Applications of Spherical Fuzzy Sets in Ternary Semigroups

Wasitthirawat Krailoet

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
2022

Applications of Spherical Fuzzy Sets in Ternary Semigroups

Wasitthirawat Krailoet

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
2022

Thesis Title Applications of Spherical Fuzzy Sets in Ternary Semigroups
Author Mister Wasitthirawat Krailoet
Major Program Mathematics

Major Advisor

....
(Assoc. Prof. Dr. Ronnason Chinram)

Co-Advisor

....
(Asst. Prof. Dr. Montakarn Petapirak)

Examining Committee:

....Chairperson
(Dr. Pattarawan Singavananda)

....Committee
(Assoc. Prof. Dr. Ronnason Chinram)

....Committee
(Asst. Prof. Dr. Montakarn Petapirak)

....Committee
(Asst. Prof. Dr. Winita Yonthanthum)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Mathematics

....
(Asst. Prof. Dr. Thakerng Wongsirichot)
Acting Dean of Graduate School

This is to certify that the work here submitted is the result of the candidate's own investigations. Due acknowledgement has been made of any assistance received.

.... Signature
(Assoc. Prof. Dr. Ronnason Chinram)
Major Advisor

.... Signature
(Asst. Prof. Dr. Montakarn Petapirak)
Co-Advisor

.... Signature
(Mister Wasitthirawat Krailoet)
Candidate

I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

.... Signature
(Mister Wasitthirawat Krailoet)
Candidate

ชื่อวิทยานิพนธ์ การประยุกต์ของเซตวิภัชนัยทรงกลมในกึ่งกรุปไตรภาค
ผู้เขียน นายวศิษฐ์ิรวรรธ ไกรเลิศ
สาขาวิชา คณิตศาสตร์
ปีการศึกษา 2565

บทคัดย่อ

กึ่งกรุปไตรภาค เป็นโครงสร้างทางพีชคณิต $(T,(\cdot))$ ซึ่งมี T เป็นเซตไม่ว่าง และมีฟังก์ชัน $(\cdot)$ จาก $T\times T\times T$ ไปยัง T เป็นการดำเนินการไตรภาค ที่สอดคล้องกับกฎการเปลี่ยนหมู่ นั่นคือ $(abc)de = a(bcd)e = ab(cde)$ สำหรับทุก $a,b,c,d,e\in T$ และเราให้ $\mathcal{S}$ เป็น *เซตย่อยวิภัชนัยทรงกลม* ของเซตเอกภพสัมพัทธ์ S นิยามโดย

$$\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >|\ x\in S\}$$

เมื่อ $\mu_{\mathcal{S}}$, $\eta_{\mathcal{S}}$ และ $\nu_{\mathcal{S}}$ เป็นเซตย่อยวิภัชนัยของ S โดยมีเงื่อนไข $0\leq(\mu_{\mathcal{S}}(x))^2+(\eta_{\mathcal{S}}(x))^2+(\nu_{\mathcal{S}}(x))^2\leq 1$ เราเรียก $\mu_{\mathcal{S}}(x)$ $\eta_{\mathcal{S}}(x)$ และ $\nu_{\mathcal{S}}(x)$ นี้ว่า *ระดับขั้นความเป็นสมาชิก ระดับขั้นความลังเล* และ*ระดับขั้นความไม่เป็นสมาชิก* ตามลำดับ

จุดประสงค์หลักของวิทยานิพนธ์นี้คือ เพื่อศึกษากึ่งกรุปย่อยไตรภาควิภัชนัยทรงกลม และไอดีลวิภัชนัยทรงกลมในกึ่งกรุปไตรภาค โดยใช้แนวคิดของกึ่งกรุปย่อยไตรภาค และไอดีลในกึ่งกรุปไตรภาค

นอกจากนี้เราได้ศึกษา "ความหยาบ" ของเซตวิภัชนัยทรงกลม และไอดีลวิภัชนัยทรงกลมในกึ่งกรุปไตรภาคด้วยเช่นกัน

Thesis Title	Applications of Spherical Fuzzy Sets in Ternary Semigroups
Author	Mister Wasitthirawat Krailoet
Major Program	Mathematics
Academic Year	2022

ABSTRACT

A *ternary semigroup* is an algebraic structure $(T, (\cdot))$ such that T is a non-empty set and $(\cdot)\colon T^3 \to T$ is a ternary operation satisfying the associative law, i.e., $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in T$, and let $\mathcal{S}$ be a *spherical fuzzy subset* of a universal set S defined by

$$\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >\mid x \in S\}$$

where $\mu_{\mathcal{S}}$, $\eta_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be three fuzzy subsets of S with the condition $0 \leq (\mu_{\mathcal{S}}(x))^2 + (\eta_{\mathcal{S}}(x))^2 + (\nu_{\mathcal{S}}(x))^2 \leq 1$. Then $\mu_{\mathcal{S}}(x)$, $\eta_{\mathcal{S}}(x)$ and $\nu_{\mathcal{S}}(x)$ are called the *degree of membership*, the *degree of hesitancy* and the *degree of non-membership*, respectively.

The main purpose of this thesis is to study spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups by using the concepts of ternary subsemigroups and ideals in ternary semigroups.

Moreover, we study roughness of spherical fuzzy sets and spherical fuzzy ideals in ternary semigroups.

Acknowledgements

This thesis can be accomplished with the guidance and support of several persons. I would like to express my deepest gratitude to my thesis advisor, Associate Professor Dr.Ronnason Chinram and my co-advisor Assistant Professor Dr.Montakarn Petapirak, for providing me an invaluable opportunity, motivation, encouragement and basic backgrounds throughout the course of this thesis.

Furthermore, I would like to express my special thanks to my thesis committee Dr.Pattarawan Singavananda and Assistant Professor Dr.Winita Yonthanthum. Their suggestions and comments are my sincere appreciation.

I feel very grateful to all my friends, senior and colleagues for encouragements.

Special thanks are given to my parents for their endless and unconditional love.

Finally, I most gratefully acknowledge Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University.

Wasitthirawat Krailoet

Contents

Abstract in Thai **V**

Abstract in English **VI**

Acknowledgements **VII**

Contents **VIII**

1 Introduction **1**

2 Preliminaries **3**

2.1 Ternary semigroups 3

2.2 Fuzzy sets 5

2.2.1 Fuzzy subsets 5

2.2.2 Fuzzy ideals in ternary semigroups 6

2.3 Spherical fuzzy sets 7

3 Spherical fuzzy sets in ternary semigroups **9**

3.1 Spherical fuzzy ideals in ternary semigroups 9

3.2 Rough spherical fuzzy sets in ternary semigroups 17

3.3 Rough spherical fuzzy ideals in ternary semigroups 22

4 Conclusions **27**

Bibliography **28**

VITAE **30**

Chapter 1

Introduction

The theory of ternary algebraic system was investigated by Lehmer ([9]) in 1932, but earlier such structures were studied by Kasner ([6]) who gave the idea of n-ary algebras. Furthermore, the ideal theory in ternary semigroups was established by Sioson ([13]).

In 1965, the notion of fuzzy sets was initiated by Zadeh ([15]). The fuzzy set is an extension of classical sets and represented by using a generalization of the indicator of classical sets that is called a membership function. Later, the concept of fuzzy set was applied to study in many algebraic structures. In 1981, Kuroki ([7]) provided some properties of fuzzy ideals.

In 2013, Iampan ([5]) gave the definition and characterized the properties of ideal extensions in ternary semigroups. After the introduction of ordinary fuzzy sets, the concept of rough sets was given by Pawlak ([11]) in 1982 which is defined depending on some equivalence relation on a universal finite set. The combination of theories of fuzzy sets and rough sets has been discussed in many research papers through all the years until 1990, when Dubois and Prade ([3]) proposed the notion of rough fuzzy sets.

In 2009, Petchkhaew and Chinram ([12]) studied fuzzy, rough and rough fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ternary semigroups. Later, in 2012, Kar and Sarkar ([8]) focused on studying fuzzy ideals of ternary semigroups and their related properties. In 2016, Wang and Zhan ([14]) established the rough semigroups and the rough fuzzy semigroups based on fuzzy ideals.

In 2019, Ashraf et al. ([1]) introduced the notion of spherical fuzzy set with applications in decision making problems, which is a generalization of the picture fuzzy sets, intuitionistic fuzzy sets and Pythagorean fuzzy sets fail when the degree of abstinence is involved, as it provides enlargement of the space of degrees of truthfulness (membership), abstinence (hesitancy) and falseness (non-membership).

Recently, in 2020, Chinram and Panityakul ([2]) introduced rough Pythagorean fuzzy ideals in ternary semigroups and gave some remarkable properties.

Our aim of this thesis is

1. to study spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups by using the concepts of ternary subsemigroups,
2. to study ideals in ternary semigroups,
3. to study roughness of spherical fuzzy sets and spherical fuzzy ideals in ternary semigroups.

Chapter 2

Preliminaries

In this chapter, we shall recall some basic definitions that will be used in this thesis.

2.1 Ternary semigroups

Definition 2.1.1. [13] A non-empty set T together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a *ternary semigroup* if

$$(abc)de = a(bcd)e = ab(cde)$$

for all $a, b, c, d, e \in T$.

Example 2.1.2. (1) The following example (Banach's Example) shows that a ternary semigroup does not necessarily reduce an ordinary semigroup. Let $T = \{-i, 0, i\}$ be a ternary semigroup under ternary multiplication over $\mathbb{C}$. We obtain that T is not a semigroup under multiplication over $\mathbb{C}$.

(2) Let $\mathbb{Z}^-$ be the set of all negative integers. Then $\mathbb{Z}^-$ is a ternary semigroup under ternary multiplication over $\mathbb{Z}$. We obtain that $\mathbb{Z}^-$ is not a semigroup under multiplication over $\mathbb{Z}$.

(3) The set of all odd permutations in S_n is a ternary semigroup under ternary composition. It is not a semigroup under composition.

For any three non-empty subsets A, B and C of a ternary semigroup T, a *product* ABC is the set of all elements $abc \in T$ where $a \in A$, $b \in B$ and

$c \in C$, i.e.,

$$ABC = \{\, abc \mid a \in A, b \in B \text{ and } c \in C \,\}.$$

In dealing with singleton sets we denote,

$$aBC := \{a\}BC = \{\, abc \mid b \in B \text{ and } c \in C \,\},$$

$$AbC := A\{b\}C = \{\, abc \mid a \in A \text{ and } c \in C \,\},$$

$$ABc := AB\{c\} = \{\, abc \mid a \in A \text{ and } b \in B \,\}.$$

Note that if $A_i = A$ for all $i = 1, 2, ..., n$, then we denote

$$\prod_{i=1}^{n} A_i = \underbrace{A \cdot \cdots \cdot A}_{n \text{ terms}} = A^n := \{\, \alpha_1\alpha_2\alpha_3 \cdots \alpha_n \mid \alpha_1, ..., \alpha_n \in A \,\}.$$

Definition 2.1.3. A non-empty subset S of a ternary semigroup T is called a *ternary subsemigroup* of T if $S^3 \subseteq S$.

Example 2.1.4. Let T be the set of all odd permutations in S_4. By Example 2.1.2.(3), we have that T is a ternary semigroup under ternary composition.

Let A, B and C be non-empty subsets of T given by

$$A = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \right\} = \{(1\ 2\ 3\ 4), (1\ 3\ 4\ 2)\},$$

$$B = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \right\} = \{(3\ 4), (1\ 2\ 4\ 3)\},$$

$$C = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \right\} = \{(2\ 4), (1\ 3)\}.$$

Then

$$\begin{aligned}
(1\ 2\ 3\ 4)BC &= \{(1\ 2\ 4\ 3), (2\ 3), (1\ 3\ 2\ 4), (1\ 2)\}, \\
(1\ 3\ 4\ 2)BC &= \{(1\ 3\ 2\ 4), (1\ 2), (2\ 4), (1\ 3)\}, \\
A(3\ 4)C &= \{(1\ 2\ 4\ 3), (2\ 3), (1\ 3\ 2\ 4)\}, (1\ 2)\}, \\
A(1\ 2\ 4\ 3)C &= \{(1\ 3\ 2\ 4), (1\ 2), (2\ 4), (1\ 3)\}, \\
AB(1\ 3) &= \{(2\ 3), (1\ 2), (1\ 3)\}, \\
AB(2\ 4) &= \{(1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (2\ 4)\}, \\
ABC &= \{(2\ 3), (2\ 4), (1\ 2), (1\ 2\ 4\ 3), (1\ 3), (1\ 3\ 2\ 4)\}, \\
A^3 &= \{(1\ 4\ 3\ 2), (2\ 3), (1\ 2), (2\ 4), (1\ 4), (1\ 3), (1\ 2\ 4\ 3)\}, \\
B^3 &= \{(3\ 4), (1\ 2\ 4\ 3), (1\ 2\ 3\ 4), (1\ 3\ 2\ 4), (1\ 4\ 3\ 2), (1\ 4\ 2\ 3), (1\ 3\ 4\ 2)\}, \\
C^3 &= \{(2\ 4), (1\ 3)\}.
\end{aligned}$$

We can see that $C^3 = \{(2\ 4), (1\ 3)\} \subseteq C$. Hence C is a ternary subsemigroup of T.

Definition 2.1.5. Let I be a non-empty subset of a ternary semigroup T. Then

1. I is called a *left ideal* of T if $TTI \subseteq I$.

2. I is called a *lateral ideal* of T if $TIT \subseteq I$.

3. I is called a *right ideal* of T if $ITT \subseteq I$.

A non-empty subset I of a ternary semigroup T is called an *ideal* of T if I is a left ideal, a lateral ideal and a right ideal of T. An ideal I of a ternary semigroup T is called a *proper ideal* if $I \neq T$.

2.2 Fuzzy sets

In 1965, the notion of fuzzy sets was initiated by Zadeh [15]. In this section, we recall the definitions and the representations of fuzzy subsets.

2.2.1 Fuzzy subsets

Definition 2.2.1. A *fuzzy subset* of a set S is a function $f : S \to [0, 1]$.

For $x \in S$, the value $f(x)$ is called the *degree of membership* of x, and the *complement* of f, denoted by f^c, is the fuzzy subset given by $f^c(x) = 1 - f(x)$.

We may denote $(S, f) := \{< x, f(x) >|\ x \in S\}$ is a *fuzzy set* of S.

Definition 2.2.2. Let f and g be any two fuzzy subsets of any set S.

1. The *intersection* of f and g is

$$(f \cap g)(a) = \min\{f(a), g(a)\}$$

for all $a \in S$.

2. The *union* of f and g is

$$(f \cup g)(a) = \max\{f(a), g(a)\}$$

for all $a \in S$.

3. $f \subseteq g$ if $f(a) \leq g(a)$ for all $a \in S$.

Definition 2.2.3. Let f and g be fuzzy subsets of a semigroup S. The *product* of f and g is defined by

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=\alpha\beta} \min\{f(\alpha), g(\beta)\} & \text{if } x \in S^2 \quad \text{for some } \alpha, \beta \in S, \\ 0 & \text{otherwise.} \end{cases}.$$

Example 2.2.4. Let f and g be fuzzy subsets of a semigroup $(\mathbb{Z}_5,\oplus)$ defined by

$$f(\bar{0}) = 0.20, \quad f(\bar{1}) = 0.12, \quad f(\bar{2}) = 0, \quad f(\bar{3}) = 0.75, \quad f(\bar{4}) = 0.06,$$

and

$$g(\bar{0}) = 1, \quad g(\bar{1}) = 0.20, \quad g(\bar{2}) = 0.50, \quad g(\bar{3}) = 0.75, \quad g(\bar{4}) = 0.99.$$

In this example, we obtain the following

- the complement of f and g:

  $$f^c(\bar{0}) = 0.80, \quad f^c(\bar{1}) = 0.88, \quad f^c(\bar{2}) = 1, \quad f^c(\bar{3}) = 0.25, \quad f^c(\bar{4}) = 0.94,$$

  and

  $$g^c(\bar{0}) = 0, \quad g^c(\bar{1}) = 0.80, \quad g^c(\bar{2}) = 0.50, \quad g^c(\bar{3}) = 0.25, \quad g^c(\bar{4}) = 0.01,$$

- the intersection of f and g:
  $(f \cap g)(\bar{0}) = 0.20$, $(f \cap g)(\bar{1}) = 0.12$, $(f \cap g)(\bar{2}) = 0$, $(f \cap g)(\bar{3}) = 0.75$, $(f \cap g)(\bar{4}) = 0.06$,

- the union of f and g:
  $(f \cup g)(\bar{0}) = 1$, $(f \cup g)(\bar{1}) = 0.20$, $(f \cup g)(\bar{2}) = 0.50$, $(f \cup g)(\bar{3}) = 0.75$, $(f \cup g)(\bar{4}) = 0.99$,

- the product $(f \circ g)(x)$:
  $(f \circ g)(\bar{0}) = 0.50$, $(f \circ g)(\bar{1}) = 0.75$, $(f \circ g)(\bar{2}) = 0.75$, $(f \circ g)(\bar{3}) = 0.75$, $(f \circ g)(\bar{4}) = 0.20$.

2.2.2 Fuzzy ideals in ternary semigroups

Definition 2.2.5. [8] A fuzzy subset f of a ternary semigroup T is called a *fuzzy ternary subsemigroup* of T if

$$f(xyz) \geq \min\{f(x), f(y), f(z)\}$$

for all $x, y, z \in T$.

Definition 2.2.6. [8] A fuzzy subset f of a ternary semigroup T is called

1. a *fuzzy left ideal* of T if $f(xyz) \geq f(z)$ for all $x, y, z \in T$,

2. a *fuzzy lateral ideal* of T if $f(xyz) \geq f(y)$ for all $x, y, z \in T$,

3. a *fuzzy right ideal* of T if $f(xyz) \geq f(x)$ for all $x, y, z \in T$,

4. a *fuzzy ideal* of T if it is a fuzzy left ideal, a fuzzy lateral ideal and a fuzzy right ideal of T, i.e.,

$$f(xyz) \geq \max\{f(x), f(y), f(z)\}$$

for all $x, y, z \in T$.

Definition 2.2.7. For any three fuzzy sets f_1, f_2 and f_3 of a ternary semigroup T. The *product* $f_1 \circ f_2 \circ f_3$ of f_1, f_2 and f_3 is defined by

$$(f_1 \circ f_2 \circ f_3)(y) = \begin{cases} \sup\limits_{y=y_1y_2y_3} \min\{f_1(y_1), f_2(y_2), f_3(y_3)\} & \text{if } y \in T^3, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the product $f_1 \circ f_2 \circ f_3$ of fuzzy subsets f_1, f_2 and f_3 of a ternary semigroup T is also a fuzzy subset of T.

Let $\mathcal{F}(T)$ be the set of all fuzzy subsets of a ternary semigroup T. Then $\mathcal{F}(T)$ is a ternary semigroup under this product.

2.3 Spherical fuzzy sets

In 2019, The spherical fuzzy set proposed by Gündogku, F.K. and Kahraman, C. [4], which is an extension of the picture fuzzy set.

Definition 2.3.1. Let S be a universal set. A *spherical fuzzy set* on S

$$\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >|\ x \in S\}$$

where $\mu_{\mathcal{S}} : S \to [0, 1]$, $\eta_{\mathcal{S}} : S \to [0, 1]$ and $\nu_{\mathcal{S}} : S \to [0, 1]$ represent the degree of membership, the degree of hesitancy and the degree of non-membership of $x \in S$ with the condition $0 \leq (\mu_{\mathcal{S}}(x))^2 + (\eta_{\mathcal{S}}(x))^2 + (\nu_{\mathcal{S}}(x))^2 \leq 1$.

We may also denote a spherical fuzzy set $\mathcal{S}$ by $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$.

Example 2.3.2. Let f be any fuzzy subset of a set S. Let $\mu_{\mathcal{S}} : S \to [0,1]$, $\eta_{\mathcal{S}} : S \to [0,1]$ and $\nu_{\mathcal{S}} : S \to [0,1]$ be defined by

$$\mu_{\mathcal{S}}(x) = f(x),\ \eta_{\mathcal{S}}(x) = 0 \text{ and } \nu_{\mathcal{S}}(x) = 1 - f(x).$$

We obtain

$$0 \leq (\mu_{\mathcal{S}}(x))^2 + (\eta_{\mathcal{S}}(x))^2 + (\nu_{\mathcal{S}}(x))^2 = (f(x))^2 + (1 - f(x))^2 \leq f(x) + 1 - f(x) = 1.$$

Therefore, $\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) > \mid x \in S\}$ is a spherical fuzzy set on S.

Definition 2.3.3. Let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$ and $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ be any two spherical fuzzy set of a universal set S.

1. The *intersection* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is

$$\mathcal{S}_1 \cap \mathcal{S}_2 = (\mu_{\mathcal{S}_1} \cap \mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_1} \cap \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_1} \cup \nu_{\mathcal{S}_2}).$$

2. The *union* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is

$$\mathcal{S}_1 \cup \mathcal{S}_2 = (\mu_{\mathcal{S}_1} \cup \mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_1} \cup \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_1} \cap \nu_{\mathcal{S}_2}).$$

3. $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if $\mu_{\mathcal{S}_1}(x) \leq \mu_{\mathcal{S}_2}(x)$, $\eta_{\mathcal{S}_1}(x) \leq \eta_{\mathcal{S}_2}(x)$ and $\nu_{\mathcal{S}_1}(x) \geq \nu_{\mathcal{S}_2}(x)$ for all $x \in S$.

Note that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are spherical fuzzy sets of a universal set S, then $\mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ are also spherical fuzzy sets of S.

Example 2.3.4. Let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$, $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ be two spherical fuzzy sets of $\mathbb{R}$ defined by

$$\mu_{\mathcal{S}_1}(x) = \left|\frac{1}{\sqrt{2}}\sin(x)\right|, \quad \eta_{\mathcal{S}_1}(x) = 0, \quad \nu_{\mathcal{S}_1}(x) = \left|\frac{1}{\sqrt{2}}\cos(x)\right|,$$

and

$$\mu_{\mathcal{S}_2}(x) = |\sin(x)|, \quad \eta_{\mathcal{S}_2}(x) = \left|\frac{1}{\sqrt{2}}\cos(x)\right|, \quad \nu_{\mathcal{S}_2}(x) = \left|\frac{1}{\sqrt{2}}\cos(x)\right|.$$

We have $\mu_{\mathcal{S}_1}(x) \leq \mu_{\mathcal{S}_2}(x)$, $\eta_{\mathcal{S}_1}(x) = 0 \leq \left|\frac{1}{\sqrt{2}}\cos(x)\right| = \eta_{\mathcal{S}_2}(x)$ and $\nu_{\mathcal{S}_1}(x) = \nu_{\mathcal{S}_2}(x)$ for all $x \in \mathbb{R}$.
Therefore, $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

Chapter 3

Spherical fuzzy sets in ternary semigroups

3.1 Spherical fuzzy ideals in ternary semigroups

In this section, we define spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups.

Definition 3.1.1. A spherical fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ on a ternary semigroup T is called a *spherical fuzzy ternary subsemigroup* of T if, for all $a, b, c \in T$

1. $\mu_{\mathcal{S}}(abc) \geq \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}$,
2. $\eta_{\mathcal{S}}(abc) \geq \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$,
3. $\nu_{\mathcal{S}}(abc) \leq \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}$.

Definition 3.1.2. A spherical fuzzy set $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ on a ternary semigroup T is called

1. a *spherical fuzzy left ideal* of T if for all $a, b, c \in T$,
$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(c),\ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(c) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(c),$$
2. a *spherical fuzzy lateral ideal* of T if for all $a, b, c \in T$,
$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(b),\ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(b) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(b),$$
3. a *spherical fuzzy right ideal* of T if for all $a, b, c \in T$,
$$\mu_{\mathcal{S}}(abc) \geq \mu_{\mathcal{S}}(a),\ \eta_{\mathcal{S}}(abc) \geq \eta_{\mathcal{S}}(a) \quad \text{and} \quad \nu_{\mathcal{S}}(abc) \leq \nu_{\mathcal{S}}(a),$$

4. a *spherical fuzzy ideal* of T if for all $a, b, c \in T$,

$$\mu_{\mathcal{S}}(abc) \geq \max\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\},$$

$$\eta_{\mathcal{S}}(abc) \geq \max\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$$

and

$$\nu_{\mathcal{S}}(abc) \leq \min\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}.$$

Example 3.1.3. Let $T = \{-i, 0, i\}$ be a ternary semigroup under ternary multiplication over $\mathbb{C}$ and $\mathcal{S} = (\mu_{\mathcal{S}}, \nu_{\mathcal{S}}, \eta_{\mathcal{S}})$ be a spherical fuzzy set on T defined by

$$\mu_{\mathcal{S}}(-i) = 0.5, \quad \eta_{\mathcal{S}}(-i) = 0, \quad \nu_{\mathcal{S}}(-i) = 0.5,$$

$$\mu_{\mathcal{S}}(0) = 0, \quad \eta_{\mathcal{S}}(0) = 1, \quad \nu_{\mathcal{S}}(0) = 0,$$

and

$$\mu_{\mathcal{S}}(i) = 0.5, \quad \eta_{\mathcal{S}}(i) = 0, \quad \nu_{\mathcal{S}}(i) = 0.5.$$

First, consider $0 = abc$ for some $a, b, c \in T$. Let $\mu_{\min} := \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}$, $\eta_{\min} := \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$ and $\nu_{\max} := \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}$. Then

a	b	c	$\mu_{\min}$	$\eta_{\min}$	$\nu_{\max}$
0	$-i$	$-i$	0	0	0.5
0	$-i$	0	0	0	0.5
0	i	i	0	0	0.5
0	0	$-i$	0	0	0.5
0	0	0	0	1	0
0	0	i	0	0	0.5
0	i	$-i$	0	0	0.5
0	i	0	0	0	0.5
0	i	i	0	0	0.5

We can see that $\mu_{\mathcal{S}}(0) \geq \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}$, $\eta_{\mathcal{S}}(0) \geq \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$ and $\nu_{\mathcal{S}}(0) \leq \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}$ for all $a, b, c \in T$.
For $-i = abc$ and $i = abc$ are similar.
Then $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of T.

Example 3.1.4. Let $T = \{-i, i\}$ be a ternary semigroup under ternary multiplication over $\mathbb{C}$ and $\mathcal{S} = (\mu_{\mathcal{S}}, \nu_{\mathcal{S}}, \eta_{\mathcal{S}})$ be a spherical fuzzy set on T defined by

$$\mu_{\mathcal{S}}(-i) = 0.5, \quad \eta_{\mathcal{S}}(-i) = 0, \quad \nu_{\mathcal{S}}(-i) = 0.5$$

and

$$\mu_{\mathcal{S}}(i) = 0.5, \quad \eta_{\mathcal{S}}(i) = 0, \quad \nu_{\mathcal{S}}(i) = 0.5.$$

We have $\mu_{\mathcal{S}}(abc) \geq \max\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\}$, $\eta_{\mathcal{S}}(abc) \geq \max\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}$ and $\nu_{\mathcal{S}}(abc) \leq \min\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}$ for all $a, b, c \in T$.
Therefore, $\mathcal{S}$ is a spherical fuzzy ideal of T.

Next, we define the product of three spherical fuzzy sets.

Definition 3.1.5. Let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$, $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ and $\mathcal{S}_3 = (\mu_{\mathcal{S}_3}, \eta_{\mathcal{S}_3}, \nu_{\mathcal{S}_3})$ be any three spherical fuzzy sets on a ternary semigroup T. The *product* $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3$ of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ is defined by

$$\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3 = ((\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3}), (\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3}), (\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3}))$$

where

$$(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x) = \begin{cases} \sup\limits_{x=abc} \min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 0, & \text{otherwise,} \end{cases}$$

$$(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x) = \begin{cases} \sup\limits_{x=abc} \min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x) = \begin{cases} \inf\limits_{x=abc} \max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\}, & \text{if } x \in T^3; \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.1.6. *Let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$, $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ and $\mathcal{S}_3 = (\mu_{\mathcal{S}_3}, \eta_{\mathcal{S}_3}, \nu_{\mathcal{S}_3})$ be any three spherical fuzzy sets on a ternary semigroup T. Then $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3$ is also a spherical fuzzy set on T.*

Proof. Assume that $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are spherical fuzzy sets of a ternary semigroup T. Let $x \in T$. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x) = 0,$$

$$(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x) = 0$$

and

$$(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x) = 1.$$

Then

$$0 \leq ((\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x))^2 + ((\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x))^2 + ((\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x))^2 = 1.$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\},$$

$$(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\}$$

and

$$(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\}.$$

Then

$$\begin{aligned}
&((\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(x))^2 + ((\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(x))^2 + ((\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(x))^2 \\
&= (\sup_{x=abc} \min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 + (\sup_{x=abc} \min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2 \\
&\quad + (\inf_{x=abc} \max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\})^2 \\
&= \sup_{x=abc} (\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 + \sup_{x=abc} (\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2 \\
&\quad + \inf_{x=abc} (\max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\})^2 \\
&\leq \sup_{x=abc} (\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 + \sup_{x=abc} (\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2 \\
&\quad + \inf_{x=abc} [1 - (\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 - (\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2] \\
&\leq \sup_{x=abc} (\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 + \sup_{x=abc} (\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2 \\
&\quad + 1 - \sup_{x=abc} (\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\})^2 - \sup_{x=abc} (\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\})^2 \\
&= 1.
\end{aligned}$$

Therefore, $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3$ is a spherical fuzzy set of T. □

Example 3.1.7. Let $T = \{-i, i\}$ be a ternary semigroup under ternary multiplication over $\mathbb{C}$, let $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$, $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ and $\mathcal{S}_3 = (\mu_{\mathcal{S}_3}, \eta_{\mathcal{S}_3}, \nu_{\mathcal{S}_3})$ be three spherical fuzzy sets on T defined by

$$\mu_{\mathcal{S}_1}(-i) = 0.5, \quad \eta_{\mathcal{S}_1}(-i) = 0, \quad \nu_{\mathcal{S}_1}(-i) = 0.5,$$

$$\mu_{\mathcal{S}_1}(i) = 0.5, \quad \eta_{\mathcal{S}_1}(i) = 0, \quad \nu_{\mathcal{S}_1}(i) = 0.5,$$

$$\mu_{\mathcal{S}_2}(-i) = 1, \quad \eta_{\mathcal{S}_2}(-i) = 0, \quad \nu_{\mathcal{S}_2}(-i) = 0,$$

$$\mu_{\mathcal{S}_2}(i) = 0, \quad \eta_{\mathcal{S}_2}(i) = 0, \quad \nu_{\mathcal{S}_2}(i) = 1,$$

and

$$\mu_{\mathcal{S}_3}(-i) = 0.8, \quad \eta_{\mathcal{S}_3}(-i) = 0.4, \quad \nu_{\mathcal{S}_3}(-i) = 0.2,$$

$$\mu_{\mathcal{S}_3}(i) = 0.9, \quad \eta_{\mathcal{S}_3}(i) = 0.1, \quad \nu_{\mathcal{S}_3}(i) = 0.4.$$

Consider $i = abc$ for some $a, b, c \in T$, then
we obtain $(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(i)$, $(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(i)$ and $(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(i)$ as follows:

a	b	c	$\mu_{\mathcal{S}_1}(a)$	$\mu_{\mathcal{S}_2}(b)$	$\mu_{\mathcal{S}_3}(c)$	$\min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\}$
$-i$	$-i$	$-i$	0.5	1	0.8	0.5
$-i$	i	i	0.5	0	0.9	0
i	$-i$	i	0.5	1	0.9	0.5
i	i	$-i$	0.5	0	0.8	0

a	b	c	$\eta_{\mathcal{S}_1}(a)$	$\eta_{\mathcal{S}_2}(b)$	$\eta_{\mathcal{S}_3}(c)$	$\min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\}$
$-i$	$-i$	$-i$	0	0	0.4	0
$-i$	i	i	0	0	0.1	0
i	$-i$	i	0	0	0.1	0
i	i	$-i$	0	0	0.4	0

a	b	c	$\nu_{\mathcal{S}_1}(a)$	$\nu_{\mathcal{S}_2}(b)$	$\nu_{\mathcal{S}_3}(c)$	$\max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\}$
$-i$	$-i$	$-i$	0.5	0	0.2	0.5
$-i$	i	i	0.5	1	0.4	1
i	$-i$	i	0.5	0	0.4	0.5
i	i	$-i$	0.5	1	0.2	1

That is

$$\begin{aligned}
(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(i) &= \sup_{i=abc} \min\{\mu_{\mathcal{S}_1}(a), \mu_{\mathcal{S}_2}(b), \mu_{\mathcal{S}_3}(c)\} = 0.5, \\
(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(i) &= \sup_{i=abc} \min\{\eta_{\mathcal{S}_1}(a), \eta_{\mathcal{S}_2}(b), \eta_{\mathcal{S}_3}(c)\} = 0, \\
(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(i) &= \inf_{i=abc} \max\{\nu_{\mathcal{S}_1}(a), \nu_{\mathcal{S}_2}(b), \nu_{\mathcal{S}_3}(c)\} = 0.5.
\end{aligned}$$

The shows of $(\mu_{\mathcal{S}_1} \circ \mu_{\mathcal{S}_2} \circ \mu_{\mathcal{S}_3})(-i)$, $(\eta_{\mathcal{S}_1} \circ \eta_{\mathcal{S}_2} \circ \eta_{\mathcal{S}_3})(-i)$ and $(\nu_{\mathcal{S}_1} \circ \nu_{\mathcal{S}_2} \circ \nu_{\mathcal{S}_3})(-i)$ are similar to the previous one.

Therefore, $\mathcal{S}_1 \circ \mathcal{S}_2 \circ \mathcal{S}_3 = \{< i, 0.5, 0, 0.5 >, < -i, 0.5, 0, 0.5 >\}$ is a product of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, as desired.

Theorem 3.1.8. *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup T. Then $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of T if and only if $\mathcal{S} \circ \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$.*

Proof. Assume that $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of T. Let $x \in T$. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\} \leq \sup_{x=abc} \mu_{\mathcal{S}}(abc) = \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\} \leq \sup_{x=abc} \eta_{\mathcal{S}}(abc) = \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\} \geq \inf_{x=abc} \nu_{\mathcal{S}}(abc) = \nu_{\mathcal{S}}(x).$$

Hence, $\mathcal{S} \circ \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$.

Conversely, let $a, b, c \in T$.

$$\begin{aligned}
\mu_{\mathcal{S}}(abc) &\geq (\mu_{\mathcal{S}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{S}})(abc) \\
&= \sup_{abc=x_1x_2x_3} \min\{\mu_{\mathcal{S}}(x_1), \mu_{\mathcal{S}}(x_2), \mu_{\mathcal{S}}(x_3)\} \\
&\geq \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{S}}(c)\},
\end{aligned}$$

$$\begin{aligned}
\eta_{\mathcal{S}}(abc) &\geq (\eta_{\mathcal{S}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{S}})(abc) \\
&= \sup_{abc=x_1x_2x_3} \min\{\eta_{\mathcal{S}}(x_1), \eta_{\mathcal{S}}(x_2), \eta_{\mathcal{S}}(x_3)\} \\
&\geq \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{S}}(c)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\mathcal{S}}(abc) &\leq (\nu_{\mathcal{S}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{S}})(abc) \\
&= \inf_{abc=x_1x_2x_3} \max\{\nu_{\mathcal{S}}(x_1), \nu_{\mathcal{S}}(x_2), \nu_{\mathcal{S}}(x_3)\} \\
&\leq \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{S}}(c)\}.
\end{aligned}$$

This implies that $\mathcal{S}$ is a spherical fuzzy ternary subsemigroup of T. □

Let $\mathcal{T} := (\mu_{\mathcal{T}}, \eta_{\mathcal{T}}, \nu_{\mathcal{T}})$ be a spherical fuzzy set on a ternary semigroup T defined by $\mu_{\mathcal{T}}(x) = 1$ and $\eta_{\mathcal{T}}(x) = \nu_{\mathcal{T}}(x) = 0$ for all $x \in T$. The following theorem holds.

Theorem 3.1.9. *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy left ideal of T, then $\mathcal{T} \circ \mathcal{T} \circ \mathcal{S} \subseteq \mathcal{S}$.*

Proof. Assume that $\mathcal{S}$ is a spherical fuzzy left ideal of T. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{S}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{S}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{S}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{T}}(a), \mu_{\mathcal{T}}(b), \mu_{\mathcal{S}}(c)\} = \sup_{x=abc} \mu_{\mathcal{S}}(c) \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{S}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{T}}(a), \eta_{\mathcal{T}}(b), \eta_{\mathcal{S}}(c)\} = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{S}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{T}}(a), \nu_{\mathcal{T}}(b), \nu_{\mathcal{S}}(c)\} = \inf_{x=abc} \nu_{\mathcal{S}}(c) \geq \nu_{\mathcal{S}}(x).$$

Hence, $\mathcal{T} \circ \mathcal{T} \circ \mathcal{S} \subseteq \mathcal{S}$. □

Theorem 3.1.10. *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy lateral ideal of T, then $\mathcal{T} \circ \mathcal{S} \circ \mathcal{T} \subseteq \mathcal{S}$.*

Proof. Assume that $\mathcal{S}$ is a spherical fuzzy lateral ideal of T. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{T}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{T}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{T}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{T}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{T}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{T}}(a), \mu_{\mathcal{S}}(b), \mu_{\mathcal{T}}(c)\} = \sup_{x=abc} \mu_{\mathcal{S}}(b) \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{T}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{T}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{T}}(a), \eta_{\mathcal{S}}(b), \eta_{\mathcal{T}}(c)\} = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{T}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{T}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{T}}(a), \nu_{\mathcal{S}}(b), \nu_{\mathcal{T}}(c)\} = \inf_{x=abc} \nu_{\mathcal{S}}(b) \geq \nu_{\mathcal{S}}(x).$$

Hence, $\mathcal{T} \circ \mathcal{S} \circ \mathcal{T} \subseteq \mathcal{S}$. □

Theorem 3.1.11. *Let $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ be a spherical fuzzy set on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy right ideal of T, then $\mathcal{S} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{S}$.*

Proof. Assume that $\mathcal{S}$ is a spherical fuzzy right ideal of T. If $x \notin T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{T}})(x) = 0 \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{T}})(x) = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{T}})(x) = 1 \geq \nu_{\mathcal{S}}(x).$$

Now, assume that $x \in T^3$, we obtain that

$$(\mu_{\mathcal{S}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{T}})(x) = \sup_{x=abc} \min\{\mu_{\mathcal{S}}(a), \mu_{\mathcal{T}}(b), \mu_{\mathcal{T}}(c)\} = \sup_{x=abc} \mu_{\mathcal{S}}(a) \leq \mu_{\mathcal{S}}(x),$$

$$(\eta_{\mathcal{S}} \circ \eta_{\mathcal{T}} \circ \eta_{\mathcal{T}})(x) = \sup_{x=abc} \min\{\eta_{\mathcal{S}}(a), \eta_{\mathcal{T}}(b), \eta_{\mathcal{T}}(c)\} = 0 \leq \eta_{\mathcal{S}}(x)$$

and

$$(\nu_{\mathcal{S}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{T}})(x) = \inf_{x=abc} \max\{\nu_{\mathcal{S}}(a), \nu_{\mathcal{T}}(b), \nu_{\mathcal{T}}(c)\} = \inf_{x=abc} \nu_{\mathcal{S}}(a) \geq \nu_{\mathcal{S}}(x).$$

Hence, $\mathcal{S} \circ \mathcal{T} \circ \mathcal{T} \subseteq \mathcal{S}$. □

3.2 Rough spherical fuzzy sets in ternary semigroups

The aims of this section is to connect rough set theory and spherical fuzzy sets of ternary semigroups.

Definition 3.2.1. An equivalence relation ρ on a ternary semigroup T is called a *congruence* if for all $x_1, x_2, x_3, y_1, y_2, y_3 \in T$

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \rho \Rightarrow (x_1x_2x_3, y_1y_2y_3) \in \rho.$$

For $x \in T$, the ρ-congruence class containing x is denoted by $[x]_\rho$.

Definition 3.2.2. A congruence ρ on T is called *complete* if

$$[y_1]_\rho[y_2]_\rho[y_3]_\rho = [y_1y_2y_3]_\rho$$

for all $y_1, y_2, y_3 \in T$.

Definition 3.2.3. Let ρ be a congruence on a ternary semigroup T and $\mathcal{S} = (\mu_\mathcal{S}, \eta_\mathcal{S}, \nu_\mathcal{S})$ be the spherical fuzzy set on a ternary semigroup T.

(1) The *lower approximation* is defined as

$$\underline{App}(\mathcal{S}) = \{< y, \underline{\mu_\mathcal{S}}(y), \underline{\eta_\mathcal{S}}(y), \underline{\nu_\mathcal{S}}(y) >\mid y \in T\},$$

where $\underline{\mu_\mathcal{S}}(y) = \inf\limits_{y' \in [y]_\rho} \mu_\mathcal{S}(y'), \underline{\eta_\mathcal{S}}(y) = \inf\limits_{y' \in [y]_\rho} \eta_\mathcal{S}(y')$ and $\underline{\nu_\mathcal{S}}(y) = \sup\limits_{y' \in [y]_\rho} \nu_\mathcal{S}(y')$.

(2) The *upper approximation* is defined as

$$\overline{App}(\mathcal{S}) = \{< y, \overline{\mu_\mathcal{S}}(y), \overline{\eta_\mathcal{S}}(y), \overline{\nu_\mathcal{S}}(y) >\mid y \in T\},$$

where $\overline{\mu_\mathcal{S}}(y) = \sup\limits_{y' \in [y]_\rho} \mu_\mathcal{S}(y'), \overline{\eta_\mathcal{S}}(y) = \sup\limits_{y' \in [y]_\rho} \eta_\mathcal{S}(y')$ and $\overline{\nu_\mathcal{S}}(y) = \inf\limits_{y' \in [y]_\rho} \nu_\mathcal{S}(y')$.

(3) The *rough spherical fuzzy set* of T is defined by

$$App(\mathcal{S}) = (\underline{App}(\mathcal{S}), \overline{App}(\mathcal{S})).$$

Example 3.2.4. Let ρ be a congruence relation on a ternary semigroup $\mathbb{Z}^-$ under usual multiplication defined by

$$(x, y) \in \rho \text{ if and only if } 2 \mid (x - y)$$

for all $x, y \in \mathbb{Z}^-$.

Let $\mu_{\mathcal{S}}(y) = \dfrac{1}{y^2}$, $\eta_{\mathcal{S}}(y) = 0$ and $\nu_{\mathcal{S}}(y) = 1 - \dfrac{1}{y^2}$ for all $y \in \mathbb{Z}^-$.

Then

$$0 \leq (\mu_{\mathcal{S}}(y))^2 + (\eta_{\mathcal{S}}(y))^2 + (\nu_{\mathcal{S}}(y))^2 = \left(\frac{1}{y^2}\right)^2 + \left(1 - \frac{1}{y^2}\right)^2 \leq \frac{1}{y^2} + \left(1 - \frac{1}{y^2}\right) = 1$$

for all $y \in \mathbb{Z}^-$, this implies that $\mathcal{S} = (\mu_{\mathcal{S}}, \eta_{\mathcal{S}}, \nu_{\mathcal{S}})$ is a spherical fuzzy set on $\mathbb{Z}^-$. Thus we obtain that

$\underline{\mu_{\mathcal{S}}}(-1) = \inf\limits_{y' \in [-1]_\rho} \mu_{\mathcal{S}}(y') = 0$, $\underline{\mu_{\mathcal{S}}}(-2) = \inf\limits_{y' \in [-2]_\rho} \mu_{\mathcal{S}}(y') = 0$,
$\underline{\eta_{\mathcal{S}}}(-1) = \inf\limits_{y' \in [-1]_\rho} \eta_{\mathcal{S}}(y') = 0$, $\underline{\eta_{\mathcal{S}}}(-2) = \inf\limits_{y' \in [-2]_\rho} \eta_{\mathcal{S}}(y') = 0$,
$\underline{\nu_{\mathcal{S}}}(-1) = \inf\limits_{y' \in [-1]_\rho} \nu_{\mathcal{S}}(y') = 1$, $\underline{\nu_{\mathcal{S}}}(-2) = \inf\limits_{y' \in [-2]_\rho} \nu_{\mathcal{S}}(y') = 1$.

Hence,

$$\underline{App}(\mathcal{S}) = \{< y, \underline{\mu_{\mathcal{S}}}(y), \underline{\eta_{\mathcal{S}}}(y), \underline{\nu_{\mathcal{S}}}(y) >|\ y \in \mathbb{Z}^-\} = \{< y, 0, 0, 1 >|\ y \in \mathbb{Z}^-\}$$

and

$\overline{\mu_{\mathcal{S}}}(-1) = \sup\limits_{y' \in [-1]_\rho} \mu_{\mathcal{S}}(y') = 1$, $\overline{\mu_{\mathcal{S}}}(-2) = \sup\limits_{y' \in [-2]_\rho} \mu_{\mathcal{S}}(y') = 0.25$,
$\overline{\eta_{\mathcal{S}}}(-1) = \sup\limits_{y' \in [-1]_\rho} \eta_{\mathcal{S}}(y') = 0$, $\overline{\eta_{\mathcal{S}}}(-2) = \sup\limits_{y' \in [-2]_\rho} \eta_{\mathcal{S}}(y') = 0$,
$\overline{\nu_{\mathcal{S}}}(-1) = \sup\limits_{y' \in [-1]_\rho} \nu_{\mathcal{S}}(y') = 0$, $\overline{\nu_{\mathcal{S}}}(-2) = \sup\limits_{y' \in [-2]_\rho} \nu_{\mathcal{S}}(y') = 0.75$.

Hence,

$$\begin{aligned} \overline{App}(\mathcal{S}) &= \{< y, \overline{\mu_{\mathcal{S}}}(y), \overline{\eta_{\mathcal{S}}}(y), \overline{\nu_{\mathcal{S}}}(y) >|\ y \in \mathbb{Z}^-\} \\ &= \{< y, 1, 0, 0 >|\ y \text{ is odd}\} \cup \{< y, 0.25, 0, 0.75 >|\ y \text{ is even}\}. \end{aligned}$$

Theorem 3.2.5. *Let ρ be a congruence on a ternary semigroup T and $\mathcal{S}_1 = (\mu_{\mathcal{S}_1}, \eta_{\mathcal{S}_1}, \nu_{\mathcal{S}_1})$ and $\mathcal{S}_2 = (\mu_{\mathcal{S}_2}, \eta_{\mathcal{S}_2}, \nu_{\mathcal{S}_2})$ be any two spherical fuzzy sets on T. The following statements hold.*

(1) If $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\overline{App}(\mathcal{S}_1) \subseteq \overline{App}(\mathcal{S}_2)$ and $\underline{App}(\mathcal{S}_1) \subseteq \underline{App}(\mathcal{S}_2)$.

(2) $\overline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) \subseteq \overline{App}(\mathcal{S}_1) \cap \overline{App}(\mathcal{S}_2)$.

(3) $\overline{App}(\mathcal{S}_1 \cup \mathcal{S}_2) = \overline{App}(\mathcal{S}_1) \cup \overline{App}(\mathcal{S}_2)$.

(4) $\underline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) = \underline{App}(\mathcal{S}_1) \cap \underline{App}(\mathcal{S}_2)$.

(5) $\underline{App}(\mathcal{S}_1) \cup \underline{App}(\mathcal{S}_2) \subseteq \underline{App}(\mathcal{S}_1 \cup \mathcal{S}_2)$.

Proof. (1) Assume that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then $\mu_{\mathcal{S}_1}(x) \leq \mu_{\mathcal{S}_2}(x)$, $\eta_{\mathcal{S}_2}(x) \leq \eta_{\mathcal{S}_1}(x)$ and $\nu_{\mathcal{S}_2}(x) \geq \nu_{\mathcal{S}_1}(x)$ for all $x \in T$. Thus for all $y \in T$, we have

$$\overline{\mu_{\mathcal{S}_1}}(y) = \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y') \leq \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y') = \overline{\mu_{\mathcal{S}_2}}(y),$$

$$\overline{\eta_{\mathcal{S}_1}}(y) = \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y') \leq \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y') = \overline{\eta_{\mathcal{S}_2}}(y)$$

and

$$\overline{\nu_{\mathcal{S}_1}}(y) = \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y') \geq \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y') = \overline{\nu_{\mathcal{S}_2}}(y).$$

This implies that $\overline{App}(\mathcal{S}_1) \subseteq \overline{App}(\mathcal{S}_2)$. Similarly, we have $\underline{App}(\mathcal{S}_1) \subseteq \underline{App}(\mathcal{S}_2)$.

(2) Since $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}_1$ and $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}_2$, by (1) we obtain that $\overline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) \subseteq \overline{App}(\mathcal{S}_1) \cap \overline{App}(\mathcal{S}_2)$.

(3) Note that

$$\overline{App}(\mathcal{S}_1) \cup \overline{App}(\mathcal{S}_2) = (\overline{\mu_{\mathcal{S}_1}} \cup \overline{\mu_{\mathcal{S}_2}}, \overline{\eta_{\mathcal{S}_1}} \cup \overline{\eta_{\mathcal{S}_2}}, \overline{\nu_{\mathcal{S}_1}} \cap \overline{\nu_{\mathcal{S}_2}})$$

and

$$\overline{App}(\mathcal{S}_1 \cup \mathcal{S}_2) = (\overline{\mu_{\mathcal{S}_1 \cup \mathcal{S}_2}}, \overline{\eta_{\mathcal{S}_1 \cup \mathcal{S}_2}}, \overline{\nu_{\mathcal{S}_1 \cup \mathcal{S}_2}}).$$

Let $y \in T$. Then

$$\begin{aligned}
(\overline{\mu_{\mathcal{S}_1}} \cup \overline{\mu_{\mathcal{S}_2}})(y) &= \max\{\overline{\mu_{\mathcal{S}_1}}(y), \overline{\mu_{\mathcal{S}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\mu_{\mathcal{S}_1}(y'), \mu_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \mu_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\mu_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y),
\end{aligned}$$

$$\begin{aligned}
(\overline{\eta_{\mathcal{S}_1}} \cup \overline{\eta_{\mathcal{S}_2}})(y) &= \max\{\overline{\eta_{\mathcal{S}_1}}(y), \overline{\eta_{\mathcal{S}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\eta_{\mathcal{S}_1}(y'), \eta_{\mathcal{S}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \eta_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\eta_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y)
\end{aligned}$$

and

$$
\begin{aligned}
(\overline{\nu_{\mathcal{S}_1}} \cap \overline{\nu_{\mathcal{S}_2}})(y) &= \min\{\overline{\nu_{\mathcal{S}_1}}(y), \overline{\nu_{\mathcal{S}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\nu_{\mathcal{S}_1}(y'), \nu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \nu_{\mathcal{S}_1 \cup \mathcal{S}_2}(y') \\
&= \overline{\nu_{\mathcal{S}_1 \cup \mathcal{S}_2}}(y).
\end{aligned}
$$

(4) Note that

$$
\underline{App}(\mathcal{S}_1) \cap \underline{App}(\mathcal{S}_2) = (\underline{\mu_{\mathcal{S}_1}} \cap \underline{\mu_{\mathcal{S}_2}}, \underline{\eta_{\mathcal{S}_1}} \cap \underline{\eta_{\mathcal{S}_2}}, \underline{\nu_{\mathcal{S}_1}} \cup \underline{\nu_{\mathcal{S}_2}})
$$

and

$$
\underline{App}(\mathcal{S}_1 \cap \mathcal{S}_2) = (\underline{\mu_{\mathcal{S}_1 \cap \mathcal{S}_2}}, \underline{\eta_{\mathcal{S}_1 \cap \mathcal{S}_2}}, \underline{\nu_{\mathcal{S}_1 \cap \mathcal{S}_2}}).
$$

Let $y \in T$. Then

$$
\begin{aligned}
(\underline{\mu_{\mathcal{S}_1}} \cap \underline{\mu_{\mathcal{S}_2}})(y) &= \min\{\underline{\mu_{\mathcal{S}_1}}(y), \underline{\mu_{\mathcal{S}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\mu_{\mathcal{S}_1}(y'), \mu_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \mu_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\
&= \underline{\mu_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y),
\end{aligned}
$$

$$
\begin{aligned}
(\underline{\eta_{\mathcal{S}_1}} \cap \underline{\eta_{\mathcal{S}_2}})(y) &= \min\{\underline{\eta_{\mathcal{S}_1}}(y), \underline{\eta_{\mathcal{S}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_1}(y'), \inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\eta_{\mathcal{S}_1}(y'), \eta_{\mathcal{S}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \eta_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\
&= \underline{\eta_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y)
\end{aligned}
$$

and

$$\begin{aligned}(\underline{\nu_{\mathcal{S}_1}} \cup \underline{\nu_{\mathcal{S}_2}})(y) &= \max\{\underline{\nu_{\mathcal{S}_1}}(y), \underline{\nu_{\mathcal{S}_2}}(y)\} \\ &= \max\{\sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_1}(y'), \sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_2}(y')\} \\ &= \sup_{y' \in [y]_\rho} \max\{\nu_{\mathcal{S}_1}(y'), \nu_{\mathcal{S}_2}(y')\} \\ &= \sup_{y' \in [y]_\rho} \nu_{\mathcal{S}_1 \cap \mathcal{S}_2}(y') \\ &= \underline{\nu_{\mathcal{S}_1 \cap \mathcal{S}_2}}(y).\end{aligned}$$

(5) Since $\mathcal{S}_1 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$, by (1) we obtain that $\underline{App}(\mathcal{S}_1) \cup \underline{App}(\mathcal{S}_2) \subseteq \underline{App}(\mathcal{S}_1 \cup \mathcal{S}_2)$. □

Theorem 3.2.6. *Let ρ be a congruence relation on a ternary semigroup T and $\mathcal{S}$ be a spherical fuzzy set on T. Then $\underline{App}(\mathcal{S})$ is also a spherical fuzzy set on T.*

Proof. Let $y \in T$. Then

$$\begin{aligned}&(\underline{\mu_{\mathcal{S}}}(y))^2 + (\underline{\eta_{\mathcal{S}}}(y))^2 + (\underline{\nu_{\mathcal{S}}}(y))^2 \\ &= (\inf_{y' \in [y]_\rho} \mu_{\mathcal{S}}(y'))^2 + (\inf_{y' \in [y]_\rho} \eta_{\mathcal{S}}(y'))^2 + (\sup_{y' \in [y]_\rho} \nu_{\mathcal{S}}(y'))^2 \\ &= \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + \sup_{y' \in [y]_\rho} (\nu_{\mathcal{S}}(y'))^2 \\ &\le \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + \sup_{y' \in [y]_\rho} (1 - (\mu_{\mathcal{S}}(y'))^2 - (\eta_{\mathcal{S}}(y'))^2) \\ &\le \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 + \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 + 1 - \inf_{y' \in [y]_\rho} (\mu_{\mathcal{S}}(y'))^2 - \inf_{y' \in [y]_\rho} (\eta_{\mathcal{S}}(y'))^2 = 1.\end{aligned}$$

This implies that $0 \le (\underline{\mu_{\mathcal{S}}}(y))^2 + (\underline{\eta_{\mathcal{S}}}(y))^2 + (\underline{\nu_{\mathcal{S}}}(y))^2 \le 1$. Therefore, $\underline{App}(\mathcal{S})$ is a spherical fuzzy set on T. □

Let $\mathcal{S}$ be a spherical fuzzy set on a ternary semigroup T. Note that $\overline{App}(\mathcal{S})$ need not be a spherical fuzzy set on T, as can be seen in the following example.

Example 3.2.7. Let $T = \{i, -i\}$ be the ternary semigroup under the ternary multiplication, $\rho = T \times T$ and $\mathcal{S}$ be a spherical fuzzy set on T defined by

$$\mu_{\mathcal{S}}(i) = 1, \eta_{\mathcal{S}}(i) = 0, \nu_{\mathcal{S}}(i) = 0 \text{ and } \mu_{\mathcal{S}}(-i) = 0, \eta_{\mathcal{S}}(-i) = 1, \nu_{\mathcal{S}}(-i) = 0.$$

Then

$$\overline{\mu_{\mathcal{S}}}(i) = \overline{\mu_{\mathcal{S}}}(-i) = 1, \overline{\eta_{\mathcal{S}}}(i) = \overline{\eta_{\mathcal{S}}}(-i) = 1, \overline{\nu_{\mathcal{S}}}(i) = \overline{\nu_{\mathcal{S}}}(-i) = 0.$$

In this example, we have that $\overline{App}(\mathcal{S})$ is not a spherical fuzzy set on T.

3.3 Rough spherical fuzzy ideals in ternary semigroups

The aims of this section is to connect rough set theory and spherical fuzzy ideals of ternary semigroups.

Theorem 3.3.1. *Let ρ be a complete congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy left ideal of T, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}
\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y) \\
&= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc) \\
&\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(c) = \inf_{c\in[y_3]_\rho} \mu_{\mathcal{S}}(c) = \underline{\mu_{\mathcal{S}}}(y_3),
\end{aligned}$$

$$\begin{aligned}
\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y) \\
&= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc) \\
&\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(c) = \inf_{c\in[y_3]_\rho} \eta_{\mathcal{S}}(c) = \underline{\eta_{\mathcal{S}}}(y_3)
\end{aligned}$$

and

$$\begin{aligned}
\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y) \\
&= \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc) \\
&\leq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(c) = \sup_{c\in[y_3]_\rho} \nu_{\mathcal{S}}(c) = \underline{\nu_{\mathcal{S}}}(y_3).
\end{aligned}$$

This implies that

$$\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\mu_{\mathcal{S}}}(y_3),\ \underline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\eta_{\mathcal{S}}}(y_3) \text{ and } \underline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \underline{\nu_{\mathcal{S}}}(y_3).$$

Then $\underline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of T. □

Theorem 3.3.2. *Let ρ be a complete congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy lateral ideal of T, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy lateral ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y)\\ &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc)\\ &\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(b) = \inf_{b\in[y_2]_\rho} \mu_{\mathcal{S}}(b) = \underline{\mu_{\mathcal{S}}}(y_2),\end{aligned}$$

$$\begin{aligned}\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(b) = \inf_{b\in[y_2]_\rho} \eta_{\mathcal{S}}(b) = \underline{\eta_{\mathcal{S}}}(y_2)\end{aligned}$$

and

$$\begin{aligned}\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &= \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(b) = \sup_{b\in[y_2]_\rho} \nu_{\mathcal{S}}(b) = \underline{\nu_{\mathcal{S}}}(y_2).\end{aligned}$$

This implies that

$$\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\mu_{\mathcal{S}}}(y_2),\ \underline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\eta_{\mathcal{S}}}(y_2) \text{ and } \underline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \underline{\nu_{\mathcal{S}}}(y_2).$$

Then $\underline{App}(\mathcal{S})$ is a spherical fuzzy lateral ideal of T. □

Theorem 3.3.3. *Let ρ be a complete congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy right ideal of T, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy right ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y)\\ &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc)\\ &\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(a) = \inf_{a\in[y_1]_\rho} \mu_{\mathcal{S}}(a) = \underline{\mu_{\mathcal{S}}}(y_1),\end{aligned}$$

$$\begin{aligned}\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(a) = \inf_{a\in[y_1]_\rho} \eta_{\mathcal{S}}(a) = \underline{\eta_{\mathcal{S}}}(y_1)\end{aligned}$$

and

$$\begin{aligned}\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &= \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(a) = \sup_{a\in[y_1]_\rho} \nu_{\mathcal{S}}(a) = \underline{\nu_{\mathcal{S}}}(y_1).\end{aligned}$$

This implies that

$$\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\mu_{\mathcal{S}}}(y_1),\ \underline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \underline{\eta_{\mathcal{S}}}(y_1) \text{ and } \underline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \underline{\nu_{\mathcal{S}}}(y_1).$$

Then $\underline{App}(\mathcal{S})$ is a spherical fuzzy right ideal of T. □

Corollary 3.3.4. *Let ρ be a complete congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy ideal of T, then $\underline{App}(\mathcal{S})$ is a spherical fuzzy ideal of T.*

Proof. This follows from Theorem 3.3.1 – 3.3.3, we obtain

$$\underline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \max\{\underline{\mu_{\mathcal{S}}}(y_1), \underline{\mu_{\mathcal{S}}}(y_2), \underline{\mu_{\mathcal{S}}}(y_3)\},$$

$$\underline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \max\{\underline{\eta_{\mathcal{S}}}(y_1), \underline{\eta_{\mathcal{S}}}(y_2), \underline{\eta_{\mathcal{S}}}(y_3)\},$$

and

$$\underline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \min\{\underline{\nu_{\mathcal{S}}}(y_1), \underline{\nu_{\mathcal{S}}}(y_2), \underline{\nu_{\mathcal{S}}}(y_3)\}.$$

Therefore, $\underline{App}(\mathcal{S})$ is a spherical fuzzy ideal of T. □

Theorem 3.3.5. *Let ρ be a congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy left ideal of T and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of T, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(c) = \sup_{c\in[y_3]_\rho} \mu_{\mathcal{S}}(c) = \overline{\mu_{\mathcal{S}}}(y_3),\end{aligned}$$

$$\begin{aligned}\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(c) = \sup_{c\in[y_3]_\rho} \eta_{\mathcal{S}}(c) = \overline{\eta_{\mathcal{S}}}(y_3)\end{aligned}$$

and

$$\begin{aligned}\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &\leq \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \inf_{c\in[y_3]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(c) = \inf_{c\in[y_3]_\rho} \nu_{\mathcal{S}}(c) = \overline{\nu_{\mathcal{S}}}(y_3).\end{aligned}$$

This implies that

$$\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\mu_{\mathcal{S}}}(y_3),\ \overline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\eta_{\mathcal{S}}}(y_3) \text{ and } \overline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \overline{\nu_{\mathcal{S}}}(y_3).$$

Then $\overline{App}(\mathcal{S})$ is a spherical fuzzy left ideal of T. □

Theorem 3.3.6. *Let ρ be a congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy lateral ideal of T and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of T, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy lateral ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(b) = \sup_{b\in[y_2]_\rho} \mu_{\mathcal{S}}(b) = \overline{\mu_{\mathcal{S}}}(y_2),\end{aligned}$$

$$\begin{aligned}\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y)\\ &\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc)\\ &\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(b) = \sup_{b\in[y_2]_\rho} \eta_{\mathcal{S}}(b) = \overline{\eta_{\mathcal{S}}}(y_2)\end{aligned}$$

and

$$\begin{aligned}\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y)\\ &\leq \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc)\\ &\leq \inf_{c\in[y_3]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(b) = \inf_{b\in[y_2]_\rho} \nu_{\mathcal{S}}(b) = \overline{\nu_{\mathcal{S}}}(y_2).\end{aligned}$$

This implies that

$$\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\mu_{\mathcal{S}}}(y_2),\ \overline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\eta_{\mathcal{S}}}(y_2) \text{ and } \overline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \overline{\nu_{\mathcal{S}}}(y_2).$$

Then $\overline{App}(\mathcal{S})$ is a spherical fuzzy lateral ideal of T. □

Theorem 3.3.7. *Let ρ be a congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy right ideal of T and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of T, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy right ideal of T.*

Proof. Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}
\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{S}}(y) \\
&\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(abc) \\
&\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{S}}(a) = \sup_{a\in[y_1]_\rho} \mu_{\mathcal{S}}(a) = \overline{\mu_{\mathcal{S}}}(y_1),
\end{aligned}$$

$$\begin{aligned}
\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) &= \sup_{y\in[y_1y_2y_3]_\rho} \eta_{\mathcal{S}}(y) \\
&\geq \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(y) = \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(abc) \\
&\geq \sup_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \eta_{\mathcal{S}}(a) = \sup_{a\in[y_1]_\rho} \eta_{\mathcal{S}}(a) = \overline{\eta_{\mathcal{S}}}(y_1)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) &= \inf_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{S}}(y) \\
&\leq \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(y) = \inf_{abc\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(abc) \\
&\leq \inf_{c\in[y_3]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{S}}(a) = \inf_{a\in[y_1]_\rho} \nu_{\mathcal{S}}(a) = \overline{\nu_{\mathcal{S}}}(y_1).
\end{aligned}$$

This implies that

$$\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\mu_{\mathcal{S}}}(y_1),\ \overline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \overline{\eta_{\mathcal{S}}}(y_1) \text{ and } \overline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \overline{\nu_{\mathcal{S}}}(y_1).$$

Then $\overline{App}(\mathcal{S})$ is a spherical fuzzy right ideal of T. □

Corollary 3.3.8. *Let ρ be a congruence relation on a ternary semigroup T. If $\mathcal{S}$ is a spherical fuzzy ideal of T and $\overline{App}(\mathcal{S})$ is a spherical fuzzy set of T, then $\overline{App}(\mathcal{S})$ is a spherical fuzzy ideal of T.*

Proof. This follows from Theorem 3.3.5 – 3.3.7, we obtain

$$\overline{\mu_{\mathcal{S}}}(y_1y_2y_3) \geq \max\{\overline{\mu_{\mathcal{S}}}(y_1), \overline{\mu_{\mathcal{S}}}(y_2), \overline{\mu_{\mathcal{S}}}(y_3)\},$$

$$\overline{\eta_{\mathcal{S}}}(y_1y_2y_3) \geq \max\{\overline{\eta_{\mathcal{S}}}(y_1), \overline{\eta_{\mathcal{S}}}(y_2), \overline{\eta_{\mathcal{S}}}(y_3)\},$$

and

$$\overline{\nu_{\mathcal{S}}}(y_1y_2y_3) \leq \min\{\overline{\nu_{\mathcal{S}}}(y_1), \overline{\nu_{\mathcal{S}}}(y_2), \overline{\nu_{\mathcal{S}}}(y_3)\}.$$

Therefore, $\overline{App}(\mathcal{S})$ is a spherical fuzzy ideal of T. □

Chapter 4

Conclusions

A *ternary semigroups* is an algebraic structure $(T, (\cdot))$ such that T is a non-empty set and $(\cdot)\colon T^3 \to T$ is a ternary operation satisfying the associative law, i.e., $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in T$, and let $\mathcal{S}$ be a *spherical fuzzy subset* of a universal set S defined by

$$\mathcal{S} := \{< x, \mu_{\mathcal{S}}(x), \eta_{\mathcal{S}}(x), \nu_{\mathcal{S}}(x) >| \; x \in S\}$$

where $\mu_{\mathcal{S}}$, $\eta_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ be three fuzzy subsets of S with the condition $0 \leq (\mu_{\mathcal{S}}(x))^2 + (\eta_{\mathcal{S}}(x))^2 + (\nu_{\mathcal{S}}(x))^2 \leq 1$. Then $\mu_{\mathcal{S}}(x)$, $\mu_{\mathcal{S}}(x)$ and $\mu_{\mathcal{S}}(x)$ are called the *degree of membership*, the *degree of hesitancy* and the *degree of non-membership*, respectively.

In Chapter 3, we define spherical fuzzy ternary subsemigroups and spherical fuzzy ideals in ternary semigroups. It is shown that the spherical fuzzy set $\mathcal{S}$ on a ternary semigroup T is a spherical fuzzy ternary subsemigroup if and only if $\mathcal{S} \circ \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$.

Furthermore, we study the relationship between rough set theory and spherical fuzzy sets of ternary semigroups. We obtain that if ρ is a congruence relation on a ternary semigroup T and $\mathcal{S}$ is a spherical fuzzy set on T, then the lower approximation is also a spherical fuzzy set on T. However, under the same assumption the upper approximation need not be a spherical fuzzy set on T. In addition, when we study the relationship between rough set theory and spherical fuzzy ideals of ternary semigroups, we obtain that if $\mathcal{S}$ is a spherical fuzzy ideal [spherical left ideal, spherical lateral ideal, spherical right ideal] of T, then so are the lower and upper approximations.

Bibliography

[1] Ashraf, S., Abdullah, T., Mahmood, T., Gahni, F., and Mahmood, T. 2019. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *Journal of Intelligent & Fuzzy Systems*, 36(3), 2829-2844.

[2] Chinram, R., and Panityakul, T. 2020. Rough Pythagorean fuzzy ideals in ternary semigroups. *Journal of Mathematics and Computer Science*, 20(4), 302-312.

[3] Dubois, D., and Prade, H. 1990. Rough fuzzy sets and fuzzy rough sets. *International Journal of General Systems*, 17(2), 191-209.

[4] Gündogku, F.K., and Kahraman, C. 2019. Spherical fuzzy sets and spherical fuzzy TOPSIS method. *Journal of Intelligent & Fuzzy Systems*, 36(1), 337-352.

[5] Iampan, A. 2013. Some properties of ideal extensions in ternary semigroups. *Iranian Journal of Mathematical Sciences and Informatics*, 8(1), 67-74.

[6] Kasner, E. 1904. An extension of the group concept. *Bulletin of the American Mathematical Society*, 10, 290-291.

[7] Kuroki, N. 1981. On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets and Systems*, 5(2), 203-215.

[8] Kar, S., and Sarkar, P. 2012. Fuzzy ideals of ternary semigroups. *Fuzzy Information and Engineering*, 4(2), 181-193.

[9] Lehmer, D.H. 1932. A ternary analogue of Abelian groups. *American Journal of Mathematics*, 54(2), 329-338.

[10] Los, J. 1955. On the extending of models I. *Fundamenta Mathematicae*, 42(1), 38-54.

[11] Pawlak, Z. 1982. Rough sets. *International Journal of Computer and Information Sciences*, 11(5), 341-356.

[12] Petchkhaew, P., and Chinram, R. 2009. Fuzzy, rough and rough fuzzy ideals in ternary semigroups. *International Journal of Pure and Applied Mathematics* , 56(1), 21-36.

[13] Siosan, F.M. 1965. Ideal theory in ternary semigroups. *Mathematica Japonica*, 10, 63-84.

[14] Wang, Q. and Zhan, J. 2016. Rough semigroups and rough fuzzy semigroups based on fuzzy ideals. *Open Mathematics*, 10(1), 1114-1121.

[15] Zadeh, L.A. 1965. Fuzzy sets. *Information and Control*, 8(3), 338-353.

[16] Zeng, S., Hussain, A., Mahmood, T., Ali, M.I., Ashraf, S., and Munir, M. 2019. Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-making. *Symmetry*, 11(4), 547.

VITAE

Name Mister Wasitthirawat Krailoet

Student ID 6310220035

Educational Attainment

Degree	Name of Institution	Year of Graduation
Bachelor of Science (Mathematics)	Prince of Songkla University	2020

List of Publications and Proceeding

Krailoet, W., Chinram, R., Petapirak, M., and Iampan, A. 2022. Applications of Spherical Fuzzy Sets in Ternary Semigroups. *International Journal of Analysis and Applications*, 20(1), 29.