}^2)^3$. Since

$$4 \equiv 12 \equiv 0 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$\begin{aligned}
&H_{\mathscr{A}}(1) = 4, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 12,\\
&H'_{\mathscr{A}}(1) = 4, H'_{\mathscr{A}}(2) = 3, H'_{\mathscr{A}}(3) = 12,
\end{aligned}$$

is

$$H_{\mathscr{A}}(x) = -51 + 147x - 144x^2 + 64x^3 - 13x^4 + x^5 \in \mathbb{Z}[x].$$

Example 53. Let $\mathscr{A} = \big((4,2),(3,3),(12,10)\big) \in (\mathbb{Z}^2)^3$. Since

$$4 \equiv 12 \equiv 0 \mod 4 \text{ and } 2 \equiv 10 \equiv 2 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 4, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 12,$$
$$H'_{\mathscr{A}}(1) = 2, H'_{\mathscr{A}}(2) = 3, H'_{\mathscr{A}}(3) = 10,$$

is

$$H_{\mathscr{A}}(x) = -27 + 79x - 70x^2 + 25x^3 - 3x^4 \in \mathbb{Z}[x].$$

Example 54. Let $\mathscr{A} = \big((5,4),(3,3),(9,12)\big) \in (\mathbb{Z}^2)^3$. Since

$$5 \equiv 9 \equiv 1 \mod 4 \text{ and } 4 \equiv 12 \equiv 0 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 5, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 9,$$
$$H'_{\mathscr{A}}(1) = 4, H'_{\mathscr{A}}(2) = 3, H'_{\mathscr{A}}(3) = 12,$$

is

$$H_{\mathscr{A}}(x) = -99 + 303x - 332x^2 + 171x^3 - 42x^4 + 4x^5 \in \mathbb{Z}[x].$$

Example 55. Let $\mathscr{A} = \big((5,2),(3,3),(9,10)\big) \in (\mathbb{Z}^2)^3$. Since

$$5 \equiv 9 \equiv 1 \mod 4 \text{ and } 2 \equiv 10 \equiv 2 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 5, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 9,$$
$$H'_{\mathscr{A}}(1) = 2, H'_{\mathscr{A}}(2) = 3, H'_{\mathscr{A}}(3) = 10,$$

is

$$H_{\mathscr{A}}(x) = -75 + 235x - 258x^2 + 132x^3 - 32x^4 + 3x^5 \in \mathbb{Z}[x].$$

Example 56. Let $\mathscr{A} = \big((6,4),(3,3),(14,12)\big) \in (\mathbb{Z}^2)^3$. Since

$$6 \equiv 14 \equiv 2 \mod 4 \text{ and } 4 \equiv 12 \equiv 0 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 6, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 14,$$
$$H'_{\mathscr{A}}(1) = 4, H'_{\mathscr{A}}(2) = 3, f'(3) = 12,$$

is

$$H_{\mathscr{A}}(x) = -67 + 195x - 188x^2 + 80x^3 - 15x^4 + x^5 \in \mathbb{Z}[x].$$

Example 57. Let $\mathscr{A} = \big((6,2),(3,3),(14,10)\big) \in (\mathbb{Z}^2)^3$. Since

$$6 \equiv 14 \equiv 2 \mod 4 \text{ and } 2 \equiv 10 \equiv 2 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 6, H_{\mathscr{A}}(2) = 3, H_{\mathscr{A}}(3) = 14,$$
$$H'_{\mathscr{A}}(1) = 2, H'_{\mathscr{A}}(2) = 3, H'_{\mathscr{A}}(3) = 10,$$

is

$$H_{\mathscr{A}}(x) = -43 + 127x - 114x^2 + 41x^3 - 5x^4 \in \mathbb{Z}[x].$$

Example 58. Let $\mathscr{A} = \big((7,4),(3,3),(11,12)\big) \in (\mathbb{Z}^2)^3$. Since

$$7 \equiv 11 \equiv 3 \mod 4 \text{ and } 4 \equiv 12 \equiv 0 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 7, \quad H_{\mathscr{A}}(2) = 3, \quad H_{\mathscr{A}}(3) = 11,$$
$$H'_{\mathscr{A}}(1) = 4, \quad H'_{\mathscr{A}}(2) = 3, \quad H'_{\mathscr{A}}(3) = 12,$$

is

$$H_{\mathscr{A}}(x) = -115 + 351x - 376x^2 + 187x^3 - 44x^4 + 4x^5 \in \mathbb{Z}[x].$$

Example 59. Let $\mathscr{A} = \big((7,2),(3,3),(11,10)\big) \in (\mathbb{Z}^2)^3$. Since

$$7 \equiv 11 \equiv 3 \mod 4 \text{ and } 2 \equiv 10 \equiv 2 \mod 4,$$

we have $\mathscr{A} \in \wp_3^{(1)}$ by Corollary 5.10.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 7, \quad H_{\mathscr{A}}(2) = 3, \quad H_{\mathscr{A}}(3) = 11,$$
$$H'_{\mathscr{A}}(1) = 2, \quad H'_{\mathscr{A}}(2) = 3, \quad H'_{\mathscr{A}}(3) = 10,$$

is

$$H_{\mathscr{A}}(x) = -91 + 283x - 302x^2 + 148x^3 - 34x^4 + 3x^5 \in \mathbb{Z}[x].$$

Corollary 5.11. *Let* $\mathscr{A} = \big((a_1^0, a_1^1, a_1^2),(a_2^0, a_2^1, a_2^2)\big) \in (\mathbb{Z}^3)^2$.
Then $\mathscr{A} \in \wp_2^{(2)}$ *if and only if* a_1^2 *and* a_2^2 *are even.*

Proof. Since $A \in \wp_2^{(2)}$ if and only if

$$[1] = a_1^0, [1,1] = a_1^1, [1,1,1] = \frac{a_1^1}{2},$$
$$[1,1,1,2] = a_2^0 - a_1^0 - a_1^1 - a_1^2,$$
$$[1,1,1,2,2] = 3a_1^0 + 2a_1^1 - 2a_2^0 + a_2^1 + a_1^2,$$
$$[1,1,1,2,2,2] = -6a_1^0 - 3a_1^1 + a_1^2 + a_2^0 - 3a_1^1 + \frac{a_1^2}{2}$$

are integers. The result follows. □

Example 60. Let $\mathscr{A} = \big((3,7,10),(4,5,6)\big) \in (\mathbb{Z}^3)^2$. Since 10 and 6 are even, by Corollary 5.11 $\mathscr{A} \in \wp_2^{(2)}$.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$H_{\mathscr{A}}(1) = 3, \quad H_{\mathscr{A}}(2) = 4,$$
$$H'_{\mathscr{A}}(1) = 7, \quad H'_{\mathscr{A}}(2) = 5,$$
$$H''_{\mathscr{A}}(1) = 10, \quad H''_{\mathscr{A}}(2) = 6,$$

is

$$H_{\mathscr{A}}(x) = 182 - 695x + 1027x^2 - 724x^3 + 245x^4 - 32x^5 \in \mathbb{Z}[x].$$

Example 61. Let $\mathscr{A} = \big((3,7,103),(4,5,11)\big) \in (\mathbb{Z}^3)^2$. Since 103 and 11 are not even, by Corollary 5.11 we obtain that $\mathscr{A} \notin \wp_2^{(2)}$.

In fact, the generalization of Hermite's formula for $\mathscr{A}$ such that

$$\begin{aligned} H_{\mathscr{A}}(1) &= 3, & H_{\mathscr{A}}(2) &= 4, \\ H'_{\mathscr{A}}(1) &= 7, & H'_{\mathscr{A}}(2) &= 5, \\ H''_{\mathscr{A}}(1) &= 103, & H''_{\mathscr{A}}(2) &= 11, \end{aligned}$$

is

$$H_{\mathscr{A}}(x) = 172 - 655x + \frac{1929}{2}x^2 - \frac{1353}{2}x^3 + \frac{455}{2}x^4 - \frac{59}{2}x^5 \notin \mathbb{Z}[x].$$

CHAPTER 6

Conclusion

In this chapter we will summarize all of the results in Chapter 3, 4 and 5.

1. Polynomial sequences over integral domain

Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and
$\overline{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Let

$$P_{n,I} = \{\overline{a} \in D^n \mid \text{there exists } f(x) \in D[x] \text{ such that } f(i_j) = a_j, \text{ for all } 1 \leq j \leq n\}$$

be the set of all (D^n, I)-polynomial sequences. We call $\overline{a}$ an element in $P_{n,I}$, a polynomial sequence over D with respect to I or a (D^n, I)-polynomial sequence.

If $I = (1, 2, 3, \ldots, n)$, then we write P_n for $P_{n,I}$ and call an element in P_n, a polynomial sequence. Then we have the results as follows.

1. $P_{n,I}$ is a group under addition.
2. If $\overline{a} \in P_{n,I}$ then $c\overline{a} \in P_{n,I}$ for any $c \in D$.
3. If $\overline{a}, \overline{b} \in P_{n,I}$ then $\overline{a} \cdot \overline{b} \in P_{n,I}$.
4. $(P_{n,I}, +, \cdot)$ is a commutative ring with identity $\overline{1}$.
5. Let $r(x), s(x) \in D_Q[x]_n$ where $r(x)$ and $s(x)$ agree at n distinct points. Then $r(x) = s(x)$.
6. Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and $\overline{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Then $\bar{a}$ is a (D^n, I)-polynomial sequence if and only if $L_{a,I}(x) \in D[x]_n$, the set of all polynomials in $D[x]$ of degree $< n$. Furthermore, $L_{a,I}(x)$ is the unique polynomial of degree $< n$ in $D_Q[x]$ that generates $\overline{a}$.

7. Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. Then $\bar{a} \in P_n$ if and only if $L_a(x) \in \mathbb{Z}[x]_n$. Furthermore, $L_a(x)$ is the unique polynomial of degree $< n$ with real coefficients that generates $\bar{a}$.

8. Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in D^n$. Let

$$N_{a,I}(x) = b_{0,I} p_{i_0}(x) + b_{1,I} p_{i_1}(x) + \cdots + b_{n-1,I} p_{i_{n-1}}(x) \in D_Q[x]$$

where

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j+1}}^{k+1} (i_{j+1} - i_m)} \in D_Q, \qquad (k = 0, 1, \ldots, n-1).$$

Then $N_{a,I}(x) = L_{a,I}(x)$.

9. Let $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. Let

$$N_a(x) = b_0 p_0(x) + b_1 p_1(x) + b_2 p_2(x) + \cdots + b_{n-1} p_{n-1}(x)$$

where

$$b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j}}{j!(k-j)!} a_{j+1}$$

for $k = 0, \ldots, n-1$. Then $N_a(x) = L_a(x)$.

10. Let $f(x) \in D[x]_n$. Then there are unique elements in D $b_{0,I}, b_{1,I}, \ldots, b_{n-1,I}$ such that $f(x) = b_{0,I} p_{i_0}(x) + b_{1,I} p_{i_1}(x) + \cdots + b_{n-1,I} p_{i_{n-1}}(x)$.

11. Let $f(x) \in \mathbb{Z}[x]_n$. Then there are unique integers $b_0, b_1, \ldots, b_{n-1}$ such that $f(x) = b_0 p_0(x) + b_1 p_1(x) + \cdots + b_{n-1} p_{n-1}(x)$.

12. Let $\bar{a} \in D^n$. Then $\bar{a} \in P_{n,I}$ if and only if

$$b_{k,I} = \sum_{j=0}^{k} \frac{a_{j+1}}{\prod\limits_{\substack{m=1 \\ m \neq j+1}}^{k+1} (i_{j+1} - i_m)} \qquad (k = 0, 1, \ldots, n-1)$$

is an element in D.

13. Let $\bar{a} \in \mathbb{Z}^n$. Then $\bar{a}$ is a polynomial sequence if and only if for all $k = 0, 1, \dots, n-1$ then

$$b_k = \sum_{j=0}^{k} \frac{(-1)^{k+j} a_{j+1}}{j!(k-j)!}$$

is an integer.

14. For any $I = (i_1) \in \mathbb{Z}$, we have $P_{1,I} = \mathbb{Z}$.

15. For any $\overline{a} = (a_1, a_2), I = (i_1, i_2) \in \mathbb{Z}^2$ where $i_1 < i_2$, we have

$$\overline{a} \in P_{2,I} \text{ if and only if } a_1 \equiv a_2 \mod (i_1 - i_2).$$

In fact if $I = (1, 2)$ then $P_2 = \mathbb{Z}^2$.

16. For any $\overline{a} = (a_1, a_2, a_3), I = (i_1, i_2, i_3) \in \mathbb{Z}^3$ where $i_1 < i_2 < i_3$, we have

$\overline{a} \in P_{3,I}$ if and only if $\dfrac{(a_3 - a_2) + m(i_2 - i_3)}{(i_1 - i_3)(i_2 - i_3)}$ and $m = \dfrac{a_1 - a_2}{i_1 - i_2}$ are integers.

In fact if $I = (1, 2, 3)$ then $P_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 \equiv a_3 \mod 2\}$.

17. Let $I = (i_1, i_2, \dots, i_n) \in D^n$ with distinct i_j, let $\bar{a} = (a_1, a_2, \dots, a_n)$ in D^n and let

$$M = \prod_{j=0}^{n-1} M_j \quad \text{where} \quad M_j = \prod_{\substack{m=1, \\ m \neq j+1}}^{n} (i_{j+1} - i_m) \text{ for all } j = 0, 1, 2, \dots, n-1.$$

Then $M\bar{a} = (Ma_1, Ma_2, \dots, Ma_n) \in P_{n,I}$.

Moreover, if the integral domain D is a unique factorization domain, then

$$M'\bar{a} = (M'a_1, M'a_2, \dots, M'a_n) \in P_{n,I} \text{ where } M' = \operatorname{lcm}\{M_j\}_{j=0}^{n-1}$$

and M' is the minimal element in D for which this is true for every sequence of length n.

18. Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. Then

$$(n-1)!a = \big((n-1)!a_1, (n-1)!a_2, \dots, (n-1)!a_n\big) \in P_{n,I}.$$

Moreover, $(n-1)!$ is the least positive integer for which is true every sequence of length n.

19. The group $P_{n,I}$ is isomorphic to $D[x]_n$.

20. For $n \geq 2$, let $I = (i_1, i_2, \ldots, i_n) \in D^n$. If

$$\prod_{m=1}^{k-1}(i_j - i_m)/\prod_{m=1}^{k-1}(i_k - i_m) \in D \quad (1 < k < j \leq n),$$

then

$$D^n/P_{n,I} \cong D/(i_2-i_1)D \oplus D/(i_3-i_1)(i_3-i_2)D \oplus \cdots \oplus D/(i_n-i_1)\cdots(i_n-i_{n-1})D.$$

21. $D^n/P_{n,I}$ is a finite abelian group of the form

$$D/d_{n-1,I}D \oplus \cdots \oplus D/d_{2,I}D \oplus D/d_{1,I}D$$

where $d_{1,I} \mid d_{2,I} \mid \cdots \mid d_{n-1,I}$.

22. If $I = (1, 2, \ldots, n)$ $(n \geq 3)$, then $\mathbb{Z}^n/P_n$ is a finite abelian group with Smith normal form

$$\mathbb{Z}/(n-1)!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z}$$

and Smith invariant $((n-1)!, \ldots, 3!, 2!)$. Moreover, $|\mathbb{Z}^n/P_n| = \prod_{i=1}^{n-1} i!$.

23. Let a, q be elements in D and $n \geq 2$. If $i_k = aq^k$ $(1 \leq k \leq n)$, then

$$D^n/P_{n,I} \cong D/aq(q-1)D \oplus D/a^2q^{1+2}(q^2-1)(q-1)D \oplus \cdots \oplus D/a^{n-1}q^{\sum_{i=1}^{n-1} i}\prod_{i=1}^{n-1}(q^i-1)D.$$

24. For $n \geq 2$, if $i_{k+1} - i_k$ is a constant $c \in D$ for all $1 \leq k \leq n-1$, then

$$D^n/P_{n,I} \cong D/c \cdot D \oplus D/2!c^2D \oplus D/3!c^3D \oplus \cdots \oplus D/(n-1)!c^{n-1}D.$$

25. For $n \geq 3$, and $I = \{1, 2, \ldots, n\}$ then

$$\mathbb{Z}^n/P_n \cong \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}.$$

2. Difference polynomial sequences

For a fixed $n \in \mathbb{N}$, we consider a sequence $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$, call $\bar{c}$ a difference polynomial sequence of length n if there exists $f(x) \in \mathbb{Z}[x]$ such that $\Delta_F f(i) = c_i$ for all $1 \leq i \leq n-1$. Denote by ΔP_n the set of all difference polynomial sequences of length $n-1$. We will characterize ΔP_n using difference of Lagrange and (implicitly) Newton interpolation polynomials, and determine the structure of $\mathbb{Z}^{n-1}/\Delta P_n$ and $P_{n-1}/\Delta P_n$.

Definition: Let $f(x) \in \mathbb{Z}[x]$. Then we define

$$\Delta_F f(x) = f(x+1) - f(x).$$

1. With the above notation, for any $f(x), g(x) \in \mathbb{Z}[x]$

   a) $\Delta_F(p_0(x)) = 0$ and $\Delta_F(p_i(x)) = ip_{i-1}(x)$ for $i \geq 1$.

   b) $\Delta_F(cf(x)) = c\Delta_F f(x)$ for any constant c.

   c) $\Delta_F(f+g) = \Delta_F f + \Delta_F g$.

2. For any $\bar{a} \in P_n$ and $N_{\bar{a}}(x) = \sum_{i=0}^{n-1} b_i p_i(x)$, then

$$\Delta_F N_{\bar{a}}(x) = \sum_{i=0}^{n-2} (i+1) b_{i+1} p_i(x).$$

3. If $\bar{a} \in P_n$ then $\Delta \bar{a} \in P_{n-1}$.

Definition: Let n be a positive integer. Let $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$. We define

$$\Delta P_n = \{\bar{c} \in \mathbb{Z}^{n-1} \mid \text{there exists } f(x) \in \mathbb{Z}[x] \text{ such that } \Delta_F f(i) = c_i, \text{ for all } 1 \leq i \leq n-1\}.$$

Definition: For any positive integer n, let

$$\Delta\mathbb{Z}[x]_n = \{f(x) \in \mathbb{Z}[x]_{n-1} | \text{ there exists } g(x) \in \mathbb{Z}[x]_n \text{ such that } \Delta_F(g(x)) = f(x)\}.$$

Then we have the results as follows.

1. $\Delta\mathbb{Z}[x]_n$ is an abelian group under addition.

2. ΔP_n is an abelian group under addition.

3. The group $\Delta\mathbb{Z}[x]_n$ is isomorphic to ΔP_n.

4. For any positive integer $n \geq 2$,

$$\mathbb{Z}^{n-1}/\Delta P_n \cong \mathbb{Z}/1!\mathbb{Z} \oplus \mathbb{Z}/2!\mathbb{Z} \oplus \mathbb{Z}/3!\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)!\mathbb{Z}.$$

5. For $n \geq 2$, $P_{n-1}/\Delta P_n \cong \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(n-1)\mathbb{Z}$.

6. Let $\bar{c} = (c_1, c_2, \ldots, c_{n-1}) \in \mathbb{Z}^{n-1}$ with $N_{\bar{c}}(x) = \sum_{i=0}^{n-2} d_i p_i(x)$ for some $d_i \in \mathbb{Q}$. Then $\bar{c} \in \Delta P_n$ if and only if $(i+1)|d_i$ for all $0 \leq i \leq n-2$.

7. Suppose $\bar{c}_1 \in \mathbb{Z}^{n-1}$. Let $\bar{c}_2 = \Delta\bar{c}_1, \bar{c}_3 = \Delta\bar{c}_2 = \Delta^2\bar{c}_1, \ldots,$ $\bar{c}_k = \Delta\bar{c}_{k-1} = \Delta^{k-1}\bar{c}_1$ for $1 \leq k \leq n-2$. There exists $\bar{a} \in P_n$ such that $\Delta^j\bar{a} = \bar{c}_j$ for $1 \leq j \leq k$ if and only if

   a) $(i+1)(i+2)\cdots(i+k)|b_i^{(k)}$ where

   $$N_{\bar{c}_k}(x) = \sum_{i=0}^{n-k-1} b_i^{(k)} p_i(x)$$

   for all $0 \leq i \leq n-k-1$ and

   b) $j!|a_1^{(j)}$ where $\Delta^j\bar{a} = (a_1^{(j)}, a_2^{(j)}, \ldots, a_{n-j}^{(j)})$ for all $1 \leq j \leq k$.

3. Differential polynomial sequences

Let D be an integral domain, $R = (r_1, r_2, \ldots, r_n)$, $I = (i_1, i_2, \ldots, i_n)$ in D^n where $i_j \neq i_m$ if $j \neq m$ and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Let

$$\wp_{n,I}^R = \{\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n} \in D \mid$ there exists $f(x) \in D[x]$ such that $f^{(m)}(i_j) = a_j^m,$ for all $1 \leq j \leq n, 0 \leq m \leq r_j\}$

be the set of all differential polynomial sequences of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I. We call $\mathscr{A}$ an element in $\wp_{n,I}^R$, a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I.

If $r_j = k$ for all $j = 1, 2, \ldots, n$, then we let $\wp_{n,I}^{(k)} = \wp_{n,I}^R$. We call $\mathscr{A}$ an element in $\wp_{n,I}^{(k)}$, a differential polynomial sequence of length n and order k with respect to I. We next define some definition as follows:

1. If $D = \mathbb{Z}$ and $I = (1, 2, \ldots, n)$, then we let $\wp_n^{(k)} = \wp_{n,I}^{(k)}$. We call $\mathscr{A}$ an element in $\wp_n^{(k)}$, a differential polynomial sequence of length n and order k.

2. If $I = (c)$ where $c \in D$, then we let $\wp_{1,c}^{(k)} = \wp_{n,I}^{(k)}$. We call $\mathscr{A}$ an element in $\wp_{1,c}^{(k)}$, a differential polynomial sequence of length 1 and order k with respect to c.

Then we get the results as following:

1. $\wp_{n,I}^R$ is an abelian group under addition.

2. If $\mathscr{A} \in \wp^R$ then $c \cdot \mathscr{A} \in \wp_{n,I}^R$ for any $c \in D$.

3. Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ and

$$p(x) := (x - i_1)^{r_1+1}(x - i_2)^{r_2+1} \cdots (x - i_n)^{r_n+1}$$

   with degree $n + \sum_{j=1}^n r_j$. Then $p^{(m)}(i_j) = 0$ for all $1 \leq j \leq n$ and $0 \leq m \leq r_j$.

4. Let $I = (i_1, i_2, \ldots, i_n) \in D^n$ where i_j's are all distinct and

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2},\right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right)\right)$$

   where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Then $\mathscr{A}$ is a differential polynomial sequence of length n and order $(r_1, r_2, \ldots, r_n)$ with respect to I if and only if $H_{\mathscr{A}}(x)$ is a polynomial in $D[x]$ of degree less than $n + \sum_{j=1}^n r_j$.

Furthermore, $H_{\mathscr{A}}(x)$ is the unique polynomial of degree $< n + \sum_{j=1}^{n} r_j$ in $D_Q[x]$ where D_Q is the quotient of D, such that

$$H_{\mathscr{A}}^{(m)}(i_j) = a_j^m \quad (1 \leq j \leq n, 0 \leq m \leq r_j).$$

5. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right) \right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Let

$$\mathscr{N}_{\mathscr{A}}(x) = [i_1] + [i_1, i_1]p_{i_1}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2]p_{i_1}^{r_1+1}(x) +$$

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, i_2, i_2]p_{i_2}(x) + \cdots + [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]p_{i_n}^{r_n}(x)$$

be a polynomial in $D_Q[x]$ where

$$p_{i_j}^q(x) = \prod_{h=1}^{j-1} (x - i_h)^{r_h+1}(x - i_j)^q, \qquad 1 \leq j \leq n, 1 \leq q \leq r_j + 1$$

and

$$[\underbrace{i_1, \ldots, i_1}_{r_1+1}, \underbrace{i_2, \ldots, i_2}_{r_2+1}, \ldots, \underbrace{i_j, \ldots, i_j}_{r_j+1}] = \sum_{k=1}^{j} \sum_{m=0}^{r_k} \frac{1}{m!} \frac{1}{(r_k - m)!} g_{i_k}^{(r_k - m)}(i_k) a_k^m \in D_Q$$

for $1 \leq j \leq n$ and

$$g_{i_k}(x) = \frac{1}{(x - i_1)^{r_1+1} \cdots (x - i_{k-1})^{r_{k-1}+1}(x - i_{k+1})^{r_{k+1}+1} \cdots (x - i_j)^{r_j+1}}$$

for $1 \leq k \leq j$. Then $\mathscr{N}_{\mathscr{A}}(x) = H_{\mathscr{A}}(x)$.

6. Let

$$\mathscr{A} = \left(\left(a_1^0, a_1^1, \ldots, a_1^{r_1}\right), \left(a_2^0, a_2^1, \ldots, a_2^{r_2}, \right) \ldots, \left(a_n^0, a_n^1, \ldots, a_n^{r_n}\right) \right)$$

where $a_1^0, a_1^1, \ldots, a_1^{r_1}, a_2^0, a_2^1, \ldots, a_2^{r_2}, \ldots, a_n^0, a_n^1, \ldots, a_n^{r_n}$ are elements in D. Then $\mathscr{A} \in \wp_{n,I}^R$ if and only if

$$[i_1], [i_1, i_1], \ldots, [\underbrace{i_1, \ldots, i_1}_{r_1+1}, \ldots, \underbrace{i_n, \ldots, i_n}_{r_n+1}]$$

are elements in D.

7. Let $\mathscr{A} = ((a^0, a^1, \ldots, a^k)) \in D^{k+1}$ and $c \in D$. Then there exists

$$\mathscr{T}(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots + b_k(x-c)^k \in D_Q[x]_{k+1}$$

where

$$b_j = \frac{a^j}{j!} \in D$$

for all $j = 0, 1, 2, \ldots, k$ such that $\mathscr{T}^{(j)}(c) = a^j$ for all $j = 0, 1, 2, \ldots, k$. Furthermore $\mathscr{A} \in \wp_c^{(k)}$ if and only if $b_j \in D$ for all $j = 0, 1, \ldots, k$.

8. If $\mathscr{A} = ((a^0, a^1, \ldots, a^k)) \in D^{k+1}$ then $k!\mathscr{A} \in \wp_{1,c}^{(k)}$.
   Moreover, $k!$ is the least positive integer for which this is true for every sequence of length 1 and order k with respect to c.

9. Let $D = \mathbb{Z}$, $I = (1, 2)$ and $\mathscr{A} = ((a_1^0, a_1^1), ((a_2^0, a_2^1)) \in (\mathbb{Z}^2)^2$. Then $\mathscr{A} \in \wp_2^{(1)}$.

10. Let $D = \mathbb{Z}$, $I = (1, 2, 3)$ and $\mathscr{A} = \big((a_1^0, a_1^1), (a_2^0, a_2^1), (a_3^0, a_3^1)\big) \in (\mathbb{Z}^2)^3$. Then $\mathscr{A} \in \wp_3^{(1)}$ if and only if either one of these holds:

    a) $a_1^1 \equiv a_3^1 \equiv 1, 3 \mod 4$ and $a_1^0 \equiv a_3^0 + 2 \mod 4$ or

    b) $a_1^1 \equiv a_3^1 \equiv 0, 2 \mod 4$ and $a_1^0 \equiv a_3^0 \mod 4.$

11. Let $\mathscr{A} = \big((a_1^0, a_1^1, a_1^2), (a_2^0, a_2^1, a_2^2)\big) \in (\mathbb{Z}^3)^2$.
    Then $\mathscr{A} \in \wp_2^{(2)}$ if and only if a_1^2 and a_2^2 are even.

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