Duality of Reciprocal Weighted Segal-Bargmann Spaces

Phraewmai Wannateeradet

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

Prince of Songkla University

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This is to certify that the work here submitted is the result of the candidate's own investigations. Due acknowledgement has been made of any assistance received.

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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ชื่อวิทยานิพนธ์ ภาวะคู่กันของปริภูมิซีกัล-บาร์กมันน์ถ่วงน้ำหนักส่วนกลับ
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บทคัดย่อ

นิยามปริภูมิซีกัล-บาร์กมันน์ถ่วงน้ำหนัก สำหรับฟังก์ชันบวก $\phi(z)$ ดังนี้

$$H_0 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, dz < \infty \right\}$$
$$H_1 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \phi(z) e^{-|z|^2} \, dz < \infty \right\}$$
$$H_{-1} := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \frac{1}{\phi(z)} \, e^{-|z|^2} \, dz < \infty \right\}$$

จุดมุ่งหมายของวิทยานิพนธ์นี้ คือการหาขอบเขตบนของ $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ และใช้ค่าขอบเขตดังกล่าว เพื่อแสดงว่า $H_1^* = H_{-1}$ ภายใต้ปริพันธ์แบบคู่กันที่เหมาะสม

Thesis Title Duality of Reciprocal Weighted Segal-Bargmann Spaces
Author Miss Phraewmai Wannateeradet
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ABSTRACT

Define weighted Segal-Bargmann spaces for a positive function $\phi(z)$ by

$$H_0 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\},$$
$$H_1 := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 \phi(z) e^{-|z|^2}\, dz < \infty \right\},$$
$$H_{-1} := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 \frac{1}{\phi(z)}\, e^{-|z|^2}\, dz < \infty \right\}.$$

The aim of this work is to establish an upper bound for $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ and use this bound together with an appropriate integral pairing to show that $H_1^* = H_{-1}$.

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Phraewmai Wannateeradet

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Introduction

1.1 Literature Review

David Hilbert, a German mathematician, was the first to introduce Hilbert spaces. Later on, it was possible to generalize linear algebra and calculus approaches from two-dimensional and three-dimensional Euclidean spaces to infinite-dimensional spaces. A Hilbert space is a vector space equipped with an inner product that allows a distance function and perpendicularity to be defined. Furthermore, Hilbert spaces are complete with respect to the norm defined by its linear product, implying that the space contains all of its limit points and allows us to use techniques from calculus.

Hilbert spaces, being a powerful mathematical tool, has widely used in functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

In mathematics, every vector space V has a corresponding dual vector space (or simply dual space) that contains all bounded linear functional on V, as well as the vector space structure of pointwise addition and scalar multiplication by constants.

Many fields of mathematics that use vector spaces, such as tensor analysis with finite-dimensional vector spaces, make use of dual vector spaces. Dual spaces are used to explain measures, distributions, and Hilbert spaces when applied to function vector spaces. As a result, in functional analysis, the dual space is a key idea.

The Segal-Bargmann space (also called a Fock space) is the holomorphic function space $HL^2(\mathbb{C},\alpha)$ where $\alpha(z) = \dfrac{1}{\pi}e^{-|z|^2}$. It is a Hilbert space of holomorphic functions on $\mathbb{C}$ with inner product given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}\,dz.$$

(See [2], [9], [12], [14]). The norm of z^k in this space can be calculated using polar coordinates as follows:

$$\|z^k\|_0^2 = \frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{-|z|^2}\,dz = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} r^{2k+1}e^{-r^2}\,drd\theta = k!.$$

Therefore, the set $\left\{\dfrac{z^k}{\sqrt{k!}}\right\}_{k=0}^{\infty}$ forms an orthonormal basis for this space and hence for every $f \in HL^2(\mathbb{C},\alpha)$, we can express f as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

when $a_k \in \mathbb{C}$. (See [9]).

We commonly weight the measure by multiplying a positive function in a weighted Segal-Bargmann space or a weighted Fock space. There are, however, various variants of these spaces. For example, the author of [14] defined and investigated a weighted Fock space associated with the perturbed Dunkl operator. The inner product in this space is given by

$$\langle f, g\rangle_Q = \int_{\mathbb{C}} f_e(z)\overline{g_e(z)}\,dm_a^Q(z) + 2(a+1)\int_{\mathbb{C}} f_o(z)\overline{g_o(z)}|z|^{-2}\,dm_{a+1}^Q(z)$$

where $a > -1/2$, $f_e(z) = \dfrac{f(z)+f(-z)}{2}$, $f_o(z) = \dfrac{f(z)-f(-z)}{2}$ and a measure $dm_a^Q(z)$ associated with a function Q. In [13] and [6], a weighted Fock space is defined as $HL^2(\mathbb{C}, e^{\phi(z)})$ for some plurisubharmonic function $\phi(z)$. In [5], the t-weighted Fock space is introduced as a space consisting of all holomorphic functions f on $\mathbb{C}^n$ such that the integral

$$\int_{\mathbb{C}^n}\left|f(z)e^{-\frac{a}{2}|z|^2}\right|^p \frac{1}{(1+|z|)^t}\,dV(z) < \infty$$

where $a > 0$, $0 < p < \infty$ and $dV(z)$ is the volume measure on $\mathbb{C}^n$. The version we use in this work is the radial weighted Segal-Bargmann space. For $h(z) := h(|z|)$, this a weighted Segal-Bargmann space consists of all holomorphic functions on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-h(z)}\, dz < \infty.$$

(See [1]).

In this work, we denote the classical Segal-Bargmann space by

$$H_0 := HL^2(\mathbb{C}, \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\}.$$

By multiplying a positive function $\phi(z)$ to the measure $d\alpha(z)$, we obtain another holomorphic function space $HL^2(\mathbb{C}, \phi\alpha)$. This new space will be referred to as a weighted Segal-Bargmann space. To make use of polar coordinates as we compute the norm $\|z^k\|_0^2$, one may assume that the function ϕ is rotation invariant as $\phi = \phi(|z|)$. Since the function $\alpha(z) = \dfrac{1}{\pi}\, e^{-|z|^2}$, the space $HL^2(\mathbb{C}, \phi\alpha)$ is a radial weighted Segal-Bargmann space.

For $m \geq 1$, let $\phi_1 = \phi(z) = e^{|z|^m}$ and $\phi_{-1} = \dfrac{1}{\phi(z)} = \dfrac{1}{e^{|z|^m}}$. Then we define the spaces H_1 and H_{-1} as follows.

$$H_1 := HL^2(\mathbb{C}, \phi_1\alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \phi(z) e^{-|z|^2}\, dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C}, \phi_{-1}\alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \frac{1}{\phi(z)}\, e^{-|z|^2}\, dz < \infty \right\}.$$

Consider

$$\frac{1}{\pi} \int_{\mathbb{C}} |z^k|^2 e^{a|z|^m} e^{-|z|^2}\, dz = 2 \int_0^\infty r^{2k+1} e^{ar^m - r^2}\, dr.$$

where $a = \pm 1$ and $m \leq 2$. Now, the integral $\displaystyle\int r^{2k+1} e^{ar^m - r^2}\, dr$ does not result in a basic function. However, the integral $\displaystyle\frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{a|z|^m} e^{-|z|^2}\, dz$ is finite as the term $e^{-|z|^2}$ dominates all other terms.

Despite the fact that the formula for $\displaystyle\|z^k\|_{a,m=1}^2 := \frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{a|z|} e^{-|z|^2}\, dz$ is implicit, the behavior of the growth of $\|z^k\|_{a,m=1}^2$ in terms of k is remarkably similar to that of $\|z^k\|_0^2$. We shall show in this work that the functions $r^{2k+1}e^{-r^2}$, $r^{2k+1}e^{-r-r^2}$ and $r^{2k+1}e^{r-r^2}$ are all concentrated towards the peaks of these functions. As with the normal distribution, definite integrals can be used to approximate it. (See Figure 1).

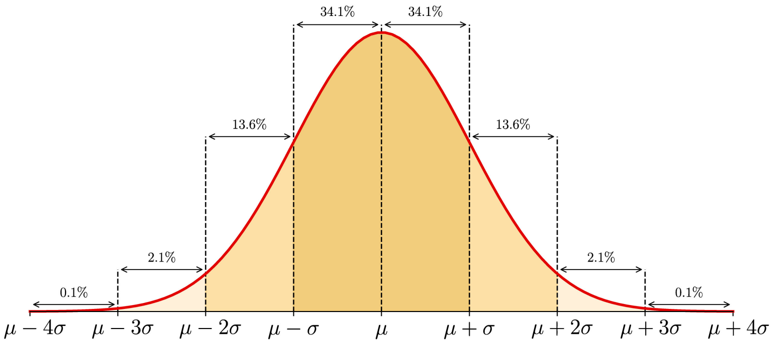

Figure 1. The graphs of the normal distribution.

The area under the curve can be approximated by the definite integral

$$\int_{-a}^{a} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

for some $a \in \mathbb{R}^+$. As a result, the norm $\|z^k\|_0^2$, $\|z^k\|_{-1}^2$ and $\|z^k\|_1^2$ can be approximated asymptotically by definite integrals.

In [3], the authors shows that the boundedness of $\dfrac{\|z^k\|_\mu^2\|z^k\|_\beta^2}{\|z^k\|_\gamma^4}$ plays an important role in a proof of the dual of a generalized Bergman spaces, $HL^2(\mathbb{B}^d,\mu)^* = HL^2(\mathbb{B}^d,\beta)$ under the integral pairing

$$\langle f, g\rangle_\gamma = \int_{\mathbb{B}^d} f(z)\overline{g(z)}c_\lambda \left(1-|z|^2\right)^{\lambda-(d+1)}\, dz$$

for $f \in HL^2(\mathbb{B}^d,\mu)$ and $g \in HL^2(\mathbb{B}^d,\beta)$.

In this work, we try to find a weight ϕ such that $H_1^* = H_{-1}$ under the integral pairing

$$\langle F, S\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{S(z)}e^{-|z|^2}\, dz$$

where $F \in H_1$ and $S \in H_{-1}$.

If we let $T_S(F) = \langle F, S\rangle_0$, then $\left|T_S(F)\right| \le \left\|\overline{S}\right\|_{-1}\left\|F\right\|_1$. Therefore, T_S is an element in H_1^*. Thus, an element S in H_{-1} defines a functional T_S on H_1^*.

Let P is an element in H_1^*, by Riesz representation, we have an element $\tilde{G}$ in H_1 such that $P(F) = \langle F, \tilde{G}\rangle_1$ and $\left\|P\right\| = \left\|\tilde{G}\right\|_1$ for all F in H_1. If there exist an element G in H_{-1}, then we have $H_1^* = H_{-1}$. We write $\tilde{G} = \sum_{i=0}^{\infty} a_i z^i$ and $G = \sum_{j=0}^{\infty} b_j$. If

$b_k = \left(\frac{\|z^k\|_1}{\|z^k\|_0}\right)^2 a_k$, then

$$\|G\|_{-1}^2 = \sum_{k=0}^{\infty} \left(\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}\right) \|z^k\|_1^2 |a_k|^2 = \sum_{k=0}^{\infty} \left(\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}\right) \|\tilde{G}\|_1^2.$$

The purpose of this work is to demonstrate that $\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is bounded and independent of k. Then, we obtain that G is an element in H_{-1}. We see that this upper bound which is independent of k is a key to prove that $H_1^* = H_{-1}$ under the integral pairing.

Finally, we obtain the following theorem

Theorem 1. *If there is a constant C which is independent of k such that*

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} < C,$$

then $H_1^* = H_{-1}$.

1.2 Procedure

This thesis consists of Chapter 1 Introduction, Chapter 2 Preliminaries, Chapter 3 Duality of reciprocal weighted Segal-Bargmann spaces and Chapter 4 Norms of monomials in Segal-Bargmann spaces.

We review the literature and research on the Segal-Bargmann space, the weighted Segal-Bargmann space and dual spaces in Chapter 1. After that, we discuss the purpose of this work.

Several definitions and basic properties of Hilbert spaces of holomorphic functions, including their dual spaces are collected in Chapter 2. Moreover, we also discuss asymptotic analysis and the Taylor series approximation.

The spaces H_0, H_1 and H_{-1} are defined in Chapter 3. In this chapter, we introduce the quantity $C_k = \frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$. We also discover the condition for establishing $H_1^* = H_{-1}$ that under the integral pairing.

Finally, in Chapter 4, we find a weight $\phi(z)$ so that C_k is bounded and independent of k. We divided this chapter into three sections. We find the weight $\phi(z) = e^{|z|}$

such that $H_1^* = H_{-1}$ under the integral pairing defined in Chapter 3. We consider a weight $\phi(z) = e^{|z|^m}$ where $m \geq 2$ and show that C_k is unbounded and depends on k in the second section. In the third section, we consider a weight $\phi(z) = e^{|z|^{1+p}}$ where $0 < p < 1$ and we prove that $H_1^* \neq H_{-1}$.

CHAPTER 2

Preliminaries

In this chapter, we collect several definitions and basic properties of Hilbert spaces of holomorphic functions, including their dual spaces. After that, we will introduce the concept of asymptotic analysis and the Taylor series approximation.

2.1 Hilbert space of holomorphic functions

The notion and properties of a Hilbert space of holomorphic functions are introduced in this section. The proof can be found in [2] for more information.

Let U be a non-empty open subset of the complex plane $\mathbb{C}$. Let $H(U)$ denote the space of holomorphic (or complex analytic) functions on U. Recall that a function of complex variables, $f : U \to \mathbb{C}$, is said to be holomorphic on U if f is differentiable at any point $z \in U$. Assume that μ is a continuous, strictly positive function on U. Let $L^2(U, \mu)$ denote the space of square-integrable functions with respect to the measure $d\mu(z)$, that is,

$$L^2(U, \mu) = \left\{ f : U \to \mathbb{C} \,\Big|\, \int_U |f(z)|^2 \, d\mu(z) < \infty \right\}.$$

Then $L^2(U, \mu)$ is a Hilbert space. We can write $HL^2(U, \mu) = H(U) \cap L^2(U, \mu)$, the space of holomorphic functions on U which are square-integrable with respect to the measure $d\mu(z)$, that is,

$$HL^2(U, \mu) = \left\{ f \in H(U) \,\Big|\, \int_U |f(z)|^2 \, d\mu(z) < \infty \right\}$$

where $d\mu(z)$ denotes Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$.

Remark 2.1.1. If f and g are continuous functions and $f = g$ μ-a.e., then $f = g$ everywhere. Therefore, we can consider the space $HL^2(U, \mu)$ as a subspace of $L^2(U, \mu)$.

Theorem 2.1. *Let $z \in U$. Then there exists a constant c_z such that*

$$|f(z)|^2 \leq c_z \|f\|^2_{L^2(U,\mu)}$$

for any $f \in HL^2(U, \mu)$.

Theorem 2.2. *$HL^2(U, \mu)$ is a closed subspace of $L^2(U, \mu)$, and therefore a Hilbert space.*

Theorem 2.3. (Hölder's inequality). *Let f, g be nonnegative measurable functions on a measure space (U, μ). Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_U fg\, d\mu \leq \left\{ \int_U f^p\, d\mu \right\}^{1/p} \cdot \left\{ \int_U g^q\, d\mu \right\}^{1/q}.$$

Theorem 2.4. (Cauchy − Schwarz inequality). *Let X be an inner product space. Then for any $x, y \in X$,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

and the equality holds if and only if x and y are linearly dependent.

Definition 2.1.1. The **Segal-Bargmann space** is the space $HL^2(\mathbb{C}, \alpha)$, where

$$\alpha(z) = \frac{1}{\pi} e^{-|z|^2}.$$

Remark 2.1.2. The Segal-Bargmann space $HL^2(\mathbb{C}, \alpha)$ is a Hilbert space of holomorphic functions on $\mathbb{C}$ with inner product given by

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\alpha(z)\, dz.$$

In these spaces, we have

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 \alpha(z)\, dz.$$

By using the polar coordinates, we obtain the norm of z^k in this space which is an element in an orthonormal basis $\left\{ \frac{z^k}{\sqrt{k!}} \right\}_{k=0}^{\infty}$ as

$$\|z^k\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |z^k|^2 e^{-|z|^2}\, dz = 2 \int_0^{\infty} r^{2k+1} e^{-r^2}\, dr = k!.$$

Theorem 2.5. $\left\{\frac{z^k}{\sqrt{k!}}\right\}_{k=0}^{\infty}$ *is an orthonormal basis for the Segal-Bargmann space.*

If f is a holomorphic function on $\mathbb{C}$, f has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

when $a_k \in \mathbb{C}$.

Theorem 2.6. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a holomorphic function on* $\mathbb{C}$*, and*

$$f_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Then $\|f_n - f\|_2 \to 0$.

Definition 2.1.2. For $\phi(z) > 0$, the space $HL^2(\mathbb{C}, \phi\,\alpha)$ is called a **weighted Segal-Bargmann space**.

Remark 2.1.3. The weighted Segal-Bargmann space $HL^2(\mathbb{C}, \phi\,\alpha)$ is a Hilbert space of holomorphic functions on $\mathbb{C}$ with inner product given by

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\phi(z)\alpha(z)\,dz.$$

In these spaces, we have

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 \phi(z)\alpha(z)\,dz.$$

2.2 Dual spaces

Any vector space V in mathematics has a corresponding dual vector space (or just dual space for short) that contains all bounded linear functionals on V as well as the vector space structure of pointwise addition and scalar multiplication by constants.

The dual space, as defined above, is defined for all vector spaces, and it is also known as the algebraic dual space to prevent misunderstanding. The continuous dual space is a subspace of the dual space that corresponds to continuous linear functionals when specified for a topological vector space.

Many branches of mathematics that use vector spaces, such as tensor analysis with finite-dimensional vector spaces, use dual vector spaces. Dual spaces are used to describe measures, distributions, and Hilbert spaces when applied to vector spaces of functions (which are typically infinite-dimensional). As a result, dual space is a crucial idea in functional analysis.

Definition 2.2.1. Let V be a vector space over a field F, where $F = \mathbb{R}$ or $\mathbb{C}$. A linear map from V to F is called a **linear functional** on V.

The **(algebraic) dual space** V^* is defined as the set of all bounded linear maps $\varphi : V \to F$ (linear functionals) given any vector space V over a field F. The dual space can be denoted $\hom(V, F)$ because linear maps are vector space homomorphisms. The dual space V^* itself becomes a vector space over F when equipped with an addition and scalar multiplication satisfying:

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a(\varphi(x))$$

for all $\varphi, \psi \in V^*$, $x \in V$, and $a \in F$. Elements of the algebraic dual space V^* are sometimes called **covectors** or **one-forms**.

The pairing of a functional φ in the dual space V^* and an element x of V is denoted by $\langle x, \varphi \rangle$. This pairing defines a nondegenerate bilinear mapping $\langle \cdot, \cdot \rangle : V^* \times V \to F$ called the natural pairing.

In linear algebra, identifying any vector space V with its dual vector space V^* is a highly important example of duality. Its elements are the linear functionals $\varphi : V \to F$, where F is the field over which V is defined. With replacing subsets of $\mathbb{R}^2$ with vector space and inclusions of such subsets by linear maps, the three properties of the dual cone are carried over to this sort of duality. That is:

- Applying the operation of taking the dual vector space twice gives another vector space V^{**}. There is always a map $V \to V^{**}$. For some V, namely precisely the finite-dimensional vector spaces, this map is an isomorphism.
- A linear map $V \to W$ gives rise to a map in the opposite direction $W^* \to V^*$.
- Given two vector spaces V and W, the maps from V to W^* correspond to the maps from W to V^*.

This duality has a particular feature in that V and V^* are isomorphic for certain objects, namely finite-dimensional vector spaces. This is, however, a fortunate coincidence, because obtaining such an isomorphism necessitates a specific decision, such as selecting a V basis. The Riesz representation theorem holds in the case where V is a Hilbert space.

The Riesz representation theorem, sometimes known as the Riesz-Fréchet representation theorem in honor of Frigyes Riesz and Maurice René Fréchet, establishes a crucial link between a Hilbert space and its continuous dual space. The two are isometrically isomorphic if the underlying field is real numbers; the two are isometrically anti-isomorphic if the underlying field is complex numbers.

Finally, we will look at the **Riesz representation theorem**, which is the last theorem in this section.

Theorem 2.7. (The Riesz representation theorem). *If φ is a bounded linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that*

$$\varphi(x) = \langle x, y \rangle$$

for each $x \in H$. Moreover, $\|\varphi\| = \|y\|$.

It follows by Theorem 2.1 and Theorem 2.2 that the pointwise evaluation is continuous. This means that for each $z \in U$, the evaluation map $T_z : HL^2(U, \mu) \to \mathbb{C}$ defined by

$$T_z(f) = f(z)$$

for any $f \in HL^2(U, \mu)$ is a continuous linear functional on $HL^2(U, \mu)$. Thus, by the Riesz representation theorem, for each $z \in \mathbb{C}$ this linear functional can be represented uniquely as inner product with some $\kappa_z \in HL^2(U, \mu)$, that is,

$$T_z(f) = \langle f, \kappa_z \rangle_{L^2(U,\mu)} = \int_U f(z)\overline{\kappa_z(z)}\mu(z)\, dz.$$

for any $f \in HL^2(U, \mu)$.

2.3 Asymptotic analysis

In mathematical analysis, **asymptotic analysis**, also known as **asymptotics**, is a method of describing limiting behavior. Suppose that we have a function $f(z)$ and we are interested in the behavior of $f(z)$ as z close to z_0. If the $\lim_{z \to z_0} f/g$ exists and is equal to 1, we say that $f(z)$ is **asymptotically equivalent** or **equal** to $g(z)$ under the limit $z \to z_0$. We write

$$f(z) \sim g(z) \text{ as } z \to z_0 \text{ if and only if } \lim_{z \to z_0} \frac{f(z)}{g(z)} = 1.$$

The symbol $\sim$ is the tilde. The relation is an equivalence relation on the set of functions of z; the functions$f$ and g are said to be asymptotically equivalent. The domain of f and g can be any set for which the limit is defined: e.g. real numbers, complex numbers, positive integers.

Proposition 2.3.1. *If $f \sim g$ and $a \sim b$, then the following hold.*

- $f^r \sim g^r$*, for every real* r.
- $\log(f) \sim \log(g)$ *if* $\lim g \neq 1$.
- $f \times a \sim g \times b$.
- $f/a \sim g/b$.

Example 2.3.1. Examples of asymptotic formulas:

- Factorial

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

this is Stirling's approximation.

- Partition function

For a positive integer n, the partition function, $p(n)$, gives the number of ways of writing the integer n as a sum of positive integers, where the order of addends is not considered.

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{\frac{2n}{3}}}.$$

- Hankel functions

$$H_\alpha^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{2\pi\alpha - \pi}{4}\right)},$$

$$H_\alpha^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{2\pi\alpha - \pi}{4}\right)}.$$

2.4 The Taylor series

In mathematics, Brook Taylor introduced the Taylor series in 1715. A function's Taylor series is an infinite sum of terms expressed in terms of the derivatives of the function at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. A Taylor series is also known as a Maclaurin series if zero is the point at which the derivatives are evaluated.

The partial sum formed by the first $n + 1$ terms of a Taylor series is a polynomial of degree n, which is known as the nth Taylor polynomial of the function. Taylor polynomials are approximations of a function, which become generally better as n increases. Taylor's theorem gives quantitative estimates on the error introduced by the use of such approximations.

Definition 2.4.1. The **Taylor series** of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots,$$

where $n!$ denotes the factorial of n.
In the more compact sigma notation, this can be written as

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where $f^n(a)$ denotes the nth derivative of f evaluated at the point a.
When $a = 0$, the series is also called a **Maclaurin series**.

Theorem 2.8. *Let $HL^2(U, \mu)$ be the space of holomorphic functions on U which are square-integrable with respect to the measure $d\mu(z)$. For every $f \in HL^2(U, \mu)$ can be written as a Taylor series.*

CHAPTER 3

Duality of reciprocal weighted Segal-Bargmann spaces

The Segal-Bargmann space is the holomorphic function space $HL^2(\mathbb{C}, \alpha)$ where α is the Gaussian function. That is $\alpha(z) = \frac{1}{\pi} e^{-|z|^2}$. In this thesis, we denote the classical Segal-Bargmann space by

$$H_0 := HL^2(\mathbb{C}, \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\}.$$

By multiplying positive functions $\phi(z)$ and $\frac{1}{\phi(z)}$ to the Gaussian measure $d\alpha(z)$, we obtain holomorphic function spaces $HL^2(\mathbb{C}, \phi\, \alpha)$ and $HL^2(\mathbb{C}, \frac{1}{\phi}\, \alpha)$. These spaces will be referred to as weighted Segal-Bargmann spaces.

Let $\phi_1 = \phi(z)$ and $\phi_{-1} = \frac{1}{\phi(z)}$. Then we define the spaces H_1 and H_{-1} as follows.

$$H_1 := HL^2(\mathbb{C}, \phi_1 \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \phi(z) e^{-|z|^2}\, dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C}, \phi_{-1} \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \frac{1}{\phi(z)}\, e^{-|z|^2}\, dz < \infty \right\}.$$

With the properties of the dual space of Hilbert spaces, it is known that for each $T \in H_1^*$, by Riesz representation, there exist a function G in H_1 such that

$$T(F) = \langle F, G \rangle_1 = \frac{1}{\pi} \int_{\mathbb{C}} F(z) \overline{G(z)} \phi(z) e^{-|z|^2}\, dz$$

for all $F \in H_1$, and the operator norm $\|T\| = \|G\|_1$.

In this thesis, we want to find a weight ϕ such that $H_1^* = H_{-1}$ under the **integral pairing**

$$\langle F, S\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{S(z)}e^{-|z|^2}\,dz$$

where $F \in H_1$ and $S \in H_{-1}$.

The purpose of this chapter is to prove the main following theorem, which requires us to find the condition for establishing $H_1^* = H_{-1}$ under integral pairing.

Theorem 1. *If there is a constant C which is independent of k such that*

$$\frac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4} < C,$$

then $H_1^* = H_{-1}$.

Proof. Let F be an element in H_1, and let S be an element in H_{-1}. We compute

$$\langle F, S\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{S(z)}e^{-|z|^2}\,dz.$$

Consider

$$\begin{aligned}\left|\langle F, S\rangle_0\right| &\le \frac{1}{\pi}\int_{\mathbb{C}} \left|F(z)\right|\left|\overline{S(z)}\right| e^{-|z|^2}\,dz \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \left|F(z)\phi(z)^{\frac{1}{2}}\right|\left|\overline{S(z)}\phi(z)^{-\frac{1}{2}}\right| e^{-|z|^2}\,dz.\end{aligned}$$

From Hölder's inequality, we get

$$\left|\langle F, S\rangle_0\right| \le \left\{\frac{1}{\pi}\int_{\mathbb{C}} \left|F(z)\right|^2\phi(z)e^{-|z|^2}\,dz\right\}^{\frac{1}{2}}\left\{\frac{1}{\pi}\int_{\mathbb{C}} \left|\overline{S(z)}\right|^2\frac{1}{\phi(z)}\,e^{-|z|^2}\,dz\right\}^{\frac{1}{2}}.$$

Now, if we let T_S be an element in H_1^* defined by $T_S(F) = \langle F, S\rangle_0$, then we have $\left|T_S(F)\right| \le \left\|\overline{S}\right\|_{-1}\left\|F\right\|_1$. It means that a functional T_S is an element in H_1^*. Therefore, an element S in H_{-1} defines a functional T_S on H_1^*.

On the other hand, let P be an element in H_1^*. Then we want to prove that there exist an element G in H_{-1} such that for each element F in H_1, $P(F) = \langle F, G\rangle_0$. Since P is an element in H_1^*, we have an element $\tilde{G}$ in H_1 such that $P(F) = \langle F, \tilde{G}\rangle_1$ and the operator norm $\left\|P\right\| = \left\|\tilde{G}\right\|_1$ for all F in H_1.

Consider

$$\langle F, G\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{G(z)}e^{-|z|^2}\,dz.$$

We can write $\tilde{G} = \sum_{i=0}^{\infty} a_i z^i$ and $G = \sum_{j=0}^{\infty} b_j z^j$. Then

$$\begin{aligned}
\langle z^k, G \rangle_0 &= \frac{1}{\pi} \int_{\mathbb{C}} z^k \overline{G(z)} e^{-|z|^2} \, dz \\
&= \frac{1}{\pi} \int_{\mathbb{C}} z^k \overline{\left(\sum_{j=0}^{\infty} b_j z^j \right)} e^{-|z|^2} \, dz \\
&= \bar{b}_k \|z^k\|_0^2.
\end{aligned}$$

Now, we obtain that $b_k = \dfrac{\overline{\langle z^k, G \rangle_0}}{\|z^k\|_0^2}$. Similarly, for $\tilde{G}$ we get $a_k = \dfrac{\overline{\langle z^k, \tilde{G} \rangle_1}}{\|z^k\|_1^2}$. Observe that, if b_k satisfies the following equality:

$$b_k = \left(\frac{\|z^k\|_1}{\|z^k\|_0} \right)^2 a_k,$$

then

$$G(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \left(\frac{\|z^k\|_1}{\|z^k\|_0} \right)^2 a_k z^k.$$

We compute the following result,

$$\begin{aligned}
\|G\|_{-1}^2 &= \langle G, G \rangle_{-1} \\
&= \sum_{k=0}^{\infty} |b_k|^2 \|z^k\|_{-1}^2 \\
&= \sum_{k=0}^{\infty} \frac{\|z^k\|_1^4}{\|z^k\|_0^4} |a_k|^2 \|z^k\|_{-1}^2 \\
&= \sum_{k=0}^{\infty} \left(\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \right) \|z^k\|_1^2 |a_k|^2 \\
&= \sum_{k=0}^{\infty} C_k \|z^k\|_1^2 |a_k|^2
\end{aligned}$$

where $C_k = \dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$.

If $C_k = \dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is bounded and independent of k, then we obtain that

$$\begin{aligned}
\|G\|_{-1}^2 &= \langle G, G\rangle_{-1} \\
&= \sum_{k=0}^{\infty} |b_k|^2 \|z^k\|_{-1}^2 \\
&= \sum_{k=0}^{\infty} C_k \|z^k\|_1^2 |a_k|^2 \\
&\leq C \sum_{k=0}^{\infty} \|z^k\|_1^2 |a_k|^2 \\
&= C \left\|\tilde{G}\right\|_1^2 \\
&< \infty
\end{aligned}$$

and hence G is an element in H_{-1}. This implies that $H_1^* = H_{-1}$ under the integral pairing. □

CHAPTER 4

Norms of monomials in Segal-Bargmann spaces

We introduced the properties of the dual space of Hilbert spaces in the previous chapter and found the condition for demonstrating that $H_1^* = H_{-1}$ under the integral pairing.

The goal of this chapter is to find a weight $\phi(z)$ such that $C_k = \dfrac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is both bounded and independent of k. Let us demonstrate this concept in the following section.

4.1 $\phi(z) = e^{|z|}$

In the classical Segal-Bargamann space,

$$H_0 := HL^2(\mathbb{C}, \alpha(z)) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\}.$$

By using the polar coordinates, we obtain the norm of z^k which is an element in an orthonormal basis $\left\{ \dfrac{z^k}{\sqrt{k!}} \right\}_{k=0}^{\infty}$ as

$$\begin{aligned} \|z^k\|_0^2 &= \frac{1}{\pi}\int_{\mathbb{C}} |z^k|^2 e^{-|z|^2}\, dz \\ &= 2\int_0^{\infty} r^{2k+1} e^{-r^2}\, dr. \end{aligned}$$

Consider the graph of $f_k(r) = r^{2k+1}e^{-r^2}$. It resembles a Gaussian-shaped wave function that propagates to the right as k increases. (See Figure 2).

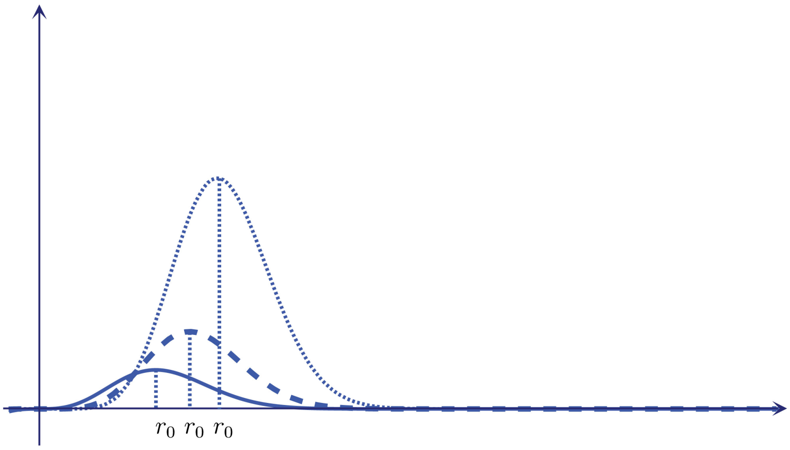

Figure 2. The graphs of $f_k(r) = r^{2k+1}e^{-r^2}$ for different k's.

We shall show in this paragraph that the function f_k behaves like a Gaussian shaped wave function in the sense that it is concentrated near its peak and has a finite width that is measured from where the function is cut off. Consequently, the integral $\int_0^\infty r^{2k+1}e^{-r^2}\,dr$ can be estimated by a definite integral $\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr$ for some $r_0 > 0$. Explicitly, we will show that

$$\int_0^\infty r^{2k+1}e^{-r^2}\,dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr \text{ as } k \to \infty$$

where $r_0 = \sqrt{\dfrac{2k+1}{2}}$ is a critical point of $r^{2k+1}e^{-r^2}$. (See Figure 3).

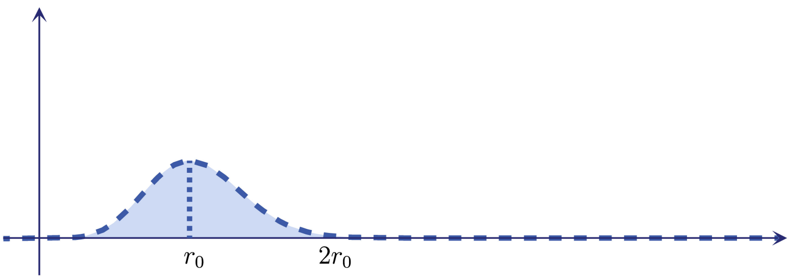

Figure 3. The graphs of $\int_0^\infty r^{2k+1}e^{-r^2}\,dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr$ as $k \to \infty$.

Let us compute some useful formulas that will be used in our study.

Lemma 4.1. *Let* $n = 2k + 1$ *where* k *is a nonnegative integer. For any* $a > 0$.

$$\int_0^\infty x^n e^{-ax^2}\,dx = \frac{k!}{2(a^{k+1})}. \tag{4.1}$$

Proof. We compute a definite integral $\int_0^\infty x^n e^{-ax^2}\,dx$ by using the gamma function. By using integration by substitution, we have

$$\begin{aligned}\int_0^\infty x^{2k+1} e^{-ax^2}\,dx &= \frac{1}{2a}\int_0^\infty \left(\frac{t}{a}\right)^k e^{-t}\,dt \\ &= \frac{1}{2a}\int_0^\infty \left(\frac{t}{a}\right)^k e^{-t}\,dt \\ &= \frac{1}{2(a^{k+1})}\int_0^\infty t^k e^{-t}\,dt.\end{aligned}$$

Since

$$\int_0^\infty t^k e^{-t}\,dt = \Gamma(k+1) = k!,$$

we obtain

$$\begin{aligned}\int_0^\infty x^{2k+1} e^{-ax^2}\,dx &= \frac{1}{2(a^{k+1})}\int_0^\infty t^k e^{-t}\,dt \\ &= \frac{1}{2(a^{k+1})}\Gamma(k+1) \\ &= \frac{1}{2(a^{k+1})}k!.\end{aligned}$$

Therefore,

$$\int_0^\infty x^n e^{-ax^2}\,dx = \frac{k!}{2(a^{k+1})}.$$

□

Lemma 4.2. *For a nonnegative integer* n *and* $a, b > 0$,

$$\int_0^b x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}\left(1 - e^{-ab}\sum_{i=0}^{n}\frac{(ab)^i}{i!}\right). \tag{4.2}$$

Proof. Integration by parts gives

$$\begin{aligned}
\int_0^b x^n e^{-ax}\,dx &= -\frac{x^n e^{-ax}}{a} - \frac{n x^{n-1} e^{-ax}}{a^2} - \cdots - \frac{n! x e^{-ax}}{a^n} - \frac{n! e^{-ax}}{a^{n+1}} \Bigg|_0^b \\
&= \frac{n!}{a^{n+1}} - \left(\frac{b^n e^{-ab}}{a} + \frac{n b^{n-1} e^{-ab}}{a^2} + \cdots + \frac{n! b e^{-ab}}{a^n} + \frac{n! e^{-ab}}{a^{n+1}} \right) \\
&= \frac{n!}{a^{n+1}} - e^{-ab} \left(\frac{b^n}{a} + \frac{n b^{n-1}}{a^2} + \cdots + \frac{n! b}{a^n} + \frac{n!}{a^{n+1}} \right) \\
&= \frac{n!}{a^{n+1}} - \frac{e^{-ab}}{a^{n+1}} \left(a^n b^n + n a^{n-1} b^{n-1} + \cdots + n! ab + n! \right) \\
&= \frac{n!}{a^{n+1}} - \frac{n! e^{-ab}}{a^{n+1}} \left(\frac{(ab)^n}{n!} + \frac{(ab)^{n-1}}{(n-1)!} + \cdots + ab + 1 \right) \\
&= \frac{n!}{a^{n+1}} \left(1 - e^{-ab} \sum_{i=0}^{n} \frac{(ab)^i}{i!} \right).
\end{aligned}$$

□

Lemma 4.3. *For* $r_0 = \sqrt{\dfrac{2k+1}{2}}$,

$$\lim_{k\to\infty} e^{-4{r_0}^2} \sum_{i=0}^{k} \frac{(4{r_0}^2)^i}{i!} = 0.$$

Proof. For $i = 0, 1, 2, \ldots, k$, we have $i + 1 < 4k + 2$ for all nonnegative integer k. Thus, $\dfrac{(4{r_0}^2)^i}{i!} < \dfrac{(4{r_0}^2)^{i+1}}{(i+1)!}$ and hence

$$\begin{aligned}
0 &< e^{-4{r_0}^2} \sum_{i=0}^{k} \frac{(4{r_0}^2)^i}{i!} \\
&< e^{-4{r_0}^2} (k+1) \frac{(4{r_0}^2)^k}{k!} \\
&= e^{-(4k+2)} (k+1) \frac{(4k+2)^k}{k!}.
\end{aligned}$$

Since $k! \sim \sqrt{2\pi k} \left(\dfrac{k}{e}\right)^k$,

$$e^{-(4k+2)} (k+1) \frac{(4k+2)^k}{k!} \sim \frac{(k+1)(4k+2)^k}{e^{3k+2}\sqrt{2\pi k}(k^k)}.$$

It is not hard to see that the limit of the last term is equal to zero.

Therefore,

$$\lim_{k\to\infty} e^{-4{r_0}^2} \sum_{i=0}^{k} \frac{(4{r_0}^2)^i}{i!} = 0.$$

□

Next, we will show that $\|z^k\|_0^2 \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr$.
From Lemma 4.1, we obtain

$$\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr = \frac{k!}{2(1^{k+1})} = \frac{k!}{2}. \tag{4.3}$$

By using integration by substitution, we have

$$\begin{aligned}\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr &= \int_0^{4{r_0}^2} r^{2k+1}e^{-s}\,\frac{ds}{2r}\\ &= \frac{1}{2}\int_0^{4{r_0}^2} s^k e^{-s}\,ds.\end{aligned}$$

Substituting $n = k$, $a = 1$, and $b = 4{r_0}^2$ into the equation (4.2), we obtain

$$\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr = \frac{k!}{2}\left(1 - e^{-4{r_0}^2}\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!}\right). \tag{4.4}$$

From equations (4.3) and (4.4), we obtain

$$\frac{\displaystyle\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} = \frac{\frac{k!}{2}}{\frac{k!}{2}\left(1 - e^{-4{r_0}^2}\displaystyle\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!}\right)}.$$

Thus,

$$\lim_{k\to\infty}\frac{\displaystyle\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} = \lim_{k\to\infty}\frac{1}{\left(1 - e^{-4{r_0}^2}\displaystyle\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!}\right)}.$$

From Lemma 4.3, we obtain

$$\lim_{k\to\infty} e^{-4{r_0}^2}\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!} = 0.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty}\frac{\displaystyle\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} &= \lim_{k\to\infty}\frac{1}{\left(1 - e^{-4{r_0}^2}\displaystyle\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!}\right)}\\ &= \frac{1}{1-0}\\ &= 1.\end{aligned}$$

Therefore,

$$\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr.$$

Hence,

$$\|z^k\|_0^2 = 2\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr \sim 2\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr.$$

4.1.1 Asymptotic behavior of the norms of monomials in weighted Segal-Bargmann spaces

In this section, we will introduce two Segal-Bargmann spaces which are weighted by the exponential growth $e^{|z|}$ and $e^{-|z|}$. Then we will estimate the norm of z^k in these spaces.

With the weight $\phi_1 = e^{|z|}$ and $\phi_{-1} = e^{-|z|} = \dfrac{1}{e^{|z|}}$, we have the following Hilbert spaces

$$H_1 := HL^2(\mathbb{C}, \phi_1\alpha(z)) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{|z|}e^{-|z|^2}\,dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C}, \phi_{-1}\alpha(z)) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 \frac{1}{e^{|z|}}\, e^{-|z|^2}\,dz < \infty \right\}.$$

In these spaces, we have

$$\|z^k\|_1^2 = 2\int_0^{\infty} r^{2k+1}e^{r-r^2}\,dr,$$

$$\|z^k\|_{-1}^2 = 2\int_0^{\infty} r^{2k+1}e^{-r-r^2}\,dr.$$

Although we can use integration by substitution and induction to find the closed form of the integral $\displaystyle\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr$, there is no elementary function whose derivative is $r^{2k+1}e^{-r-r^2}$ or $r^{2k+1}e^{r-r^2}$. However, if we consider the graphs of $f_{k,-1}(r) = r^{2k+1}e^{-r-r^2}$ and $f_{k,1}(r) = r^{2k+1}e^{r-r^2}$ compared with that of $f_k(r) = r^{2k+1}e^{-r^2}$. We can see that they are also concentrated near their peaks and have finite widths. (See Figure 4).

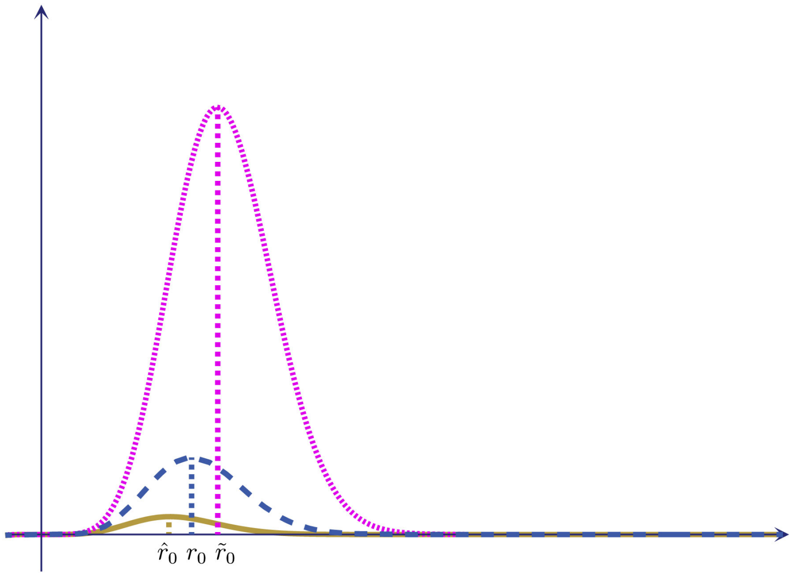

Figure 4. The graphs of $f_{k,-1}(r)$, $f_{k,1}(r)$ and $f_k(r)$.

So it makes sense to estimate those integrals by definite integrals.

In this subsection, we are interested in the behavior of $\dfrac{\|z^k\|_{-1}^2}{\|z^k\|_0^2}$ and $\dfrac{\|z^k\|_1^2}{\|z^k\|_0^2}$ as $k \to \infty$. Consider

$$
\begin{aligned}
\frac{\|z^k\|_{-1}^2}{\|z^k\|_0^2} &= \frac{\displaystyle\int_0^\infty r^{2k+1} e^{-r-r^2}\, dr}{\displaystyle\int_0^\infty r^{2k+1} e^{-r^2}\, dr} \\
&\sim \frac{\displaystyle\int_0^\infty r^{2k+1} e^{-r-r^2}\, dr}{\displaystyle\int_0^{2r_0} r^{2k+1} e^{-r^2}\, dr} \\
&= \frac{\displaystyle\int_0^{2\hat{r}_0} r^{2k+1} e^{-r-r^2}\, dr}{\displaystyle\int_0^{2r_0} r^{2k+1} e^{-r^2}\, dr} + \frac{\displaystyle\int_{2\hat{r}_0}^\infty r^{2k+1} e^{-r-r^2}\, dr}{\displaystyle\int_0^{2r_0} r^{2k+1} e^{-r^2}\, dr}
\end{aligned}
$$

where $\hat{r}_0 = \dfrac{-1+\sqrt{16k+9}}{4}$ is a critical point of $r^{2k+1}e^{-r-r^2}$.

Consider

$$\frac{\displaystyle\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \le \frac{\displaystyle\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} = \frac{\displaystyle\int_0^{\infty} r^{2k+1}e^{-r^2}\,dr - \int_0^{2\hat{r}_0} r^{2k+1}e^{-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}.$$

By using integration by substitution and substituting $n = k$, $a = 1$, and $b = 4\hat{r}_0^2$ into the equation (4.2), we obtain

$$\int_0^{2\hat{r}_0} r^{2k+1}e^{-r^2}\,dr = \frac{k!}{2}\left(1 - e^{-4\hat{r}_0^2}\sum_{i=0}^{k}\frac{(4\hat{r}_0^2)^i}{i!}\right). \tag{4.5}$$

From equations (4.3), (4.4) and (4.5), we obtain

$$\frac{\displaystyle\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \le \frac{\dfrac{k!}{2} - \dfrac{k!}{2}\left(1 - e^{-4\hat{r}_0^2}\displaystyle\sum_{i=0}^{k}\frac{(4\hat{r}_0^2)^i}{i!}\right)}{\dfrac{k!}{2}\left(1 - e^{-4{r_0}^2}\displaystyle\sum_{i=0}^{k}\frac{(4{r_0}^2)^i}{i!}\right)}.$$

Obviously, $\dfrac{\displaystyle\int_{2\hat{r}_0}^{\infty} r^{2k+1}e^{-r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \to 0$ as $k \to \infty$.

Therefore, we obtain the following proposition

Proposition 4.1.1. *Let $k = 0, 1, 2, 3, \ldots$. Then*

$$\frac{\|z^k\|_{-1}^2}{\|z^k\|_0^2} \sim \frac{\displaystyle\int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \tag{4.6}$$

where $r_0 = \sqrt{\dfrac{2k+1}{2}}$, and $\hat{r}_0 = \dfrac{-1+\sqrt{16k+9}}{4}$.

Now, let $\tilde{r}_0 = \dfrac{1+\sqrt{16k+9}}{4}$ be a critical point of $r^{2k+1}e^{r-r^2}$.

Since $\displaystyle\int_0^\infty r^{2k+1}e^{-r^2}\,dr \sim \int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr$, we have

$$\frac{\|z^k\|_1^2}{\|z^k\|_0^2} \sim \frac{\displaystyle\int_0^\infty r^{2k+1}e^{r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}$$

$$= \frac{\displaystyle\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} + \frac{\displaystyle\int_{2\tilde{r}_0}^\infty r^{2k+1}e^{r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}.$$

If we can show that

$$\frac{\displaystyle\int_{2\tilde{r}_0}^\infty r^{2k+1}e^{r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \le \frac{\displaystyle\int_{2\tilde{r}_0}^\infty (r-1)^{2k+1}e^{-(r-1)^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr} \to 0 \text{ as } k \to \infty$$

then, we obtain

$$\frac{\|z^k\|_1^2}{\|z^k\|_0^2} \sim \frac{\displaystyle\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr}{\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr}.$$

Let r be an element in an interval $(2\tilde{r}_0, \infty)$.

The function $\dfrac{e}{e^r}$ is decreasing and

$$\frac{e}{e^r} \to 0 \text{ as } r \to \infty;$$

On the other hand, the function $\dfrac{(r-1)^{2k+1}}{r^{2k+1}}$ is increasing and

$$\frac{(r-1)^{2k+1}}{r^{2k+1}} \to 1 \text{ as } k \to \infty.$$

Then

$$\frac{e}{e^r} \le \left(\frac{r-1}{r}\right)^{2k+1}$$

for all $r \ge 2\tilde{r}_0$.

We also obtain that

$$r^{2k+1} \leq e^{r-1}(r-1)^{2k+1}$$

for all $r \geq 2\tilde{r}_0$.

Therefore,

$$\int_{2\tilde{r}_0}^{\infty} r^{2k+1} e^{r-r^2}\, dr \leq \int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2}\, dr.$$

By using integration by substitution and equations (4.2) and (4.1), we have

$$\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2}\, dr = \frac{k!}{2} - \frac{k!}{2}\left(1 - e^{-(2\tilde{r}_0-1)^2} \sum_{i=0}^{k} \frac{(2\tilde{r}_0-1)^{2i}}{i!}\right). \tag{4.7}$$

From equations (4.4) and (4.7), we obtain

$$\begin{aligned}
\frac{\displaystyle\int_{2\tilde{r}_0}^{\infty} (r-1)^{2k+1} e^{-(r-1)^2}\, dr}{\displaystyle\int_{0}^{2r_0} r^{2k+1} e^{-r^2}\, dr} &= \frac{\displaystyle\frac{k!}{2}\left(1 - 1 + e^{-(2\tilde{r}_0-1)^2} \sum_{i=0}^{k} \frac{(2\tilde{r}_0-1)^{2i}}{i!}\right)}{\displaystyle\frac{k!}{2}\left(1 - e^{-4{r_0}^2} \sum_{i=0}^{k} \frac{(4{r_0}^2)^i}{i!}\right)} \\
&= \frac{\displaystyle e^{-(2\tilde{r}_0-1)^2} \sum_{i=0}^{k} \frac{(2\tilde{r}_0-1)^{2i}}{i!}}{\displaystyle 1 - e^{-4{r_0}^2} \sum_{i=0}^{k} \frac{(4{r_0}^2)^i}{i!}} \\
&\to 0
\end{aligned}$$

as $k \to \infty$.

Therefore, we obtain the following proposition

Proposition 4.1.2. *Let* $k = 0, 1, 2, 3, \ldots$. *Then*

$$\frac{\|z^k\|_1^2}{\|z^k\|_0^2} \sim \frac{\displaystyle\int_0^{2\tilde{r}_0} r^{2k+1} e^{r-r^2}\, dr}{\displaystyle\int_0^{2r_0} r^{2k+1} e^{-r^2}\, dr} \tag{4.8}$$

where $r_0 = \sqrt{\dfrac{2k+1}{2}}$, *and* $\tilde{r}_0 = \dfrac{1+\sqrt{16k+9}}{4}$.

4.1.2 The boundedness of C_k

Recall that $C_k = \dfrac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4}$. The boundedness of C_k plays an important role in a proof of the dual of reciprocal weighted Segal-Bargmann spaces, $H_1^* = H_{-1}$ under the integral pairing

$$\langle F, S\rangle_0 = \frac{1}{\pi}\int_{\mathbb{C}} F(z)\overline{S(z)}e^{-|z|^2}\,dz$$

where $F \in H_1$ and $S \in H_{-1}$.

In this subsection, we will show that $C_k = \dfrac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically equivalent to some constant. From Proposition 4.1.1 and Proposition 4.1.2, we obtain

$$\frac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim \frac{\displaystyle\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr \int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}\,dr}{\left(\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr\right)^2}. \tag{4.9}$$

First, we consider the definite integral

$$\begin{aligned}\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr &= \int_0^{2r_0} e^{-r^2+(2k+1)\ln r}\,dr\\ &= \int_0^{2r_0} e^{f(r)}\,dr\end{aligned}$$

where $f(r) = -r^2 + (2k+1)\ln r$.

Obviously, $r_0 = \sqrt{\dfrac{2k+1}{2}}$ is the critical point of $f(r)$.

With the function $f(r) = -r^2 + (2k+1)\ln r$, the Taylor series expansion of $f(r)$ about $r = r_0$ is given by

$$f(r) = \sum_{n=0}^{\infty}\frac{f^n(r_0)}{n!}(r-r_0)^n$$

with the interval of convergence $(0, 2r_0)$. Thus,

$$\int_0^{2r_0} e^{f(r)}\,dr = \int_0^{2r_0} e^{f(r_0)+f'(r_0)(r-r_0)+\frac{f''(r_0)(r-r_0)^2}{2!}+\sum_{n=3}^{\infty}\frac{f^n(r_0)(r-r_0)^n}{n!}}\,dr.$$

We have $f'(r_0) = 0$ and $f''(r_0) = -4$. If we consider $k \to \infty$, then $f^m(r_0) \to 0$ for all $m \geq 3$.

Therefore,

$$\int_0^{2r_0} e^{f(r)}\,dr = e^{f(r_0)}\int_0^{2r_0} e^{-2(r-r_0)^2}\,dr = e^{f(r_0)}\int_{-r_0}^{r_0} e^{-2u^2}\,du \tag{4.10}$$

where $u = r - r_0$.

Next, we consider the definite integral

$$\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr = \int_0^{2\tilde{r}_0} e^{r-r^2+(2k+1)\ln r}\,dr$$
$$= \int_0^{2\tilde{r}_0} e^{\tilde{f}(r)}\,dr$$

where $\tilde{f}(r) = r - r^2 + (2k+1)\ln r$ and $\tilde{r}_0 = \dfrac{1+\sqrt{16k+9}}{4}$.

By the Taylor series expansion of $\tilde{f}(r)$ about $r = \tilde{r}_0$, we obtain

$$\int_0^{2\tilde{r}_0} e^{\tilde{f}(r)}\,dr = \int_0^{2\tilde{r}_0} e^{\tilde{f}(\tilde{r}_0)+\tilde{f}'(\tilde{r}_0)(r-\tilde{r}_0)+\frac{\tilde{f}''(\tilde{r}_0)(r-\tilde{r}_0)^2}{2!}+\sum_{n=3}^{\infty}\frac{\tilde{f}^n(\tilde{r}_0)(r-\tilde{r}_0)^n}{n!}}\,dr.$$

For each $m \geq 3$, we have $\tilde{f}^m(\tilde{r}_0) \to 0$ as $k \to \infty$. Also, $\tilde{f}''(\tilde{r}_0) \to -4$ as $k \to \infty$.

Hence,

$$\int_0^{2\tilde{r}_0} e^{\tilde{f}(r)}\,dr = e^{\tilde{f}(\tilde{r}_0)}\int_0^{2\tilde{r}_0} e^{-2(r-\tilde{r}_0)^2}\,dr = e^{\tilde{f}(\tilde{r}_0)}\int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2}\,d\tilde{u} \tag{4.11}$$

where $\tilde{u} = r - \tilde{r}_0$.

Similarly,

$$\int_0^{2\hat{r}_0} e^{\hat{f}(r)}\,dr = e^{\hat{f}(\hat{r}_0)}\int_0^{2\hat{r}_0} e^{-2(r-\hat{r}_0)^2}\,dr = e^{\hat{f}(\hat{r}_0)}\int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2}\,d\hat{u} \tag{4.12}$$

where $\hat{f}(r) = -r - r^2 + (2k+1)\ln r$, $\hat{r}_0 = \dfrac{-1+\sqrt{16k+9}}{4}$ and $\hat{u} = r - \hat{r}_0$.

Substituting equation (4.10), (4.11) and (4.12) into the equation (4.9), we obtain

$$\frac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim \frac{\displaystyle\int_0^{2\tilde{r}_0} r^{2k+1}e^{r-r^2}\,dr \int_0^{2\hat{r}_0} r^{2k+1}e^{-r-r^2}\,dr}{\left(\displaystyle\int_0^{2r_0} r^{2k+1}e^{-r^2}\,dr\right)^2}$$
$$= \frac{\left(e^{\tilde{f}(\tilde{r}_0)}\displaystyle\int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2}\,d\tilde{u}\right)\left(e^{\hat{f}(\hat{r}_0)}\displaystyle\int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2}\,d\hat{u}\right)}{\left(e^{f(r_0)}\displaystyle\int_{-r_0}^{r_0} e^{-2u^2}\,du\right)^2}$$
$$= e^{\tilde{f}(\tilde{r}_0)+\hat{f}(\hat{r}_0)-2f(r_0)}\,\frac{\left(\displaystyle\int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2}\,d\tilde{u}\right)\left(\displaystyle\int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2}\,d\hat{u}\right)}{\left(\displaystyle\int_{-r_0}^{r_0} e^{-2u^2}\,du\right)^2}.$$

Observe that

$$\lim_{k\to\infty}\frac{r_0}{\tilde{r}_0} = \lim_{k\to\infty}\frac{\sqrt{k+\frac{1}{2}}}{\frac{1}{4}+\sqrt{k+\frac{9}{16}}} = 1,$$

and

$$\lim_{k\to\infty} \frac{\tilde{r}_0}{\hat{r}_0} = \lim_{k\to\infty} \frac{\frac{1}{4} + \sqrt{k + \frac{9}{16}}}{-\frac{1}{4} + \sqrt{k + \frac{9}{16}}} = 1.$$

Therefore, $r_0 \sim \tilde{r}_0 \sim \hat{r}_0$ as $k \to \infty$. Thus,

$$\int_{-r_0}^{r_0} e^{-2u^2}\, du \sim \int_{-\tilde{r}_0}^{\tilde{r}_0} e^{-2\tilde{u}^2}\, d\tilde{u} \sim \int_{-\hat{r}_0}^{\hat{r}_0} e^{-2\hat{u}^2}\, d\hat{u}.$$

Therefore,

$$\frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} \sim e^{\tilde{f}(\tilde{r}_0) + \hat{f}(\hat{r}_0) - 2f(r_0)}.$$

Consider

$$\begin{aligned} f(r_0) &= -r_0^2 + (2k+1)\ln r_0 \\ &= -\left(k + \frac{1}{2}\right) + \frac{(2k+1)}{2} \ln\left(k + \frac{1}{2}\right) \\ 2f(r_0) &= -2k - 1 + (2k+1)\ln\left(k + \frac{1}{2}\right). \end{aligned}$$

Also,

$$\begin{aligned} \hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) &= \left(\hat{r}_0 - \hat{r}_0^2 + (2k+1)\ln \hat{r}_0\right) + \left(-\tilde{r}_0 - \tilde{r}_0^2 + (2k+1)\ln \tilde{r}_0\right) \\ &= (\hat{r}_0 - \tilde{r}_0) - \left(\hat{r}_0^2 + \tilde{r}_0^2\right) + (2k+1)\ln\left(\hat{r}_0 \cdot \tilde{r}_0\right) \\ &= \frac{2}{4} - \left(\frac{8k+5}{4}\right) + (2k+1)\ln\left(k + \frac{1}{2}\right) \\ &= \frac{1}{4} - 2k - 1 + (2k+1)\ln\left(k + \frac{1}{2}\right). \end{aligned}$$

It is easy to see that $\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) = \dfrac{1}{4} + 2f(r_0)$.

This yields

$$\begin{aligned} \frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} &\sim e^{\hat{f}(\hat{r}_0) + \tilde{f}(\tilde{r}_0) - 2f(r_0)} \\ &= e^{\frac{1}{4}}. \end{aligned}$$

Now, we obtain that $\dfrac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}$ is asymptotically equivalent to a constant $e^{\frac{1}{4}}$.

It implies that C_k is bounded and independent of k.

Finally, we obtain the following theorem

Theorem 4.4. *Let a weighted* $\phi(z) = e^{|z|}$ *and*

$$C_k = \frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4}.$$

Then C_k *is asymptotically equivalent to a constant* $C = e^{\frac{1}{4}}$ *and hence* $H_1^* = H_{-1}$ *under the integral pairing*

$$\langle F, S \rangle_0 = \frac{1}{\pi} \int_{\mathbb{C}} F(z)\overline{S(z)} e^{-|z|^2}\, dz$$

where $F \in H_1$ *and* $S \in H_{-1}$.

4.2 $\phi(z) = e^{|z|^m}, m \geq 2$

In this section, we extend the power of $|z|$ from 1 to m when $m \geq 2$.

With a weight $\phi(z) = e^{a|z|^2}$, we have the following spaces

$$H_0 := HL^2(\mathbb{C}, \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\},$$

$$H_1 := HL^2(\mathbb{C}, e^{a|z|^2}\alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-(1-a)|z|^2}\, dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C}, \frac{1}{e^{a|z|^2}}\alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-(1+a)|z|^2}\, dz < \infty \right\}.$$

Observe that, if $a \geq 1$ that is $1 - a \leq 0$, then the integral $\int_{\mathbb{C}} |f(z)|^2 e^{-(1-a)|z|^2}\, dz$ is finite if and only if f is bounded and hence f is a constant function. Thus, this integral is an infinite integral for all $f(z) \neq 0$ which implies that $H_1 = \{0\}$. In the same way, if $a \leq -1$ then the space H_{-1} contains only a zero function because the integral $\int_{\mathbb{C}} |f(z)|^2 e^{-(1+a)|z|^2}\, dz$ is an infinite integral for all $f(z) \neq 0$. Throughout this section, let a be an element in the open interval $(-1, 1)$.

Let us recall some useful the formula that will be used in our study.

$$\int_0^\infty x^{2k+1} e^{-ax^2}\, dx = \frac{k!}{2(a^{k+1})} \tag{4.13}$$

where k is a nonnegative integer, and $a > 0$.

From equation (4.13), we obtain

$$\int_0^\infty r^{2k+1} e^{-r^2}\, dr = \frac{k!}{2},$$
$$\int_0^\infty r^{2k+1} e^{-(1-a)r^2}\, dr = \frac{k!}{2(1-a)^{k+1}},$$
$$\int_0^\infty r^{2k+1} e^{-(1+a)r^2}\, dr = \frac{k!}{2(1+a)^{k+1}}.$$

Consider

$$C_k = \frac{\|z^k\|_1^2 \|z^k\|_{-1}^2}{\|z^k\|_0^4} = \frac{\left(\int_0^\infty r^{2k+1} e^{-(1-a)r^2}\, dr\right)\left(\int_0^\infty r^{2k+1} e^{-(1+a)r^2}\, dr\right)}{\left(\int_0^\infty r^{2k+1} e^{-r^2}\, dr\right)^2}$$
$$= \frac{1}{(1-a^2)^{k+1}}.$$

Since $a \in (-1, 1)$, we have $0 < 1 - a^2 < 1$.

It means that $C_k = \dfrac{1}{(1-a^2)^{k+1}}$ tends to infinity as k tends to infinity.

In case $\phi(z) = e^{a|z|^2}$, C_k is obtained, which is unbounded and depends on k.

It should be noted that if we consider a weight $\phi(z) = e^{a|z|^m}$ when $m > 2$ and a is an arbitrary element in real number, then we obtain

$$H_0 := HL^2(\mathbb{C}, \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz < \infty \right\},$$
$$H_1 := HL^2(\mathbb{C}, e^{a|z|^m} \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{a|z|^m - |z|^2}\, dz < \infty \right\},$$
$$H_{-1} := HL^2(\mathbb{C}, \frac{1}{e^{a|z|^m}} \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-a|z|^m - |z|^2}\, dz < \infty \right\}.$$

We see that $H_1 = \{0\}$ when $a > 0$ and $H_{-1} = \{0\}$ when $a < 0$.

Finally, C_k is unbounded when $\phi(z) = e^{|z|^m}, m \geq 2$.

4.3 $\phi(z) = e^{|z|^{1+p}}, 0 < p < 1$

In previous section, we consider a weight $\phi(z) = e^{|z|^m}$ where $m \geq 2$. In that case, C_k is obtained, which is unbounded and depends on k. In this section, we consider a weight $\phi(z) = e^{|z|^{1+p}}$ where $0 < p < 1$. It means that we are interested in the power of $|z|$ that lies between 1 and 2.

With a weight $\phi(z) = e^{|z|^{1+p}}$, we have the following Hilbert spaces

$$H_0 := HL^2(\mathbb{C}, \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_0^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, dz < \infty \right\},$$

$$H_1 := HL^2(\mathbb{C}, e^{|z|^{1+p}} \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_1^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{|z|^{1+p} - |z|^2} \, dz < \infty \right\},$$

$$H_{-1} := HL^2(\mathbb{C}, \frac{1}{e^{|z|^{1+p}}} \alpha) = \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \|f\|_{-1}^2 = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^{1+p} - |z|^2} \, dz < \infty \right\}.$$

In these spaces, we have

$$\|z^k\|_0^2 = 2 \int_0^\infty r^{2k+1} e^{-r^2} \, dr,$$

$$\|z^k\|_1^2 = 2 \int_0^\infty r^{2k+1} e^{r^{1+p} - r^2} \, dr,$$

$$\|z^k\|_{-1}^2 = 2 \int_0^\infty r^{2k+1} e^{-r^{1+p} - r^2} \, dr.$$

We set the following functions,

$$f_k(r) = r^{2k+1} e^{-r^2}$$

$$F_{k,1}(r) = r^{2k+1} e^{r^{1+p} - r^2}$$

$$F_{k,-1}(r) = r^{2k+1} e^{-r^{1+p} - r^2}$$

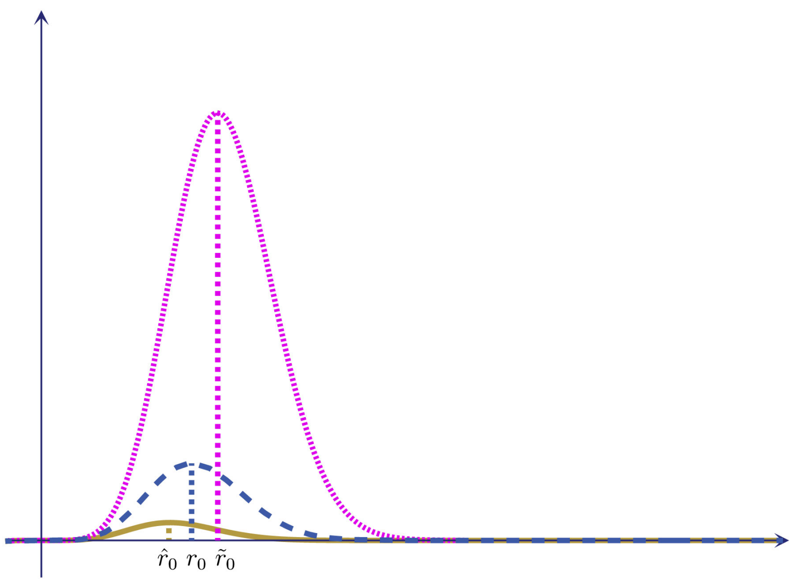

Figure 4. The graphs of $f_{k,-1}(r)$, $f_{k,1}(r)$ and $f_k(r)$.

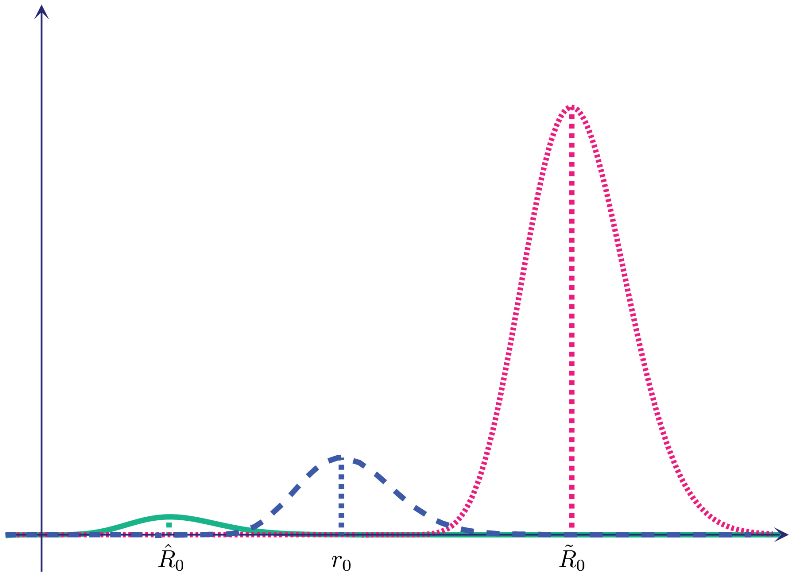

Figure 5. The graphs of $F_{k,-1}(r)$, $F_{k,1}(r)$ and $f_k(r)$.

According to the graph (Figure 5), the solid line (——) is created by a function $F_{k,-1}(r)$, the dashed line (- - -) is created by a function $f_k(r)$, and the dotted line (· · ·) is created by a function $F_{k,1}(r)$.

In the case of $\phi(z) = e^{|z|}$, we compare the graphs of $f_{k,-1}(r) = r^{2k+1}e^{-r-r^2}$ and $f_{k,1}(r) = r^{2k+1}e^{r-r^2}$ to that of $f_k(r) = r^{2k+1}e^{-r^2}$. We can see that they are similarly concentrated toward their peaks and have finite widths. There peaks appear at r_0, $\tilde{r}_0$ and $\hat{r}_0$. These critical points are asymtotically equal $r_0 \sim \tilde{r}_0 \sim \hat{r}_0$. (See Figure 4).

If we consider the graphs of $F_{k,-1}(r)$ and $F_{k,1}(r)$ compared with that of $f_k(r)$. We can see that they are also concentrated near their peaks. Unlike the case $\phi(z) = e^{|z|}$, their critical points r_0, $\tilde{R}_0$ and $\hat{R}_0$ are not asymtotically equal. They are seperated apart as $k \to \infty$. (See Figure 5). Thus, we need to show that

$$\int_0^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr \sim \int_{r_0}^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr,$$
$$\int_0^\infty r^{2k+1}e^{-r^{1+p}-r^2}\,dr \sim \int_0^{r_0} r^{2k+1}e^{-r^{1+p}-r^2}\,dr.$$

(See Figure 6 and Figure 7).

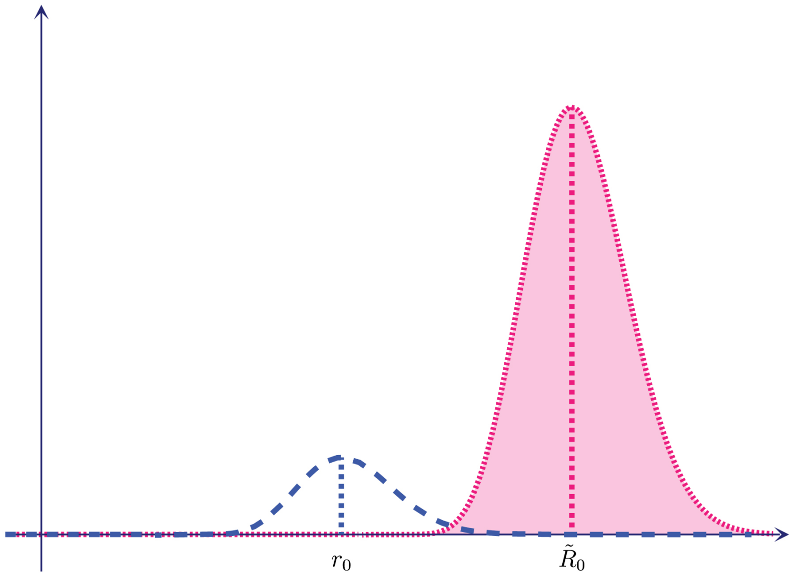

Figure 6. The graphs of $\int_0^\infty F_{k,1}(r)\,dr \sim \int_{r_0}^\infty F_{k,1}(r)\,dr$.

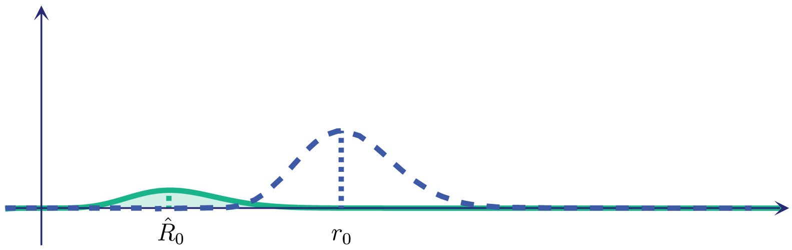

Figure 7. The graphs of $\int_0^\infty F_{k,-1}(r)\,dr \sim \int_0^{r_0} F_{k,-1}(r)\,dr$.

We consider the definite integral

$$\int_0^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr = \int_0^\infty e^{r^{1+p}-r^2+(2k+1)\ln r}\,dr = \int_0^\infty e^{F_1(r)}\,dr$$

where $F_1(r) = r^{1+p} - r^2 + (2k+1)\ln r$, and

$$\int_0^\infty r^{2k+1}e^{-r^{1+p}-r^2}\,dr = \int_0^\infty e^{-r^{1+p}-r^2+(2k+1)\ln r}\,dr = \int_0^\infty e^{F_{-1}(r)}\,dr$$

where $F_{-1}(r) = -r^{1+p} - r^2 + (2k+1)\ln r$. Then

$$F_1'(r) = (1+p)r^p - 2r + (2k+1)\frac{1}{r},$$

$$F_{-1}'(r) = -(1+p)r^p - 2r + (2k+1)\frac{1}{r}.$$

Recall that $r_0 = \sqrt{\frac{2k+1}{2}}$ is the critical point of $f_k(r) = r^{2k+1}e^{-r^2}$.

Assume that $\tilde{R}_0$ is the critical point of $F_1(r)$ and $\hat{R}_0$ is the critical point of $F_{-1}(r)$.

Observe that $\hat{R}_0 < r_0 < \tilde{R}_0$.

Let $\epsilon := \epsilon_k > 0$. Consider

$$F_1'(r_0+\epsilon) = (1+p)(r_0+\epsilon)^p - 2(r_0+\epsilon) + (2k+1)\frac{1}{(r_0+\epsilon)}$$
$$= \frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}+\epsilon\right)^{1+p} - 4\epsilon\left(\epsilon+\sqrt{4k+2}\right)}{\sqrt{4k+2}+2\epsilon}.$$

Also,

$$F_{-1}'(r_0-\epsilon) = -(1+p)(r_0-\epsilon)^p - 2(r_0-\epsilon) + (2k+1)\frac{1}{(r_0-\epsilon)}$$
$$= \frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}-\epsilon\right)^{1+p} + 4\epsilon\left(\epsilon-\sqrt{4k+2}\right)}{-\sqrt{4k+2}+2\epsilon}.$$

Choose $\epsilon = k^x$ where $x < \frac{1}{2}$. Then

$$F_1'(r_0+\epsilon) = \frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}+k^x\right)^{1+p} - 4k^x\left(k^x+\sqrt{4k+2}\right)}{\sqrt{4k+2}+2k^x},$$
$$F_{-1}'(r_0-\epsilon) = \frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}-k^x\right)^{1+p} + 4k^x\left(k^x-\sqrt{4k+2}\right)}{-\sqrt{4k+2}+2k^x}.$$

If $k \to \infty$, then we have the following equations

$$\frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}+k^x\right)^{1+p} - 4k^x\left(k^x+\sqrt{4k+2}\right)}{\sqrt{4k+2}+2k^x} \sim \frac{(2p+2)(\sqrt{k})^{1+p} - 4k^x(\sqrt{k})}{2\sqrt{k}}$$
$$= \frac{(2p+2)(\sqrt{k})k^{\frac{p}{2}} - 4k^x(\sqrt{k})}{2\sqrt{k}},$$

$$\frac{(2p+2)\left(\frac{1}{2}\sqrt{4k+2}-k^x\right)^{1+p} + 4k^x\left(k^x-\sqrt{4k+2}\right)}{-\sqrt{4k+2}+2k^x} \sim \frac{(2p+2)(\sqrt{k})^{1+p} + 4k^x(-2\sqrt{k})}{-2\sqrt{k}}$$
$$= \frac{(2p+2)(\sqrt{k})k^{\frac{p}{2}} + 4k^x(-2\sqrt{k})}{-2\sqrt{k}}.$$

If $x < \dfrac{p}{2}$, then

$$\frac{(2p+2)(\sqrt{k})k^{\frac{p}{2}} - 4k^x(\sqrt{k})}{2\sqrt{k}} > 0,$$

$$\frac{(2p+2)(\sqrt{k})k^{\frac{p}{2}} + 4k^x(-2\sqrt{k})}{-2\sqrt{k}} < 0.$$

We see that $F_1'(r_0+\epsilon) > 0$ and $F_{-1}'(r_0-\epsilon) < 0$.

It implies that $r_0+\epsilon < \tilde{R}_0$ and $r_0-\epsilon > \hat{R}_0$, respectively.

Since $\epsilon \to \infty$ as $k \to \infty$, we obtain $\left|r_0 - \tilde{R}_0\right| \to \infty$ and $\left|r_0 - \hat{R}_0\right| \to \infty$ as $k \to \infty$.

Therefore,

$$\int_0^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr \sim \int_{r_0}^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr, \tag{4.14}$$

$$\int_0^\infty r^{2k+1}e^{-r^{1+p}-r^2}\,dr \sim \int_0^{r_0} r^{2k+1}e^{-r^{1+p}-r^2}\,dr. \tag{4.15}$$

Next, we are interested in C_k. With a weight $\phi(z) = e^{|z|^{1+p}}$, we try to show that C_k is infinite.

Since $C_k = \dfrac{\|z^k\|_1^2\|z^k\|_{-1}^2}{\|z^k\|_0^4}$,

$$C_k = \frac{\left(\int_0^\infty r^{2k+1}e^{-r^{1+p}-r^2}\,dr\right)\left(\int_0^\infty r^{2k+1}e^{r^{1+p}-r^2}\,dr\right)}{\left(\int_0^\infty r^{2k+1}e^{-r^2}\,dr\right)^2}$$

$$= \frac{\left(\int_0^\infty F_{k,-1}(r)\,dr\right)\left(\int_0^\infty F_{k,1}(r)\,dr\right)}{\left(\int_0^\infty f_k(r)\,dr\right)^2}.$$

Now, we are ready to start our proof that C_k is infinite, we start by assuming it is not, and then we try to come up with a contradiction.

Suppose that C_k is finite. Consider

$$\begin{aligned}
&\int_0^\infty F_{k,-1}(r)\,dr\int_0^\infty F_{k,1}(r)\,dr\\
&= \left(\int_0^{r_0} F_{k,-1}(r)\,dr + \int_{r_0}^\infty F_{k,-1}(r)\,dr\right)\left(\int_0^{r_0} F_{k,1}(r)\,dr + \int_{r_0}^\infty F_{k,1}(r)\,dr\right)\\
&= \left(\int_0^{r_0} F_{k,-1}(r)\,dr\int_0^{r_0} F_{k,1}(r)\,dr\right) + \left(\int_0^{r_0} F_{k,-1}(r)\,dr\int_{r_0}^\infty F_{k,1}(r)\,dr\right)\\
&\quad + \left(\int_{r_0}^\infty F_{k,-1}(r)\,dr\int_0^{r_0} F_{k,1}(r)\,dr\right) + \left(\int_{r_0}^\infty F_{k,-1}(r)\,dr\int_{r_0}^\infty F_{k,1}(r)\,dr\right).
\end{aligned}$$

Now, we write

$$C_k = \frac{A+B+C+D}{E^2}$$

where

$$\begin{aligned}
A &= \int_0^{r_0} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr,\\
B &= \int_0^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr,\\
C &= \int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr,\\
D &= \int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr,
\end{aligned}$$

and

$$E = \int_0^{\infty} f_k(r)\,dr.$$

First, we consider

$$\frac{C}{E^2} = \frac{\displaystyle\int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr}{\left(\displaystyle\int_0^{\infty} f_k(r)\,dr\right)^2}$$

Since

$$\frac{\displaystyle\int_{r_0}^{\infty} F_{k,-1}(r)\,dr}{\displaystyle\int_0^{\infty} f_k(r)\,dr} \to 0 \text{ as } k \to 0$$

and

$$\frac{\displaystyle\int_0^{r_0} F_{k,1}(r)\,dr}{\displaystyle\int_0^{\infty} f_k(r)\,dr} \to 0 \text{ as } k \to 0,$$

we obtain

$$\frac{C}{E^2} = \frac{\left(\int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr\right)}{\left(\int_0^{\infty} f_k(r)\,dr\right)^2} \to 0 \tag{4.16}$$

as $k \to 0$. Consider

$$\begin{aligned}&\left(\int_0^{r_0} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr\right)^{\frac{1}{2}} \\ &= \left(\int_0^{r_0} \left(F_{k,-1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \left(\int_0^{r_0} \left(F_{k,1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}}.\end{aligned}$$

From Hölder's inequality, we obtain

$$\left(\int_0^{r_0} \left(F_{k,-1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \left(\int_0^{r_0} \left(F_{k,1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \geq \int_0^{r_0} \left(F_{k,-1}(r)F_{k,1}(r)\right)^{\frac{1}{2}} dr.$$

Thus,

$$\left(\int_0^{r_0} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr\right)^{\frac{1}{2}} \geq \int_0^{r_0} \left(F_{k,-1}(r)F_{k,1}(r)\right)^{\frac{1}{2}} dr.$$

It is not hard to see that $F_{k,-1}(r)F_{k,1}(r) = (f_k(r))^2$. Therefore,

$$\left(\int_0^{r_0} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr\right)^{\frac{1}{2}} \geq \int_0^{r_0} \left((f_k(r))^2\right)^{\frac{1}{2}} dr$$

and hence

$$A = \int_0^{r_0} F_{k,-1}(r)\,dr \int_0^{r_0} F_{k,1}(r)\,dr \geq \left(\int_0^{r_0} f_k(r)\,dr\right)^2. \tag{4.17}$$

Next, we consider

$$\begin{aligned}&\left(\int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr\right)^{\frac{1}{2}} \\ &= \left(\int_{r_0}^{\infty} \left(F_{k,-1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \left(\int_{r_0}^{\infty} \left(F_{k,1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}}.\end{aligned}$$

From Hölder's inequality, we obtain

$$\left(\int_{r_0}^{\infty} \left(F_{k,-1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \left(\int_{r_0}^{\infty} \left(F_{k,1}(r)^{\frac{1}{2}}\right)^2 dr\right)^{\frac{1}{2}} \geq \int_{r_0}^{\infty} \left(F_{k,-1}(r)F_{k,1}(r)\right)^{\frac{1}{2}} dr.$$

Thus,

$$\left(\int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr\right)^{\frac{1}{2}} \geq \int_{r_0}^{\infty} (F_{k,-1}(r)F_{k,1}(r))^{\frac{1}{2}}\,dr$$
$$= \int_{r_0}^{\infty} \left((f_k(r))^2\right)^{\frac{1}{2}}\,dr.$$

Hence,

$$D = \int_{r_0}^{\infty} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr \geq \left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2. \tag{4.18}$$

From equations (4.16), (4.17) and (4.18), we obtain

$$\int_{0}^{\infty} F_{k,-1}(r)\,dr \int_{0}^{\infty} F_{k,1}(r)\,dr$$
$$\geq \left(\int_{0}^{r_0} f_k(r)\,dr\right)^2 + \int_{0}^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr + \left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2.$$

Consider

$$C_k = \frac{\int_{0}^{\infty} F_{k,-1}(r)\,dr \int_{0}^{\infty} F_{k,1}(r)\,dr}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}$$
$$\geq \frac{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2} + \frac{\int_{0}^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2} + \frac{\left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}$$
$$= 1 + \frac{\int_{0}^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2} + \frac{\left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}.$$

From equations (4.14) and (4.15), we obtain

$$\frac{\int_{0}^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2} \geq 1 + \frac{\int_{0}^{r_0} F_{k,-1}(r)\,dr \int_{r_0}^{\infty} F_{k,1}(r)\,dr}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2} + \frac{\left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}.$$

Therefore,

$$C_k \geq 1 + C_k + \frac{\left(\int_{r_0}^{\infty} f_k(r)\,dr\right)^2}{\left(\int_{0}^{r_0} f_k(r)\,dr\right)^2}.$$

Since $\dfrac{\left(\displaystyle\int_{r_0}^{\infty} f_k(r)\, dr\right)^2}{\left(\displaystyle\int_0^{r_0} f_k(r)\, dr\right)^2}$ is positive, the quantity C_k is infinite.

It means that C_k is unbounded when $\phi(z) = e^{|z|^{1+p}}$, $0 < p < 1$.

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