Representation of Integers in the Form $x^2 + ky^2 - lz^2$

Nattaporn Thongngam

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

Prince of Songkla University

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ชื่อวิทยานิพนธ์ การเขียนจำนวนเต็มในรูป $x^2+ky^2-lz^2$
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บทคัดย่อ

สำหรับจำนวนเต็มบวก k และ l เราจะเรียก l ว่า $k-$special ถ้าทุกจำนวนเต็ม n สามารถเขียนในรูป $x^2+ky^2-lz^2$ โดยที่ x, y และ z เป็นจำนวนเต็มที่ไม่เป็นศูนย์

ในการศึกษาครั้งนี้เราจะหาเงื่อนไขจำเป็นและเงื่อนไขเพียงพอสำหรับ $1-$special และหาเงื่อนไขของจำนวนเต็มบวกคี่ l ที่จะเป็น $k-$special และ $2k-$special

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ABSTRACT

For positive integers k and l, we call l a $k-$special if every integer can be represented in the form $x^2+ky^2-lz^2$ where x, y, and z are non-zero integers. In this thesis, we find the necessary and sufficient conditions for $1-$special and find the conditions for an odd positive integer l to be $k-$special and $2k-$special.

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CHAPTER 1

Introduction

In number theory, the representation of integers as sums of squares are concerned by many mathematicians. For example, in 1640, Fermat [9] proved that every prime number p of type $p = 4k + 1$ can be represented as a sum of two squares of integers. This implies that a positive integer n can be written as a sum of two squares of integers if and only if all prime factors of n of the form $4k + 3$ have even exponents in the prime factorization of n. In 1770, Lagrange [9] showed that every positive integer n can be written as

$$w^2 + x^2 + y^2 + z^2 \tag{1.1}$$

where w, x, y, and z are integers. In 1798, Lagrange [2] proved that a positive integer can be represented in the form

$$x^2 + y^2 + z^2 \tag{1.2}$$

where x, y, and z are integers if and only if it is not of the form $4^a(8b+7)$ for integers $a, b \geq 0$. In connection with Lagrange's four-square theorem, in 1917, Ramanujan [7] determined all positive integers a, b, c, and d such that every natural number n is representable in the form

$$aw^2 + bx^2 + cy^2 + dz^2. \tag{1.3}$$

Finally, he found 54 quadruples (a, b, c, d) with $1 \leq a \leq b \leq c \leq d$. In 2005, Panaitopol [5] showed that there exist no natural numbers a, b, and c such that all even positive integers can be expressed in the form

$$ax^2 + by^2 + cz^2 \tag{1.4}$$

and he proved that for each odd natural number there exist non-zero integers x, y and z in (1.4) if and only if 3 triples (a, b, c) with $1 \leq a \leq b \leq c$ are $(1, 1, 2), (1, 2, 3)$, or $(1, 2, 4)$. However, if we allow c in (1.4) to be negative, then the representation is possible . In 2015, Nowicki [4] showed that if all natural numbers are representable in the form

$$x^2 + y^2 - cz^2, \tag{1.5}$$

then c is of the form q or $2q$, where either $q = 1$ or q is a product of primes of the form $4m+1$. In the same year, Lam [3] proved its sufficiency. In 2021, Prugsapitak and Thongngam [6] proved that if k is not divisible by 4, then all integers can be written as

$$x^2 + ky^2 - z^2, \tag{1.6}$$

where x, y, and z are non-zero integers. In what follows, we study the representation of integers of the form

$$x^2 + ky^2 - lz^2 \tag{1.7}$$

for given positive integers k and l, where $xyz \neq 0$.

To obtain the result that we mentioned above, we separate our work into three chapters as follows:

In Chapter 2, we review definitions and theorems, which use throughout the dissertation.

In Chapter 3, we first define $k-$special. Let k be a positive integer. We say that a positive integer l is $k-$special if all integers n can be expressed in the term $n = x^2 + ky^2 - lz^2$ where x, y, and z are non-zero integers. We find the necessary and sufficient conditions for representing all integers in the form $x^2 + ky^2 - z^2$ where x, y, and z are non-zero integers. For an odd positive integer k, we find the conditions of an odd positive integer l to be $k-$special and we proved that there are infinitely many $k-$special. Moreover, we show that if k is odd, then $4m$ is not $k-$special.

In Chapter 4, for a positive odd integer k, we find the conditions of an odd positive integer l to be $2k-$special and we proved that there are infinitely many $2k-$special. Moreover, we show some properties of $k-$special when $k \equiv 2 \pmod 8$ and $k \equiv 2 \pmod 4$.

CHAPTER 2

Preliminaries

In this chapter, we recall some definitions, theorems and examples that will be used throughout our study.

Definition 2.1 ([8])**.** If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$. If a divides b, we also say that a is a divisor or factor of b and that b is a multiple of a. If a divides b we write $a \mid b$, and if a does not divides b we write $a \nmid b$.

Theorem 2.1 ([8])**.** *If $a, b,$ and c are integers with $a \mid b$ and $b \mid c$, then $a \mid c$.*

Theorem 2.2 ([8])**.** *If $a, b, m,$ and n are integers, and if $c \mid a$ and $c \mid b$, then $c \mid (ma + nb)$.*

Theorem 2.3 ([8])**.** ***(The Division Algorithm)*** *If a and b are integers such that $b > 0$, then there are unique integers q and r such that*

$$a = bq + r \text{ with } 0 \leq r < b.$$

Definition 2.2 ([8])**.** The greatest common divisor of a and b, which are not both 0, is the largest integer that divides both a and b. We denote the greatest common divisor of a and b by $\gcd(a, b)$.

Definition 2.3 ([8])**.** The integers a and b, with $a \neq 0$ and $b \neq 0$, are relative prime if a and b have the greatest common divisor $\gcd(a, b) = 1$.

Definition 2.4 ([8])**.** A prime is an integer greater than 1 that is divisible by no positive integers other than 1 and itself.

Definition 2.5 ([8])**.** An integer greater than 1 that is not prime is called composite.

Definition 2.6 ([8])**.** If a and b are integers, then a linear combination of a and b is a sum of the form $ma + nb$, where both m and n are integers.

Theorem 2.4 ([8])**.** *The greatest common divisor of the integers a and b, not both 0, is the least positive integer that is a linear combination of a and b.*

Corollary 2.5 ([8])**.** *The integers a and b are relatively prime integers if and only if there are integers m and n such that $ma + nb = 1$.*

Theorem 2.6 ([8])**.** *(**The Euclidean Algorithm**) Let $r_0 = a$ and $r_1 = b$ be integers such that $a \geq b > 0$. If the division algorithm is successively applied to obtain $r_j = r_{j+1}q_{j+1} + r_{j+2}$, with $0 < r_{j+2} < r_{j+1}$ for $j = 0, 1, 2, ..., n-2$ and $r_{n+1} = 0$, then $\gcd(a, b) = r_n$, the last non-zero remainder.*

Definition 2.7 ([8])**.** Let m be a positive integer. If a and b are integers, we say that a is congruent to b modulo m if $m \mid (a - b)$.

If a is congruent to b modulo m, we write $a \equiv b \pmod{m}$. If $m \nmid (a - b)$, we write $a \not\equiv b \pmod{m}$, and we say that a and b are incongruent modulo m.

Theorem 2.7 ([8])**.** *If a and b are integers, then $a \equiv b \pmod{m}$ if and only if there is an integer k such that $a = b + km$.*

Theorem 2.8 ([8])**.** *Let m be a positive integer. Congruences modulo m satisfy the following properties:*

1. *Reflexive property: If a is an integer, then $a \equiv a \pmod{m}$.*

2. *Symmetric property: If a and b are integers such that $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.*

3. *Transitive property: If a, b, and c are integers with $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.*

Theorem 2.9 ([8])**.** *If a, b, c, d, and m are integers with $m > 0$, such that $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, then*

1. $a + c \equiv b + c \pmod{m}$,

2. $a - c \equiv b - c \pmod{m}$,

3. $ac \equiv bc \pmod{m}$,

4. $a + c \equiv b + d \pmod{m}$,

5. $a - c \equiv b - d \pmod{m}$,

6. $ac \equiv bd \pmod{m}$.

Theorem 2.10 ([8])**.** *If a, b, c, and m are positive integers such that $m > 0$, $d = \gcd(c, m)$, and $ac \equiv bc \pmod{m}$, then $a \equiv b \left(\bmod \dfrac{m}{d}\right)$.*

Definition 2.8 ([8])**.** If m is a positive integer, we say that an integer a is a quadratic residue of m if $\gcd(a, m) = 1$ and the congruence $x^2 \equiv a \pmod{m}$ has a solution. If the congruence $x^2 \equiv a \pmod{m}$ has no solution, we say that a is a quadratic nonresidue of m.

Definition 2.9 ([8])**.** Let p be an odd prime and a be an integer not divisible by p. The Legendre symbol $\left(\dfrac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p; \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Definition 2.10 ([1])**.** A nonzero, nonunit elelment p of an integral domain D is called a prime if $p \mid ab$, where $a, b \in D$, implies that $p \mid a$ or $p \mid b$.

Example 1. 2 is not a prime in $\mathbb{Z} + \mathbb{Z}\sqrt{-5}$ as $2 \mid (1 + \sqrt{-5})(1 - \sqrt{-5})$ yet $2 \nmid 1 \pm \sqrt{-5}$.

Definition 2.11 ([1])**. (Element integral over a domain)** Let A and B be integral domains with $A \subseteq B$. The element $b \in B$ is said to be integral over A if it satisfies a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $a_0, a_1, \ldots, a_{n-1}$.

Definition 2.12 ([1])**. (Algebraic integer)** A complex number which is integral over $\mathbb{Z}$ is called an algebraic integer.

Definition 2.13 ([1])**. (Element algebraic over field)** Let A and B be integral domains with $A \subseteq B$. Suppose that A is a field and $b \in B$ is integral over A; then b is said to be algebraic over A.

Definition 2.14 ([1]). **(Algebraic number)** A complex number that is algebraic over $\mathbb{Q}$ is called an algebraic number.

Theorem 2.11. *A rational number is an algebraic integer if and only if α is an integer.*

Proof. Let α be a rational number. Suppose that α is an algebraic integer. Let f be the monic polynomial in $\mathbb{Z}[x]$ of least degree having α as a root, i.e., $f(\alpha) = 0$. So $f(x) = (x - \alpha)h(x)$ for some $h(x) \in \mathbb{Q}[x]$. Since $f(x)$ is irreducible over $\mathbb{Q}$, $h(x) = 1$ or -1. So $x - \alpha \in \mathbb{Z}[x]$. Therefore α is an integer as desired. For the converse, it is easy to see that if α is an integer then α is a rational number. $\square$

We next prove the lemma that we will use in the proof of our main results.

Lemma 2.12. *Let a, b, c, d be integers. If $\frac{an+n}{cn+d}$ is an integer for some integer n, then $cn + d \mid ad - bc$.*

Proof. Suppose $\frac{an+b}{cn+d}$ is an integer for some integer n. Then $\frac{c(an+b)}{cn+d}$ is also an integer.

Thus

$$\begin{aligned} \frac{c(an+b)}{cn+d} &= \frac{acn + ad + (bc - ad)}{cn+d} \\ &= a - \frac{ad - bc}{cn+d} \\ \frac{ad - bc}{cn+d} &= a - \frac{c(an+b)}{cn+d}. \end{aligned}$$

Since a and $\frac{c(an+b)}{cn+d}$ are integers, $\frac{ad-bc}{cn+d}$ is also an integer.

Hence $cn + d \mid ad - bc$. $\square$

Lemma 2.13. *Let m be a rational. If m^2 is an integer, then m is an integer.*

Proof. Suppose that m^2 is an integer. Thus m is an algebraic integer because it is a root of $x^2 - m^2 = 0$. Since a rational number is an algebraic integer if and only if it is an integer, m is an integer as desired. $\square$

Lemma 2.14. *If x and y can both be represented as $a^2 + 2b^2$, for some integers a and b, then xy can be written of this form.*

Proof. Suppose $x = a^2 + 2b^2$ and $y = c^2 + 2d^2$ for some $a, b, c, d \in \mathbb{Z}$. We have

$$\begin{aligned} xy &= (a^2 + 2b^2)(c^2 + 2d^2) \\ &= a^2c^2 + 2a^2d^2 + 2b^2c^2 + 4b^2d^2 \\ &= (ac - 2bd)^2 + 2(ad + bc)^2, \end{aligned}$$

which is of the desired form. □

Lemma 2.15 ([1]). *Let p be a prime of the form $8k+1$ or $8k+3$, then $p = x^2+2y^2$ for some $x, y \in \mathbb{Z}$.*

Lemma 2.16. *If $n \in \mathbb{Z}$ is of the form x^2+2y^2 for some integers x and y, then all primes p of the form $8k+5$ or $8k+7$ have even exponent in the prime factorization.*

Proof. Let $n = x^2 + 2y^2$ for some integers x and y. We have

$$n = x^2 + 2y^2 = (x + y\sqrt{-2})(x - y\sqrt{-2}).$$

Let p be a prime of the form $8k + 5$ or $8k + 7$ and $p|n$. Since -2 is quadratic nonresidue modulo p, we can see that p is a prime in $\mathbb{Z}[\sqrt{-2}]$. Thus $p|x + y\sqrt{-2}$ or $p|x - y\sqrt{-2}$. If $p|x + y\sqrt{-2}$, then $p|x - y\sqrt{-2}$. Thus $p|2x$ and $p|2y$. Since p is odd, we have $p|x$ and $p|y$. Similarly, we can show that if $p|x - y\sqrt{-2}$, then $p|x$ and $p|y$. Thus $p|x+y\sqrt{-2}$ and $p^2|n$. Write $x = px_1$ and $y = py_1$ for some integers x_1 and y_1. Thus $n = p^2x_1^2+2p^2y_1^2 = p^2(x_1^2+2y_1^2)$. So $\frac{n}{p^2} = x_1^2+2y_1^2$. If $p \nmid x_1^2+2y_1^2$, then $p^2||n$. If $p|x_1^2 + 2y_1^2$ then $p^2|x_1^2 + 2y_1^2$. We can continue this process and thus p has even multiplicity in the prime factorization of n. □

Lemma 2.17. *A positive integer n can be written as $x^2 + 2y^2$ for some integers x and y if and only if all primes of the form $8k + 5$ or $8k + 7$ have even exponent in the prime factorization of n.*

Proof. Let n be a positive integer of the form $n = x^2 + 2y^2$ for some integers x and y. Let p be a prime of the form $8k + 5$ and $8k + 7$. By Lemma 2.16, if $p|n$, then p has even multiplicity in the prime factorization of n. Conversely, we know that $2 = 0^2 + 2(1^2)$. Let p be a prime divisor of n. If $p \equiv 5, 7 \pmod 8$, then its exponent is even and we have $p^2 = p^2+2(0^2)$. If $p \equiv 1, 3 \pmod 8$, then by Lemma 2.15 we have $p = a^2 + 2b^2$ for some integers a and b. Thus by Lemma 2.14 any product of integer of the form $x^2 + 2y^2$ is still an integer of the form $x^2 + 2y^2$. □

In 2005, L. Panaitopol [5] expressed natural numbers as sums of three squares as follows:

Theorem 2.18 ([5]). *Consider integers $a, b,$ and c satisfying $1 \leq a \leq b \leq c$. There exist for each odd natural number n non-negative integers $x, y,$ and z such that*

$$n = ax^2 + by^2 + cz^2$$

if and only if (a, b, c) are $(1, 1, 2), (1, 2, 3),$ or $(1, 2, 4)$.

Theorem 2.19 ([5]). *There exist no natural numbers $a, b,$ and c such that every even natural number n has the representation*

$$n = ax^2 + by^2 + cz^2$$

in which $x, y,$ and z are integers.

In 2015, A. Nowicki [4] and P. C. H. Lam [3] provided necessary and sufficient conditions for representing all integers in the form $x^2 + y^2 - cz^2$ as follows:

Definition 2.15 ([4]). Let c be a positive integer. We say that c is special if for every integer k there exist non-zero integers $x, y,$ and z such that $x^2 + y^2 - cz^2 = k$.

Theorem 2.20 ([3]). *If c is of the form q or $2q$, where either $q = 1$ or q is a product of prime numbers of the form $4k + 1$, then c is special.*

Theorem 2.21 ([4]). *Every special number is of the form q or $2q$, where either $q = 1$ or q is a product of prime numbers of the form $4k + 1$.*

Theorem 2.22 ([4]). *There are infinitely many special numbers.*

CHAPTER 3

$k-$Special Numbers

In this chapter, we first define $k-$special where k is a positive integer. We say that a positive integer l is $k-$special if for all integers n there exist non-zero integers x, y, and z such that

$$n = x^2 + ky^2 - lz^2.$$

We provide the necessary and sufficient conditions for 1 to be k-special and we find the condition for an odd positive integer l to be $k-$special for a given odd positive integer k. Moreover, we provide some properties of $k-$special.

Definition 3.1. Let k and l be positive integers. Let $[x, y, z]_{k,l}$ denote the number $x^2 + ky^2 - lz^2$ where x, y, and z are integers and we say that l is ***k*-special** if for every integer n there exist non-zero integers x, y, and z such that $n = [x, y, z]_{k,l}$.

A. Nowicki [4] showed that 1 is $1-$special by giving the following identities.

Lemma 3.1 ([4])**.** *1 is $1-$special.*

Proof. It is easy to see that

$$2j - 1 = 2^2 + (j - 2)^2 - (j - 3)^2,$$
$$2j = j^2 + 1^2 - (j - 1)^2$$

for $j \in \mathbb{Z}$. However, one of the variables $j - 3, j - 2, j - 1$, and j becomes zero if $j = 3, 2, 1$, and 0 respectively. So we can use the representations $0 = [3, 4, 5]_{1,1}$, $2 = [3, 3, 4]_{1,1}$, $3 = [6, 4, 7]_{1,1}$, and $5 = [5, 4, 6]_{1,1}$. □

Theorem 3.2. *Let k be a positive integer. If k is divisible by 4, then 1 is not $k-$special.*

Proof. Let k be divisible by 4. Assume that 1 is $k-$special. Then there exist non-zero integers x, y, and z such that

$$x^2 + ky^2 - z^2 = 2.$$

So we have $x^2 - z^2 \equiv 2 \pmod 4$. Since quadratic residues modulo 4 are 0 and 1, we deduce that $x^2 - z^2 \equiv 0, 1, 3 \pmod 4$. This is a contradiction. Hence 1 is not $k-$special. □

Theorem 3.3. *Let k be a positive integer. If k is not divisible by 4, then 1 is $k-$special.*

Proof. Let k be a positive integer not divisible by 4. We will show that for any integer n there exist non-zero integers x, y, and z such that

$$n = x^2 + ky^2 - z^2,$$

i.e., $x^2 - z^2 = (x - z)(x + z) = n - ky^2$.

We now consider the following four cases on the value of n:

Case 1. Suppose $n \equiv 0 \pmod 4$. Thus $n = 4j$ for some integer j.
We next find non-zero integers x, y, and z such that $(x - z)(x + z) = 4j - ky^2$.
We choose $y = 4j + 2$.
Then

$$\begin{aligned}(x - z)(x + z) &= 4j - k(4j + 2)^2 \\ &= 2(2j - k(2j + 1)(4j + 2)).\end{aligned}$$

Let $x - z = 2$ and $x + z = 2j - k(2j + 1)(4j + 2)$.
Then $2x = 2 + 2j - k(2j + 1)(4j + 2)$ and $2z = 2j - k(2j + 1)(4j + 2) - 2$.
So we obtain $x = 1 + j - k(2j + 1)^2$ and $z = j - k(2j + 1)^2 - 1$.
We next show that x, y, and z are non-zero.
Since $y \equiv 2 \pmod 4$, it implies that $y \neq 0$.
If $z = 0$, then

$$\begin{aligned}j - k(2j + 1)^2 - 1 &= 0 \\ k(2j + 1)^2 &= j - 1 \\ k &= \frac{j - 1}{(2j + 1)^2}.\end{aligned}$$

Since $k > 0$, we deduce that $j > 1$. Thus $1 \leq j - 1 < (2j + 1)^2$.
This implies that $\frac{j-1}{(2j+1)^2}$ is not an integer. Therefore z is non-zero.

If $x = 0$, then

$$\begin{aligned} 1 + j - k(2j+1)^2 &= 0 \\ k(2j+1)^2 &= 1 + j \\ k &= \frac{1+j}{(2j+1)^2}. \end{aligned}$$

Since $k > 0$, we deduce that $j > -1$. If $j > 0$, then $\frac{1+j}{(2j+1)^2}$ is not an integer. If $j = 0$, then $k = 1$ and $n = 0$. We need to provide a presentation for 0, namely $0 = [x, y, z]_{1,1}$ where $xyz \neq 0$. We can write 0 as $0 = 3^2 + 4^2 - 5^2$.

Case 2. Suppose $n \equiv 1 \pmod 4$. Thus $n = 4j + 1$ for some integer j. We next find non-zero integers x, y, and z such that $(x-z)(x+z) = 4j + 1 - ky^2$. We choose $y = 2(4j + 1)$.

Then

$$\begin{aligned} (x-z)(x+z) &= 4j + 1 - 4k(4j+1)^2 \\ &= (4j+1)(1 - 4k(4j+1)). \end{aligned}$$

Let $x - z = 4j + 1$ and $x + z = 1 - 4k(4j + 1)$.

Then $2x = 4j + 2 - 4k(4j + 1)$ and $2z = -4k(4j + 1) - 4j$.

So we obtain $x = 1 + 2j - 2k(4j + 1)$ and $z = -2j - 2k(4j + 1)$.

We next show that x, y, and z are non-zero.

Since $y \equiv 2 \pmod 8$, we have $y \neq 0$.

If $x = 0$, then

$$\begin{aligned} 1 + 2j - 2k(4j+1) &= 0 \\ 2k(4j+1) &= 2j + 1 \\ k &= \frac{2j+1}{2(4j+1)}. \end{aligned}$$

Since $2j + 1$ is odd and $2(4j + 1)$ is even, we deduce that $\frac{2j+1}{2(4j+1)}$ is not an integer. So this is a contradiction.

If $z = 0$, then

$$\begin{aligned} -2j - 2k(4j+1) &= 0 \\ 2k(4j+1) &= -2j \\ k &= \frac{-j}{4j+1}. \end{aligned}$$

Since k is an integer and by Lemma 2.12, we deduce that $4j+1 \mid (-1)(1)-(0)(4)$. This implies that $4j+1 \mid -1$. Thus $4j+1 = -1, 1$ and hence $j = 0$.
If $j = 0$, then $k = 0$. This contradicts the fact that k is a positive integer.
Thus z is non-zero.

Case 3. Suppose $n \equiv 2 \pmod 4$. Thus $n = 4j+2$ for some integer j.

Subcase 3.1. Suppose $k \equiv 2 \pmod 4$. Thus $k = 4r+2$ for some non-negative integer r. We next find non-zero integers x, y, and z such that

$$(x-z)(x+z) = 4j+2-(4r+2)y^2.$$

We choose $y = 2j+1$.
Then

$$\begin{aligned}(x-z)(x+z) &= 4j+2-(4r+2)(2j+1)^2 \\ &= (4j+2)(1-(2r+1)(2j+1)) \\ &= (4j+2)(-4rj-2j-2r).\end{aligned}$$

Let $x-z = 4j+2$ and $x+z = -4rj-2j-2r$.
Then $2x = -4rj+2j-2r+2$ and $2z = -4rj-6j-2r-2$.
So we obtain $x = j-r-2rj+1$ and $z = -3j-r-2rj-1$.
We next show that x, y, and z are non-zero.
Since y is odd, we have $y \neq 0$.
If $x = 0$, then

$$\begin{aligned}j-r-2rj+1 &= 0 \\ r(2j+1) &= j+1 \\ r &= \frac{j+1}{2j+1}.\end{aligned}$$

Since r is an integer and by Lemma 2.12, we deduce that $2j+1 \mid (1)(1)-(1)(2)$. This implies that $2j+1 \mid -1$. Thus $2j+1 = -1, 1$ and hence $j = 0$ and $j = -1$.
If $j = 0$, then $r = \frac{j+1}{2j+1} = 1$. So $k = 6$ and $n = 2$. We need to provide a presentation for 2, namely $2 = [x, y, z]_{6,1}$ where $xyz \neq 0$. So we can use the representation $2 = [12, 3, 14]_{6,1}$.
If $j = -1$, then $r = \frac{j+1}{2j+1} = 0$. So $k = 2$ and $n = -2$. We need to provide a presentation for -2, namely $-2 = [x, y, z]_{2,1}$ where $xyz \neq 0$. We can write -2 as

$-2 = [4, 3, 6]_{2,1}$.

If $z = 0$, then

$$\begin{aligned} -3j - r - 2rj - 1 &= 0 \\ r(2j+1) &= -3j - 1 \\ r &= \frac{-3j-1}{2j+1}. \end{aligned}$$

Since r is an integer and by Lemma 2.12, we deduce that $2j+1 \mid (-3)(1)-(-1)(2)$. This implies that $2j+1 \mid -1$. Thus $2j+1 = -1, 1$ and hence $j = 0$ and $j = -1$. If $j = 0$, then $r = -1$. This contradicts the fact that r is a non-negative integer. If $j = -1$, then $r = -2$. This contradicts the facts that r is a non-negative integer. Thus z is non-zero.

Subcase 3.2. Suppose $k \equiv 1 \pmod 2$. Thus $k = 2r+1$ for some non-negative integer r. We next find non-zero integers x, y, and z such that

$$(x-z)(x+z) = 4j + 2 - (2r+1)y^2.$$

We choose $y = 2j + 1$.

Thus

$$\begin{aligned} (x-z)(x+z) &= 4j + 2 - (2r+1)(2j+1)^2 \\ &= (2j+1)(2 - (2r+1)(2j+1)) \\ &= (2j+1)(1 - 4rj - 2r - 2j). \end{aligned}$$

Let $x - z = 2j + 1$ and $x + z = 1 - 4rj - 2r - 2j$.

Then $2x = -4rj - 2r + 2$ and $2z = -4rj - 2r - 4j$.

So we obtain $x = 1 - r - 2rj$ and $z = -r - 2j - 2rj$.

We next show that x, y, and z are non-zero.

Since y is odd, it implies that y is non-zero.

If $x = 0$, then

$$\begin{aligned} 1 - r - 2rj &= 0 \\ r(2j+1) &= 1 \\ r &= \frac{1}{2j+1}. \end{aligned}$$

Since r is an integer and by Lemma 2.12, we deduce that $2j+1 \mid (0)(1)-(1)(2)$. This implies that $2j+1 \mid -2$. Thus $2j+1 = -1, 1, 2, -2$ and hence $j = 0$ and $j = -1$.

If $j = 0$, then $r = 1$. We obtain $k = 3$ and $n = 2$. We will use the representation $2 = 12^2 + 3(3)^2 - 13^2$.

If $j = -1$, then $r = -1$. This contradicts the fact that r is a non-negative integer.

If $z = 0$, then

$$\begin{aligned} -r - 2j - 2rj &= 0 \\ r(2j+1) &= -2j \\ r &= \frac{-2j}{2j+1}. \end{aligned}$$

Since r is an integer and by Lemma 2.12, we deduce that $2j+1 \mid (-2)(1)-(0)(2)$. This implies that $2j+1 \mid -2$. Thus $2j+1 = -1, 1, 2, -2$ and hence $j = 0$ and $j = -1$.

If $j = 0$, then $r = 0$. We obtain $k = 1$ and $n = 2$. We will use the representation $2 = 3^2 + 1(3)^2 - 4^2$ instead.

If $j = -1$, then $r = -2$. We obtain $k = -3$. This contradicts the fact that $k > 0$.

Case 4. Let $n \equiv 3 \pmod 4$. Thus $n = 4j+3$ for some integer j.

We next find non-zero integers x, y, and z such that $(x-z)(x+z) = 4j+3-ky^2$.

We choose $y = 2(4j+3)$.

Thus

$$\begin{aligned} (x-z)(x+z) &= 4j+3-4k(4j+3)^2 \\ &= (4j+3)(1-4k(4j+3)). \end{aligned}$$

Let $x - z = 4j+3$ and $x + z = 1-4k(4j+3)$.

Then $2x = 4j+4-4k(4j+3)$ and $2z = -4k(4j+3)-4j-2$.

So we obtain $x = 2j-6k-8kj+2$ and $z = -2j-6k-8kj-1$.

We next show that x, y, and z are non-zero.

Since $y \equiv 6 \pmod 8$, we have that y is non-zero.

If $x = 0$, then

$$\begin{aligned} 2j - 6k - 8kj + 2 &= 0 \\ k(8j+6) &= 2j+2 \\ k &= \frac{j+1}{4j+3}. \end{aligned}$$

Since k is an integer and by Lemma 2.12, we deduce that $4j + 3 \mid (1)(3) - (1)(1)$. This implies that $4j + 3 \mid -2$. Thus $4j + 3 = -1, 1, -2, 2$ and hence $j = -1$.
If $j = -1$, then $k = 0$. This contradicts the fact that $k > 0$.
If $z = 0$, then

$$\begin{aligned} -2j - 6k - 8kj - 1 &= 0 \\ k(8j+6) &= -(2j+1) \\ k &= \frac{-(2j+1)}{8j+6}. \end{aligned}$$

Since $2j + 1$ is odd and $8j + 6$ is even, we deduce that $\frac{-(2j+1)}{8j+6}$ is not an integer.
Both cases imply that x and z are non-zero. □

In conclusion, we have proved the following theorem.

Theorem 3.4. *Let k be a positive integer. Then 1 is $k-$special if and only if k is not divisible by 4.*

We next provide examples when 1 is $k-$special where $k \le 20$ and k is not divisible by 4.

Example 2. We show that 1 is $1-$special and next we will show that 1 is $k-$special for $2 \le k \le 20$ by giving the following identities:

- 1 is $2-$special.

$$\begin{aligned} [8j^2 + 3j + 1, 4j + 2, 8j^2 + 7j + 3]_{2,1} &= 4j, \\ [14j + 3, 8j + 2, 18j + 4]_{2,1} &= 4j + 1, \\ [j + 1, 2j + 1, 3j + 1]_{2,1} &= 4j + 2, \\ [14j + 10, 8j + 6, 18j + 13]_{2,1} &= 4j + 3, \end{aligned}$$

and $[4,3,6]_{2,1}=-2$.

• 1 is 3−special.

$$[12j^2+11j+2,4j+2,12j^2+11j+4]_{3,1}=4j,$$
$$[22j+5,8j+2,26j+6]_{3,1}=4j+1,$$
$$[2j,2j+1,4j+1]_{3,1}=4j+2,$$
$$[22j+16,8j+6,26j+19]_{3,1}=4j+3,$$

and $[12,3,13]_{2,1}=2$.

• 1 is 5−special.

$$[20j^2+19j+4,4j+2,20j^2+19j+6]_{5,1}=4j,$$
$$[38j+9,8j+2,42j+10]_{5,1}=4j+1,$$
$$[4j+1,2j+1,6j+2]_{5,1}=4j+2,$$
$$[38j+28,8j+6,42j+31]_{5,1}=4j+3.$$

• 1 is 6−special.

$$[24j^2+23j+5,4j+2,24j^2+23j+7]_{6,1}=4j,$$
$$[46j+11,8j+2,50j+12]_{6,1}=4j+1,$$
$$[j,2j+1,5j+2]_{6,1}=4j+2,$$
$$[46j+34,8j+6,50j+37]_{6,1}=4j+3,$$

and $[12,3,14]_{6,1}=2$.

• 1 is 7−special.

$$[28j^2+27j+6,4j+2,28j^2+27l+8]_{7,1}=4j,$$
$$[54j+13,8j+2,58j+14]_{7,1}=4j+1,$$
$$[6j+2,2j+1,8j+3]_{7,1}=4j+2,$$
$$[54j+40,8j+6,58j+43]_{7,1}=4j+3.$$

• 1 is 9−special.

$$[36j^2+35j+8,4j+2,36j^2+35j+10]_{9,1}=4j,$$
$$[70j+17,8j+2,74j+18]_{9,1}=4j+1,$$
$$[8j+3,2j+1,10j+4]_{9,1}=4j+2,$$
$$[70j+52,8j+6,74j+55]_{9,1}=4j+3.$$

- 1 is 10−special.

$$
\begin{aligned}
[40j^2+39j+9, 4j+2, 40j^2+39j+11]_{10,1} &= 4j,\\
[78j+19, 8j+2, 82j+20]_{10,1} &= 4j+1,\\
[3j+1, 2j+1, 7j+3]_{10,1} &= 4j+2,\\
[78j+58, 8j+6, 74j+61]_{10,1} &= 4j+3.
\end{aligned}
$$

- 1 is 11−special.

$$
\begin{aligned}
[44j^2+43j+10, 4j+2, 44j^2+43j+12]_{11,1} &= 4j,\\
[86j+21, 8j+2, 90j+22]_{11,1} &= 4j+1,\\
[10j+4, 2j+1, 12j+5]_{11,1} &= 4j+2,\\
[86j+64, 8j+6, 90j+67]_{11,1} &= 4j+3.
\end{aligned}
$$

- 1 is 13−special.

$$
\begin{aligned}
[52j^2+51j+12, 4j+2, 52j^2+51j+14]_{13,1} &= 4j,\\
[102j+25, 8j+2, 106j+26]_{13,1} &= 4j+1,\\
[12j+5, 2j+1, 14j+6]_{13,1} &= 4j+2,\\
[102j+76, 8j+6, 106j+79]_{13,1} &= 4j+3.
\end{aligned}
$$

- 1 is 14−special.

$$
\begin{aligned}
[56j^2+55j+13, 4j+2, 56j^2+55j+15]_{14,1} &= 4j,\\
[110j+27, 8j+2, 114j+28]_{14,1} &= 4j+1,\\
[5j+2, 2j+1, 9j+4]_{14,1} &= 4j+2,\\
[110j+82, 8j+6, 114j+85]_{14,1} &= 4j+3.
\end{aligned}
$$

- 1 is 15−special.

$$
\begin{aligned}
[60j^2+59j+14, 4j+2, 60j^2+59j+16]_{15,1} &= 4j,\\
[118j+29, 8j+2, 122j+30]_{15,1} &= 4j+1,\\
[14j+6, 2j+1, 16j+7]_{15,1} &= 4j+2,\\
[118j+88, 8j+6, 122j+91]_{15,1} &= 4j+3.
\end{aligned}
$$

- 1 is 17−special.

$$[68j^2+67j+16, 4j+2, 68j^2+67j+18]_{17,1} = 4j,$$
$$[134j+33, 8j+2, 138j+34]_{17,1} = 4j+1,$$
$$[16j+7, 2j+1, 18j+8]_{17,1} = 4j+2,$$
$$[134j+100, 8j+6, 138j+103]_{17,1} = 4j+3.$$

- 1 is 18−special.

$$[72j^2+71j+17, 4j+2, 72j^2+71j+19]_{18,1} = 4j,$$
$$[142j+35, 8j+2, 146j+36]_{18,1} = 4j+1,$$
$$[7j+3, 2j+1, 11j+5]_{18,1} = 4j+2,$$
$$[142j+106, 8j+6, 146j+109]_{18,1} = 4j+3.$$

- 1 is 19−special.

$$[76j^2+75j+18, 4j+2, 76j^2+75j+20]_{19,1} = 4j,$$
$$[150j+37, 8j+2, 154j+38]_{19,1} = 4j+1,$$
$$[18j+8, 2j+1, 20j+9]_{19,1} = 4j+2,$$
$$[150j+112, 8j+6, 154j+115]_{19,1} = 4j+3.$$

We next provide some properties of $k-$special.

Theorem 3.5. *Let k be a positive integer. Then k is $k-$special if and only if $k=1$.*

Proof. It is known that 1 is 1-special by Lemma 3.1. Now if $k=2$, then for any integer n there exist non-zero integers x, y, and z such that $n = x^2+2y^2-2z^2$. Since $x^2 \equiv 0,1,4 \pmod 8$, by a direct calculation $2y^2 \equiv 0,2 \pmod 8$ and $-2z^2 \equiv 0,-2 \pmod 8$, we have $x^2+2y^2-2z^2 \not\equiv 5 \pmod 8$. Thus 2 is not 2−special. For $k>2$, if k is $k-$special then for any integer n there exist non-zero integers x, y, and z such that $n = x^2+ky^2-kz^2$. Since $k>2$, there exists a non-quadratic residue modulo k, namely k'. Thus $k' \equiv x^2 \pmod k$. This is a contradiction. □

Next, we apply P. C. H. Lam's method [3] to identify $k-$special numbers when k is odd.

Theorem 3.6. *Let l and k be odd positive integers. If $l = x^2 + ky^2$ for some positive integers x and y and $\gcd(x, ky) = 1$, then l is $k-$special.*

Proof. Suppose $l = x^2 + ky^2$ for some positive integers x and y where $\gcd(x, ky) = 1$. Since $\gcd(x, ky) = 1$, there exist integers α_0 and β_0 such that $x\alpha_0 + ky\beta_0 = 1$.

For any positive integer n, We define $\alpha_n = \alpha_0 + nky$ and $\beta_n = \beta_0 - nx$. Consider

$$\begin{aligned} x\alpha_n + ky\beta_n &= x(\alpha_0 + nky) + ky(\beta_0 - nx) \\ &= x\alpha_0 + xnky + ky\beta_0 - knyx \\ &= x\alpha_0 + ky\beta_0. \end{aligned}$$

So (α_n, β_n) is a solution of $x\alpha_0 + ky\beta_0 = 1$.

Let $a_n = xj + \alpha_n$, $b_n = yj + \beta_n$ and $c = j$, where j is an integer which will be selected later. Thus

$$\begin{aligned} a_n^2 + kb_n^2 - lc_n^2 &= a_n^2 + kb_n^2 - (x^2 + ky^2)c_n^2 \\ &= (xj + \alpha_n)^2 + k(yj + \beta_n)^2 - (x^2 + ky^2)j^2 \\ &= x^2j^2 + 2xj\alpha_n + \alpha_n^2 + ky^2j^2 + 2kyj\beta_n + k\beta_n^2 - x^2j^2 - ky^2j^2 \\ &= 2xj\alpha_n + \alpha_n^2 + 2kyj\beta_n + k\beta_0^2 \\ &= 2j(x\alpha_n + ky\beta_n) + \alpha_n^2 + k\beta_n^2. \end{aligned}$$

We have

$$\begin{aligned} \alpha_n^2 + k\beta_n^2 &= (\alpha_0 + nky)^2 + k(\beta_0 - nx)^2 \\ &= \alpha_0^2 + 2nk\alpha_0 y + n^2k^2y^2 + k\beta_0^2 - 2kn\beta_0 x + kn^2x^2 \\ &\equiv \alpha_0^2 + k\beta_0^2 + n^2k^2y^2 + kn^2x^2 \pmod 2 \\ &\equiv \begin{cases} \alpha_0^2 + k\beta_0^2 \pmod 2 & \text{if } n \text{ is even,} \\ \alpha_0^2 + k\beta_0^2 + y^2 + x^2 \pmod 2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Since l and k are odd, we can see that x and y have different parities. Thus

$$\alpha_n^2 + k\beta_n^2 \equiv \begin{cases} \alpha_0^2 + k\beta_0^2 \pmod 2 & \text{if } n \text{ is even,} \\ \alpha_0^2 + k\beta_0^2 + 1 \pmod 2 & \text{if } n \text{ is odd.} \end{cases}$$

For any non-negative integer r, We obtain the following identities

$$a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2 = 2j_{2r} + \alpha_{2r}^2 + k\beta_{2r}^2 \equiv 2j_{2r} + \alpha_0^2 + k\beta_0^2 \pmod 2$$
$$a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2 = 2j_{2r+1} + \alpha_{2r+1}^2 + k\beta_{2r+1}^2 \equiv 2j_{2r+1} + \alpha_0^2 + k\beta_0^2 + 1 \pmod 2.$$

We can see that all integers can be represented in the form $a^2 + kb^2 - lc^2$ by using both identities.

Case 1. $\alpha_0^2 + k\beta_0^2 \equiv 0 \pmod 2$. We first consider an even integer. Let m be an even integer. We choose a suitable value of j_{2r} such that

$$m = 2j_{2r} + \alpha_{2r}^2 + k\beta_{2r}^2 = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$$

where $a_{2r} = xj_{2r} + \alpha_{2r}$, $b_{2r} = yj_{2r} + \beta_{2r}$ and $c_{2r} = j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r} = 0$ if and only if m is one of the following values: $\alpha_{2r}^2 + k\beta_{2r}^2, \alpha_{2r}^2 + k\beta_{2r}^2 - \frac{2\alpha_{2r}}{x}$ or $\alpha_{2r}^2 + 2k\beta_{2r}^2 - \frac{2\beta_{2r}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r} = \lim_{r\to\infty} (-\beta_{2r}) = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r}^2 + k\beta_{2r}^2 - \frac{2\alpha_{2r}}{x} > m$, $\alpha_{2r} > 0$, and $\beta_{2r} < 0$. Thus we obtain a representation for m, namely $m = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r} \neq 0$.

Next, we consider a representation for an odd integer m. Let m be an odd integer. We choose a suitable value of an integer j_{2r+1} such that

$$m = 2j_{2r+1} + \alpha_{2r+1}^2 + k\beta_{2r+1}^2 = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$$

where $a_{2r+1} = xj_{2r+1} + \alpha_{2r+1}$, $b_{2r+1} = yj_{2r+1} + \beta_{2r+1}$ and $c_{2r+1} = j_{2r+1}$. We can see that $a_{2r+1}b_{2r+1}c_{2r+1} = 0$ if and only if m is one of the following values: $\alpha_{2r+1}^2 + k\beta_{2r+1}^2$, $\alpha_{2r+1}^2 + k\beta_{2r+1}^2 - \frac{2\alpha_{2r+1}}{x}$ or $\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{2\beta_{2r+1}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} (-\beta_{2r+1}) = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r+1}^2 + k\beta_{2r+1}^2 - \frac{2\alpha_{2r+1}}{x} > m$, $\alpha_{2r+1} > 0$, and $\beta_{2r+1} < 0$. Therefore we obtain a representation for m, namely $m = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$ where $a_{2r+1}b_{2r+1}c_{2r+1} \neq 0$.

Case 2. $\alpha_0^2 + k\beta_0^2 \equiv 1 \pmod 2$. We first find a representation for an even integer m. Let m be an even integer. We choose a suitable value of an integer j_{2r+1} such that

$$m = 2j_{2r+1} + \alpha_{2r+1}^2 + k\beta_{2r+1}^2 = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$$

where $a_{2r+1} = xj_{2r+1} + \alpha_{2r+1}$, $b_{2r+1} = yj_{2r+1} + \beta_{2r+1}$ and $c_{2r+1} = j_{2r+1}$. We can see that $a_{2r+1}b_{2r+1}c_{2r+1} = 0$ if and only if m is one of the following values: $\alpha_{2r+1}^2 + k\beta_{2r+1}^2, \alpha_{2r+1}^2 + k\beta_{2r+1}^2 - \frac{2\alpha_{2r+1}}{x}$ or $\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{2\beta_{2r+1}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} (-\beta_{2r+1}) = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r+1}^2 + k\beta_{2r+1}^2 - \frac{2\alpha_{2r+1}}{x} > m$, $\alpha_{2r+1} > 0$, and $\beta_{2r+1} < 0$. Therefore we obtain a representation for m, namely $m = a_{2r+1}^2 + kb_{2r+1}^2 - lc_{2r+1}^2$ where $a_{2r+1}b_{2r+1}c_{2r+1} \neq 0$.

Next, let m be an odd integer. We choose a suitable value of an integer j_{2r} such that

$$m = 2j_{2r} + \alpha_{2r}^2 + k\beta_{2r}^2 = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$$

where $a_{2r} = xj_{2r}+\alpha_{2r}$, $b_{2r} = yj_{2r}+\beta_{2r}$ and $c_{2r} = j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r} = 0$ if and only if m is one of the following values: $\alpha_{2r}^2 + k\beta_{2r}^2, \alpha_{2r}^2 + k\beta_{2r}^2 - \frac{2\alpha_{2r}}{x}$ or $\alpha_{2r}^2 + 2k\beta_{2r}^2 - \frac{2\beta_{2r}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r} = \lim_{r\to\infty} (-\beta_{2r}) = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r} > 0$, $\alpha_{2r}^2 + k\beta_{2r}^2 - \frac{2\alpha_{2r}}{x} > m$ and $\beta_{2r} < 0$. Therefore we obtain a representation for m, namely $m = a_{2r}^2 + kb_{2r}^2 - lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r} \neq 0$. Thus l is $k-$special. $\square$

We next provide examples how to obtain representation for any integer n of the form $x^2 + ky^2 - lz^2$ for $(k,l) = (3,7), (3,13), (3,19)$, and $(3,49)$.

Example 3. Let $l = 7$. Then $l = 2^2 + 3(1^2)$. So that $x = 2$ and $y = 1$. Since $\gcd(x, 3y) = \gcd(2,3) = 1$, there exist integers $\alpha_0 = -1$ and $\beta_0 = 1$ such that $2(-1) + 3(1)(1) = 1$. Using the notation in Theorem 3.6, we obtain $\alpha_1 = 2$ and $\beta_1 = -1$.

Thus the identities are given by

$$(2k-1)^2 + 3(k+1)^2 - 7k^2 = 2k+4,$$
$$(2k+2)^2 + 3(k-1)^2 - 7k^2 = 2k+7.$$

So all integers except $2, 4, 5, 7$, and 9 can be written in the form $x^2 + 3y^2 - 7z^2$ where $xyz \neq 0$. Thus we have to find new representations for $2, 4, 5, 7$, and 9.

We define $\alpha_2 = \alpha_0 + 6y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 9y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 5, \beta_2 = -3, \alpha_3 = 8$ and $\beta_3 = -5$.

Then we obtain new identities given by

$$(2k+5)^2 + 3(k-3)^2 - 7k^2 = 2k + 52,$$
$$(2k+8)^2 + 3(k-5)^2 - 7k^2 = 2k + 139.$$

Thus

$$2 = 45^2 + 3(28^2) - 7(25^2),$$
$$4 = 43^2 + 3(27^2) - 7(24^2),$$
$$5 = 126^2 + 3(72^2) - 7(67^2),$$
$$7 = 124^2 + 3(71^2) - 7(66^2),$$
$$9 = 122^2 + 3(70^2) - 7(65^2).$$

Hence 7 is 3−special as desired.

Example 4. Let $l = 13$. Then $l = 1^2 + 3(2^2)$. So that $x = 1$ and $y = 2$. Since $\gcd(x, 3y) = \gcd(1, 6) = 1$, there exist integers $\alpha_0 = -5$ and $\beta_0 = 1$ such that $1(-5) + 3(2)(1) = 1$. Using the notation in Theorem 3.6, we obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

Thus the identities are given by

$$(k-5)^2 + 3(2k+1)^2 - 13k^2 = 2k + 28,$$
$$(k+1)^2 + 3(k)^2 - 13k^2 = 2k + 1.$$

So all integers except $-1, 1, 28$, and 38 can be written in the form $x^2 + 3y^2 - 13z^2$ where $xyz \neq 0$. Thus we have to find new representation of $-1, 1, 28$, and 38.

We define $\alpha_2 = \alpha_0 + 6y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 9y$ and $\beta_3 = \beta_0 - 3x$,i.e., $\alpha_2 = 7, \beta_2 = -1, \alpha_3 = 13$ and $\beta_3 = -2$.

Then we obtain new identities given by

$$(k+7)^2 + 3(2k-1)^2 - 13k^2 = 2k + 52,$$
$$(k+13)^2 + 3(2k-2)^2 - 13k^2 = 2k + 181.$$

Thus

$$28 = 5^2 + 3(25^2) - 13(12^2),$$
$$-1 = 78^2 + 3(184^2) - 13(91^2),$$
$$1 = 77^2 + 3(182^2) - 13(90^2).$$

We again find the representation of 38. We define $\alpha_4 = \alpha_0 + 12y$ and $\beta_4 = \beta_0 - 4x$, i.e., $\alpha_4 = 19$ and $\beta_4 = -3$. Thus the new identity is given by

$$(k + 19)^2 + 3(2k - 3)^2 - 13k^2 = 2k + 388$$

and $38 = 156^2 + 3(353)^2 - 13(175^2)$.

Hence 13 is $3-$special as desired.

Example 5. Let $l = 19$. Then $l = 4^2 + 3(1^2)$. So that $x = 4$ and $y = 1$. Since $\gcd(x, 3y) = \gcd(4, 3) = 1$, there exist integers $\alpha_0 = 1$ and $\beta_0 = -1$ such that $4(1) + 3(1)(-1) = 1$. Using the notation in Theorem 3.6, we obtain $\alpha_1 = 4$ and $\beta_1 = -5$.

Thus the identities are given by

$$(4k + 1)^2 + 3(k - 1)^2 - 19k^2 = 2k + 4,$$
$$(4k + 4)^2 + 3(k - 5)^2 - 19k^2 = 2k + 91.$$

So all integers except $6, 4, 89, 101,$ and 91 can be written in the form $x^2 + 3y^2 - 19z^2$ where $xyz \neq 0$. Thus we have to find new representation of $6, 4, 89, 101,$ and 91. We define $\alpha_2 = \alpha_0 + 6y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 9y$ and $\beta_3 = \beta_0 - 3x$,i.e., $\alpha_2 = 7, \beta_2 = -9, \alpha_3 = 10$ and $\beta_3 = -13$.

Then we obtain new identities given by

$$(4k + 7)^2 + 3(k - 9)^2 - 19k^2 = 2k + 292,$$
$$(4k + 10)^2 + 3(k - 13)^2 - 19k^2 = 2k + 607.$$

Thus

$$6 = 565^2 + 3(152^2) - 19(143^2),$$
$$4 = 569^2 + 3(153^2) - 19(144^2),$$
$$89 = 1026^2 + 3(272^2) - 19(259^2),$$
$$101 = 1002^2 + 3(266^2) - 19(253^2),$$
$$91 = 1022^2 + 3(271^2) - 19(258^2).$$

Hence 19 is $3-$special as desired.

Example 6. Let $l = 49$. Then $l = 1^2 + 3(4^2)$. So that $x = 1$ and $y = 4$. Since $\gcd(x, 3y) = \gcd(1, 12) = 1$, there exist integers $\alpha_0 = -11$ and $\beta_0 = 1$ such that $1(-11) + 3(4)(1) = 1$. Using the notation in Theorem 3.6, we obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

Thus the identities are given by

$$(k-11)^2 + 3(4k+1)^2 - 49k^2 = 2k + 124,$$
$$(k+1)^2 + 3(4k)^2 - 49k^2 = 2k + 1.$$

So all integers except $146, 124, -1,$ and 1 can be written in the form $x^2+3y^2-49z^2$ where $xyz \neq 0$. Thus we have to find new representation of $146, 124, -1,$ and 1.

We define $\alpha_2 = \alpha_0 + 6y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 9y$ and $\beta_3 = \beta_0 - 3x$,i.e., $\alpha_2 = 13, \beta_2 = -1, \alpha_3 = 25$ and $\beta_3 = -2$.

Then we obtain new identities are given by

$$(k+13)^2 + 3(4k-1)^2 - 49k^2 = 2k + 172,$$
$$(k+25)^2 + 3(4k-2)^2 - 49k^2 = 2k + 637.$$

Thus

$$124 = 11^2 + 3(97^2) - 49(24^2),$$
$$-1 = 294^2 + 3(1278)^2) - 49(319^2),$$
$$1 = 293^2 + 3(1274^2) - 49(318^2).$$

We again find the representation of 146. We define $\alpha_4 = \alpha_0+12y$ and $\beta_4 = \beta_0-4x$, i.e., $\alpha_4 = 37$ and $\beta_4 = -3$. Thus the new identity is given by

$$(k+37)^2 + 3(4k-3)^2 - 49k^2 = 2k + 1396$$

and $146 = 588^2 + 3(2503)^2 - 49(625^2)$.

Hence 49 is $3-$special as desired.

We present an odd integer l which is $k-$special for some odd integer k where $l < 50$ by providing the following identities:

- 5 is $1-$special.

$$[2k+1, k-1, k]_{1,5} = 2k + 2,$$

$$[2k+2, k-3, k]_{1,5} = 2k + 13,$$

$[27,20,15]_{1,5} = 4$, $[29,21,16]_{1,5} = 2$, $[50,34,27]_{1,5} = 11$, $[42,30,23]_{1,5} = 19$, and $[48,33,26]_{1,5} = 13$.

• 7 is 3−special.

$$[2k-1, k+1, k]_{3,7} = 2k+4,$$
$$[2k+2, k-1, k]_{3,7} = 2k+7,$$

$[45,28,25]_{3,7} = 2$, $[43,27,24]_{3,7} = 4$, $[126,72,67]_{3,7} = 5$, $[124,71,66]_{3,7} = 7$, and $[122,70,65]_{3,7} = 9$.

• 9 is 5−special.

$$[2k-2, k+1, k]_{5,9} = 2k+9,$$
$$[2k+3, k-1, k]_{5,9} = 2k+14,$$

$[92,53,50]_{5,9} = 9$, $[94,54,51]_{5,9} = 7$, $[90,52,49]_{5,9} = 11$, $[265,144,139]_{5,9} = 16$, and $[267,145,140]_{5,9} = 14$.

• 11 is 7−special.

$$[2k-3, k+1, k]_{7,11} = 2k+16,$$
$$[2k+4, k-1, k]_{7,11} = 2k+23,$$

$[159,88,85]_{7,11} = 14$, $[157,87,84]_{7,11} = 16$, $[462,245,240]_{7,11} = 19$, $[456,242,237]_{7,11} = 25$, and $[458,243,238]_{7,11} = 23$.

• 13 is 3−special.

$$[k-5, 2k+1, k]_{3,13} = 2k+28,$$
$$[k+1, k, k]_{3,13} = 2k+1,$$

$[5,25,12]_{3,13} = 28$, $[78,184,91]_{3,13} = -1$, $[77,182,90]_{3,13} = 1$, and $[156,353,175]_{3,13} = 38$.

• 15 is 11−special.

$$[2k-5, k+1, k]_{11,15} = 2k+316,$$
$$[2k+6, k-1, k]_{11,15} = 2k+47,$$

$[337,180,177]_{11,15} = 34$, $[335,179,176]_{11,15} = 36$, $[990,514,509]_{11,15} = 41$, $[982,510,505]_{11,15} = 49$, and $[984,511,506]_{11,15} = 47$.

• 17 is $13-$special.

$$[2k-6,k+1,k]_{13,17}=2k+49,$$
$$[2k+7,k-1,k]_{13,17}=2k+52,$$

$[442,234,231]_{13,17}=55$, $[450,238,235]_{13,17}=47$, $[448,237,234]_{13,17}=49$, $[1327,685,680]_{13,17}=54$, and $[1329,686,681]_{13,17}=52$.

• 19 is $15-$special.

$$[2k-7,k+1,k]_{15,19}=2k+64,$$
$$[2k+8,k-1,k]_{15,19}=2k+79,$$

$[579,304,301]_{15,19}=62$, $[577,303,300]_{15,19}=64$, $[1710,879,874]_{15,19}=71$, $[700,874,869]_{15,19}=81$, and $[1702,875,870]_{15,19}=79$.

• 21 is $5-$special.

$$[k-9,2k+1,k]_{5,21}=2k+86,$$
$$[k+1,2k,k]_{5,21}=2k+1,$$

$[420,905,451]_{5,21}=104$, $[1,21,10]_{5,21}=106$, $[9,41,20]_{5,21}=86$, $[210,464,231]_{5,21}=-1$, and $[209,462,230]_{5,21}=1$.

• 23 is $7-$special.

$$[4k+2,k-1,k]_{7,23}=2k+11,$$
$$[4k+9,k-5,k]_{7,23}=2k+256,$$

$[1604,414,405]_{7,23}=13$, $[1608,415,406]_{7,23}=11$, $[2869,736,723]_{7,23}=266$, and $[2889,741,728]_{7,23}=256$.

• 25 is $1-$special.

$$[4k+1,3k-1,k]_{1,25}=2k+2,$$
$$[4k+4,3k-5,k]_{1,25}=2k+41,$$

$[249,201,64]_{1,25}=2$, $[450,358,115]_{1,25}=39$, and $[446,355,114]_{1,25}=41$.

• 27 is $23-$special.

$$[2k-11,k+1,k]_{23,27}=2k+144,$$
$$[2k+12,k-1,k]_{23,27}=2k+167,$$

$[1255, 648, 645]_{23,27} = 1420$, $[1253, 647, 644]_{23,27} = 144$, $[3726, 1897, 1892]_{23,27} = 155$, $[3712, 1890, 1885]_{23,27} = 169$, and $[3714, 1891, 1886]_{23,27} = 167$.

• 29 is 5−special.

$$[3k - 3, 2k + 1, k]_{5,29} = 2k + 14,$$
$$[3k + 7, 2k - 2, k]_{5,29} = 2k + 69,$$

$[580, 403, 199]_{5,29} = 16$, $[583, 405, 200]_{5,29} = 14$, $[1440, 986, 489]_{5,29} = 71$, and $[1443, 988, 490]_{5,29} = 69$.

• 31 is 3−special.

$$[2k - 4, 3k + 1, k]_{3,31} = 2k + 19,$$
$$[2k + 5, 3k - 1, k]_{3,31} = 2k + 28,$$

$[186, 303, 100]_{3,31} = 23$, $[190, 309, 102]_{3,31} = 19$, and $[553, 869, 288]_{3,31} = 28$.

• 33 is 29−special.

$$[2k - 14, k + 1, k]_{29,33} = 2k + 225,$$
$$[2k + 15, k - 1, k]_{29,33} = 2k + 254,$$

$[1914, 982, 979]_{29,33} = 239$, $[1930, 990, 987]_{29,33} = 223$, $[1928, 989, 986]_{29,33} = 225$, $[5725, 2904, 2899]_{29,33} = 256$, and $[5727, 2905, 2900]_{29,33} = 254$.

• 35 is 31−special.

$$[2k - 15, k + 1, k]_{31,35} = 2k + 256,$$
$$[2k + 16, k - 1, k]_{31,35} = 2k + 287,$$

$[2187, 1120, 1117]_{31,35} = 254$, $[2185, 1119, 1116]_{31,35} = 256$, $[6510, 3299, 3294]_{31,35} = 271$, $[6492, 3290, 3285]_{31,35} = 289$, and $[6494, 3291, 3286]_{31,35} = 287$.

• 37 is 3−special.

$$[5k - 1, 2k + 1, k]_{3,37} = 2k + 4,$$
$$[5k + 5, 2k - 4, k]_{3,37} = 2k + 73,$$

$[889, 369, 180]_{3,37} = 4$, $[1998, 820, 403]_{3,37} = 71$, $[1983, 814, 400]_{3,37} = 77$, and $[1993, 818, 402]_{3,37} = 73$.

• 39 is 35−special.

$$[2k-17,k+1,k]_{35,39}=2k+324,$$
$$[2k+18,k-1,k]_{35,39}=2k+359,$$

$[2749,1404,1401]_{35,39}=322$, $[2747,1403,1400]_{35,39}=324$, $[8190,4144,4139]_{35,39}=341$, $[8170,4134,4129]_{35,39}=361$, and $[8172,4135,4130]_{35,39}=359$.

• 41 is 5−special.

$$[6k+1,k-1,k]_{5,41}=2k+6,$$
$$[6k+6,k-7,k]_{5,41}=2k+281,$$

$[2863,492,479]_{5,41}=8$, $[2869,493,480]_{5,41}=6$, $[5330,910,891]_{5,41}=279$, $[5282,902,883]_{5,41}=295$, and $[5324,909,890]_{5,41}=281$.

• 43 is 3−special.

$$[4k-2,3k+1,k]_{3,43}=2k+7,$$
$$[4k+7,3k-3,k]_{3,43}=2k+76,$$

$[776,601,198]_{3,43}=7$, $[1795,1376,455]_{3,43}=78$, and $[1799,1379,456]_{3,43}=76$.

• 45 is 41−special.

$$[2k-20,k+1,k]_{41,45}=2k+441,$$
$$[2k+21,k-1,k]_{41,45}=2k+482,$$

$[3690,1879,1876]_{41,45}=461$, $[3712,1890,1887]_{41,45}=439$, $[3710,1889,1886]_{41,45}=441$, $[11047,5580,5575]_{41,45}=484$, and $[11049,5581,5576]_{41,45}=482$.

• 47 is 43−special.

$$[2k-21,k+1,k]_{43,47}=2k+484,$$
$$[2k+22,k-1,k]_{43,47}=2k+527,$$

$[4065,2068,2065]_{43,47}=482$, $[4063,2067,2064]_{43,47}=484$, $[12126,6122,6117]_{43,47}=505$, $[12102,6110,6105]_{43,47}=529$, and $[12104,6111,6106]_{43,47}=527$.

• 49 is 3−special.

$$[k-11,4k+1,k]_{3,49}=2k+172,$$
$$[k+1,4k,k]_{3,49}=2k+1,$$

$[11, 97, 24]_{3,49} = 124$, $[294, 1278, 319]_{3,49} = -1$, $[293, 1274, 318]_{3,49} = 1$, and $[588, 2503, 625]_{3,49} = 146$.

We now present some results obtained from Theorem 3.6.

Corollary 3.7. *Let k be an odd positive integer. There are infinitely many $k-$special numbers.*

Proof. For any odd integer k, we can always choose infinitely many integers x and y such that $\gcd(2x, ky) = 1$. By Theorem 3.6, we have that $l = (2x)^2 + ky^2$ is $k-$special. □

Theorem 3.8. *Let k and l be positive integers. If k is odd, then $4l$ is not $k-$special.*

Proof. Assume that $4l$ is $k-$special. For any integer n, there exist non-zero integers x, y, and z such that

$$x^2 + ky^2 - 4lz^2 = n$$
$$x^2 + ky^2 \equiv n \pmod 4.$$

We now consider the following two cases on the values of k.

Case 1. $k \equiv 1 \pmod 4$. Then

$$x^2 + y^2 \equiv n \pmod 4.$$

We can see that $x^2 + y^2 \not\equiv 3 \pmod 4$. This is a contradiction.

Case 2. $k \equiv 3 \pmod 4$. Then

$$x^2 + 3y^2 \equiv n \pmod 4.$$

We can see that $x^2 + 3y^2 \not\equiv 2 \pmod 4$. This is a contradiction.

Therefore $4l$ is not $k-$special. □

CHAPTER 4

$2k-$Special Numbers

In this chapter, we provide conditions for an integer l to be $2k-$special where k is odd. Furthermore, we show that there are infinitely many $2k-$special. Moreover, we provide some properties of $k-$special when $k \equiv 2 \pmod 4$ and $k \equiv 2 \pmod 8$.

Theorem 4.1. *Let l be a positive integer. If l is $2-$special, then $l = x^2 + 2y^2$ for some integers x and y.*

Proof. Let l be $2-$special. Then there exist non-zero integers x, y, and z such that $x^2 + 2y^2 - lz^2 = 2lc^2$ where $c \in \mathbb{Z}$. So $x^2 + 2y^2 = l(2c^2 + z^2)$. By Lemma 2.17,

$$l(2c^2 + z^2) = \prod_{p_i \equiv 5,7 \pmod 8} {p_i}^{a_i} \prod_{q_i \not\equiv 5,7 \pmod 8} {q_i}^{b_i}$$

where p_i, q_i are primes, b_i is a non-negative integer and a_i is even for all i.
By Lemma 2.17,

$$2c^2 + z^2 = \prod_{p_i \equiv 5,7 \pmod 8} {p_i}^{a'_i} \prod_{q_i \not\equiv 5,7 \pmod 8} {q_i}^{b'_i}$$

where b'_i is a non-negative integer and a'_i is even for all i,

$$l = \prod_{p_i \equiv 5,7 \pmod 8} {p_i}^{a_i - a'_i} \prod_{q_i \not\equiv 5,7 \pmod 8} {q_i}^{b_i - b'_i}.$$

So we have $a_i - a'_i$ is even for all i.
Hence again by Lemma 2.17, l is of the form $x^2 + 2y^2$. □

Example 7. From Theorem 3.4, 1 is $2-$special because $1 = 1^2 + 2(0^2)$.

The converse of the above theorem is not true. As we will see in the next theorem that 8 is not $2-$special.

Theorem 4.2. *Let k be an odd integer. If l is divisible by 8, then l is not $2k-$special.*

Proof. Let l be divisible by 8. Suppose on the contrary that l is $2k-$special. Then

$$x^2 + 2ky^2 - lz^2 = 5$$

for some non-zero integers x, y, and z. So $x^2 + 2ky^2 = lz^2 + 5$.
This implies that $x^2 + 2ky^2 \equiv 5 \pmod 8$. Since $x^2 \equiv 0, 1, 4 \pmod 8$ and $2ky^2 \equiv 0, 2k \pmod 8$, it is easy to see that $x^2 + 2ky^2 \equiv 0, 1, 2k, 2k + 1, 2k + 4 \pmod 8$. Since k is odd, we can see that $x^2+2ky^2 \not\equiv 5 \pmod 8$. This is a contradiction. □

Next, we apply P. C. H. Lam's method [3] to identify $2k-$special numbers when k is odd.

Theorem 4.3. *Let k and l be odd positive integers. If l can be written as x^2+2ky^2 for some positive integers x and y where* $\gcd(x, 2ky) = 1$, *then l is $2k-$special.*

Proof. Let l be an odd positive integer and $l = x^2 + 2ky^2$ where $\gcd(x, 2ky) = 1$.
Case 1. We first find the representation of odd numbers of the form $x^2+2ky^2-lz^2$ where x, y, and z are non-zero integers.
Since $\gcd(x, 2ky) = 1$, there exist integers α_0 and β_0 such that

$$x\alpha_0 + 2ky\beta_0 = 1.$$

For any positive integer n, let $\alpha_n = \alpha_0 + 2kny$ and $\beta_n = \beta_0 - nx$.
Consider

$$\begin{aligned} x\alpha_n + 2ky\beta_n &= x(\alpha_0 + 2kny) + 2ky(\beta_0 - nx) \\ &= x\alpha_0 + 2kxny + 2ky\beta_0 - 2kynx \\ &= x\alpha_0 + 2ky\beta_0. \end{aligned}$$

So (α_n, β_n) is another solution of $x\alpha_0 + 2ky\beta_0 = 1$.
Let $a_n = xj + \alpha_n$, $b_n = yj + \beta_n$ and $c_n = j$, where j is an integer which will be selected later. Thus

$$\begin{aligned} a_n^2 + 2kb_n^2 - lc_n^2 &= (xj + \alpha_n)^2 + 2k(yj + \beta_n)^2 - (x^2 + 2ky^2)j^2 \\ &= x^2j^2 + 2xj\alpha_n + \alpha_n^2 + 2ky^2j^2 + 4kyj\beta_n + 2k\beta_n^2 - x^2j^2 - 2ky^2j^2 \\ &= 2xj\alpha_n + 4kyj\beta_n + \alpha_n^2 + 2k\beta_n^2 \\ &= 2j(x\alpha_n + 2ky\beta_n) + \alpha_n^2 + 2k\beta_n^2. \end{aligned}$$

Since x is odd and $x\alpha_n + 2ky\beta_n = 1$, we can see that α_n is odd.
Then we obtain the identity of odd given by

$$(xj_i + \alpha_i)^2 + 2k(yj_i + \beta_i)^2 - (x^2 + 2ky^2)j_i^2 = 2j_i + \alpha_i^2 + 2k\beta_i^2,$$

for any non-negative integer i. We can use these identities to represent odd integers. Let m be an odd integer. For any non-negative integer n, we choose a suitable value of an integer j_n such that

$$m = 2j_n + \alpha_n^2 + 2k\beta_n^2 = a_n^2 + 2kb_n^2 - lc_n^2$$

where $a_n = xj_n + \alpha_n$, $b_n = yj_n + \beta_n$ and $c_n = j_n$. We can see that $a_nb_nc_n = 0$ if and only if m is one of the following values; $\alpha_n^2 + 2k\beta_n^2, \alpha_n^2 + 2k\beta_n^2 - \frac{2\alpha_n}{x}$ or $\alpha_n^2 + 2k\beta_n^2 - \frac{2\beta_n}{y}$. Since

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (-\beta_n) = \infty,$$

there exists a non-negative integer n such that $\alpha_n > 0$, $\alpha_n^2 + 2k\beta_n^2 - \frac{2\alpha_n}{x} > m$ and $\beta_n < 0$. Therefore we obtain a representation for m, namely $m = a_n^2 + 2kb_n^2 - lc_n^2$ where $a_nb_nc_n \neq 0$.

Case 2. We next find the representation of even numbers of the form $x^2+2ky^2-lz^2$ where x, y, and z are non-zero integer.

Since $\gcd(x, 2ky) = 1$, $\gcd(x, ky) = 1$. Then there exist integers α_0 and β_0 such that

$$x\alpha_0 + ky\beta_0 = 1.$$

For any positive integer n, let $\alpha_n = \alpha_0 + kny$ and $\beta_n = \beta_0 - nx$. Then

$$\begin{aligned} x\alpha_n + kny\beta_n &= x(\alpha_0 + kny) + ky(\beta_0 - nx) \\ &= x\alpha_0 + kxny + ky\beta_0 - kynx \\ &= x\alpha_0 + ky\beta_0. \end{aligned}$$

So (α_n, β_n) is another solution of $x\alpha_0 + ky\beta_0 = 1$.

Let $a_n = xj + 2\alpha_n$, $b_n = yj + \beta_n$ and $c = j$ where j is an integer which will be selected later. Thus

$a_n^2+2kb_n^2-lc_n^2$

$$\begin{aligned}
&= (xj+2\alpha_n)^2+2k(yj+\beta_n)^2-(x^2+2ky^2)j^2\\
&= x^2j^2+4xj\alpha_n+4\alpha_n^2+2ky^2j^2+4yjk\beta_n+2k\beta_n^2-x^2j^2-2y^2ky^2j^2\\
&= 4j(x\alpha_n+ky\beta_n)+4\alpha_n^2+2k\beta_n^2.
\end{aligned}$$

Since x is odd, we have

$$\begin{aligned}
4\alpha_n^2+2k\beta_n^2 &= 4(\alpha_0+kny)^2+2k(\beta_0-nx)^2\\
&= 4\alpha_0^2+8kny\alpha_0+4k^2n^2y^2+2k\beta_0^2-4knx\beta_0+2kn^2x^2\\
&\equiv 2k\beta_0^2+2kn^2x^2 \pmod 4\\
&\equiv 2k\beta_0^2+2kn^2 \pmod 4\\
&\equiv \begin{cases} 2k\beta_0^2 \pmod 4 & \text{if } n \text{ is even,}\\ 2k(\beta_0^2+1) \pmod 4 & \text{if } n \text{ is odd.}\end{cases}
\end{aligned}$$

For any non-negative integer r, we obtain two identities of even given by

$$(xj_{2r}+2\alpha_{2r})^2+2k(yj_{2r}+\beta_{2r})^2-(x^2+2ky^2)j_{2r}^2=4j_{2r}+4\alpha_{2r}^2+2k\beta_{2r}^2,$$
$$(xj_{2r+1}+2\alpha_{2r+1})^2+2k(yj_{2r+1}+\beta_{2r+1})^2-(x^2+2ky^2)j_{2r+1}^2=4j_{2r+1}+4\alpha_{2r+1}^2+2k\beta_{2r+1}^2.$$

Let m be an even integer. we can write m as follows:

Subcase 2.1. $m\equiv 0 \pmod 4$ and $\beta_0^2\equiv 0 \pmod 4$. We choose a suitable value of an integer j_{2r} such that

$$m=4j_{2r}+4\alpha_{2r}^2+2k\beta_{2r}^2=a_{2r}^2+2kb_{2r}^2-lc_{2r}^2$$

where $a_{2r}=xj_{2r}+2\alpha_{2r}$, $b_{2r}=yj_{2r}+\beta_{2r}$ and $c_{2r}=j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r}=0$ if and only if m is one of the following values: $4\alpha_{2r}^2+2k\beta_{2r}^2, 4\alpha_{2r}^2+2k\beta_{2r}^2-\frac{8\alpha_{2r}}{x}$ or $4\alpha_{2r}^2+2k\beta_{2r}^2-\frac{4\beta_{2r}}{y}$. Since

$$\lim_{r\to\infty}\alpha_{2r}=\lim_{r\to\infty}(-\beta_{2r})=\infty,$$

there exists a non-negative integer r such that $\alpha_{2r}>0$, $4\alpha_{2r}^2+2k\beta_{2r}^2-\frac{8\alpha_{2r}}{x}>m$ and $\beta_{2r}<0$. Therefore we obtain a representation for m, namely $m=a_{2r}^2+2kb_{2r}^2-lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r}\neq 0$.

Subcase 2.2. $m\equiv 2 \pmod 4$ and $\beta_0^2\equiv 0 \pmod 4$. We choose a suitable value of an integer j_{2r+1} such that

$$m=4j_{2r+1}+4\alpha_{2r+1}^2+2k\beta_{2r+1}^2=a_{2r+1}^2+2kb_{2r+1}^2-lc_{2r+1}^2$$

where $a_{2r+1} = xj_{2r+1} + 2\alpha_{2r+1}$, $b_{2r+1} = yj_{2r+1} + \beta_{2r+1}$ and $c_{2r+1} = j_{2r+1}$. We can see that $a_{2r+1}b_{2r+1}c_{2r+1} = 0$ if and only if m is one of the following values: $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2, 4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{8\alpha_{2r+1}}{x}$ or $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{4\beta_{2r+1}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} (-\beta_{2r+1}) = \infty,$$

there exists a non-negative integer r such that $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{8\alpha_{2r+1}}{x} > m$, $\alpha_{2r+1} > 0$, and $\beta_{2r+1} < 0$. Therefore we obtain a representation for m, namely $m = a_{2r+1}^2 + 2kb_{2r+1}^2 - lc_{2r+1}^2$ where $a_{2r+1}b_{2r+1}c_{2r+1} \neq 0$.

Subcase 2.3. $m \equiv 0 \pmod 4$ and $\beta_0^2 \equiv 1 \pmod 4$. We choose a suitable value of an integer j_{2r+1} such that

$$m = 4j_{2r+1} + 4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 = a_{2r+1}^2 + 2kb_{2r+1}^2 - lc_{2r+1}^2$$

where $a_{2r+1} = xj_{2r+1} + 2\alpha_{2r+1}$, $b_{2r+1} = yj_{2r+1} + \beta_{2r+1}$ and $c_{2r+1} = j_{2r+1}$. We can see that $a_{2r+1}b_{2r+1}c_{2r+1} = 0$ if and only if m is one of the following values: $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2, 4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{8\alpha_{2r+1}}{x}$ or $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{4\beta_{2r+1}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r+1} = \lim_{r\to\infty} (-\beta_{2r+1}) = \infty,$$

there exists a non-negative integer r such that $4\alpha_{2r+1}^2 + 2k\beta_{2r+1}^2 - \frac{8\alpha_{2r+1}}{x} > m$, $\alpha_{2r+1} > 0$, and $\beta_{2r+1} < 0$. Thus we obtain a representation for m, namely $m = a_{2r+1}^2 + 2kb_{2r+1}^2 - lc_{2r+1}^2$ where $a_{2r+1}b_{2r+1}c_{2r+1} \neq 0$.

Subcase 2.4. $m \equiv 2 \pmod 4$ and $\beta_0^2 \equiv 1 \pmod 4$. We choose a suitable value of an integer j_{2r} such that

$$m = 4j_{2r} + 4\alpha_{2r}^2 + 2k\beta_{2r}^2 = a_{2r}^2 + 2kb_{2r}^2 - lc_{2r}^2$$

where $a_{2r} = xj_{2r}+2\alpha_{2r}$, $b_{2r} = yj_{2r}+\beta_{2r}$ and $c_{2r} = j_{2r}$. We can see that $a_{2r}b_{2r}c_{2r} = 0$ if and only if m is one of the following values: $4\alpha_{2r}^2 + 2k\beta_{2r}^2, 4\alpha_{2r}^2 + 2k\beta_{2r}^2 - \frac{8\alpha_{2r}}{x}$ or $4\alpha_{2r}^2 + 2k\beta_{2r}^2 - \frac{4\beta_{2r}}{y}$. Since

$$\lim_{r\to\infty} \alpha_{2r} = \lim_{r\to\infty} (-\beta_{2r}) = \infty,$$

there exists a non-negative integer r such that $\alpha_{2r} > 0$, $4\alpha_{2r}^2 + 2k\beta_{2r}^2 - \frac{8\alpha_{2r}}{x} > m$ and $\beta_{2r} < 0$. Hence we again obtain a representation for m, namely $m = a_{2r}^2 + 2kb_{2r}^2 - lc_{2r}^2$ where $a_{2r}b_{2r}c_{2r} \neq 0$. Therefore, l is $2k-$special as desired. $\square$

We next provide examples of how to obtain representations for any integer n of the form $x^2+2ky^2-lz^2$ where $(k,l) = (2,3), (2,9), (2,11), (2,17), (2,19)$, $(6,7)$, $(10,11)$, and $(10,19)$.

Example 8. We will show that 3 is 2−special. So we have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 2y^2 - 3z^2$. We will use the notation in Theorem 4.3. Write $3 = 1^2 + 2(1^2)$. Then $x = y = 1$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 2y) = \gcd(1, 2) = 1$, there exist $\alpha_0 = -1$ and $\beta_0 = 1$ such that $1(-1) + 2(1) = 1$.

A representation for odd integers is given by

$$(j-1)^2 + 2(j+1)^2 - 3j^2 = 2j + 3.$$

The above identity gives a representation for odd integers $n \neq 1, 3$, and 5 of the form $x^2 + 2y^2 - 3z^2$ where $xyz \neq 0$. We next define $\alpha_1 = \alpha_0 + 2y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -1 + 2 = 1$ and $\beta_1 = 1 - 1 = 0$.

Then a representation for odd integers is

$$(j+1)^2 + 2(j)^2 - 3j^2 = 2j + 1.$$

We can use this identity to represent 3 and 5. So we can write 3 and 5 as follows:

$$3 = 2^2 + 2(1^2) - 3(1^2),$$
$$5 = 3^2 + 2(2^2) - 3(2^2).$$

We next find a new representation for 1. We define $\alpha_2 = \alpha_0 + 4y$ and $\beta_2 = \beta_0 - 2x$, i.e., $\alpha_2 = -1 + 4 = 3$ and $\beta_2 = 1 - 2 = -1$.

Then a new representation for odd integers is

$$(j+3)^2 + 2(j-1)^2 - 3j^2 = 2j + 11.$$

So the representation for 1 is $2^2 + 2(6^2) - 3(5^2) = 1$.

Case 2. We next find the representation for even integers.

Since $\gcd(x, y) = \gcd(1, 1) = 1$, there exist $\alpha_0 = 2$ and $\beta_0 = -1$ such that $1(2) + (1)(-1) = 1$. We also obtain $\alpha_1 = 3$ and $\beta_1 = -2$.

The representations for even integers are given by

$$(j+4)^2 + 2(j-1)^2 - 3j^2 = 4j + 18,$$
$$(j+6)^2 + 2(j-2)^2 - 3j^2 = 4j + 44.$$

The above identity gives a representation for even integers $n \neq 2, 22, 18, 20, 52$, and 44 of the form $x^2 + 2y^2 - 3z^2$ where $xyz \neq 0$. We next define $\alpha_2 = \alpha_0 + 2y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 3y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 4, \beta_2 = -3, \alpha_3 = 5$ and $\beta_3 = -4$. Then the new representations for even integers are given by

$$(j+8)^2 + 2(j-3)^2 - 3j^2 = 4j + 82,$$
$$(j+10)^2 + 2(j-4)^2 - 3j^2 = 4j + 132.$$

Thus

$$2 = 12^2 + 2(23^2) - 3(20^2),$$
$$22 = 7^2 + 2(18^2) - 3(15^2),$$
$$18 = 8^2 + 2(19^2) - 3(16^2),$$
$$20 = 18^2 + 2(32^2) - 3(28^2),$$
$$52 = 10^2 + 2(24^2) - 3(20^2),$$
$$44 = 12^2 + 2(26^2) - 3(22^2).$$

Therefore from both cases we can conclude that 3 is 2−special.

Example 9. We will show that 9 is 2−special. So we have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 2y^2 - 9z^2$. We will use the notation in Theorem 4.3. Write $9 = 1^2 + 2(2^2)$. Then $x = 1$ and $y = 2$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 2y) = \gcd(1, 4) = 1$, there exist $\alpha_0 = -3$ and $\beta_0 = 1$ such that $1(-3) + 2(2)(1) = 1$. A representation for odd integers is given by

$$(j-3)^2 + 2(2j+1)^2 - 9j^2 = 2j + 11.$$

The above identity gives a representation for odd integers $n \neq 11$ and 17 of the form $x^2 + 2y^2 - 9z^2$ where $xyz \neq 0$. We next define $\alpha_1 = \alpha_0 + 2y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -3 + 4 = 1$ and $\beta_1 = 1 - 1 = 0$.

Then a representation for odd integers is

$$(j+1)^2 + 2(2j)^2 - 9j^2 = 2j+1.$$

We can use this identity to represent 17 and 11. So we can write 17 and 11 as follows:

$$17 = 9^2 + 2(16^2) - 9(8^2),$$
$$11 = 5^2 + 2(10^2) - 9(5^2).$$

Case 2. We next find the representation for even integers.

Since $\gcd(x,y) = \gcd(1,2) = 1$, there exist $\alpha_0 = -1$ and $\beta_0 = 1$ such that $1(-1) + (2)(1) = 1$.

We also obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

The representations for even integers are given by

$$(j-2)^2 + 2(2j+1)^2 - 9j^2 = 4j+6,$$
$$(j+2)^2 + 2(2j)^2 - 9j^2 = 4j+4.$$

The above identities give a representation for even integers $n \neq 14, 6, -4$, and 4 of the form $x^2 + 2y^2 - 9z^2$ where $xyz \neq 0$.

We next define $\alpha_2 = \alpha_0 + 2y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 3y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 3, \beta_2 = -1, \alpha_3 = 5$ and $\beta_3 = -2$.

Then the representations for even integers are

$$(j+6)^2 + 2(2j-1)^2 - 9j^2 = 4j+38,$$
$$(j+10)^2 + 2(2j-2)^2 - 9j^2 = 4j+108.$$

Thus

$$6 = 2^2 + 2(17^2) - 9(8^2),$$
$$-4 = 18^2 + 2(58^2) - 9(28^2),$$
$$4 = 16^2 + 2(54^2) - 9(26^2).$$

We next find a representation for 6.

We define $\alpha_4 = \alpha_0 + 4y$ and $\beta_4 = \beta_0 - 4x$, i.e., $\alpha_4 = 7$ and $\beta_4 = -3$.

Then a new representation for even integers is

$$(j+14)^2 + 2(2j-3)^2 - 9j^2 = 4j+214.$$

So the representation for 14 is $14 = 36^2 + 2(103)^2 - 9(50^2)$.

Therefore 9 is $2-$special.

Example 10. We will show that 11 is $2-$special. We have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 2y^2 - 11z^2$. We will use the notation in Theorem 4.3. Write $11 = 3^2 + 2(1^2)$. Then $x = 3$ and $y = 1$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 2y) = \gcd(3, 2) = 1$, there exist $\alpha_0 = 1$ and $\beta_0 = -1$ such that $3(1) + 2(1)(-1) = 1$.

A representation for odd integers is given by

$$(3j+1)^2 + 2(j-1)^2 - 11j^2 = 2j + 3.$$

The above identity gives a representation for odd integers $n \neq 3$ and 5 of the form $x^2 + 2y^2 - 11z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 2y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = 1 + 2 = 3$ and $\beta_1 = -1 - 3 = -4$.

Then a representation for odd integers is

$$(3j+3)^2 + 2(j-4)^2 - 11j^2 = 2j + 41.$$

We can use this identity to represent 3 and 5. So we can write 3 and 5 as follows:

$$3 = 54^2 + 2(23^2) - 11(19^2),$$
$$5 = 51^2 + 2(22^2) - 11(18^2).$$

Case 2. We will find the representation for even integers.

Since $\gcd(x, y) = \gcd(3, 1) = 1$, there exist $\alpha_0 = 1$ and $\beta_0 = -2$ such that $3(1) + 1(-2) = 1$. We also obtain $\alpha_1 = 2$ and $\beta_1 = -5$.

The representations for even integers are given by

$$(3j+2)^2 + 2(j-2)^2 - 11j^2 = 4j + 12,$$
$$(3j+4)^2 + 2(j-5)^2 - 11j^2 = 4j + 66.$$

The above identities give a representation for even integers $n \neq 20, 12, 86$, and 66 of the form $x^2 + 2y^2 - 11z^2$ where $xyz \neq 0$..

We next define $\alpha_2 = \alpha_0 + 2y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 3y$ and $\beta_3 = \beta_0 - 3x$,i.e., $\alpha_2 = 3, \beta_2 = -8, \alpha_3 = 4$ and $\beta_3 = -11$.

Then the new representations for even integers are

$$(3j+6)^2 + 2(j-8)^2 - 11j^2 = 4j + 164,$$
$$(3j+8)^2 + 2(j-11)^2 - 11j^2 = 4j + 306.$$

Thus

$$20 = 102^2 + 2(44^2) - 11(36^2),$$
$$12 = 108^2 + 2(46^2) - 11(38^2),$$
$$86 = 157^2 + 2(66^2) - 11(55^2),$$
$$66 = 172^2 + 2(71^2) - 11(60^2).$$

Therefore 11 is 2−special.

Example 11. We will show that 17 is 2−special. We have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 2y^2 - 17z^2$. We will use the notation in Theorem 4.3. Write $17 = 3^2 + 2(2^2)$. Then $x = 3$ and $y = 2$.

Case 1. We will fine the representation for odd integers.

Since $\gcd(x, 2y) = \gcd(3, 4) = 1$, there exist $\alpha_0 = -1$ and $\beta_0 = 1$ such that $3(-1) + 2(2)(1) = 1$.

A representation for odd integers is given by

$$(3j-1)^2 + 2(2j+1)^2 - 17j^2 = 2j + 3.$$

The above identity gives a representation for odd integers $n \neq 3$ of the form $x^2 + 2y^2 - 17z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 2y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -1 + 4 = 3$ and $\beta_1 = 1 - 3 = -2$.

Then a new representation for integers is

$$(3j+3)^2 + 2(2j-2)^2 - 17j^2 = 2j + 17.$$

So we can write $3 = 18^2 + 2(1^2) - 17(7^2)$.

Case 2. We will find the representation for even integers.

Since $\gcd(x, y) = \gcd(3, 2) = 1$, there exist $\alpha_0 = 1$ and $\beta_0 = -1$ such that $3(1) + 2(-1) = 1$. We also obtain $\alpha_1 = 3$ and $\beta_1 = -4$.
The representations for even integers are given by

$$(3j + 2)^2 + 2(2j - 1)^2 - 17j^2 = 4j + 6,$$
$$(3j + 6)^2 + 2(2j - 4)^2 - 17j^2 = 4j + 68.$$

The above identities give a representation for even integers $n \neq 6, 60, 76,$ and 68 of the form $x^2 + 2y^2 - 17z^2$ where $xyz \neq 0$.
We next define $\alpha_2 = \alpha_0 + 2y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 3y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 5, \beta_2 = -7, \alpha_3 = 7$ and $\beta_3 = -10$.
Then the representation for even integers are

$$(3j + 10)^2 + 2(2j - 7)^2 - 17j^2 = 4j + 198,$$
$$(3j + 14)^2 + 2(2j - 10)^2 - 17j^2 = 4j + 396.$$

Thus

$$6 = 134^2 + 2(103^2) - 17(48^2),$$
$$60 = 238^2 + 2(178^2) - 17(84^2),$$
$$76 = 226^2 + 2(170^2) - 17(80^2),$$
$$68 = 232^2 + 2(174^2) - 17(82^2).$$

Therefore 17 is $2-$special.

Example 12. We will show that 19 is $2-$special. We have to show that for any integer n, there exist non-zero integers $x, y,$ and z such that $n = x^2 + 2y^2 - 19z^2$. We will use the notation in Theorem 4.3. Write $11 = 1^2 + 2(3^2)$. Then $x = 1$ and $y = 3$.
Case 1. We will find the representation for odd integers.
Since $\gcd(x, 2y) = \gcd(1, 6) = 1$, there exist $\alpha_0 = -5$ and $\beta_0 = 1$ such that $1(-5) + 2(3)(1) = 1$.
A representation for odd integers is given by

$$(j - 5)^2 + 2(3j + 1)^2 - 19j^2 = 2j + 27.$$

The above identity gives a representation for odd integers $n \neq 27$ and 37 of the form $x^2 + 2y^2 - 19z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 2y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -5 + 6 = 1$ and $\beta_1 = 1 - 1 = 0$.

Then a new representation for odd integers is

$$(j+1)^2 + 2(3j)^2 - 19j^2 = 2j + 1.$$

We can use this identity to represent 27 and 37. So we can write 27 and 37 as follows:

$$27 = 14^2 + 2(39^2) - 19(13^2),$$
$$37 = 19^2 + 2(54^2) - 19(18^2).$$

Case 2. we will find the representation for even integers.

Since $\gcd(x, y) = \gcd(1, 3) = 1$, there exist $\alpha_0 = -2$ and $\beta_0 = 1$ such that $1(-2) + 3(1) = 1$. We also obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

The representations for even integers are given by

$$(j-4)^2 + 2(3j-1)^2 - 19j^2 = 4j + 18,$$
$$(j+2)^2 + 2(3j)^2 - 19j^2 = 4j + 4.$$

The above identities give a representation for even integers $n \neq 34, 18, -4$, and 4 of the form $x^2 + 2y^2 - 19z^2$ where $xyz \neq 0$.

We define $\alpha_2 = \alpha_0 + 2y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 3y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 4, \beta_2 = -1, \alpha_3 = 7$ and $\beta_3 = -2$.

Thus the representations for even integers are

$$(j+8)^2 + 2(3j-1)^2 - 19j^2 = 4j + 66,$$
$$(j+14)^2 + 2(3j-2)^2 - 19j^2 = 4j + 204.$$

Thus

$$18 = 4^2 + 2(37^2) - 19(12^2),$$
$$-4 = 38^2 + 2(158^2) - 19(52^2),$$
$$4 = 36^2 + 2(152^2) - 19(50^2).$$

Next, we have to find the representation for 34. Then we define $\alpha_4 = \alpha_0 + 4y$ and $\beta_4 = \beta_0 - 4x$, i.e., $\alpha_4 = 10$ and $\beta_4 = -3$.

So the new representation for even integers which congruent 2 modulo 4 is

$$(j + 20)^2 + 2(3j - 3)^2 - 19j^2 = 4j + 418.$$

Thus $34 = (76^2) + 2(291^2) - 19(96^2)$.

Therefore 19 is $2-$special.

Example 13. We will show that 7 is $6-$special. We have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 6y^2 - 7z^2$. We will use the notation in theorem 4.3. Write $7 = 1^2 + 6(1^2)$. Then $x = 1$ and $y = 1$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 6y) = \gcd(1, 6) = 1$, there exist $\alpha_0 = -5$ and $\beta_0 = 1$ such that $1(-5) + (6)(1) = 1$.

A representation for odd integers is given by

$$(j - 5)^2 + 6(j + 1)^2 - 7j^2 = 2j + 31.$$

The above identity gives a representation for odd integers $n \neq 29, 31$, and 41 of the form $x^2 + 6y^2 - 7z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 6y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -5 + 6 = 1$ and $\beta_1 = 1 - 1 = 0$.

Then a new representation for odd integers is

$$(j + 1)^2 + 6(j)^2 - 7j^2 = 2j + 1.$$

So we can write $29 = 15^2 + 6(14^2) - 7(14^2)$, $31 = 16^2 + 6(15^2) - 7(15^2)$, and $41 = 21^2 + 6(20^2) - 7(20^2)$.

Case 2. we will find the representation for even integers.

Since $\gcd(x, 3y) = \gcd(1, 3) = 1$, there exist $\alpha_0 = -2$ and $\beta_0 = 1$ such that $1(-2) + 3(1) = 1$. We also obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

The representations for even integers are given by

$$(j - 4)^2 + 6(j + 1)^2 - 7j^2 = 4j + 22,$$

$$(j + 2)^2 + 6(j)^2 - 7j^2 = 4j + 4.$$

The above identities give a representation for even integers $n \neq 38, 18, 22, -4$, and 4 of the form $x^2 + 6y^2 - 7z^2$ where $xyz \neq 0$.

We next define $\alpha_2 = \alpha_0 + 6y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 9y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 4, \beta_2 = -1, \alpha_3 = 7$ and $\beta_3 = -2$.

Then the representation for even integers are

$$(j+8)^2 + 6(j-1)^2 - 7j^2 = 4j + 70,$$
$$(j+14)^2 + 6(j-2)^2 - 7j^2 = 4j + 220.$$

Thus

$$18 = 5^2 + 6(14^2) - 7(13^2),$$
$$22 = 4^2 + 6(13^2) - 7(12^2),$$
$$-4 = 42^2 + 6(58^2) - 7(56^2),$$
$$4 = 40^2 + 6(56^2) - 7(54^2).$$

We next find a representation for 38.

We define$\alpha_4 = \alpha_0 + 12y$ and $\beta_4 = \beta_0 - 4x$, i.e., $\alpha_4 = 10$ and $\beta_4 = -3$. Then a new representation for even integers is

$$(j+20)^2 + 6(j-3)^2 - 7j^2 = 4j + 454.$$

Thus $38 = (84^2) + 6(107^2) - 7(104^2)$.

Therefore 7 is $6-$special.

Example 14. We will show that 11 is $10-$special. We have to show that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 10y^2 - 11z^2$. We will use the notation in Theorem 4.3. Thus $11 = 1^2 + 10(1^2)$. Then $x = 1$ and $y = 1$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 10y) = \gcd(1, 10) = 1$, there exist $\alpha_0 = -9$ and $\beta_0 = 1$ such that $1(-9) + (10)(1) = 1$.

A representation for odd integers are given by

$$(j-9)^2 + 10(j+1)^2 - 11j^2 = 2j + 91.$$

The above identity gives a representation for odd integers $n \neq 89, 91$, and 109 of the form $x^2 + 10y^2 - 11z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 10y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -9 + 10 = 1$ and $\beta_1 = 1 - 1 = 0$.

Then a new representation for odd integers is

$$(j+1)^2 + 10(j)^2 - 11j^2 = 2j + 1.$$

So we can write $89 = 45^2 + 10(44^2) - 11(44^2)$, $91 = 46^2 + 10(45^2) - 11(45^2)$, and $109 = 55^2 + 10(54^2) - 11(54^2)$.

Case 2. we will find the representation for even integers.

Since $\gcd(x, 5y) = \gcd(1, 5) = 1$, there exist $\alpha_0 = -4$ and $\beta_0 = 1$ such that $1(-4) + 5(1) = 1$. We also obtain $\alpha_1 = 1$ and $\beta_1 = 0$.

The representations for even integers are given by

$$(j-8)^2 + 10(j+1)^2 - 11j^2 = 4j + 74,$$
$$(j+2)^2 + 10(j)^2 - 11j^2 = 4j + 4.$$

The above identity gives a representation for even integers $n \neq 106, 70, 74, -4$, and 4 of the form $x^2 + 10y^2 - 11z^2$ where $xyz \neq 0$.

We next define $\alpha_2 = \alpha_0 + 10y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 15y$ and $\beta_3 = \beta_0 - 3x$, i.e., $\alpha_2 = 6, \beta_2 = -1, \alpha_3 = 11$ and $\beta_3 = -2$.

Then the representation for even integers are

$$(j+12)^2 + 10(j-1)^2 - 11j^2 = 4j + 154,$$
$$(j+22)^2 + 10(j-2)^2 - 11j^2 = 4j + 524.$$

Thus

$$70 = 9^2 + 10(22^2) - 11(21^2),$$
$$74 = 8^2 + 10(21^2) - 11(20^2),$$
$$-4 = 110^2 + 10(134^2) - 11(132^2),$$
$$4 = 108^2 + 10(132^2) - 11(130^2).$$

We next find the representation for 106.

We define $\alpha_4 = \alpha_0 + 20y$ and $\beta_4 = \beta_0 - 4x$, i.e., $\alpha_4 = 16$ and $\beta_4 = -3$. So the representation of even integers which congruent 2 modulo 4 is

$$(j+32)^2 + 10(j-3)^2 - 11j^2 = 4j + 1114.$$

So we can write $106 = (220^2) + 10(255^2) - 11(252^2)$.

Therefore 11 is 10−special.

Example 15. We will show that 19 is 10−special. We have toshow that for any integer n, there exist non-zero integers x, y, and z such that $n = x^2 + 10y^2 - 19z^2$. We will use the notation in Theorem 4.3. Write $11 = 3^2 + 10(1^2)$. Then $x = 3$ and $y = 1$.

Case 1. We will find the representation for odd integers.

Since $\gcd(x, 10y) = \gcd(3, 10) = 1$, there exist $\alpha_0 = -3$ and $\beta_0 = 1$ such that $3(-3) + (10)(1) = 1$.

A representation foe odd integers is given by

$$(3j-3)^2 + 10(j+1)^2 - 19j^2 = 2j + 19.$$

The above identity gives a representation for odd integers $n \neq 21, 17$, and 19 of the form $x^2 + 10y^2 - 19z^2$ where $xyz \neq 0$.

We next define $\alpha_1 = \alpha_0 + 10y$ and $\beta_1 = \beta_0 - x$, i.e., $\alpha_1 = -3 + 10 = 7$ and $\beta_1 = 1 - 3 = -2$.

Then a new representation for odd integers is

$$(3j+7)^2 + 10(j-2)^2 - 19j^2 = 2j + 89.$$

So we can write $21 = 95^2 + 10(36^2) - 19(34^2)$, $17 = 101^2 + 10(38^2) - 19(36^2)$, and $19 = 98^2 + 10(37^2) - 19(35^2)$.

Case 2. we will find the representation for even integers.

Since $\gcd(x, 5y) = \gcd(3, 5) = 1$, there exist $\alpha_0 = 2$ and $\beta_0 = -1$ such that $3(2) + 5(-1) = 1$. We also obtain $\alpha_1 = 7$ and $\beta_1 = -4$.

The representations for even integers are given by

$$(3j-4)^2 + 10(j-1)^2 - 19j^2 = 4j + 26,$$

$$(3j+14)^2 + 10(j-4)^2 - 19j^2 = 4j + 356.$$

The above identities give a representation for even integers $n \neq 30, 26, 372$, and 356 of the form $x^2 + 10y^2 - 19z^2$ where $xyz \neq 0$.

We define $\alpha_2 = \alpha_0 + 10y, \beta_2 = \beta_0 - 2x, \alpha_3 = \alpha_0 + 15y$ and $\beta_3 = \beta_0 - 3x$, i.e.,

$\alpha_2 = 12, \beta_2 = -7, \alpha_3 = 17$ and $\beta_3 = -10$.

Then the representations for even integers are

$$(3j+24)^2 + 10(j-7)^2 - 19j^2 = 4j + 1066,$$
$$(3j+34)^2 + 10(j-10)^2 - 19j^2 = 4j + 2156.$$

Thus

$$30 = 753^2 + 10(266^2) - 19(259^2),$$
$$26 = 756^2 + 10(267^2) - 19(260^2),$$
$$372 = 1304^2 + 10(456^2) - 19(446^2),$$
$$356 = 1316^2 + 10(460^2) - 19(450^2).$$

Therefore 19 is 10−special.

We now present examples of 2−special. We show that l is 2−special for $l < 50$ by giving identities to represent any integer n of the form $n = x^2 + 2y^2 - kz^2$ where $xyz \neq 0$.

• 3 is 2−special.

$$[j-1, j+1, j]_{2,3} = 2j + 3,$$
$$[j+4, j-1, j]_{2,3} = 4j + 18,$$
$$[j+6, j-2, j]_{2,3} = 4j + 44,$$

$[2,6,5]_{2,3} = 1$, $[12,23,20]_{2,3} = 2$, $[2,1,1]_{2,3} = 3$, $[3,2,2]_{2,3} = 5$, $[8,19,16]_{2,3} = 18$, $[18,32,28]_{2,3} = 20$, $[7,18,15]_{2,3} = 22$, $[12,26,22]_{2,3} = 44$, and $[10,24,20]_{2,3} = 52$.

• 9 is 2−special.

$$[j-3, 2j+1, j]_{2,9} = 2j + 11,$$
$$[j-2, 2j+1, j]_{2,9} = 4j + 6,$$
$$[j+2, 2j, j]_{2,9} = 4j + 4,$$

$[18,58,28]_{2,9} = -4$, $[16,54,26]_{2,9} = 4$, $[5,10,5]_{2,9} = 11$, $[2,17,8]_{2,9} = 6$, $[9,16,8]_{2,9} = 17$, and $[36,103,50]_{2,9} = 14$.

• 11 is 2−special.

$$[3j+1, j-1, j]_{2,11} = 2j+3,$$

$$[3j+2, j-2, j]_{2,11} = 4j+12,$$

$$[3j+4, j-5, j]_{2,11} = 4j+66,$$

$[54, 23, 19]_{2,11} = 3$, $[51, 22, 18]_{2,11} = 5$, $[108, 46, 38]_{2,11} = 12$, $[102, 44, 36]_{2,11} = 20$, $[172, 71, 60]_{2,11} = 66$, and $[157, 66, 55]_{2,11} = 86$.

• 17 is 2−special.

$$[3j-1, 2j+1, j]_{2,17} = 2j+3,$$

$$[3j+2, 2j-1, j]_{2,17} = 4j+6,$$

$$[3j+6, 2j-4, j]_{2,17} = 4j+68,$$

$[18, 1, 7]_{2,17} = 3$, $[134, 103, 48]_{2,17} = 6$, $[238, 178, 84]_{2,17} = 60$, $[232, 174, 82]_{2,17} = 68$, and $[226, 170, 80]_{2,17} = 76$.

• 19 is 2−special.

$$[j-5, 3j+1, j]_{2,19} = 2j+27,$$

$$[j-4, 3j-1, j]_{2,19} = 4j+18,$$

$$[j+2, 3j, j]_{2,19} = 4j+4,$$

$[38, 158, 52]_{2,19} = -4$, $[36, 152, 50]_{2,19} = 4$, $[4, 37, 12]_{2,19} = 18$, $[14, 39, 13]_{2,19} = 27$, $[76, 291, 96]_{2,19} = 34$, and $[19, 54, 18]_{2,19} = 37$.

• 27 is 2−special.

$$[5j+1, j-2, j]_{2,27} = 2j+9,$$

$$[5j+2, j-4, j]_{2,27} = 4j+36,$$

$$[5j+4, j-9, j]_{2,27} = 4j+178,$$

$[232, 54, 47]_{2,27} = 13$, $[242, 56, 49]_{2,27} = 9$, $[484, 112, 98]_{2,27} = 36$, $[464, 108, 94]_{2,27} = 52$, $[752, 171, 152]_{2,27} = 178$, and $[707, 162, 143]_{2,27} = 214$.

• 33 is 2−special.

$$[j-7, 4j+1, j]_{2,33} = 2j+51,$$

$$[j-6, 4j+1, j]_{2,33} = 4j+38,$$

$$[j+2, 4j, j]_{2,33} = 4j+4,$$

$[66, 338, 84]_{2,33} = -4$, $[64, 330, 82]_{2,33} = 4$, $[6, 65, 16]_{2,33} = 38$, $[26, 100, 25]_{2,33} = 51$, $[132, 635, 158]_{2,33} = 62$, and $[33, 128, 32]_{2,33} = 65$.

• 41 is 2−special.

$$[3j + 3, 4j - 1, j]_{2,41} = 2j + 11,$$

$$[3j - 2, 4j + 1, j]_{2,41} = 4j + 6,$$

$$[3j + 6, 4j - 2, j]_{2,41} = 4j + 44,$$

$[166, 245, 60]_{2,41} = 6$, $[205, 292, 72]_{2,41} = 9$, $[202, 288, 71]_{2,41} = 11$, and $[410, 584, 144]_{2,41} = 36$.

Corollary 4.4. *Let k be an odd positive integer. There are infinitely many $2k-$special numbers.*

Proof. For any odd integer k, we can always choose infinitely many integers x and y such that $\gcd(x, 2ky) = 1$. By Theorem 4.3, we have that $l = x^2 + 2ky^2$ is $2k-$special. □

Theorem 4.5. *Let k be a positive integer. If 4 is $k-$special, then $k \equiv 2 \pmod 4$.*

Proof. Let 4 be k-special. Then $x^2 + ky^2 - 4z^2 = n$ for all integers n, there exist integers x, y, and z such that $x^2 + ky^2 \equiv n \pmod 4$.

Case 1. $k \equiv 0 \pmod 4$. Then $x^2 \equiv n \pmod 4$. If $n \equiv -1 \pmod 4$, then $x^2 \equiv -1 \pmod 4$. We know that the Legendre symbol $\left(\frac{-1}{4}\right) = -1$. Hence there is no integer x such that $x^2 + ky^2 - 4z^2 = n$ where $n \equiv -1 \pmod 4$.

Case 2. $k \equiv 1 \pmod 4$. Then $x^2 + y^2 \equiv n \pmod 4$. We have $x^2 \equiv 0, 1 \pmod 4$ and $y^2 \equiv 0, 1 \pmod 4$. If $n \equiv -1 \pmod 4$, then $x^2 + y^2 \equiv -1 \pmod 4$. We know that $x^2 + y^2 \equiv 0, 1$, and $2 \pmod 4$.
Hence there is no integers x and y such that $x^2 + y^2 - 4z^2 = n$ where $n \equiv -1 \pmod 4$.

Case 3. $k \equiv 3 \pmod 4$. Then $x^2 + 3y^2 \equiv n \pmod 4$. We have $x^2 \equiv 0, 1 \pmod 4$ and $y^2 \equiv 0, 1 \pmod 4$. Then $x^2 + 3y^2 \equiv 0, 1$, and $3 \pmod 4$. Moreover, if $n \equiv 2 \pmod 4$, then $x^2 + 3y^2 \equiv 2 \pmod 4$. This is a contradiction. □

Theorem 4.6. *Let k and l be positive integers. If $k \equiv 2 \pmod 8$ and $l \not\equiv 2 \pmod 4$, then $4l$ is not $k-$special.*

Proof. Suppose that $k \equiv 2 \pmod 8$. Then $k = 8m + 2$ for some $m \in \mathbb{Z}$. Suppose on the contrary that $4l$ is k-special.

For any integer n, there exist non-zero integers x, y, and z such that

$$x^2 + (8m+2)y^2 - 4lz^2 = 2n.$$

Since $2n$ is even, we can see that x is even. Let $x = 2x'$ for any integer x'. Thus

$$4x'^2 + (8m+2)y^2 - 4lz^2 = 2n$$
$$2x'^2 + (4m+1)y^2 - 2lz^2 = n.$$

If n is odd, then y is odd. Since y is odd, $y^2 \equiv 1 \pmod 8$.

We consider $2x'^2 - 2lz^2 = n - (4m+1)y^2$.

We now consider the following three cases on the values of l.

Case 1. $l \equiv 0 \pmod 4$. Thus $l = 4r$ for some $r \in \mathbb{Z}$. Then

$$2x'^2 - 2(4r)z^2 \equiv n - (4m+1) \pmod 8$$
$$2x'^2 \equiv n - (4m+1) \pmod 8.$$

Subcase 1.1. m is even. Then

$$2x'^2 \equiv n - 1 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 \equiv 0, 2 \pmod 8$. If $n \equiv 5, 7 \pmod 8$, then $2x'^2 \equiv 4, 6 \pmod 8$. This is a contradiction.

Subcase 1.2. m is odd. Then

$$2x'^2 \equiv n - 5 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 \equiv 0, 2 \pmod 8$. If $n \equiv 1, 3 \pmod 8$, then $2x'^2 \equiv 4, 6 \pmod 8$. This is a contradiction.

Case 2. $l \equiv 1 \pmod 4$. Thus $l = 4r + 1$ for some $r \in \mathbb{Z}$. Then

$$2x'^2 - 2(4r+1)z^2 \equiv n - (4m+1) \pmod 8$$
$$2x'^2 - 2z^2 \equiv n - (4m+1) \pmod 8.$$

Subcase 2.1. m is even. Then

$$2x'^2 - 2z^2 \equiv n - 1 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 - 2z^2 \equiv 0, 2, 6 \pmod 8$. If $n \equiv 5 \pmod 8$, then $2x'^2 - 2z^2 \equiv 4 \pmod 8$. This is a contradiction.

Subcase 2.2. m is odd. Then

$$2x'^2 - 2z^2 \equiv n - 5 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 - 2z^2 \equiv 0, 2, 6 \pmod 8$. If $n \equiv 1 \pmod 8$, then $2x'^2 - 2z^2 \equiv 4 \pmod 8$. This is a contradiction.

Case 3. $l \equiv 3 \pmod 4$. Thus $l = 4r + 3$ for some $r \in \mathbb{Z}$. Then

$$\begin{aligned} 2x'^2 - 2(4r+3)z^2 &\equiv n - (4m+1) \pmod 8 \\ 2x'^2 - 6z^2 &\equiv n - (4m+1) \pmod 8. \end{aligned}$$

Subcase 3.1. m is even. Then

$$2x'^2 - 6z^2 \equiv n - 1 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 - 6z^2 \equiv 0, 2, 4 \pmod 8$. If $n \equiv 7 \pmod 8$, then $2x'^2 - 2z^2 \equiv 6 \pmod 8$. This is a contradiction.

Subcase 3.2. m is odd. Then

$$2x'^2 - 6z^2 \equiv n - 5 \pmod 8.$$

Since the quadratic residues modulo 8 are $0, 1$, and 4, we have $2x'^2 - 6z^2 \equiv 0, 2, 4 \pmod 8$. If $n \equiv 3 \pmod 8$, then $2x'^2 - 6z^2 \equiv 6 \pmod 8$. This is a contradiction.

Therefore $4l$ is not $k-$special.

$\square$

CHAPTER 5

Conclusion

Let k be a positive integer. We define a positive integer l to be $k-$special if for every integer n there exist non-zero integers a, b, and c such that

$$n = a^2 + kb^2 - lc^2.$$

In Chapter 3, we first show that 1 is $k-$special when k is not divisible by 4. We next widen the scope of k and l. We show that k is $k-$special if and only if $k = 1$. We let k and l be odd positive integers and show that l is $k-$special if $l = x^2 + ky^2$ for some positive integers x and y and $\gcd(x, ky) = 1$. Moreover, there are infinitely many $k-$special when k is an odd integer. Furthermore, we prove that for any positive odd integer k, $4l$ is not $k-$special.

In Chapter 4, we show that if l is $2-$special, then $l = x^2 + 2y^2$ for some integers x and y but the converse is not true. We provide conditions of l to be $2k$-special where k is odd. That is $l = x^2 + 2ky^2$ for some positive integers x and y and $\gcd(x, 2ky) = 1$. Moreover, we show that there are infinitely many $2k-$special when k is an odd integer. Furthermore, we prove that if 4 is $k-$special, then $k \equiv 2 \pmod 4$. However, for any positive integer l, $4l$ is not $k-$special if $k \equiv 2 \pmod 8$ and $l \not\equiv 2 \pmod 4$.

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