A Novel of Ideals and Fuzzy Ideals of Γ-Semigroups

Anusorn Simuen

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
Prince of Songkla University
2021

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ชื่อวิทยานิพนธ์ A Novel of Ideals and Fuzzy Ideals of Γ-Semigroups
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บทคัดย่อ

กึ่งกรุปแกมมาเป็นโครงสร้างทางพีชคณิตซึ่งเป็นหนึ่งในการวางนัยทั่วไปของกึ่งกรุป ให้ S และ Γ เป็นเซตไม่ว่าง ดังนั้น S จะถูกเรียกว่า *กึ่งกรุปแกมมา* ถ้ามีฟังก์ชัน $S \times \Gamma \times S \to S$ นิยามโดย $(a, \gamma, b) \mapsto a\gamma b$ สอดคล้องกับสัจพจน์ $(a\alpha b)\beta c = a\alpha(b\beta c)$ ทุก $a, b, c \in S$ และ $\alpha, \beta \in \Gamma$ และเรียก f ว่า *เซตย่อยวิภัชนัย* ของเซต S ถ้า f เป็นฟังก์ชันที่ส่งสมาชิกจาก S ไปยังช่วงปิด $[0, 1]$ แนวคิดทั้งสองนี้เป็นแนวคิดที่น่าสนใจที่จะนำมาศึกษาร่วมกัน

ในการศึกษาครั้งนี้เรานิยามเกือบควอซี-Γ-ไอดีล (almost quasi-Γ-ideal) ในกึ่งกรุปแกมมาและศึกษาสมบัติบางประการที่เกี่ยวข้อง เช่น ยูเนียนของสองเกือบควอซี-Γ-ไอดีลก็เป็นเกือบควอซี-Γ-ไอดีลแต่อินเตอร์เซกชันของมันไม่จำเป็นต้องเป็นเกือบควอซี-Γ-ไอดีลเสมอไป เรายังได้นิยามและศึกษาเกือบควอซี-Γ-ไอดีลวิภัชนัย (fuzzy almost quasi-Γ-ideal) ในกึ่งกรุปแกมมา ศึกษาความสัมพันธ์บางประการระหว่างเกือบควอซี-Γ-ไอดีลและเกือบควอซี-Γ-ไอดีลวิภัชนัยในกึ่งกรุปแกมมา ยิ่งไปกว่านั้นเราก็ได้เสนอแนวคิดและสมบัติของเกือบไบ-Γ-ไอดีล (almost bi-Γ-ideal) และเกือบไบ-Γ-ไอดีลวิภัชนัย (fuzzy almost bi-Γ-ideal) ในกึ่งกรุปแกมมาด้วย นอกจากนี้เราได้ศึกษาสมบัติและความสัมพันธ์ระหว่างเกือบไบ-Γ-ไอดีลและเกือบไบ-Γ-ไอดีลวิภัชนัย

สุดท้ายเราได้นิยามไอดีล (ideal) ไอดีลวิภัชนัย (fuzzy ideal) เกือบไอดีล (almost ideal) และ เกือบไอดีลวิภัชนัย (fuzzy almost ideal) ชนิดใหม่ของกึ่งกรุปแกมมาโดยใช้สมาชิกในเซตแกมมาพร้อมทั้งศึกษาสมบัติต่าง ๆ ของไอดีล ไอดีลวิภัชนัย เกือบไอดีล และ เกือบไอดีลวิภัชนัยเหล่านี้และแสดงความสัมพันธ์ระหว่างไอดีลเหล่านี้กับไอดีลวิภัชนัยของมัน

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ABSTRACT

A Γ-semigroup is an algebraic structure considered as a generalization of a semigroup. Let S and Γ be nonempty sets. Then S is called a Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$ defined as $(a, \gamma, b) \mapsto a\gamma b$ satisfying the axiom $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$ and we say that f is a *fuzzy subset* of a set S if f is a function from S into the closed interval $[0, 1]$. These two concepts are interesting to study together.

In this study, we define almost quasi-Γ-ideals of a Γ-semigroup and study some properties of them such as the union of two almost quasi-Γ-ideals is an almost quasi-Γ-ideal but their intersection need not always be an almost quasi-Γ-ideal. We also define and study fuzzy almost quasi-Γ-ideals of a Γ-semigroup. We give some relationships between almost quasi-Γ-ideals and fuzzy almost quasi-Γ-ideals of Γ-semigroups. Moreover, almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals of Γ-semigroups will be defined and we give properties of them. In addition, we investigate relationships between almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals.

Finally, we define new types of ideals, fuzzy ideals, almost ideals and fuzzy almost ideals of Γ-semigroups by using elements of Γ. We investigate properties of them and show the relationships between these ideals and their fuzzifications.

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Anusorn Simuen

Contents

Chapter 1

Introduction

The classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, *the class of tall men* does not constitute a class or a set in the usual mathematical sense of these terms. Yet, the fact remains that such a imprecise definition of classes plays an important role in human thinking, particularly in the domains of pattern recognition, and communication of information. Consequently, Zadeh ([22]) gave a basic definition of fuzzy subsets in 1965. Fuzzy subsets are interesting content and are studied by many authors in many algebraic structures see ([12]), and ([13]).

After that, Kuroki ([8, 9, 10, 11]) studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in semigroups, and fuzzy semigroups. Bi-ideals in semigroups were introduced by Good and Hughes ([4]) in 1952, and quasi-ideals in semigroups was introduced by Steinfeld in ([16, 17]). In 1981, Grosek and Satko ([5, 6, 7]) defined almost ideals (A-ideals) in semigroups and studied some of their properties. Moreover, Bogdanovic also defined and studied almost bi-ideals in semigroups in ([1]).

The algebraic structures of classical semigroups having one operation are interesting to many mathematicians. In 1984, Sen ([14]) introduced the concept of Γ-semigroup generalizing the classical semigroups to be the set with several operations in the set Γ. Later, in ([15]), Sen and Saha established several properties of Γ-semigroups. In 2006, Chinram ([2]) introduced the concept and provided some properties of quasi-Γ-ideals in Γ-semigroups. Next, in 2007, Chinram and Jirojkul ([3]) gave the notions and some properties of bi-Γ-ideals in Γ-semigroups.

Recently, Wattanatripop and Changphas ([19]) defined the concept of almost left ideals (left A-ideals) and almost right ideals (right A-ideals) of a Γ-semigroup. Wattanatripop, Chinram, and Changphas ([21]) defined the notion of almost quasi-ideals (quasi-A-ideals) and fuzzy almost ideals (fuzzy A-ideals) by using the concept of almost ideals and quasi-ideals of semigroups, and almost ideals and fuzzy ideals of semigroups, respectively. They defined fuzzy almost bi-ideals in semigroups and gave some relationships between almost bi-ideals and fuzzy almost bi-ideals of semigroups in ([20]). Moreover, Suebsung, Wattanatripop and Chinram studied almost (m, n)-ideals and fuzzy almost (m, n)-ideals of semigroups in ([18]).

In this thesis, we define the concepts of almost quasi-Γ-ideals, almost bi-Γ-ideals, (α, β)-ideals for some $\alpha, \beta \in \Gamma$, fuzzy almost quasi-Γ-ideals, fuzzy almost bi-Γ-ideals, and fuzzy (α, β)-ideals for some $\alpha, \beta \in \Gamma$ in Γ-semigroups.

Chapter 2

Preliminaries

In this chapter, we collect the definitions and examples which will be used later in the study of this thesis.

2.1 Γ-Semigroups

M.K. Sen gave the concept of Γ-semigroups in 1986.

Definition 2.1.1. ([15]) Let S and Γ be nonempty sets. We call S a Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$ written as $(a, \gamma, b) \mapsto a\gamma b$ satisfying the axiom $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Remark 2.1.2. (1) If $|\Gamma| = 1$, then the definition of Γ-semigroup concides with that of semigroups.

(2) Every semigroup $(S, \cdot)$ can be considered as a Γ-semigroup where $\Gamma := \{\cdot\}$.

(3) If S is a Γ-semigroup, then for each $\alpha \in \Gamma$, (S, α) is a semigroup.

Let S be a Γ-semigroup. For nonempty subsets A, B of S, let

$$A\Gamma B = \{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}.$$

If $x \in S$ and $\alpha \in \Gamma$, we let $A\Gamma x = A\Gamma\{x\}$, $x\Gamma A = \{x\}\Gamma A$ and $A\alpha B = A\{\alpha\}B$.

Definition 2.1.3. Let S be a Γ-semigroup. A nonempty subset A of S is called

(1) a *sub* Γ*-semigroup* of S if $A\Gamma A \subseteq A$,

(2) a *left* Γ*-ideal* of S if $S\Gamma A \subseteq A$,

(3) a *right* Γ*-ideal* of S if $A\Gamma S \subseteq A$,

(4) a Γ*-ideal* of S if it is both a left Γ-ideal and a right Γ-ideal of S,

(5) a *quasi*-Γ*-ideal* of S if $S\Gamma A \cap A\Gamma S \subseteq A$,

(6) a *bi*-Γ*-ideal* of S if $A\Gamma S\Gamma A \subseteq A$,

(7) an *almost left* Γ*-ideal* of S if $s\Gamma A \cap A \neq \emptyset$ for all $s \in S$,

(8) an *almost right* Γ*-ideal* of S if $A\Gamma s \cap A \neq \emptyset$ for all $s \in S$,

(9) an *almost* Γ*-ideal* of S if it is both an almost left Γ-ideal and an almost right Γ-ideal of S.

2.2 Fuzzy Subsets

In 1965, Zadeh introduced the fundamental fuzzy set concept in [22]. Since then, fuzzy sets have been applied in various fields. Now, we recall the definitions and some notations relevant to fuzzy subsets.

Definition 2.2.1. ([22]) A *fuzzy subset* of a set S is a function from S into the closed interval $[0, 1]$.

Any subset of a set S can be considered as a fuzzy subset of S by indicating either membership or nonmembership in a subset of S.

For any two fuzzy subsets f and g of a set S,

(1) $f \cap g$ is a fuzzy subset of S defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} \quad \text{for all } x \in S.$$

(2) $f \cup g$ is a fuzzy subset of S defined by

$$(f \cup g)(x) = \max\{f(x), g(x)\} \quad \text{for all } x \in S.$$

(3) f^c is a fuzzy subset of S defined by

$$f^c(x) = 1 - f(x) \quad \text{for all } x \in S.$$

(4) $f \circ g$ is a fuzzy subset of S defined by

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=ab} \min\{f(a), g(b)\} & \text{if } x \in S^2, \\ 0 & \text{otherwise.} \end{cases}$$

(5) $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

Example 2.2.2. Consider the Γ-semigroup $\mathbb{Z}_4$ where $\Gamma = \{\overline{0}\}$. Let f and g be fuzzy subsets of $\mathbb{Z}_4$ defined by

$$f(\overline{0}) = 0.1, f(\overline{1}) = 0.3, f(\overline{2}) = 0.2 \text{ and } f(\overline{3}) = 0.5,$$

$$g(\overline{0}) = 0, g(\overline{1}) = 0.4, g(\overline{2}) = 0.6 \text{ and } g(\overline{3}) = 0.5.$$

We have $(f \cap g)(\overline{1}) = 0.3$, $(f \cup g)(\overline{1}) = 0.4$, $f^c(\overline{1}) = 0.7$, $(f \circ g)(\overline{0}) = 0.4$, $(f \circ g)(\overline{1}) = 0.5$, $(f \circ g)(\overline{2}) = 0.5$ and $(f \circ g)(\overline{3}) = 0.3$.

Theorem 2.2.3. *If f is a fuzzy subset of a set S, then $(f^c)^c = f$.*

Theorem 2.2.4. *Let f, g and h be fuzzy subsets of a set S.*

(1) *If $f \subseteq h$ and $g \subseteq h$, then $f \cup g \subseteq h$.*

(2) *If $f \subseteq g$ and $f \subseteq h$, then $f \subseteq g \cap h$.*

Definition 2.2.5. ([12]) The *characteristic mapping* of a subset A of S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1 & x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The set S can be considered as a fuzzy subset of itself and we write $S = C_S$, that is $S(x) = 1$ for all $x \in S$.

Example 2.2.6. Let $S = \{a, b, c\}$ and $A = \{a\}$. Then $C_A(a) = 1, C_A(b) = 0$ and $C_A(c) = 0$.

Theorem 2.2.7. *Let A and B be subsets of a set S. Then*

(1) $C_{A\cap B} = C_A \cap C_B$

(2) $C_{A\cup B} = C_A \cup C_B$

(3) $C_{A^c} = (C_A)^c$

(4) *If $A \subseteq B$, then $C_A \subseteq C_B$.*

Definition 2.2.8. ([12]) Let f be a fuzzy subset of a set S.

(1) For $t \in [0,1]$, the *level set* or *t-cut* f_t is defined by

$$f_t = \{x \in S \mid f(x) \geq t\}.$$

(2) The *support* of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

Definition 2.2.9. ([13]) For $x \in S$ and $t \in (0,1]$, a *fuzzy point* x_t of a set S is a fuzzy subset of S defined by

$$x_t(y) = \begin{cases} t & y = x, \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Fuzzy ideals in semigroups

In 1981, Kuroki introduced the notion of fuzzy ideals in semigroups as follows:

Definition 2.3.1. ([12]) A fuzzy subset f of a semigroup S is called

(1) a *fuzzy subsemigroup* of S if $f(ab) \geq \min\{f(a), f(b)\}$ for all $a, b \in S$,

(2) a *fuzzy left ideal* of S if $f(ab) \geq f(b)$ for all $a, b \in S$,

(3) a *fuzzy right ideal* of S if $f(ab) \geq f(a)$ for all $a, b \in S$,

(4) a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.

Theorem 2.3.2. ([12]) *Let A be a nonempty subset of a semigroup S. Then*

(1) *A is a subsemigroup of S if and only if C_A is a fuzzy subsemigroup of S.*

(2) *A is a left ideal of S if and only if C_A is a fuzzy left ideal of S.*

(3) *A is a right ideal of S if and only if C_A is a fuzzy right ideal of S.*

(4) *A is an ideal of S if and only if C_A is a fuzzy ideal of S.*

Theorem 2.3.3. ([12]) *Let f be a fuzzy subset of a semigroup S such that $f_t \neq \emptyset$ where $t \in (0, 1]$. Then*

(1) *f is a fuzzy subsemigroup of S if and only if f_t is a subsemigroup of S.*

(2) *f is a fuzzy left ideal of S if and only if f_t is a left ideal of S.*

(3) *f is a fuzzy right ideal of S if and only if f_t is a right ideal of S.*

(4) *f is a fuzzy ideal of S if and only if f_t is an ideal of S.*

Theorem 2.3.4. ([12]) *Let f be a fuzzy subset of a semigroup S. Then*

(1) *f is a fuzzy subsemigroup of S if and only if $f \circ f \subseteq f$.*

(2) *f is a fuzzy left ideal of S if and only if $S \circ f \subseteq f$.*

(3) *f is a fuzzy right ideal of S if and only if $f \circ S \subseteq f$.*

(4) *f is a fuzzy ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.*

Next, Kuroki defined fuzzy quasi-ideals and fuzzy bi-ideals in semigroups as follows:

Definition 2.3.5. ([12]) A fuzzy subset f of a semigroup S is called

(1) a *fuzzy quasi-ideal* of S if $(f \circ S) \cap (S \circ f) \subseteq f$,

(2) a *fuzzy bi-ideal* of S if $f \circ S \circ f \subseteq f$.

Finally, fuzzy almost ideals and fuzzy almost bi-ideals in semigroups was studied by Wattanatripop, Chinram and Changphas.

Definition 2.3.6. ([21, 20]) Let f be a fuzzy subset of a semigroup S such that $f \neq 0$. Then f is called

(1) a *fuzzy almost left ideal* of S if $(C_s \circ f) \cap f \neq 0$ for all $s \in S$,

(2) a *fuzzy almost right ideal* of S if $(f \circ C_s) \cap f \neq 0$ for all $s \in S$,

(3) a *fuzzy almost quasi-ideal* of S if $(f \circ C_s) \cap (C_s \circ f) \cap f \neq 0$ for all $s \in S$,

(4) a *fuzzy almost bi-ideal* of S if $(f \circ C_s \circ f) \cap f \neq 0$ for all $s \in S$.

Let S be a Γ-semigroup and $\mathcal{F}(S)$ be the set of all fuzzy subsets of S. For each $\alpha \in \Gamma$, define a binary operation $\circ_\alpha$ on $\mathcal{F}(S)$ by

$$(f \circ_\alpha g)(x) = \begin{cases} \sup\limits_{x=a\alpha b} \{\min\{f(a), g(b)\}\} & \text{if } x \in S\alpha S, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma^\star := \{\circ_\alpha \mid \alpha \in \Gamma\}$. Then $(\mathcal{F}(S), \Gamma^\star)$ is a Γ-semigroup.

Chapter 3

Almost quasi-Γ-ideals and Fuzzy almost quasi-Γ-ideals of Γ-semigroups

In this chapter, we give the concepts of almost quasi-Γ-ideals and fuzzy almost quasi-Γ-ideals of Γ-semigroups. Moreover, we provide some properties and some relationships between almost quasi-Γ-ideals and their fuzzifications.

3.1 Almost quasi-Γ-ideals of Γ-semigroups

We begin this section with the following definition of an almost quasi-Γ-ideal of a Γ-semigroup.

Definition 3.1.1. Let S be a Γ-semigroup. A nonempty subset Q of S is called an *almost quasi-Γ-ideal* of S if $s\Gamma Q \cap Q\Gamma s \cap Q \neq \emptyset$ for all $s \in S$.

Example 3.1.2. Consider the Γ-semigroup $S = \{a, b, c\}$ with $\Gamma = \{\alpha, \beta\}$ and the multiplication tables:

α	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

β	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

.

Let $Q = \{a, b\}$. We see that

$$a\Gamma Q \cap Q\Gamma a \cap Q = \{a, b\} \cap \{a, b\} \cap \{a, b\} = \{a, b\},$$
$$b\Gamma Q \cap Q\Gamma b \cap Q = \{a, b\} \cap \{a, b\} \cap \{a, b\} = \{a, b\},$$
$$c\Gamma Q \cap Q\Gamma c \cap Q = \{a, b, c\} \cap \{a, b, c\} \cap \{a, b\} = \{a, b\}.$$

Then $s\Gamma Q \cap Q\Gamma s \cap Q \neq \emptyset$ for all $s \in S$. Thus Q is an almost quasi-Γ-ideal of S. Similarly, we can see that $\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}$ and $\{a,b,c\}$ are almost quasi-Γ-ideals of S.

Proposition 3.1.3. *If Q is a quasi-Γ-ideal of a Γ-semigroup S, then either $s\Gamma Q \cap Q\Gamma s = \emptyset$ for some $s \in S$ or Q is an almost quasi-Γ-ideal of S.*

Proof. Assume that Q is a quasi-Γ-ideal of S and $s\Gamma Q \cap Q\Gamma s \neq \emptyset$ for all $s \in S$. Let $s \in S$. By assumption, $s\Gamma Q \cap Q\Gamma s \neq \emptyset$. Since $s\Gamma Q \cap Q\Gamma s \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$, we get $s\Gamma Q \cap Q\Gamma s \cap Q = s\Gamma Q \cap Q\Gamma s \neq \emptyset$. Hence, Q is an almost quasi-Γ-ideal of S. □

Theorem 3.1.4. *Every almost quasi-Γ-ideal of a Γ-semigroup S is an almost left Γ-ideal and an almost right Γ-ideal of S.*

Proof. Assume that Q is an almost quasi-Γ-ideal of a Γ-semigroup S. Let $s \in S$. So $s\Gamma Q \cap Q\Gamma s \cap Q \neq \emptyset$. Since $s\Gamma Q \cap Q\Gamma s \cap Q \subseteq s\Gamma Q \cap Q$ and $s\Gamma Q \cap Q\Gamma s \cap Q \subseteq Q\Gamma s \cap Q$, we have $s\Gamma Q \cap Q \neq \emptyset$ and $Q\Gamma s \cap Q \neq \emptyset$. Hence, Q is an almost left Γ-ideal and an almost right Γ-ideal of S. □

The following example shows that an almost left Γ-ideal of a Γ-semigroup S need not be an almost quasi-Γ-ideal of S.

Example 3.1.5. Consider a Γ-semigroup $S = \{a,b,c,d,e\}$ with $\Gamma = \{a\}$ and

a	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	e
c	a	c	b	d	e
d	d	d	d	a	a
e	d	d	d	a	a

Let $L = \{b,c,d\}$. We have L is an almost left Γ-ideal of S but $(e\Gamma L \cap L\Gamma e) \cap L = \emptyset$. Thus L is not an almost quasi-Γ-ideal of S.

Theorem 3.1.6. *If Q is an almost quasi-Γ-ideal of a Γ-semigroup S and $Q \subseteq H \subseteq S$, then H is an almost quasi-Γ-ideal of S.*

Proof. Assume that Q is an almost quasi-Γ-ideal of a Γ-semigroup S with $Q \subseteq H \subseteq S$. Let $s \in S$. Since $\emptyset \neq s\Gamma Q \cap Q\Gamma s \cap Q \subseteq s\Gamma H \cap H\Gamma s \cap H$, we obtain $s\Gamma H \cap H\Gamma s \cap H \neq \emptyset$. Therefore, H is an almost quasi-Γ-ideal of S. □

Corollary 3.1.7. *The union of two almost quasi-Γ-ideals of a Γ-semigroup S is an almost quasi-Γ-ideal of S.*

Proof. Let Q_1 and Q_2 be any two almost quasi-Γ-ideals of a Γ-semigroup S. Since $Q_1 \subseteq Q_1 \cup Q_2$, it follow from Theorem 3.1.6 that $Q_1 \cup Q_2$ is an almost quasi-Γ-ideal of S. $\square$

From Example 3.1.5, put $Q_1 := \{a, b\}$ and $Q_2 := \{b, c, d\}$. We see that Q_1 and Q_2 are almost quasi-Γ- ideals of S, while $Q_1 \cap Q_2 = \{b\}$ is not. Thus the intersection of two almost quasi-Γ-ideals of a Γ-semigroup S need not always be an almost quasi-Γ-ideal of S.

Theorem 3.1.8. *Let S be a Γ-semigroup. Then S has no proper almost quasi-Γ-ideal if and only if for any $a \in S$, there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \subseteq \{a\}$.*

Proof. Assume that S has no proper almost quasi-Γ-ideal and let $a \in S$. Then $S \setminus \{a\}$ is not an almost quasi-Γ-ideal. Then there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \cap S \setminus \{a\} = \emptyset$. Therefore, $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \subseteq \{a\}$.

Conversely, assume that for any $a \in S$ there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \subseteq \{a\}$. Let A be a proper subset of S. Then $A \subseteq S \setminus \{a\} \subseteq S$ for some $a \in S$. By assumption, there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \subseteq \{a\}$, so $\Big(s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a\Big) \cap (S \setminus \{a\}) \subseteq \{a\} \cap (S \setminus \{a\}) = \emptyset$. Hence, A is not an almost quasi-Γ-ideal of S. Consequently, S has no proper almost quasi-Γ-ideals. $\square$

3.2 Fuzzy almost quasi-Γ-ideals of Γ-semigroups

In this section, we define fuzzy almost quasi-Γ-ideals in Γ-semigroups and give some relationships between almost quasi-Γ-ideals and fuzzy almost quasi-Γ-ideals of Γ-semigroups.

Definition 3.2.1. Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$. Then f is called a *fuzzy almost quasi-Γ-ideal* of S if for each fuzzy point x_t of S, there exist $\alpha, \beta \in \Gamma$ such that $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \neq 0$.

Theorem 3.2.2. *Let f be a fuzzy almost quasi-Γ-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost quasi-Γ-ideal of S.*

Proof. Let x_t be a fuzzy point of S. Since f is a fuzzy almost quasi-Γ-ideal of S, there exist $\alpha, \beta \in \Gamma$ such that $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \neq 0$. Since $f \subseteq g$, we get $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \subseteq (x_t \circ_\alpha g) \cap (g \circ_\beta x_t) \cap g$. This implies that $(x_t \circ_\alpha g) \cap (g \circ_\beta x_t) \cap g \neq 0$. Therefore, g is a fuzzy almost quasi-Γ-ideal of S. □

Corollary 3.2.3. *Let f and g be fuzzy almost quasi-Γ-ideals of a Γ-semigroup S. Then $f \cup g$ is a fuzzy almost quasi-Γ-ideal of S.*

Proof. Since $f \subseteq f \cup g$, by Theorem 3.2.2, $f \cup g$ is a fuzzy almost quasi-Γ-ideal of S. □

Example 3.2.4. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}\}$ and $\overline{a}\gamma\overline{b} = \overline{a}+\gamma+\overline{b}$. Let $f : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$f(\overline{0}) = 0, f(\overline{1}) = 0.4, f(\overline{2}) = 0, f(\overline{3}) = 0.2, f(\overline{4}) = 0.3$$

and $g : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$g(\overline{0}) = 0, g(\overline{1}) = 0.3, g(\overline{2}) = 0.5, g(\overline{3}) = 0, g(\overline{4}) = 0.9.$$

We see that for each fuzzy point x_t of S,

$$[(x_t \circ_{\overline{0}} f) \cap (f \circ_{\overline{0}} x_t) \cap f](\overline{4}) \neq 0 \text{ and } [(x_t \circ_{\overline{0}} g) \cap (g \circ_{\overline{0}} x_t) \cap g](\overline{4}) \neq 0.$$

Then f and g are fuzzy almost quasi-Γ-ideals of $\mathbb{Z}_5$. Also we have

$$(f \cap g)(\overline{0}) = 0, (f \cap g)(\overline{1}) = 0.3, (f \cap g)(\overline{2}) = 0, (f \cap g)(\overline{3}) = 0, (f \cap g)(\overline{4}) = 0.3.$$

Then $[(x_t \circ_{\overline{0}} (f \cap g)) \cap ((f \cap g) \circ_{\overline{0}} x_t) \cap (f \cap g)](x) = 0$ for all $x \in \mathbb{Z}_5$, so $f \cap g$ is not a fuzzy almost quasi-Γ-ideal of $\mathbb{Z}_5$.

Theorem 3.2.5. *Let Q be a nonempty subset of a Γ-semigroup S. Then Q is an almost quasi-Γ-ideal of S if and only if C_Q is a fuzzy almost quasi-Γ-ideal of S.*

Proof. Assume that Q is an almost quasi-Γ-ideal of S and let x_t be a fuzzy point of S. Then $x\Gamma Q \cap Q\Gamma x \cap Q \neq \emptyset$. Thus there exists $y \in x\Gamma Q \cap Q\Gamma x$ and $y \in Q$. Thus $y \in x\alpha Q \cap Q\beta x$ for some $\alpha, \beta \in \Gamma$. So $[(x_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta x_t)](y) \neq 0$ and $C_Q(y) = 1$. Hence, $(x_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta x_t) \cap C_Q \neq 0$. Therefore, C_Q is a fuzzy almost quasi-Γ-ideal of S.

Conversely, assume that C_Q is a fuzzy almost quasi-Γ-ideal of S. Let $s \in S$. Then $(s_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta s_t) \cap C_Q \neq 0$ for some $\alpha, \beta \in \Gamma$. Thus there exists $x \in S$ such that $[(s_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta s_t) \cap C_Q](x) \neq 0$. Hence, $x \in s\Gamma Q \cap Q\Gamma s \cap Q$. So $s\Gamma Q \cap Q\Gamma s \cap Q \neq \emptyset$. Consequently, Q is an almost quasi-Γ-ideal of S. $\square$

Theorem 3.2.6. *Let f be a fuzzy subset of a Γ-semigroup S. Then f is a fuzzy almost quasi-Γ-ideal of S if and only if $supp(f)$ is an almost quasi-Γ-ideal of S.*

Proof. Assume that f is a fuzzy almost quasi-Γ-ideal of S. Let s_t be a fuzzy point of S. Then $(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f \neq 0$ for some $\alpha, \beta \in \Gamma$. Hence, there exists $x \in S$ such that $[(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f](x) \neq 0$. So there exist $y_1, y_2 \in S$ such that $x = s\alpha y_1 = y_2 \beta s, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is $x, y_1, y_2 \in supp(f)$. Thus $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t)](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Therefore, $(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)} \neq 0$. Hence, $C_{supp(f)}$ is a fuzzy almost quasi-Γ-ideal of S. By Theorem 3.2.5, $supp(f)$ is an almost quasi-Γ-ideal of S.

Conversely, assume that $supp(f)$ is an almost quasi-Γ-ideal of S. By Theorem 3.2.5, $C_{supp(f)}$ is a fuzzy almost quasi-Γ-ideal of S. Then for each fuzzy point s_t of S, we have $(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)} \neq 0$ for some $\alpha, \beta \in \Gamma$. Thus there exists $x \in S$ such that $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)}](x) \neq 0$. Hence, $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t)](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Then there exist $y_1, y_2 \in S$ such that $x = s\alpha y_1 = y_2 \beta s, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. Thus $(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f \neq 0$. Therefore, f is a fuzzy almost quasi-Γ-ideal of S. $\square$

Corollary 3.2.7. *A Γ-semigroup S has no proper almost quasi-Γ-ideals if and only if $supp(f) = S$ for each fuzzy almost quasi-Γ-ideal f of S,*

Proof. Assume S has no proper almost quasi-Γ-ideals and let f be a fuzzy almost quasi-Γ-ideal of S. By Theorem 3.2.6, $supp(f)$ is an almost quasi-Γ-ideal of S. Thus $supp(f) = S$.

Conversely, let Q be any fuzzy almost quasi-Γ-ideal of S. By Theorem 3.2.5, C_Q is a fuzzy almost quasi-Γ-ideal of S. By assumption, $supp(C_Q) = S$.

Since $supp(C_Q) = Q$, we get $Q = S$. This implies that S has no proper almost quasi-Γ-ideals. □

Finally, we define minimal fuzzy almost quasi-Γ-ideals in Γ-semigroups and give a relationship between minimal almost quasi-Γ-ideals and minimal fuzzy almost quasi-Γ-ideals of Γ-semigroups.

Definition 3.2.8. A fuzzy almost quasi-Γ-ideal f of a Γ-semigroup S is *minimal* if for each fuzzy almost quasi-Γ-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 3.2.9. *Let Q be a nonempty subset of a Γ-semigroup S. Then Q is a minimal almost quasi-Γ-ideal of S if and only if C_Q is a minimal fuzzy almost quasi-Γ-ideal of S.*

Proof. Assume that Q is a minimal almost quasi-Γ-ideal of S. By Theorem 3.2.5, C_Q is a fuzzy almost quasi-Γ-ideal of S. Let g be a fuzzy almost quasi-Γ-ideal of S such that $g \subseteq C_Q$. Then $supp(g) \subseteq supp(C_Q) = Q$. Since $g \subseteq C_{supp(g)}$, we have $C_{supp(g)}$ is a fuzzy almost quasi-Γ-ideal of S. By Theorem 3.2.5, $supp(g)$ is an almost quasi-Γ-ideal of S. Since Q is minimal, $supp(g) = Q = supp(C_Q)$. Therefore, C_Q is minimal.

Conversely, assume that C_Q is a minimal fuzzy almost quasi-Γ-ideal of S. Let Q' be an almost quasi-Γ-ideal of S such that $Q' \subseteq Q$. Then $C_{Q'}$ is a fuzzy almost quasi-Γ-ideal of S such that $C_{Q'} \subseteq C_Q$. Since C_Q is minimal, it follows that $supp(C_{Q'}) = supp(C_Q)$. Hence, $Q' = supp(C_{Q'}) = supp(C_Q) = Q$. Therefore, Q is minimal. □

Chapter 4

Almost bi-Γ-ideals and Fuzzy almost bi-Γ-ideals of Γ-semigroups

In this chapter, we define an almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals of Γ-semigroups. Moreover we give some properties and some relationships between almost bi-Γ-ideals and their fuzzifications.

4.1 Almost bi-Γ-ideals of Γ-semigroups

Firstly, we define almost bi-Γ-ideals of Γ-semigroups as follows:

Definition 4.1.1. A nonempty subset B of a Γ-semigroup S is called an *almost bi-Γ-ideal* of S if $B\Gamma s\Gamma B \cap B \neq \emptyset$ for all $s \in S$.

Example 4.1.2. Consider the Γ-semigroup $S = \{a, b, c, d\}$ with $\Gamma = \{\alpha, \beta\}$ and the multiplication table:

α	a	b	c	d
a	a	c	c	a
b	c	a	a	c
c	c	a	a	c
d	a	c	c	a

β	a	b	c	d
a	c	a	a	c
b	a	c	c	a
c	a	c	c	a
d	c	a	a	c

.

We see that the almost bi-Γ-ideals of S are $\{a, c\}, \{a, b, c\}$, and $\{a, c, d\}$.

Theorem 4.1.3. *If B is an almost bi-Γ-ideal of a Γ-semigroup S and $B \subseteq C \subseteq S$, then C is an almost bi-Γ-ideal of S.*

Proof. Let B be an almost bi-Γ-ideal of a Γ-semigroup S and $B \subseteq C \subseteq S$. Then $B\Gamma s\Gamma B \cap B \neq \emptyset$ for all $s \in S$ and $B\Gamma s\Gamma B \cap B \subseteq C\Gamma s\Gamma C \cap C$, which implies that $C\Gamma s\Gamma C \cap C \neq \emptyset$ for all $s \in S$. Therefore, C is an almost bi-Γ-ideal of S. $\square$

Corollary 4.1.4. *The union of two almost bi-Γ-ideals of a Γ-semigroup S is also an almost bi-Γ-ideal of S.*

Proof. Let A and B be almost bi-Γ-ideals of S. Since $A \subseteq A \cup B \subseteq S$, it follows from Theorem 4.1.3 that $A \cup B$ is an almost bi-Γ-ideal of S. $\square$

The intersection of two almost bi-Γ-ideals of a Γ-semigroup S need not always be an almost bi-Γ-ideal of S as shown in the following example.

Example 4.1.5. Consider the Γ-semigroup $\mathbb{Z}_8$ with $\Gamma = \{\overline{0}, \overline{1}, \overline{2}\}$. Let $A = \{\overline{2}, \overline{3}\}$ and $B = \{\overline{4}, \overline{6}\}$. Clearly, A and B are almost bi-Γ-ideals of $\mathbb{Z}_8$ but $A \cap B = \emptyset$, so it is not an almost bi-Γ-ideal of $\mathbb{Z}_8$.

Theorem 4.1.6. *A Γ-semigroup S has a proper almost bi-Γ-ideal if and only if there exists an element $a \in S$ such that $S \setminus \{a\}$ is an almost bi-Γ-ideal of S.*

Proof. Let B be a proper almost bi-Γ-ideal of a Γ-semigroup S and let $a \in S \setminus B$. Then $B \subseteq S \setminus \{a\} \subset S$. By Theorem 4.1.3, $S \setminus \{a\}$ is an almost bi-Γ-ideal of S.

Conversely, let $a \in S$ such that $S \setminus \{a\}$ is an almost bi Γ-ideal of S. Since $S \setminus \{a\} \subset S$, we get $S \setminus \{a\}$ is a proper almost bi-Γ-ideal of S. $\square$

Theorem 4.1.7. *A Γ-semigroup S has no proper almost bi-Γ-ideal if and only if for each $a \in S$, there exists $s_a \in S$ such that $(S \setminus \{a\})\Gamma s_a\Gamma(S \setminus \{a\}) \subseteq \{a\}$.*

Proof. Assume that S has no proper almost bi-Γ-ideal and let $a \in S$. By Theorem 4.1.6, $S \setminus \{a\}$ is not an almost bi-Γ-ideal of S. Thus there exists $s_a \in S$ such that $(S \setminus \{a\})\Gamma s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$. This implies that $(S \setminus \{a\})\Gamma s_a\Gamma(S \setminus \{a\}) \subseteq \{a\}$.

Conversely, suppose S has a proper almost bi-Γ-ideal B. Let $a \in S \setminus B$. Since $B \subseteq S \setminus \{a\} \subset S$, we get $S \setminus \{a\}$ is an almost bi-Γ-ideal of S by Theorem 4.1.3. Then $(S \setminus \{a\})\Gamma s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\}) \neq \emptyset$ for all $s_a \in S$, so $(S \setminus \{a\})\Gamma s_a\Gamma(S \setminus \{a\}) \nsubseteq \{a\}$ for all $s_a \in S$. $\square$

4.2 Fuzzy almost bi-Γ-ideals of Γ-semigroups

We define fuzzy almost bi-Γ-ideals in Γ-semigroups as follows:

Definition 4.2.1. Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$. We call f a *fuzzy almost bi-*Γ*-ideal* of S if for each fuzzy point x_t of S, there exist $\alpha, \beta \in \Gamma$ such that $(f \circ_\alpha x_t \circ_\beta f) \cap f \neq 0$.

Theorem 4.2.2. *Let f be a fuzzy almost bi-*Γ*-ideal of a* Γ*-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost bi-*Γ*-ideal of S.*

Proof. Let x_t be a fuzzy point of S. Since f is a fuzzy almost bi-Γ-ideal of S, there exist $\alpha, \beta \in \Gamma$ such that $(f \circ_\alpha x_t \circ_\beta f) \cap f \neq 0$. Since $f \subseteq g$, we get $(f \circ_\alpha x_t \circ_\beta f) \cap f \subseteq (g \circ_\alpha x_t \circ_\beta g) \cap g$, so $(g \circ_\alpha x_t \circ_\beta g) \cap g \neq 0$. Thus g is a fuzzy almost bi-Γ-ideal of S. □

Corollary 4.2.3. *Let f and g be fuzzy almost bi-*Γ*-ideals of a* Γ*-semigroup S. Then $f \cup g$ is a fuzzy almost bi-*Γ*-ideal of S.*

Proof. Since $f \subseteq f \cup g$, we obtain that $f \cup g$ is a fuzzy almost bi-Γ-ideal of S by Theorem 4.2.2. □

The intersection of two fuzzy almost bi-Γ-ideals of a Γ-semigroup S need not be a fuzzy almost bi-Γ-ideal of S. We can see in the following example.

Example 4.2.4. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}\}$ and $\overline{a}\gamma\overline{b} = \overline{a}+\gamma+\overline{b}$. Let $f : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$f(\overline{0}) = 0, f(\overline{1}) = 0.5, f(\overline{2}) = 0, f(\overline{3}) = 0.1 \text{ and } f(\overline{4}) = 0.1$$

and $g : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$g(\overline{0}) = 0, g(\overline{1}) = 0.2, g(\overline{2}) = 0.1, g(\overline{3}) = 0 \text{ and } g(\overline{4}) = 0.2.$$

Then $[(f \circ_{\overline{0}} x_t \circ_{\overline{0}} f) \cap f](\overline{4}) \neq 0$ and $[(g \circ_{\overline{0}} x_t \circ_{\overline{0}} g) \cap g](\overline{4}) \neq 0$. So f and g are fuzzy almost bi-Γ-ideals of $\mathbb{Z}_5$. We see that

$$(f \cap g)(\overline{0}) = 0, (f \cap g)(\overline{1}) = 0.2, (f \cap g)(\overline{2}) = 0, (f \cap g)(\overline{3}) = 0, (f \cap g)(\overline{4}) = 0.1.$$

Then $[(f \cap g) \circ_{\overline{0}} x_t \circ_{\overline{0}} (f \cap g) \cap (f \cap g)](x) = 0$ for all $x \in \mathbb{Z}_5$, so $f \cap g$ is not a fuzzy almost bi-Γ-ideal of $\mathbb{Z}_5$.

Next, we give some relationships between almost bi-Γ-ideals and fuzzy almost bi-Γ-ideals of Γ-semigroups.

Theorem 4.2.5. *Let B be a nonempty subset of a Γ-semigroup S. Then B is an almost bi-Γ-ideal of S if and only if C_B is a fuzzy almost bi-Γ-ideal of S.*

Proof. Suppose that B is an almost bi-Γ-ideal of S. Let x_t be a fuzzy point of S. Then $B\Gamma x\Gamma B \cap B \neq \emptyset$. Thus there exists $y \in B$ such that $y \in B\alpha x\beta B$ for some $\alpha, \beta \in \Gamma$. This implies that $(C_B \circ_\alpha x_t \circ_\beta C_B)(y) = 1$ and $C_B(y) = 1$. Hence, $(C_B \circ_\alpha x_t \circ_\beta C_B) \cap C_B \neq 0$. Therefore, C_B is a fuzzy almost bi-Γ-ideal of S.

Conversely, suppose that C_B is a fuzzy almost bi-Γ-ideal of S. Let $s \in S$. Then there exist $\alpha, \beta \in \Gamma$ such that $(C_B \circ_\alpha s_t \circ_\beta C_B) \cap C_B \neq 0$, so $[(C_B \circ_\alpha s_t \circ_\beta C_B) \cap C_B](y) \neq 0$ for some $y \in S$. Thus $y \in B$ and $y = a\alpha s\beta b$ for some $a, b \in B$ and $\alpha, \beta \in \Gamma$. Hence, $y \in B\Gamma s\Gamma B \cap B$. So $B\Gamma s\Gamma B \cap B \neq \emptyset$. Consequently, B is an almost bi-Γ-ideal of S. □

Theorem 4.2.6. *Let f be a fuzzy subset of a Γ-semigroup S. Then f is a fuzzy almost bi-Γ-ideal of S if and only if $supp(f)$ is an almost bi-Γ-ideal of S.*

Proof. Suppose that f is a fuzzy almost bi-Γ-ideal of S. Let s_t be a fuzzy point of S. Then there exist $\alpha, \beta \in \Gamma$ such that $(f \circ_\alpha s_t \circ_\beta f) \cap f \neq 0$. So we obtain that $[(f \circ_\alpha s_t \circ_\beta f) \cap f](x) \neq 0$ for some $x \in S$. Thus there exist $y_1, y_2 \in S$ such that $x = y_1\alpha s\beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. So $x, y_1, y_2 \in supp(f)$. Thus we get $[C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)}](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Therefore, $(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)} \neq 0$. Hence, $C_{supp(f)}$ is a fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.5, $supp(f)$ is an almost bi-Γ-ideal of S.

Conversely, suppose that $supp(f)$ is an almost bi-Γ-ideal of S. By Theorem 4.2.5, $C_{supp(f)}$ is a fuzzy almost bi-Γ-ideal of S. Let x_t be a fuzzy point of S. Then $(C_{supp(f)} \circ_\alpha x_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)} \neq 0$ for some $\alpha, \beta \in \Gamma$. Then there exists $x \in S$ such that $[(C_{supp(f)} \circ_\alpha x_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)}](x) \neq 0$. Hence, $(C_{supp(f)} \circ_\alpha x_t \circ_\beta C_{supp(f)})(x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Then there exist $y_1, y_2 \in S$ such that $x = y_1\alpha x\beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This implies that $(f \circ_\alpha x_t \circ_\beta f) \cap f \neq 0$. Therefore, f is a fuzzy almost bi-Γ-ideal of S. □

Corollary 4.2.7. *A* Γ*-semigroup* S *has no proper almost bi-*Γ*-ideals if and only if* $supp(f) = S$ *for each fuzzy almost bi-*Γ*-ideal* f *of* S*.*

Proof. Assume S has no proper almost bi-Γ-ideals and let f be a fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.6, $supp(f)$ is an almost bi-Γ-ideal of S. Then $supp(f) = S$.

Conversely, let B be any fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.5, C_B is a fuzzy almost bi-Γ-ideal of S. By assumption, $supp(C_B) = S$. Since $supp(C_B) = B$, we get $B = S$. This implies that S has no proper almost bi-Γ-ideals. □

Next, we define minimal fuzzy almost bi-Γ-ideals in Γ-semigroups and give a relationship between minimal almost bi-Γ-ideals and minimal fuzzy almost bi-Γ-ideals of Γ-semigroups.

Definition 4.2.8. A fuzzy almost bi-Γ-ideal f of a Γ-semigroup S is called *minimal* if for each fuzzy almost bi-Γ-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 4.2.9. *Let* B *be a non-empty subset of a* Γ*-semigroup* S*. Then* B *is a minimal almost bi-*Γ*-ideal of* S *if and only if* C_B *is a minimal fuzzy almost bi-*Γ*-ideal of* S*.*

Proof. Suppose that B is a minimal almost bi-Γ-ideal of S. By Theorem 4.2.5, C_B is a fuzzy almost bi-Γ-ideal of S. Let g be a fuzzy almost bi-Γ-ideal of S such that $g \subseteq C_B$. Then $supp(g) \subseteq supp(C_B) = B$. Since $g \subseteq C_{supp(g)}$, it follows from Theorem 4.2.2 that, $C_{supp(g)}$ is a fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.6, $supp(g)$ is an almost bi-Γ-ideal of S. Since B is a minimal, $supp(g) = B = supp(C_B)$. Therefore, C_B is minimal.

Conversely, suppose that C_B is a minimal fuzzy almost bi-Γ-ideal of S. Let B' be an almost bi-Γ-ideal of S such that $B' \subseteq B$. Then $C_{B'}$ is a fuzzy almost bi-Γ-ideal of S such that $C_{B'} \subseteq C_B$. Since C_B is minimal, $supp(C_{B'}) = supp(C_B)$. Hence, $B' = supp(C_{B'}) = supp(C_B) = B$. Therefore, B is minimal. □

Next, we give a relationship between α-prime almost bi-Γ-ideals and α-prime fuzzy almost bi-Γ-ideals.

Definition 4.2.10. Let S be a Γ-semigroup and $\alpha \in \Gamma$.

(1) An almost bi-Γ-ideal A of S is said to be *α-prime* if for all $x, y \in S$,

$$x\alpha y \in A \text{ implies } x \in A \text{ or } y \in A.$$

(2) A fuzzy almost bi-Γ-ideal f of S is said to be *α-prime* if

$$f(x\alpha y) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S.$$

Theorem 4.2.11. *Let A be a nonempty subset of S. Then A is an α-prime almost bi-Γ-ideal of S if and only if C_A is an α-prime fuzzy almost bi-Γ-ideal of S.*

Proof. Suppose that A is an α-prime almost bi-Γ-ideal of S. By Theorem 4.2.5, C_A is a fuzzy almost bi-Γ-ideal of S. Let $x, y \in S$. We consider two cases:

Case 1: $x\alpha y \in A$. Since A is an α-prime, we obtain that $x \in A$ or $y \in A$.
Then $C_A(x\alpha y) = 1 \leq \max\{C_A(x), C_A(y)\}$.

Case 2: $x\alpha y \notin A$. Then $C_A(x\alpha y) = 0 \leq \max\{C_A(x), C_A(y)\}$.
Thus C_A is an α-prime fuzzy almost bi-Γ-ideal of S.

Conversely, suppose that C_A is an α-prime fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.5, A is an almost bi-Γ-ideal of S. Let $x, y \in S$ such that $x\alpha y \in A$. Then $C_A(x\alpha y) = 1$. By assumption, $C_A(x\alpha y) \leq \max\{C_A(x), C_A(y)\}$. So $\max\{C_A(x), C_A(y)\} = 1$. Hence, $x \in A$ or $y \in A$. Thus A is an α-prime almost bi-Γ-ideal of S. □

Finally, we give a relationship between α-semiprime almost bi-Γ-ideals and α-semiprime fuzzy almost bi-Γ-ideals.

Definition 4.2.12. Let S be a Γ-semigroup and $\alpha \in \Gamma$.

(1) An almost bi-Γ-ideal A of S is said to be *α-semiprime* if for all $x \in S$,

$$x\alpha x \in A \text{ implies } x \in A.$$

(2) A fuzzy almost bi-Γ-ideal f is said to be *α-semiprime* if

$$f(x\alpha x) \leq f(x) \text{ for all } x \in S.$$

Theorem 4.2.13. *Let A be a nonempty subset of S. Then A is an α-semiprime almost bi-Γ-ideal of S if and only if C_A is an α-semiprime fuzzy almost bi-Γ-ideal of S.*

Proof. Suppose that A is an α-semiprime almost bi-Γ-ideal of S. By Theorem 4.2.5, C_A is a fuzzy almost bi-Γ-ideal of S. Let $x \in S$. We consider two cases:
Case 1: $x\alpha x \in A$. Since A is an α-prime, we obtain that $x \in A$. So $C_A(x) = 1$. Hence, $C_A(x\alpha x) = C_A(x)$.

Case 2: $x\alpha x \notin A$. Then $C_A(x\alpha x) = 0 \leq C_A(x)$.
Thus C_A is an α-semiprime fuzzy almost bi-Γ-ideal of S.

Conversely, suppose that C_A is an α-semiprime fuzzy almost bi-Γ-ideal of S. By Theorem 4.2.5, A is an almost bi-Γ-ideal of S. Let $x \in S$ such that $x\alpha x \in A$. Then $C_A(x\alpha x) = 1$. By assumption, $C_A(x\alpha x) \leq C_A(x)$. Thus $C_A(x) = 1$, so $x \in A$. Consequently, A is an α-semiprime almost bi-Γ-ideal of S. □

Chapter 5

New types of ideals and fuzzy ideals of Γ-semigroups

In this chapter, we define new types of ideals, fuzzy ideals, almost ideals and fuzzy almost ideals of Γ-semigroups by using an element of Γ.

5.1 New Types of Ideals

In this section, we will focus on (α,β)-ideals, (α,β)-quasi-ideals and (α,β)-bi-ideals of Γ-semigroups for $\alpha,\beta \in \Gamma$.

5.1.1 (α,β)-ideals

Firstly, we will define (α,β)-ideals of Γ-semigroups as follows:

Definition 5.1.1. Let S be a Γ-semigroup and $\alpha,\beta \in \Gamma$, A nonempty subset A of S is called

(1) a *left α-ideal* of S if $S\alpha A \subseteq A$,

(2) a *right β-ideal* of S if $A\beta S \subseteq A$,

(3) an *(α,β)-ideal* of S if it is both a left α-ideal and a right β-ideal of S,

(4) an *α-ideal* of S if it is an (α,α)-ideal of S.

Every left ideal [right ideal, ideal] of a Γ-semigroup S is a left α-ideal [right β-ideal, (α,β)-ideal] of S for all $\alpha,\beta \in \Gamma$. However, the converse is not generally true. We can see in the following example.

Example 5.1.2. Let $S = \Gamma = \mathbb{N}$ and $(a, \gamma, b) \mapsto a + \gamma + b$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup. Let $A = \{1\} \cup \{6, 7, 8, 9, \ldots\}$. It is easy to show that A is a left 4-ideal but not a left ideal of S.

Theorem 5.1.3. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. The following statements are true.*

(1) *If L is a left α-ideal of S, then L is a left ideal of a semigroup (S, α).*

(2) *If R is a right β-ideal of S, then R is a right ideal of a semigroup (S, β).*

(3) *If I is an α-ideal of S, then I is an ideal of a semigroup (S, α).*

Proof. (1) Since S is a Γ-semigroup, we get that S is a semigroup under α for $\alpha \in \Gamma$. Let L be a left α-ideal of S. Then $S\alpha L \subseteq L$. Hence, L is a left ideal of a semigroup (S, α).

The proofs of (2) and (3) are similar to the proof of (1). □

Theorem 5.1.4. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. If L is a left α-ideal and R is a right β-ideal of S, then $L\gamma R$ is an (α, β)-ideal of S for all $\gamma \in \Gamma$.*

Proof. Let L be a left α-ideal and R a right β-ideal of S. Let $\gamma \in \Gamma$. Clearly, $L\gamma R \neq \emptyset$. We have $S\alpha(L\gamma R) = (S\alpha L)\gamma R \subseteq L\gamma R$ and $(L\gamma R)\beta S = L\gamma(R\beta S) \subseteq L\gamma R$. Therefore, $L\gamma R$ is an (α, β)-ideal of S. □

Theorem 5.1.5. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If L_1 and L_2 are left α-ideals of S, then $L_1 \cup L_2$ is a left α-ideal of S*

(2) *If R_1 and R_2 are right β-ideals of S, then $R_1 \cup R_2$ is a right β-ideal of S.*

(3) *If I_1 and I_2 are (α, β)-ideals of S, then $I_1 \cup I_2$ is an (α, β)-ideal of S.*

Proof. (1) Let L_1 and L_2 be two left α-ideals of S. It is clear that $L_1 \cup L_2 \neq \emptyset$. We have $S\alpha(L_1 \cup L_2) = S\alpha L_1 \cup S\alpha L_2 \subseteq L_1 \cup L_2$. Hence, $L_1 \cup L_2$ is a left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

Theorem 5.1.6. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If L_i is a left α-ideal of S for all $i \in I$ and $\bigcap_{i\in I} L_i \neq \emptyset$, then $\bigcap_{i\in I} L_i$ is a left α-ideal of S.*

(2) *If R_i is a right β-ideal of S for all $i \in I$ and $\bigcap_{i\in I} R_i \neq \emptyset$, then $\bigcap_{i\in I} R_i$ is a right β-ideal of S.*

(3) *If I_i is an (α, β)-ideal of S for all $i \in I$ and $\bigcap_{i\in I} I_i \neq \emptyset$, then $\bigcap_{i\in I} I_i$ is an (α, β)-ideal of S.*

Proof. (1) Assume that $\bigcap_{i\in I} L_i \neq \emptyset$. Since for each $i \in I$, L_i is a left α-ideal of S, we get $S\alpha(\bigcap_{i\in I} L_i) \subseteq S\alpha L_i \subseteq L_i$ for all $i \in I$, so $S\alpha(\bigcap_{i\in I} L_i) \subseteq \bigcap_{i\in I} L_i$. Therefore, $\bigcap_{i\in I} L_i$ is a left α-ideal of S.

The proofs of (2) and (3) are similar to (1). □

For a nonempty subset A of a Γ-semigroup S, let $(A)_{l(\alpha)}$ be the intersection of all left α-ideals of S containing A and define $(A)_{r(\beta)}$ and $(A)_{i(\alpha,\beta)}$ similarly. Then $(A)_{l(\alpha)}$ $[(A)_{r(\beta)}, (A)_{i(\alpha,\beta)}]$ is the smallest left α-ideal [right β-ideal, (α, β)-ideal] of S containing A which is called the left α-ideal [right β-ideal, (α, β)-ideal] of S generated by A.

Theorem 5.1.7. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

(1) $(A)_{l(\alpha)} = A \cup S\alpha A$.

(2) $(A)_{r(\beta)} = A \cup A\beta S$.

(3) $(A)_{i(\alpha,\beta)} = A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S$.

Proof. (1) Let $L = A \cup S\alpha A$. Clearly, $A \subseteq L$. Also, we have

$$S\alpha L = S\alpha(A \cup S\alpha A) = S\alpha A \cup S\alpha S\alpha A = S\alpha A \subseteq L.$$

Then L is a left α-ideal of S containing A. Next, let C be any left α-ideal of S containing A. Since C is a left α-ideal of S and $A \subseteq C$, we get $S\alpha A \subseteq S\alpha C \subseteq C$. Therefore, $L = A \cup S\alpha A \subseteq C$. Hence, L is the smallest α-ideal of S containing A. Consequently, $(A)_{l(\alpha)} = L = A \cup S\alpha A$, as required.

The proof of (2) is similar to (1).

(3) Let $I = A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S$. Clearly, $A \subseteq I$. We have

$$S\alpha I = S\alpha(A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S) \subseteq S\alpha A \cup S\alpha A\beta S \subseteq I,$$

$$I\beta S = (A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S)\beta S \subseteq A\beta S \cup S\alpha A\beta S \subseteq I.$$

Then I is an (α, β)-ideal of S containing A. Next, let C be any (α, β)-ideal of S containing A. Since C is an (α, β)-ideal of S and $A \subseteq C$, we obtain that $S\alpha A \subseteq S\alpha C \subseteq C$, $A\beta S \subseteq C\beta S \subseteq C$ and $S\alpha A\beta S \subseteq S\alpha C\beta S \subseteq S\alpha C \subseteq C$. Thus $I = A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S \subseteq C$. Hence, I is the smallest (α, β)-ideal of S containing A. Therefore, $(A)_{i(\alpha,\beta)} = A \cup S\alpha A \cup A\beta S \cup S\alpha A\beta S$, as required. □

5.1.2 (α, β)-quasi-ideals

We define (α, β)-quasi-ideals of Γ-semigroups as follows:

Definition 5.1.8. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset Q of S is called

(1) an (α, β)-*quasi-ideal* of S if $S\alpha Q \cap Q\beta S \subseteq Q$.

(2) an α-*quasi-ideal* of S if it is an (α, α)-quasi-ideal of S.

Theorem 5.1.9. *Let S be a Γ-semigroup and Q_i an (α, β)-quasi-ideal of S for all $i \in I$. If $\bigcap_{i \in I} Q_i \neq \emptyset$, then $\bigcap_{i \in I} Q_i$ is an (α, β)-quasi-ideal of S.*

Proof. Assume that $\bigcap_{i \in I} Q_i \neq \emptyset$. Since for each $i \in I$, Q_i is an (α, β)-quasi-ideal of S, we get $(S\alpha \bigcap_{i \in I} Q_i) \cap (\bigcap_{i \in I} Q_i \beta S) \subseteq S\alpha Q_i \cap Q_i\beta S \subseteq Q_i$ for all $i \in I$. So $(S\alpha \bigcap_{i \in I} Q_i) \cap (\bigcap_{i \in I} Q_i \beta S) \subseteq \bigcap_{i \in I} Q_i$, and hence, $\bigcap_{i \in I} Q_i$ is an (α, β)-quasi-ideal of S. □

For a nonempty subset A of a Γ-semigroup S, let $(A)_{q(\alpha,\beta)}$ be the intersection of all (α, β)-quasi-ideals of S containing A. Then $(A)_{q(\alpha,\beta)}$ is the smallest (α, β)-quasi-ideal of S containing A which is called the (α, β)-quasi-ideal of S generated by A.

Theorem 5.1.10. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

$$(A)_{q(\alpha,\beta)} = A \cup (S\alpha A \cap A\beta S).$$

Proof. Let $Q = A \cup (S\alpha A \cap A\beta S)$. Clearly, $A \subseteq Q$. We have

$$\begin{aligned} S\alpha Q \cap Q\beta S &= S\alpha[A \cup (S\alpha A \cap A\beta S)] \cap [A \cup (S\alpha A \cap A\beta S)]\beta S \\ &= [S\alpha A \cup S\alpha(S\alpha A \cap A\beta S)] \cap [A\beta S \cup (S\alpha A \cap A\beta S)\beta S] \\ &\subseteq [S\alpha A \cup (S\alpha A \cap S\alpha A\beta S)] \cap [A\beta S \cup (S\alpha A\beta S \cap A\beta S)] \\ &= S\alpha A \cap A\beta S \\ &\subseteq Q. \end{aligned}$$

Then Q is an (α, β)-quasi-ideal of S containing A. Let C be any (α, β)-quasi-ideal of S containing A. Since C is an (α, β)-quasi-ideal of S and $A \subseteq C$, we have that $S\alpha A \cap A\beta S \subseteq S\alpha C \cap C\beta S \subseteq C$. It follows that, $Q = A \cup (S\alpha A \cap A\beta S) \subseteq C$. Hence, Q is the smallest (α, β)-quasi-ideal of S containing A. Therefore, $(A)_{q(\alpha,\beta)} = Q = A \cup (S\alpha A \cap A\beta S)$, as required. □

Theorem 5.1.11. *Let S be a Γ-semigroup. If L is a left α-ideal and R is a right β-ideal of S such that $L \cap R \neq \emptyset$, then $L \cap R$ is an (α, β)-quasi-ideal of S.*

Proof. Let L be a left α-ideal and R a right β-ideal of S such that $L \cap R \neq \emptyset$. Then $S\alpha(L \cap R) \cap (L \cap R)\beta S \subseteq S\alpha L \cap R\beta S \subseteq L \cap R$. Hence, $L \cap R$ is an (α, β)-quasi-ideal of S. □

Corollary 5.1.12. *Let S be a Γ-semigroup. Let L be a left α-ideal and R a right α-ideal of S. Then $L \cap R$ is an α-quasi-ideal of S.*

Proof. We have $R\alpha L \subseteq S\alpha L \subseteq L$ and $R\alpha L \subseteq R\alpha S \subseteq R$. Then $R\alpha L \subseteq L \cap R$, which implies that $L \cap R \neq \emptyset$. By Theorem 5.1.11, $L \cap R$ is an α-quasi-ideal of S. □

Theorem 5.1.13. *Every (α, β)-quasi-ideal Q of a Γ-semigroup S is the intersection of a left α-ideal and a right β-ideal of S.*

Proof. Let Q be an (α, β)-quasi-ideal of S. Then $S\alpha Q \cap Q\beta S \subseteq Q$. Let $L = (Q)_{l(\alpha)} = Q \cup S\alpha Q$ and $R = (Q)_{r(\beta)} = Q \cup Q\beta S$. Thus L is a left α-ideal and R is a right β-ideal of S. We have that $L \cap R = (Q \cup S\alpha Q) \cap (Q \cup Q\beta S) = Q \cup (S\alpha Q \cap Q\beta S) = Q$. Therefore, $Q = L \cap R$. □

Definition 5.1.14. A Γ-semigroup S is said to be (α, β)-*quasi-simple* if S does not contain any proper (α, β)-quasi-ideals.

A Γ-semigroup S is said to be α-*quasi-simple* if S is (α, α)-quasi-simple.

Theorem 5.1.15. *Let S be a Γ-semigroup. Then S is α-quasi-simple if and only if $S\alpha s \cap s\alpha S = S$ for all $s \in S$.*

Proof. Assume that S is α-quasi-simple. Let $s \in S$. We claim that $S\alpha s \cap s\alpha S$ is an α-quasi-ideal of S. Since $s\alpha s \in S\alpha s \cap s\alpha S$, we get $S\alpha s \cap s\alpha S \neq \emptyset$. We have

$$\begin{aligned} S\alpha(S\alpha s \cap s\alpha S) \cap (S\alpha s \cap s\alpha S)\alpha S &\subseteq S\alpha(S\alpha s) \cap (s\alpha S)\alpha S \\ &= (S\alpha S)\alpha s \cap s\alpha(S\alpha S) \\ &\subseteq S\alpha s \cap s\alpha S. \end{aligned}$$

Then $S\alpha s \cap s\alpha S$ is an α-quasi-ideal of S. Since S is α-quasi-simple, we have $S = S\alpha s \cap s\alpha S$.

Conversely, assume that $S\alpha s \cap s\alpha S = S$ for all $s \in S$. Let Q be an α-quasi-ideal of S and $q \in Q$. By assumption, $S = S\alpha q \cap q\alpha S$. Since Q is an α-quasi-ideal of S, we obtain $S = S\alpha q \cap q\alpha S \subseteq S\alpha Q \cap Q\alpha S \subseteq Q$. Then $Q = S$. Therefore, S is α-quasi-simple. □

Definition 5.1.16. An (α, β)-quasi-ideal Q of a Γ-semigroup S is said to be *minimal* if $C \subseteq Q$ implies $C = Q$ for any (α, β)-quasi-ideal C of S.

Theorem 5.1.17. *Let S be a Γ-semigroup and Q an (α, β)-quasi-ideal of S. If Q is (α, β)-quasi-simple, then Q is a minimal (α, β)-quasi-ideal of S.*

Proof. Assume that Q is (α, β)-quasi-simple. Let C be an (α, β)-quasi-ideal of S such that $C \subseteq Q$. Then $Q\alpha C \cap C\beta Q \subseteq S\alpha C \cap C\beta S \subseteq C$. Thus C is an (α, β)-quasi-ideal of Q. Since Q is (α, β)-quasi-simple, $C = Q$. Hence, Q is a minimal (α, β)-quasi-ideal of S. □

5.1.3 (α, β)-bi-ideals

We will define (α, β)-bi-ideals of Γ-semigroups as follows:

Definition 5.1.18. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset B of S is called

(1) an (α, β)-*bi-ideal* of S if $B\alpha S\beta B \subseteq B$,

(2) an α-*bi-ideal* of S if it is an (α, α)-bi-ideal of S.

Theorem 5.1.19. *Every (α, β)-quasi-ideal of a Γ-semigroup S is a (β, α)-bi-ideal of S.*

Proof. Let Q be an (α, β)-quasi-ideal of S. Then $Q\beta S\alpha Q \subseteq Q\beta S \cap S\alpha Q \subseteq Q$. Hence, Q is a (β, α)-bi-ideal of S. □

Theorem 5.1.20. *Let S be a Γ-semigroup and B_i an (α, β)-bi-ideal of S for every $i \in I$. If $\bigcap_{i \in I} B_i \neq \emptyset$, then $\bigcap_{i \in I} B_i$ is an (α, β)-bi-ideal of S.*

Proof. Assume that $\bigcap_{i \in I} B_i \neq \emptyset$. Since for each $i \in I$, B_i is an (α, β)-bi-ideal of S, we get $(\bigcap_{i \in I} B_i)\alpha S\beta(\bigcap_{i \in I} B_i) \subseteq B_i\alpha S\beta B_i \subseteq B_i$ for all $i \in I$. So $(\bigcap_{i \in I} B_i)\alpha S\beta(\bigcap_{i \in I} B_i) \subseteq \bigcap_{i \in I} B_i$, and hence, $\bigcap_{i \in I} B_i$ is an (α, β)-bi-ideal of S. □

For a nonempty subset A of a Γ-semigroup S, let $(A)_{b(\alpha,\beta)}$ be the intersection of all (α, β)-bi-ideals of S containing A. Then $(A)_{b(\alpha,\beta)}$ is the smallest (α, β)-bi-ideal of S containing A which is called the (α, β)-bi-ideal of S generated by A.

Theorem 5.1.21. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

$$(A)_{b(\alpha,\beta)} = A \cup A\alpha S\beta A.$$

Proof. Let $B = A \cup A\alpha S\beta A$. Clearly, $A \subseteq B$. We have that

$$\begin{aligned}
B\alpha S\beta B &= (A \cup A\alpha S\beta A)\alpha S\beta(A \cup A\alpha S\beta A) \\
&= [A\alpha S\beta(A \cup A\alpha S\beta A)] \cup [(A\alpha S\beta A)\alpha S\beta(A \cup A\alpha S\beta A)] \\
&= [(A\alpha S\beta A) \cup (A\alpha S\beta A\alpha S\beta A)] \cup [(A\alpha S\beta A)\alpha S\beta A \cup (A\alpha S\beta A)\alpha S\beta A\alpha S\beta A)] \\
&\subseteq A\alpha S\beta A \\
&\subseteq B.
\end{aligned}$$

Then B is an (α,β)-bi-ideal of S containing A. Let C be any (α,β)-bi-ideal of S containing A. Since C is an (α,β)-bi-ideal of S and $A \subseteq C$, it follows that $A\alpha S\beta A \subseteq C\alpha S\beta C \subseteq C$. Thus $B = A \cup A\alpha S\beta A \subseteq C$. Hence, B is the smallest (α,β)-bi-ideal of S containing A. Consequently, $(A)_{b(\alpha,\beta)} = B = A \cup A\alpha S\beta A$. □

Theorem 5.1.22. *Let S be a Γ-semigroup, $\emptyset \neq A \subseteq S$, $\alpha,\beta \in \Gamma$ and B an (α,β)-bi-ideal of S. Then $B\alpha A$ and $A\beta B$ are (α,β)-bi-ideals of S.*

Proof. Since B is an (α,β)-bi-ideal of S, we have

$$(B\alpha A)\alpha S\beta(B\alpha A) \subseteq (B\alpha S\beta B)\alpha A \subseteq B\alpha A,$$

$$(A\beta B)\alpha S\beta(A\beta B) \subseteq A\beta(B\alpha S\beta B) \subseteq A\beta B.$$

Then $B\alpha A$ and $A\beta B$ are (α,β)-bi-ideals of S. □

Theorem 5.1.23. *Let S be a Γ-semigroup and $\alpha,\beta \in \Gamma$ and let B_1, B_2 and B_3 be nonempty subsets of S.*

(1) *If B_1 or B_3 is an (α,β)-bi-ideal of S, then $B_1\alpha B_2\beta B_3$ is an (α,β)-bi-ideal of S.*

(2) *If B_2 is an (β,α)-bi-ideal of S, then $B_1\alpha B_2\beta B_3$ is an (α,β)-bi-ideal of S.*

Proof. (1) Assume that B_1 or B_3 is an (α,β)-bi-ideal of S. By Theorem 5.1.22, $B_1\alpha B_2\beta B_3$ is an (α,β)-bi-ideal of S.

(2) Suppose that B_2 is a (β,α)-bi-ideal of S. Then

$$\begin{aligned}(B_1\alpha B_2\beta B_3)\alpha S\beta(B_1\alpha B_2\beta B_3) &= B_1\alpha(B_2\beta B_3\alpha S\beta B_1\alpha B_2)\beta B_3\\ &\subseteq B_1\alpha(B_2\beta S\alpha B_2)\beta B_3\\ &\subseteq B_1\alpha B_2\beta B_3.\end{aligned}$$

Therefore, $B_1\alpha B_2\beta B_3$ is an (α,β)-bi-ideal of S. □

Definition 5.1.24. Let $\alpha, \beta \in \Gamma$. A Γ-semigroup S is said to be (α, β)-*bi-simple* if S does not contain any proper (α, β)-bi-ideals.

Theorem 5.1.25. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. Then S is (α, β)-bi-simple if and only if $s\alpha S\beta s = S$ for all $s \in S$.*

Proof. Assume that S is (α, β)-bi-simple. Let $s \in S$. We claim that $s\alpha S\beta s$ is an (α, β)-bi-ideal of S. We have $s\alpha s\beta s \in s\alpha S\beta s$. This implies $s\alpha S\beta s \neq \emptyset$. Moreover, $(s\alpha S\beta s)\alpha S\beta(s\alpha S\beta s) = s\alpha(S\beta s\alpha S\beta s\alpha S)\beta s \subseteq s\alpha S\beta s$. Therefore, $s\alpha S\beta s$ is an (α, β)-bi-ideal of S. Since S is (α, β)-bi-simple, $S = s\alpha S\beta s$.

Conversely, assume that $s\alpha S\beta s = S$ for all $s \in S$. Let B be an (α, β)-bi-ideal of S and $b \in B$. By assumption, $S = b\alpha S\beta b$. Since B is an (α, β)-bi-ideal of S, we get $S = b\alpha S\beta b \subseteq B\alpha S\beta B \subseteq B$. Then $B = S$. Therefore, S is (α, β)-bi-simple. □

Definition 5.1.26. An (α, β)-bi-ideal B of a Γ-semigroup S is said to be *minimal* if $C \subseteq B$ implies $C = B$ for any (α, β)-bi-ideal C of S.

Theorem 5.1.27. *Let S be a Γ-semigroup and B an (α, β)-bi-ideal of S. If B is (α, β)-bi-simple, then B is a minimal (α, β)-bi-ideal of S.*

Proof. Assume that B is (α, β)-bi-simple. Let C be an (α, β)-bi-ideal of S such that $C \subseteq B$. Then $C\alpha B\beta C \subseteq C\alpha S\beta C \subseteq C$. Therefore, C is an (α, β)-bi-ideal of B. Since B is (α, β)-bi-simple, $C = B$. Then B is a minimal (α, β)-bi-ideal of S. □

5.2 New Types of Fuzzy Ideals

In this section, we will study fuzzy (α,β)-ideals, fuzzy (α,β)-quasi-ideals and fuzzy (α,β)-bi-ideals of Γ-semigroups for $\alpha,\beta\in\Gamma$.

5.2.1 Fuzzy (α,β)-ideals

We will define fuzzy (α,β)-ideals of Γ-semigroups as follows:

Definition 5.2.1. Let f be a fuzzy subset of a Γ-semigroup S such that $f\neq 0$ and $\alpha,\beta\in\Gamma$. Then f is called

(1) a *fuzzy left α-ideal* of S if $f(x\alpha y)\geq f(y)$ for all $x,y\in S$,

(2) a *fuzzy right β-ideal* of S if $f(x\beta y)\geq f(x)$ for all $x,y\in S$,

(3) a *fuzzy (α,β)-ideal* of S if it is both a fuzzy left α-ideal and a fuzzy right β-ideal of S.

The following theorem shows the relationships between (α,β)-ideals and fuzzy (α,β)-ideals.

Theorem 5.2.2. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha,\beta\in\Gamma$. Then the following statements hold*

(1) *A is a left α-ideal of S if and only if C_A is a fuzzy left α-ideal of S.*

(2) *A is a right β-ideal of S if and only if C_A is a fuzzy right β-ideal of S.*

(3) *A is an (α,β)-ideal of S if and only if C_A is a fuzzy (α,β)-ideal of S.*

Proof. (1) Suppose that A is a left α-ideal of S. Let $x,y\in S$. If $y\in A$, then $x\alpha y\in S\alpha A\subseteq A$, so $C_A(x\alpha y)=1$ and hence, $C_A(x\alpha y)\geq C_A(y)$. If $y\notin A$, then $C_A(y)=0\leq C_A(x\alpha y)$. Therefore, C_A is a fuzzy left α-ideal of S.

Conversely, assume that C_A is a fuzzy left α-ideal of S. Let $x\in S$ and $y\in A$. Then $C_A(y)=1$. Since $C_A(x\alpha y)\geq C_A(y)$, we have $x\alpha y\in A$. Hence, A is a left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

Theorem 5.2.3. *Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$ and $\alpha, \beta \in \Gamma$. Then the following statements hold.*

(1) *f is a fuzzy left α-ideal of S if and only if $S \circ_\alpha f \subseteq f$.*

(2) *f is a fuzzy right β-ideal of S if and only if $f \circ_\beta S \subseteq f$.*

(3) *f is a fuzzy (α, β)-ideal of S if and only if $S \circ_\alpha f \subseteq f$ and $f \circ_\beta S \subseteq f$.*

Proof. (1) Let $x \in S$. If $x \notin S\alpha S$, then $(S \circ_\alpha f)(x) = 0$, so $(S \circ_\alpha f)(x) \leq f(x)$. If $x \in S\alpha S$, since f is a fuzzy left α-ideal of S, we have

$$\begin{aligned}(S \circ_\alpha f)(x) &= \sup_{x=y\alpha z} \{\min\{S(y), f(z)\}\} \\ &= \sup_{x=y\alpha z} \{\min\{1, f(z)\}\} \\ &= \sup_{x=y\alpha z} \{f(z)\} \\ &\leq f(x).\end{aligned}$$

We conclude that $S \circ_\alpha f \subseteq f$.

Conversely, assume that $S \circ_\alpha f \subseteq f$. Let $y, z \in S$ and let $x = y\alpha z$. Then

$$\begin{aligned}f(y\alpha z) &= f(x) \\ &\geq (S \circ_\alpha f)(x) \\ &= \sup_{x=u\alpha v} \{\min\{S(u), f(v)\}\} \\ &\geq \min\{S(y), f(z)\} \\ &= \min\{1, f(z)\} \\ &= f(z).\end{aligned}$$

Hence, f is a fuzzy left α-ideal of S.

The proofs of (2) and (3) can be seen in similar fashion. $\square$

Theorem 5.2.4. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If f and g are fuzzy left α-ideals of S, then $f \cup g$ is a fuzzy left α-ideal of S and $f \cap g$ is a fuzzy left α-ideal of S if $f \cap g \neq 0$.*

(2) *If f and g are fuzzy right β-ideals of S, then $f \cup g$ is a fuzzy right β-ideal of S and $f \cap g$ is a fuzzy right β-ideal of S if $f \cap g \neq 0$.*

(3) *If f and g are fuzzy (α, β)-ideals of S, then $f \cup g$ is a fuzzy (α, β)-ideal of S and $f \cap g$ is a fuzzy (α, β)-ideal of S if $f \cap g \neq 0$.*

Proof. (1) Assume that f and g are fuzzy left α-ideals of S. Let $x, y \in S$. Then

$$\begin{aligned}(f \cap g)(x\alpha y) &= \min\{f(x\alpha y), g(x\alpha y)\} \\ &\geq \min\{f(y), g(y)\} \\ &= (f \cap g)(y), \\ (f \cup g)(x\alpha y) &= \max\{f(x\alpha y), g(x\alpha y)\} \\ &\geq \max\{f(y), g(y)\} \\ &= (f \cup g)(y).\end{aligned}$$

Hence, $f \cap g$ and $f \cup g$ are fuzzy left α-ideals of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

Theorem 5.2.5. *Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$ and $\alpha, \beta \in \Gamma$. The following statements are true.*

(1) *f is a fuzzy left α-ideal of S if and only if f_t is a left α-ideal of S for all $t \in (0, 1]$ provide $f_t \neq \emptyset$.*

(2) *f is a fuzzy right β-ideal of S if and only if f_t is a right β-ideal of S for all $t \in (0, 1]$ provide $f_t \neq \emptyset$.*

(3) *f is a fuzzy (α, β)-ideal of S if and only if f_t is an (α, β)-ideal of S for all $t \in (0, 1]$ provide $f_t \neq \emptyset$.*

Proof. (1) Suppose that f is a fuzzy left α-ideal of S. Then $f(x\alpha y) \geq f(y)$ for all $x, y \in S$. Let $t \in (0, 1]$ such that $f_t \neq \emptyset$. Let $x \in f_t$ and $s \in S$. Since $f(x) \geq t$, $f(s\alpha x) \geq f(x) \geq t$. Thus $s\alpha x \in f_t$. Hence, f_t is a left α-ideal of S.

Conversely, assume that f_t is a left α-ideal of S for all $t \in (0, 1]$ with $f_t \neq \emptyset$. Let $x, y \in S$. If $f(y) = 0$, then $f(x\alpha y) \geq f(y)$. Assume that $f(y) > 0$. Let $t = f(y)$. Then $t \in (0, 1]$ and $y \in f_t$, so that $f_t \neq \emptyset$. By assumption f_t is a left

α-ideal of S. Since $y \in f_t$ and $x \in S$, we have $x\alpha y \in f_t$. Then $f(x\alpha y) \geq t = f(y)$. Hence, f is a fuzzy left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

5.2.2 Fuzzy (α, β)-quasi-ideals

We will define fuzzy (α, β)-quasi-ideals of Γ-semigroups as follows:

Definition 5.2.6. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A fuzzy subset $f \neq 0$ of S is called a *fuzzy* (α, β)*-quasi-ideal* of S if $(S \circ_\alpha f) \cap (f \circ_\beta S) \subseteq f$.

A fuzzy subset $f \neq 0$ of S is called a *fuzzy* α*-quasi-ideal* of S if f is a fuzzy (α, α)-quasi-ideal of S.

Theorem 5.2.7. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. Then f is a fuzzy (α, β)-quasi-ideal of S if and only if $f = g \cap h$ where g is a fuzzy left α-ideal of S and h is a fuzzy right β-ideal of S.*

Proof. Let f be a fuzzy (α, β)-quasi-ideal of a Γ-semigroup S. Let $g = f \cup (S \circ_\alpha f)$ and $h = f \cup (f \circ_\beta S)$. Then

$$S \circ_\alpha g = S \circ_\alpha (f \cup (S \circ_\alpha f)) = (S \circ_\alpha f) \cup (S \circ_\alpha (S \circ_\alpha f)) \subseteq S \circ_\alpha f \subseteq g$$

and also $h \circ_\beta S \subseteq h$. Thus g is a fuzzy left α-ideal of S and h is a fuzzy right β-ideal of S. We claim that $f = g \cap h$. We see that, $f \subseteq (f \cup (S \circ_\alpha f)) \cap (f \cup (f \circ_\beta S)) = g \cap h$ and conversely, $g \cap h = (f \cup (S \circ_\alpha f)) \cap (f \cup (f \circ_\beta S)) \subseteq f \cup ((S \circ_\alpha f) \cap (f \circ_\beta S)) \subseteq f$. Therefore, $f = g \cap h$.

Conversely, let $f = g \cap h$ where g is a fuzzy left α-ideal of S and h is a fuzzy right β-ideal of S. Since $h \circ_\alpha g \subseteq g \cap h = f$, we get $f \neq 0$. We have

$$(S \circ_\alpha f) \cap (f \circ_\beta S) = (S \circ_\alpha (g \cap h)) \cap ((g \cap h) \circ_\beta S) \subseteq (S \circ_\alpha g) \cap (h \circ_\beta S) \subseteq g \cap h = f.$$

Hence, f is a fuzzy (α, β)-quasi-ideal of S. □

Theorem 5.2.8. *Let Q be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then Q is an (α, β)-quasi-ideal of S if and only if C_Q is a fuzzy (α, β)-quasi-ideal of S.*

Proof. Assume that Q is an (α, β)-quasi-ideal of a Γ-semigroup S. Let $y \in S$. If $y \notin S\alpha Q \cap Q\beta S$, then $[(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S)](y) = 0 \leq C_Q(y)$. If $y \in S\alpha Q \cap Q\beta S$, then $y \in Q$, so $[(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S)](y) = 1 = C_Q(y)$. Hence, $(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S) \subseteq C_Q$. Therefore, C_Q is a fuzzy (α, β)-quasi-ideal of S.

Conversely, assume that C_Q is a fuzzy (α, β)-quasi-ideal of S. Then $(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S) \subseteq C_Q$. Let $x \in S\alpha Q \cap Q\beta S$. Then $[(S \circ_\alpha C_Q) \cap (C_Q \circ_\beta S)](x) = 1$ and this implies that $C_Q(x) = 1$. So $S\alpha Q \cap Q\beta S \subseteq Q$. Consequently, Q is an (α, β)-quasi-ideal of S. □

5.2.3 Fuzzy (α, β)-bi-ideals

We define fuzzy (α, β)-bi-ideals of Γ-semigroups as follows:

Definition 5.2.9. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A fuzzy subset $f \neq 0$ of S is called a *fuzzy (α, β)-bi-ideal* of S if $f \circ_\alpha S \circ_\beta f \subseteq f$.

Theorem 5.2.10. *Let S be a Γ-semigroup, $\alpha, \beta \in \Gamma$, g be a fuzzy subset of S and f be a fuzzy (α, β)-bi-ideal of S. Then $f \circ_\alpha g$ and $g \circ_\beta f$ are fuzzy (α, β)-bi-ideals of S if $f \circ_\alpha g \neq 0$ and $g \circ_\beta f \neq 0$.*

Proof. Since f is a fuzzy (α, β)-bi-ideal of S, we get

$$(f \circ_\alpha g) \circ_\alpha S \circ_\beta (f \circ_\alpha g) = f \circ_\alpha (g \circ_\alpha S) \circ_\beta (f \circ_\alpha g) \subseteq (f \circ_\alpha S \circ_\beta f) \circ_\alpha g \subseteq f \circ_\alpha g,$$

$$(g \circ_\beta f) \circ_\alpha S \circ_\beta (g \circ_\beta f) = g \circ_\beta f \circ_\alpha (S \circ_\beta g) \circ_\beta f) \subseteq g \circ_\beta (f \circ_\alpha S \circ_\beta f) \subseteq g \circ_\beta f.$$

Then $f \circ_\alpha g$ and $g \circ_\beta f$ are fuzzy (α, β)-bi-ideals of S. □

Theorem 5.2.11. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$ and let f_1, f_2 and f_3 be fuzzy subsets of S such that $f_1 \circ_\alpha f_2 \circ_\beta f_3 \neq 0$.*

(1) *If f_1 or f_3 is a fuzzy (α, β)-bi-ideal of S, then $f_1 \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy bi-(α, β)-ideal of S.*

(2) *If f_2 is fuzzy (β, α)-bi-ideal of S, then $f_1 \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy bi-(α, β)-ideal of S.*

Proof. (1) Assume that f_1 or f_3 is a fuzzy (α, β)-bi-ideal of S. By Theorem 5.2.10, $f_1 \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy (α, β)-bi-ideal of S.

(2) Suppose that f_2 is a fuzzy (β, α)-bi-ideal of S. Then

$$(f_1 \circ_\alpha f_2 \circ_\beta f_3) \circ_\alpha S \circ_\beta (f_1 \circ_\alpha f_2 \circ_\beta f_3) \subseteq f_1 \circ_\alpha (f_2 \circ_\beta S \circ_\alpha f_2) \circ_\beta f_3 \subseteq f_1 \circ_\alpha f_2 \circ_\beta f_3.$$

Hence, $f \circ_\alpha f_2 \circ_\beta f_3$ is a fuzzy (α, β)-bi-ideal of S. □

Theorem 5.2.12. *Let S be a Γ-semigroup, B a nonempty subset of S and $\alpha, \beta \in \Gamma$. Then B is an (α, β)-bi-ideal of S if and only if C_B is a fuzzy (α, β)-bi-ideal of S.*

Proof. Assume that B is an (α, β)-bi-ideal of S. Then $B\alpha S\beta B \subseteq B$. Let $z \in S$. If $z \notin B\alpha S\beta B$, then $(C_B \circ_\alpha S \circ_\beta C_B)(z) = 0 \leq C_B(z)$. If $z \in B\alpha S\beta B$, then $z \in B$, so $(C_B \circ_\alpha S \circ_\beta C_B)(z) = 1 = C_B(z)$. This shows that $C_B \circ_\alpha S \circ_\beta C_B \subseteq C_B$. Hence, C_B is a fuzzy (α, β)-bi-ideal of S.

Conversely, assume that C_B is a fuzzy (α, β)-bi-ideal of S. Then $C_B \circ_\alpha S \circ_\beta C_B \subseteq C_B$. Let $z \in B\alpha S\beta B$. Then $1 = (C_B \circ_\alpha S \circ_\beta C_B)(z) \leq C_B(z)$ which implies that $C_B(z) = 1$, so $z \in B$. Hence, $B\alpha S\beta B \subseteq B$. Consequently, B is an (α, β)-bi-ideal of S. □

Theorem 5.2.13. *Let f and g be two fuzzy (α, β)-bi-ideals of a Γ-semigroup S. If $f \cap g \neq 0$, then $f \cap g$ is a fuzzy (α, β)-bi-ideal of S.*

Proof. Assume that $f \cap g \neq 0$. Since f and g are fuzzy (α, β)-bi-ideals of S, we obtain $(f \cap g) \circ_\alpha S \circ_\beta (f \cap g) \subseteq (f \circ_\alpha S \circ_\beta f) \cap (g \circ_\alpha S \circ_\beta g) \subseteq f \cap g$. Hence, $f \cap g$ is a fuzzy (α, β)-bi-ideal of S. □

5.3 New Types of Almost Ideals

In this section, we study almost (α, β)-ideals, almost (α, β)-quasi-ideals and almost (α, β)-bi-ideals of Γ-semigroups for $\alpha, \beta \in \Gamma$.

5.3.1 Almost (α, β)-ideals

We define almost (α, β)-ideals of Γ-semigroups as follows:

Definition 5.3.1. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset A of S is called

(1) an *almost left α-ideal* of S if $s\alpha A \cap A \neq \emptyset$ for all $s \in S$,

(2) an *almost right β-ideal* of S if $A\beta s \cap A \neq \emptyset$ for all $s \in S$,

(3) an *almost (α, β)-ideal* of S if it is both an almost left α-ideal and an almost right β-ideal of S.

Theorem 5.3.2. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If L is a left α-ideal of S, then L is an almost left α-ideal of S.*

(2) *If R is a right β-ideal of S, then R is an almost right β-ideal of S.*

(3) *If I is an (α, β)-ideal of S, then I is an almost (α, β)-ideal of S.*

Proof. (1) Let L be a left α-ideal of S. Then $s\alpha L \subseteq L$ for all $s \in S$, so $s\alpha L \cap L = s\alpha L \neq \emptyset$ for all $s \in S$. Thus L is an almost left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). $\square$

Theorem 5.3.3. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If L is an almost left α-ideal of S and $L \subseteq H \subseteq S$, then H is an almost left α-ideal of S.*

(2) *If R is an almost right β-ideal of S and $R \subseteq H \subseteq S$, then H is an almost right β-ideal of S.*

(3) *If I is an almost (α, β)-ideal of S and $I \subseteq H \subseteq S$, then H is an almost (α, β)-ideal of S.*

Proof. (1) Let L be an almost left α-ideal of S and $L \subseteq H \subseteq S$. Let $s \in S$. Since $\emptyset \neq s\alpha L \cap L \subseteq s\alpha H \cap H$, we have $s\alpha H \cap H \neq \emptyset$. Therefore, H is an almost left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

Corollary 5.3.4. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *If L_1 and L_2 are almost left α-ideals of S, then $L_1 \cup L_2$ is an almost left α-ideal of S.*

(2) *If R_1 and R_2 are almost right β-ideals of S, then $R_1 \cup R_2$ is an almost right β-ideal of S.*

(3) *If I_1 and I_2 are almost (α, β)-ideals of S, then $I_1 \cup I_2$ is an almost (α, β)-ideal of S.*

Proof. (1) Let L_1 and L_2 be almost left α-ideals of S. Since $L_1 \subseteq L_1 \cup L_2$, by Theorem 5.3.3 (1), $L_1 \cup L_2$ is an almost left α-ideal of S.

The proof of (2) is similar to (1) and the proof of (3) follows from (1) and (2). □

Example 5.3.5. Consider a Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha, \beta\}$ and the multiplication tables:

α	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

β	a	b	c	d	e
a	b	c	d	e	a
b	c	d	e	a	b
c	d	e	a	b	c
d	e	a	b	c	d
e	a	b	c	d	e

.

We have that $\{a,b,d\}$ and $\{a,c,d\}$ are almost left α-ideals of S. However, $\{a,b,d\} \cap \{a,c,d\} = \{a,d\}$ is not an almost left α-ideal of S.

The intersection of two almost left α-ideals [almost right β-ideals, almost (α,β)-ideals] of a Γ-semigroup S need not always be an almost left α-ideal [almost right β-ideal, almost (α,β)-ideal] of S.

Definition 5.3.6. Let S be Γ-semigroup and $\alpha,\beta \in \Gamma$. Then S is said to be

(1) *almost left α-simple* if S does not contain any proper almost left α-ideals,

(2) *almost right β-simple* if S does not contain any proper almost right β-ideals.

Theorem 5.3.7. *Let S be a Γ-semigroup and $\alpha,\beta \in \Gamma$.*

(1) *S is almost left α-simple if and only if for each $a \in S$, there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \subseteq \{a\}$.*

(2) *S is almost right β-simple if and only if for each $a \in S$, there exists $s_a \in S$ such that $(S \setminus \{a\})\beta s_a \subseteq \{a\}$.*

Proof. (1) Assume that S is almost left α-simple. Then S has no proper almost left α-ideals. Let $a \in S$. Then $S \setminus \{a\}$ is not an almost left α-ideal of S. Thus there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$, this implies that $s_a\alpha(S \setminus \{a\}) \subseteq \{a\}$.

Conversely, suppose that for each $a \in S$, there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \subseteq \{a\}$. Let L be a proper subset of S. Then $L \subseteq S \setminus \{a\}$ for some $a \in S$. By assumption, there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \subseteq \{a\}$. Thus $s_a\alpha(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$. Hence, L is not an almost left α-ideal of S. Then S has no proper left almost α-ideals. Therefore, S is almost left α-simple.

The proof of (2) is similar to (1). □

5.3.2 Almost (α, β)-quasi-ideals

We define almost (α, β)-quasi-ideals of Γ-semigroups as follows:

Definition 5.3.8. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset Q of S is called an *almost* (α, β)*-quasi-ideal* of S if $s\alpha Q \cap Q\beta s \cap Q \neq \emptyset$ for all $s \in S$.

Proposition 5.3.9. *Every (α, β)-quasi-ideal of a Γ-semigroup S is either $s\alpha Q \cap Q\beta s = \emptyset$ for some $s \in S$ or an almost (α, β)-quasi-ideal of S.*

Proof. Let Q be an (α, β)-quasi-ideal of S and $s\alpha Q \cap Q\beta s \neq \emptyset$ for all $s \in S$. Let $s \in S$. Then $s\alpha Q \cap Q\beta s \neq \emptyset$. Since $s\alpha Q \cap Q\beta s \subseteq S\alpha Q \cap Q\beta S \subseteq Q$, we get that $s\alpha Q \cap Q\beta s \cap Q = s\alpha Q \cap Q\beta s \neq \emptyset$. Hence, Q is an almost (α, β)-quasi-ideal of S. □

Theorem 5.3.10. *Every almost (α, β)-quasi-ideal of a Γ-semigroup S is an almost left α-ideal and almost right β-ideal of S.*

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S. Let $s \in S$. Then $\emptyset \neq s\alpha Q \cap Q\beta s \cap Q \subseteq s\alpha Q \cap Q$ and $\emptyset \neq s\alpha Q \cap Q\beta s \cap Q \subseteq Q\beta s \cap Q$. Thus $s\alpha Q \cap Q \neq \emptyset$ and $Q\beta s \cap Q \neq \emptyset$. Hence, Q is an almost left α-ideal and an almost right β-ideal of S. □

Theorem 5.3.11. *If Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S and $Q \subseteq H \subseteq S$, then H is an almost (α, β)-quasi-ideal of S.*

Proof. Assume that Q is an almost (α, β)-quasi-ideal of a Γ-semigroup S and $Q \subseteq H \subseteq S$. Let $s \in S$. Then $\emptyset \neq s\alpha Q \cap Q\beta s \cap Q \subseteq s\alpha H \cap H\beta s \cap H$ for all $s \in S$. This implies that $s\alpha H \cap H\beta s \cap H \neq \emptyset$ for all $s \in S$. Therefore, H is an almost (α, β)-quasi-ideal of S. □

Corollary 5.3.12. *The union of two almost (α, β)-quasi-ideals of a Γ-semigroup S is an almost (α, β)-quasi-ideal of S.*

Proof. Let Q_1 and Q_2 be almost (α, β)-quasi-ideals of S. Since $Q_1 \subseteq Q_1 \cup Q_2$, by Theorem 5.3.11, $Q_1 \cup Q_2$ is an almost (α, β)-quasi-ideal of S. □

From Example 5.3.5, we have that $\{b, c, e\}$ and $\{b, d, e\}$ are almost (α, β)-quasi-ideals of S but their intersection is not. Then the intersection of almost (α, β)-quasi-ideals need not always be an almost (α, β)-quasi-ideal of S.

Definition 5.3.13. A Γ-semigroup S is said to be *almost* (α, β)*-quasi-simple* if S does not contain any proper almost (α, β)-quasi-ideals.

Theorem 5.3.14. *A Γ-semigroup S is almost (α, β)-quasi-simple if and only if for each $a \in S$, there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \cap (S \setminus \{a\})\beta s_a \subseteq \{a\}$.*

Proof. Assume that S is almost (α, β)-quasi-simple and let $a \in S$. Then $S \setminus \{a\}$ is not an almost (α, β)-quasi-ideal of S. Then there exists $s_a \in S$ such that $s_a\alpha(S\setminus\{a\})\cap(S\setminus\{a\})\beta s_a\cap(S\setminus\{a\}) = \emptyset$. Therefore, $s_a\alpha(S\setminus\{a\})\cap(S\setminus\{a\})\beta s_a \subseteq \{a\}$.

Conversely, assume that for each $a \in S$ there exists $s_a \in S$ such that $s_a\beta(S \setminus \{a\}) \cap (S \setminus \{a\})\beta s_a \subseteq \{a\}$. Let A be a proper subset of S. Then $A \subseteq S \setminus \{a\}$ for some $a \in S$. By assumption, there exists $s_a \in S$ such that $s_a\alpha(S \setminus \{a\}) \cap (S \setminus \{a\})\beta s_a \subseteq \{a\}$, so $s_a\alpha(S \setminus \{a\}) \cap (S \setminus \{a\})\beta s_a \cap (S \setminus \{a\}) = \emptyset$. Then A is not an almost (α, β)-quasi-ideal of S. Thus S has no proper almost (α, β)-quasi-ideal. Hence, S is almost (α, β)-quasi-simple. ☐

5.3.3 Almost (α, β)-bi-ideals

We define almost (α, β)-bi-ideals of Γ-semigroups as follows:

Definition 5.3.15. Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. A nonempty subset B of S is called an *almost (α, β)-bi-ideal* of S if $B\alpha s\beta B \cap B \neq \emptyset$ for all $s \in S$.

Theorem 5.3.16. *If B is an almost (α, β)-bi-ideal of a Γ-semigroup S and $B \subseteq C \subseteq S$, then C is an almost (α, β)-bi-ideal of S.*

Proof. Let B be an almost (α, β)-bi-ideal of a Γ-semigroup S and $B \subseteq C \subseteq S$. Since $B\alpha s\beta B \cap B \neq \emptyset$ for all $s \in S$ and $B\alpha s\beta B \cap B \subseteq C\alpha s\beta C \cap C$, it follows that $C\alpha s\beta C \cap C \neq \emptyset$ for all $s \in S$. Therefore, C is an almost (α, β)-bi-ideal of S. ☐

Corollary 5.3.17. *The union of two almost (α, β)-bi-ideals of a Γ-semigroup S is also an almost (α, β)-bi-ideal of S.*

Proof. Let B_1 and B_2 be almost (α, β)-bi-ideals of S. Since $B_1 \subseteq B_1 \cup B_2$, by Theorem 5.3.16, $B_1 \cup B_2$ is an almost (α, β)-bi-ideal of S. ☐

From Example 5.3.5, we have that $\{b, c, e\}$ and $\{b, d, e\}$ are almost (α, β)-bi-ideals of S but their intersection is not. Then the intersection of almost (α, β)-bi-ideals need not always be an almost (α, β)-bi-ideal of S.

Theorem 5.3.18. *A Γ-semigroup S has a proper almost (α, β)-bi-ideal if and only if there exists an element $a \in S$ such that $S \setminus \{a\}$ is an almost (α, β)-bi-ideal of S.*

Proof. Let B be a proper almost (α, β)-bi-ideal of S and let $a \in S \setminus B$. Then $B \subseteq S \setminus \{a\} \subset S$. By Theorem 5.3.16, $S \setminus \{a\}$ is an almost (α, β)-bi-ideal of S.

Conversely, let $a \in S$ such that $S \setminus \{a\}$ is an almost (α, β)-bi-ideal. Since $S \setminus \{a\} \subset S$, we have $S \setminus \{a\}$ is a proper almost (α, β)-bi-ideal of S. □

Definition 5.3.19. A Γ-semigroup S is said to be *almost* (α, β)*-bi-simple* if S does not contain any proper almost (α, β)-bi-ideals.

Theorem 5.3.20. *A* Γ*-semigroup* S *is almost* (α, β)*-bi-simple if and only if for each* $a \in S$*, there exists* $s_a \in S$ *such that* $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \subseteq \{a\}$.

Proof. Assume that S is almost (α, β)-bi-simple. Let $a \in S$. Then $S \setminus \{a\}$ is not an almost (α, β)-bi-ideal of S. Thus $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$ for some $s_a \in S$. Hence, $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \subseteq \{a\}$.

Conversely, suppose that for each $a \in S$, there exists $s_a \in S$ such that $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \subseteq \{a\}$. Let B be a proper subset of S. Then $B \subseteq S \setminus \{a\}$ for some $a \in S$. By assumption, $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \subseteq \{a\}$ for some $s_a \in S$, so $(S \setminus \{a\})\alpha s_a \beta(S \setminus \{a\}) \cap (S \setminus \{a\}) = \emptyset$. Thus B is not an almost (α, β)-bi-ideal of S. Hence, S has no proper almost (α, β)-bi-ideals. Therefore, S is almost (α, β)-bi-simple. □

5.4 New Types of Fuzzy Almost Ideals

In this section, we study fuzzy almost (α, β)-ideals, fuzzy almost (α, β)-quasi-ideals and fuzzy almost (α, β)-bi-ideals of Γ-semigroups for $\alpha, \beta \in \Gamma$.

5.4.1 Fuzzy almost (α, β)-ideals

We define fuzzy almost (α, β)-ideals of Γ-semigroups as follows:

Definition 5.4.1. Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$ and $\alpha, \beta \in \Gamma$. Then f is called

(1) a *fuzzy almost left α-ideal* of S if $(x_t \circ_\alpha f) \cap f \neq 0$ for each fuzzy point x_t of S,

(2) a *fuzzy almost right β-ideal* of S if $(f \circ_\beta x_t) \cap f \neq 0$ for each fuzzy point x_t of S,

(3) a *fuzzy almost (α, β)-ideal* of S if it is both a fuzzy almost left α-ideal and a fuzzy almost right β-ideal of S.

Theorem 5.4.2. *Let f and g be fuzzy subsets of a Γ-semigroup S and $\alpha, \beta \in \Gamma$ such that $f \subseteq g$.*

(1) *If f is a fuzzy almost left α-ideal of S, then g is a fuzzy almost left α-ideal of S.*

(2) *If f is a fuzzy almost right β-ideal of S, then g is a fuzzy almost right β-ideal of S.*

(3) *If f is a fuzzy almost (α, β)-ideal of S, then g is a fuzzy almost (α, β)-ideal of S.*

Proof. (1) Let x_t be a fuzzy point of S. Since f is a fuzzy almost left α-ideal of S, $(x_t \circ_\alpha f) \cap f \neq 0$. Since $f \subseteq g$, we have $(x_t \circ_\alpha f) \cap f \subseteq (x_t \circ_\alpha g) \cap g$. This implies that $(x_t \circ_\alpha g) \cap g \neq 0$. Therefore, g is a fuzzy almost left α-ideal of S.

The proof of (2) and (3) are similar to the proof of (1). $\square$

Corollary 5.4.3. *Let Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$. The following statements hold.*

(1) *The union of two fuzzy almost left α-ideals of S is a fuzzy almost left α-ideal of S.*

(2) *The union of two fuzzy almost right β-ideals of S is a fuzzy almost right β-ideal of S.*

(3) *The union of two fuzzy almost (α, β)-ideals of S is a fuzzy almost (α, β)-ideal of S.*

Proof. (1) Let f and g be fuzzy almost left α-ideals of S. Since $f \subseteq f \cup g$, by Theorem 5.4.2 (1), $f \cup g$ is a fuzzy almost left α-ideal of S.

The proofs of (2) and (3) can be seen in similar fashion. □

Theorem 5.4.4. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

(1) *A is an almost left α-ideal of S if and only if C_A is a fuzzy almost left α-ideal of S.*

(2) *A is an almost right β-ideal of S if and only if C_A is a fuzzy almost right β-ideal of S.*

(3) *A is an almost (α, β)-ideal of S if and only if C_A is a fuzzy almost (α, β)-ideal of S.*

Proof. (1) Assume that A is an almost left α-ideal of S. Let x_t be a fuzzy point of S. Then $x\alpha A \cap A \neq \emptyset$. Thus there exists $y \in x\alpha A$ and $y \in A$. So $(x_t \circ_\alpha C_A)(y) = 1$ and $C_A(y) = 1$. Hence, $(x_t \circ_\alpha C_A) \cap C_A \neq 0$. Therefore, C_A is a fuzzy almost left α-ideal of S.

Conversely, assume that C_A is a fuzzy almost left α-ideal of S. Let $x \in S$. Then $(x_t \circ_\alpha C_A) \cap C_A \neq 0$, so $[(x_t \circ_\alpha C_A) \cap C_A](a) \neq 0$ for some $a \in S$. Hence, $a \in x\alpha A \cap A$. So $x\alpha A \cap A \neq \emptyset$. Consequently, A is an almost left α-ideal of S.

The proofs of (2) and (3) are similar to (1). □

Theorem 5.4.5. *Let f be a fuzzy subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

(1) *f is a fuzzy almost left α-ideal of S if and only if $supp(f)$ is an almost left α-ideal of S.*

(2) *f is a fuzzy almost right β-ideal of S if and only if $supp(f)$ is an almost right β-ideal of S.*

(3) *f is a fuzzy almost (α, β)-ideal of S if and only if $supp(f)$ is an almost (α, β)-ideal of S.*

Proof. (1) Assume that f is a fuzzy almost left α-ideal of a Γ-semigroup S. Let x_t be a fuzzy point of S. Then $(x_t \circ_\alpha f) \cap f \neq 0$. Hence, there exists $a \in S$ such that $[(x_t \circ_\alpha f) \cap f](a) \neq 0$. So there exists $b \in S$ such that $a = x\alpha b, f(a) \neq 0, f(b) \neq 0$. That is $a, b \in supp(f)$. Thus $(x_t \circ_\alpha C_{supp(f)})(a) \neq 0$ and $C_{supp(f)}(a) \neq 0$. Therefore, $(x_t \circ_\alpha C_{supp(f)}) \cap C_{supp(f)} \neq 0$. Hence, $C_{supp(f)}$ is a fuzzy almost left α-ideal of S. By Theorem 5.4.4 (1), $supp(f)$ is an almost left α-ideal of S.

Conversely, assume that $supp(f)$ is an almost left α-ideal of S. By Theorem 5.4.4 (1), $C_{supp(f)}$ is a fuzzy almost left α-ideal of S. Let x_t be a fuzzy point of S. Then $(x_t \circ_\alpha C_{supp(f)}) \cap C_{supp(f)} \neq 0$. Thus there exists $a \in S$ such that $[(x_t \circ_\alpha C_{supp(f)}) \cap C_{supp(f)}](a) \neq 0$. Hence, $(x_t \circ_\alpha C_{supp(f)})(a) \neq 0$ and $C_{supp(f)}(a) \neq 0$. Then there exists $y \in S$ such that $a = x\alpha y, f(a) \neq 0$ and $f(y) \neq 0$. This means that $(x_t \circ_\alpha f) \cap f \neq 0$. Therefore, f is a fuzzy almost left α-ideal of S.

The proof of (2) and (3) are similar to (1). □

Corollary 5.4.6. *Let S be a Γ-semigroup and $\alpha, \beta \in \Gamma$.*

(1) *S has no proper almost left α-ideals if and only if $supp(f) = S$ for each fuzzy almost left α-ideal f of S.*

(2) *S has no proper almost right β-ideals if and only if $supp(f) = S$ for each fuzzy almost right β-ideal f of S.*

(3) *S has no proper (α, β)-ideals if and only if $supp(f) = S$ for each fuzzy almost (α, β)-ideal of S.*

Proof. Assume S has no proper almost left α-ideals and let f be a fuzzy almost left α-ideal of S. By Theorem 5.4.5 (1), $supp(f)$ is an almost left α-ideal of S. Thus $supp(f) = S$.

Conversely, let L be any fuzzy almost left α-ideal of S. By Theorem 5.4.4 (1), C_L is a fuzzy almost left α-ideal of S. By assumption, $supp(C_L) = S$.

Since $supp(C_L) = L$, we get $L = S$. This implies that S has no proper almost left α-ideals.

The proofs of (2) and (3) can be seen in similar fashion. $\square$

Definition 5.4.7. A fuzzy almost left α-ideal f of a Γ-semigroup S is *minimal* if for all fuzzy almost left α-ideal g of S such that $g \subseteq f$, we obtain $supp(g) = supp(f)$.

Theorem 5.4.8. *Let A be a nonempty subset of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. Then*

(1) *A is a minimal almost left α-ideal of S if and only if C_A is a minimal fuzzy almost left α-ideal of S.*

(2) *A is a minimal almost right β-ideal of S if and only if C_A is a minimal fuzzy almost right β-ideal of S.*

(3) *A is a minimal almost (α, β)-ideal of S if and only if C_A is a minimal fuzzy almost (α, β)-ideal of S.*

Proof. (1) Assume that A is a minimal almost left α-ideal of a Γ-semigroup S. By Theorem 5.4.4 (1), C_A is a fuzzy almost left α-ideal of S. Let g be a fuzzy almost left α-ideal of S such that $g \subseteq C_A$. Then $supp(g) \subseteq supp(C_A) = A$. By Theorem 5.4.5 (1), $supp(g)$ is an almost left α-ideal of S. Since A is minimal, $supp(g) = A = supp(C_A)$. Therefore, C_A is minimal.

Conversely, assume that C_A is a minimal fuzzy almost left α-ideal of S. By Theorem 5.4.4 (1), A is an almost left α-ideal of S. Let L be an almost left α-ideal of S such that $L \subseteq A$. By Theorem 5.4.4 (1), C_L is a fuzzy almost left α-ideal of S such that $C_L \subseteq C_A$. Since C_A is minimal, it follows that, $L = supp(C_L) = supp(C_A) = A$. Therefore, A is minimal.

The proofs of (2) and (3) can be seen in similar fashion. $\square$

5.4.2 Fuzzy almost (α, β)-quasi-ideals

We define fuzzy almost (α, β)-quasi-ideals of Γ-semigroups as follows:

Definition 5.4.9. Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$ and $\alpha, \beta \in \Gamma$. We call f a *fuzzy almost* (α, β)*-quasi-ideal* of S if $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \neq 0$ for each fuzzy points x_t of S.

Theorem 5.4.10. *Let f be a fuzzy almost (α, β)-quasi-ideal of a Γ-semigroup S and g a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost (α, β)-quasi-ideal of S.*

Proof. Let x_t be a fuzzy point of S. Since f is a fuzzy almost (α, β)-quasi-ideal of S, we get $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \neq 0$. Since $f \subseteq g$, we have that $(x_t \circ_\alpha f) \cap (f \circ_\beta x_t) \cap f \subseteq (x_t \circ_\alpha g) \cap (g \circ_\beta x_t) \cap g$. This implies that $(x_t \circ_\alpha g) \cap (g \circ_\beta x_t) \cap g \neq 0$. Therefore, g is a fuzzy almost (α, β)-quasi-ideal of S. □

Corollary 5.4.11. *Let f and g be fuzzy almost (α, β)-quasi-ideals of a Γ-semigroup S. Then $f \cup g$ is a fuzzy almost (α, β)-quasi-ideal of S.*

Proof. Since $f \subseteq f \cup g$, by Theorem 5.4.10, $f \cup g$ is a fuzzy almost (α, β)-quasi-ideal of S. □

Example 5.4.12. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}, \overline{2}, \overline{4}\}$ and $\overline{a}\gamma\overline{b} = \overline{a} + \gamma + \overline{b}$ where $\overline{a}, \overline{b} \in \mathbb{Z}_5$ and $\gamma \in \Gamma$. Let $f : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$f(\overline{0}) = 0, f(\overline{1}) = 0.6, f(\overline{2}) = 0, f(\overline{3}) = 0.4 \text{ and } f(\overline{4}) = 0.4$$

and $g : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$g(\overline{0}) = 0, g(\overline{1}) = 0.3, g(\overline{2}) = 0.6, g(\overline{3}) = 0 \text{ and } g(\overline{4}) = 0.8.$$

We have f and g are fuzzy almost $(\overline{0}, \overline{0})$-quasi-ideals of $\mathbb{Z}_5$ but $f \cap g$ is not a fuzzy almost $(\overline{0}, \overline{0})$-quasi-ideal of $\mathbb{Z}_5$. Thus the intersection of two fuzzy almost (α, β)-quasi-ideals of a Γ-semigroup S need not be a fuzzy almost (α, β)-quasi-ideal of S.

Theorem 5.4.13. *Let Q be a nonempty subset of a Γ-semigroup S. Then Q is an almost (α,β)-quasi-ideal of S if and only if C_Q is a fuzzy almost (α,β)-quasi-ideal of S.*

Proof. Assume that Q is an almost (α,β)-quasi-ideal of S and let x_t be a fuzzy point of S. Then $x\alpha Q \cap Q\beta x \cap Q \neq \emptyset$. Thus there exists $y \in x\alpha Q \cap Q\beta x$ and $y \in Q$. So $[(x_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta x_t)](y) \neq 0$ and $C_Q(y) = 1$. It follows that $(x_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta x_t) \cap C_Q \neq 0$. Therefore, C_Q is a fuzzy almost (α,β)-quasi-ideal of S.

Conversely, assume that C_Q is a fuzzy almost (α,β)-quasi-ideal of S. Let $s \in S$. Then $(s_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta s_t) \cap C_Q \neq 0$. Then there exists $x \in S$ such that $[(s_t \circ_\alpha C_Q) \cap (C_Q \circ_\beta s_t) \cap C_Q](x) \neq 0$. Hence, $x \in s\alpha Q \cap Q\beta s \cap Q$. So $s\alpha Q \cap Q\beta s \cap Q \neq \emptyset$. Consequently, Q is an almost (α,β)-quasi-ideal of S. □

Theorem 5.4.14. *Let f be a fuzzy subset of a Γ-semigroup S. Then f is a fuzzy almost (α,β)-quasi-ideal of S if and only if $supp(f)$ is an almost (α,β)-quasi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α,β)-quasi-ideal of S. Let s_t be a fuzzy point of S. Then $(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f \neq 0$. So there exists $x \in S$ such that $[(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f](x) \neq 0$. Thus there exist $y_1, y_2 \in S$ such that $x = s\alpha y_1 = y_2\beta s, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is, $x, y_1, y_2 \in supp(f)$. Thus $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t)](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Therefore, $(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)} \neq 0$. Hence, $C_{supp(f)}$ is a fuzzy almost (α,β)-quasi-ideal of S. By Theorem 5.4.13, $supp(f)$ is an almost (α,β)-quasi-ideal of S.

Conversely, assume that $supp(f)$ is an almost (α,β)-quasi-ideal of S. By Theorem 5.4.13, $C_{supp(f)}$ is a fuzzy almost (α,β)-quasi-ideal of S. Then for each fuzzy point s_t of S, we have $(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)} \neq 0$. Then there exists $x \in S$ such that $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t) \cap C_{supp(f)}](x) \neq 0$. Hence, $[(s_t \circ_\alpha C_{supp(f)}) \cap (C_{supp(f)} \circ_\beta s_t)](x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Then there exist $y_1, y_2 \in S$ such that $x = s\alpha y_1 = y_2\beta s, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This means that $(s_t \circ_\alpha f) \cap (f \circ_\beta s_t) \cap f \neq 0$. Therefore, f is a fuzzy almost (α,β)-quasi-ideal of S. □

Corollary 5.4.15. *A* Γ*-semigroup* S *has no proper almost* (α, β)*-quasi-ideals if and only if* $supp(f) = S$ *for each fuzzy almost* (α, β)*-quasi-ideal* f *of* S*.*

Proof. Assume S has no proper almost (α, β)-quasi-ideals and let f be a fuzzy almost (α, β)-quasi-ideal of S. By Theorem 5.4.14, $supp(f)$ is an almost (α, β)-quasi-ideal of S. Then $supp(f) = S$.

Conversely, let Q be any fuzzy almost (α, β)-quasi-ideal of S. By Theorem 5.4.13, C_Q is a fuzzy almost (α, β)-quasi-ideal of S. By assumption, $supp(C_Q) = S$. Since $supp(C_Q) = Q$, we get $Q = S$. This implies that S has no proper almost (α, β)-quasi-ideals. □

Next, we define minimal fuzzy almost (α, β)-quasi-ideals of Γ-semigroups and give a relationship between minimal almost (α, β)-quasi-ideals and minimal fuzzy almost (α, β)-quasi-ideals of Γ-semigroups.

Definition 5.4.16. A fuzzy almost (α, β)-quasi-ideal f of a Γ-semigroup S is *minimal* if for each fuzzy almost (α, β)-quasi-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 5.4.17. *Let* Q *be a nonempty subset of a* Γ*-semigroup* S*. Then* Q *is a minimal almost* (α, β)*-quasi-ideal of* S *if and only if* C_Q *is a minimal fuzzy almost* (α, β)*-quasi-ideal of* S*.*

Proof. Assume that Q is a minimal almost (α, β)-quasi-ideal of S. By Theorem 5.4.13, C_Q is a fuzzy almost (α, β)-quasi-ideal of S. Let g be a fuzzy almost (α, β)-quasi-ideal of S such that $g \subseteq C_Q$. We obtain that $supp(g) \subseteq supp(C_Q) = Q$. Since $g \subseteq C_{supp(g)}$, it follows from Theorem 5.4.10 that $C_{supp(g)}$ is a fuzzy almost (α, β)-quasi-ideal of S. By Theorem 5.4.13, $supp(g)$ is an almost (α, β)-quasi-ideal of S. Since Q is minimal, $supp(g) = Q = supp(C_Q)$. Therefore, C_Q is minimal.

Conversely, assume that C_Q is a minimal fuzzy almost (α, β)-quasi-ideal of S. Let Q' be an almost (α, β)-quasi-ideal of S such that $Q' \subseteq Q$. Then $C_{Q'}$ is a fuzzy almost (α, β)-quasi-ideal of S such that $C_{Q'} \subseteq C_Q$. Since C_Q is minimal, $Q' = supp(C_{Q'}) = supp(C_Q) = Q$. Therefore, Q is minimal. □

5.4.3 Fuzzy almost (α, β)-bi-ideals

We define fuzzy almost (α, β)-bi-ideals of Γ-semigroups as follows:

Definition 5.4.18. Let f be a fuzzy subset of a Γ-semigroup S such that $f \neq 0$ and $\alpha, \beta \in \Gamma$. We call f a *fuzzy almost (α, β)-bi-ideal* of S if $(f \circ_\alpha x_t \circ_\beta f) \cap f \neq 0$ for each fuzzy point x_t of S.

Theorem 5.4.19. *Let f be a fuzzy almost (α, β)-bi-ideal of a Γ-semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost (α, β)-bi-ideal of S.*

Proof. Let x_t be a fuzzy point of S. Since f is a fuzzy almost (α, β)-bi-ideal of S, $(f \circ_\alpha x_t \circ_\beta f) \cap f \neq 0$. Since $f \subseteq g$, we have $(f \circ_\alpha x_t \circ_\beta f) \cap f \subseteq (g \circ_\alpha x_t \circ_\beta g) \cap g$, this implies that $(g \circ_\alpha x_t \circ_\beta g) \cap g \neq 0$. Therefore, g is a fuzzy almost (α, β)-bi-ideal of S. □

Corollary 5.4.20. *Let f and g be fuzzy almost (α, β)-bi-ideals of a Γ-semigroup S. Then $f \cup g$ is a fuzzy almost (α, β)-bi-ideal of S.*

Proof. Since $f \subseteq f \cup g$, it follows from Theorem 5.4.19 that $f \cup g$ is a fuzzy almost (α, β)-bi-ideal of S. □

Example 5.4.21. Consider the Γ-semigroup $\mathbb{Z}_5$ where $\Gamma = \{\overline{0}, \overline{4}\}$ and $\overline{a}\gamma\overline{b} = \overline{a} + \gamma + \overline{b}$ where $\overline{a}, \overline{b} \in \mathbb{Z}_5$ and $\gamma \in \Gamma$. Let $f : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$f(\overline{0}) = 0, f(\overline{1}) = 0.7, f(\overline{2}) = 0, f(\overline{3}) = 0.6 \text{ and } f(\overline{4}) = 0.4$$

and $g : \mathbb{Z}_5 \to [0, 1]$ be defined by

$$g(\overline{0}) = 0, g(\overline{1}) = 0.1, g(\overline{2}) = 0.6, g(\overline{3}) = 0 \text{ and } g(\overline{4}) = 0.8.$$

We have f and g are fuzzy almost $(\overline{0}, \overline{4})$-bi-ideals of $\mathbb{Z}_5$ but their intersection is not. Thus the intersection of two fuzzy almost (α, β)-bi-ideals of a Γ-semigroup S need not always be a fuzzy almost (α, β)-bi-ideal of S.

Theorem 5.4.22. *Let B be a nonempty subset of a Γ-semigroup S. Then B is an almost (α, β)-bi-ideal of S if and only if C_B is a fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that B is an almost (α, β)-bi-ideal of a Γ-semigroup S. Let x_t be a fuzzy point of S. Then $B\alpha x\beta B \cap B \neq \emptyset$. Thus there exists $y \in B\alpha x\beta B$ and $y \in B$. So $(C_B \circ_\alpha x_t \circ_\beta C_B)(y) = 1$ and $C_B(y) = 1$. Hence, $(C_B \circ_\alpha x_t \circ_\beta C_B) \cap C_B \neq 0$. Therefore, C_B is a fuzzy almost (α, β)-bi-ideal of S.

Conversely, assume that C_B is a fuzzy almost (α, β)-bi-ideal of S. Let $s \in S$. Then $(C_B \circ_\alpha s_t \circ_\beta C_B) \cap C_B \neq 0$. Thus there exists $x \in S$ such that $[(C_B \circ_\alpha s_t \circ_\beta C_B) \cap C_B](x) \neq 0$. Hence, $x \in B\alpha s\beta B \cap B$. So $B\alpha s\beta B \cap B \neq \emptyset$. Consequently, B is an almost (α, β)-bi-ideal of S. □

Theorem 5.4.23. *Let f be a fuzzy subset of a Γ-semigroup S. Then f is a fuzzy almost (α, β)-bi-ideal of S if and only if $supp(f)$ is an almost (α, β)-bi-ideal of S.*

Proof. Assume that f is a fuzzy almost (α, β)-bi-ideal of S. Let s_t be a fuzzy point of S. Then $(f \circ_\alpha s_t \circ_\beta f) \cap f \neq 0$. Hence, $[(f \circ_\alpha s_t \circ_\beta f) \cap f](x) \neq 0$ for some $x \in S$. So there exist $y_1, y_2 \in S$ such that $x = y_1\alpha s\beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is, $x, y_1, y_2 \in supp(f)$. Thus $(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)})(x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Therefore, $(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)} \neq 0$. Therefore, $C_{supp(f)}$ is a fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.22, $supp(f)$ is an almost (α, β)-bi-ideal of S.

Conversely, assume that $supp(f)$ is an almost (α, β)-bi-ideal of S. By Theorem 5.4.22, $C_{supp(f)}$ is a fuzzy almost (α, β)-bi-ideal of S. Let x_t be a fuzzy point of S. Then $(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)} \neq 0$. So there exists $x \in S$ such that $[(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)}) \cap C_{supp(f)}](x) \neq 0$. So $(C_{supp(f)} \circ_\alpha s_t \circ_\beta C_{supp(f)})(x) \neq 0$ and $C_{supp(f)}(x) \neq 0$. Then $x = y_1\alpha s\beta y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$ for some $y_1, y_2 \in S$. This means $(f \circ_\alpha s_t \circ_\beta f) \cap f \neq 0$. Therefore, f is a fuzzy almost (α, β)-bi-ideal of S. □

Corollary 5.4.24. *A Γ-semigroup S has no proper almost bi-Γ-ideals if and only if $supp(f) = S$ for each fuzzy almost bi-Γ-ideal f of S.*

Proof. Assume S has no proper almost (α, β)-bi-ideals and let f be a fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.23, $supp(f)$ is an almost (α, β)-bi-ideal of S. Then $supp(f) = S$.

Conversely, let B be any fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.22, C_B is a fuzzy almost (α, β)-bi-ideal of S. By assumption, we obtain that $supp(C_B) = S$. Since $supp(C_B) = B$, we get $B = S$. This implies that S has no proper almost (α, β)-bi-ideals. □

Next, we define minimal fuzzy almost (α, β)-bi-ideals in Γ-semigroups and give a relationship between minimal almost (α, β)-bi-ideals and minimal fuzzy almost (α, β)-bi-ideals of Γ-semigroups.

Definition 5.4.25. A fuzzy almost (α, β)-bi-ideal f of a Γ-semigroup S is *minimal* if for all fuzzy almost (α, β)-bi-ideal g of S such that $g \subseteq f$, we have $supp(g) = supp(f)$.

Theorem 5.4.26. *Let B be a nonempty subset of a Γ-semigroup S. Then B is a minimal almost (α, β)-bi-ideal of S if and only if C_B is a minimal fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that B is a minimal almost (α, β)-bi-ideal of a Γ-semigroup S. By Theorem 5.4.22, C_B is a fuzzy almost (α, β)-bi-ideal of S. Let g be a fuzzy almost (α, β)-bi-ideal of S such that $g \subseteq C_B$. Then $supp(g) \subseteq supp(C_B) = B$. Since $g \subseteq C_{supp(g)}$, by Theorem 5.4.19, $C_{supp(g)}$ is a fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.23, $supp(g)$ is an almost (α, β)-bi-ideal of S. Since B is minimal, $supp(g) = B = supp(C_B)$. Therefore, C_B is minimal.

Conversely, assume that C_B is a minimal fuzzy almost (α, β)-bi-ideal of S. Let B' be an almost (α, β)-bi-ideal of S such that $B' \subseteq B$. Then $C_{B'}$ is a fuzzy almost (α, β)-bi-ideal of S such that $C_{B'} \subseteq C_B$. Since C_B is minimal, we get $B' = supp(C_{B'}) = supp(C_B) = B$. Therefore, B is minimal. □

Next, we give a relationship between α-prime almost (α, β)-bi-ideals and α-prime fuzzy almost (α, β)-bi-ideals.

Definition 5.4.27. Let S be a Γ-semigroup and $\gamma \in \Gamma$.

(1) An almost (α, β)-bi-ideal A of S is said to be *γ-prime* if for all $x, y \in S$,

$$x\gamma y \in A \text{ implies } x \in A \text{ or } y \in A.$$

(2) A fuzzy almost (α, β)-bi-ideal f of S is said to be *γ-prime* if

$$f(x\gamma y) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S.$$

Theorem 5.4.28. *Let A be a nonempty subset of a Γ-semigroup S. Then A is a γ-prime almost (α, β)-bi-ideal of S if and only if C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that A is a γ-prime almost (α, β)-bi-ideal of S. By Theorem 5.4.22, C_A is a fuzzy almost (α, β)-bi-ideal of S. Let $x, y \in S$. We consider two cases:
Case 1: $x\gamma y \in A$. Since A is γ-prime, it follows that $x \in A$ or $y \in A$.
Then $\max\{C_A(x), C_A(y)\} = 1 \geq C_A(x\gamma y)$.

Case 2: $x\gamma y \notin A$. Then $C_A(x\gamma y) = 0 \leq \max\{C_A(x), C_A(y)\}$.
Thus C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S.

Conversely, assume that C_A is a γ-prime fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.22, A is an almost (α, β)-bi-ideal of S. Let $x, y \in S$ be such that $x\gamma y \in A$. This implies that $1 = C_A(x\gamma y) \leq \max\{C_A(x), C_A(y)\}$, so $\max\{C_A(x), C_A(y)\} = 1$. Thus $x \in A$ or $y \in A$. Hence, A is a γ-prime almost (α, β)-bi-ideal of S. □

Finally, we give a relationship between γ-semiprime almost (α, β)-bi-ideals and γ-semiprime fuzzy almost (α, β)-bi-ideals.

Definition 5.4.29. Let S be a Γ-semigroup and $\alpha \in \Gamma$.

(1) An almost (α, β)-bi-ideal A of S is said to be a γ-*semiprime* if for all $x \in S$,

$$x\gamma x \in A \text{ implies } x \in A.$$

(2) A fuzzy almost (α, β)-bi-ideal f of S is said to be a γ-*semiprime* if

$$f(x\gamma x) \leq f(x) \text{ for all } x \in S.$$

Theorem 5.4.30. *Let A be a nonempty subset of S. Then A is a γ-semiprime almost (α, β)-bi-ideal of S if and only if C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S.*

Proof. Assume that A is a γ-semiprime almost (α, β)-bi-ideal of S. By Theorem 5.4.22, C_A is a fuzzy almost (α, β)-bi-ideal of S. Let $x \in S$. We consider two cases:
Case 1: $x\gamma x \in A$. Since A is γ-prime, we obtain $x \in A$. So $C_A(x) = 1$.
Hence, $C_A(x) = C_A(x\gamma x)$.

Case 2: $x\gamma x \notin A$. Then $C_A(x\gamma x) = 0 \leq C_A(x)$.
Thus C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S.

Conversely, assume that C_A is a γ-semiprime fuzzy almost (α, β)-bi-ideal of S. By Theorem 5.4.22, A is an almost (α, β)-bi-ideal of S. Let $x \in S$ be such that $x\gamma x \in A$. Then $C_A(x\gamma x) = 1$. By assumption, $C_A(x\gamma x) \leq C_A(x)$. Then $C_A(x) = 1$, so $x \in A$. Thus A is a γ-semiprime almost (α, β)-bi-ideal of S. □

Chapter 6

Conclusions

A Γ-semigroup is an algebraic structure that considered as a generalization of a semigroup. Let S and Γ be nonempty sets. Also, f is a *fuzzy subset* of a set S is a function from S into the closed interval $[0,1]$. These two concepts are interesting to study together.

In Chapter 3, we define almost quasi-Γ-ideals and fuzzy almost quasi-Γ-ideals of Γ-semigroups. The union of two almost quasi-Γ-ideals [fuzzy almost quasi-Γ-ideals] is also an almost quasi-Γ-ideal [a fuzzy almost quasi-Γ-ideal]. However, the intersection of two almost quasi-Γ-ideals [fuzzy almost quasi-Γ-ideals] need not be an almost quasi-Γ-ideal [a fuzzy almost quasi-Γ-ideal]. A Γ-semigroup has no proper almost quasi-Γ-ideal if and only if for any $a \in S$ there exists $s_a \in S$ such that $s_a\Gamma(S \setminus \{a\}) \cap (S \setminus \{a\})\Gamma s_a \subseteq \{a\}$. Moreover, we investigate some relationships between almost quasi-Γ-ideals and fuzzy almost quasi-Γ-ideals. Also, f is a fuzzy almost quasi-Γ-ideal of a Γ-semigroup if and only if $supp(f)$ is an almost quasi-Γ-ideal of a Γ-semigroup.

In Chapter 4, we define almost bi-Γ-ideals and their fuzzifications of Γ-semigroups. We show that a subset of Γ-semigroup containing an almost bi-Γ-ideal is almost bi-Γ-ideal. The union of two almost bi-Γ-ideals is also an almost bi-Γ-ideal. However, this does not hold in general true for their intersection. Similarly, we have that the union of two fuzzy almost bi-Γ-ideals is also a fuzzy almost bi-Γ-ideal but it is not generally true in case the intersection. Moreover, the relationships between almost bi-Γ-ideals and their fuzzification are shown in Section 4.2.

In Chapter 5, we define new types of ideals and fuzzy ideals by using elements in Γ in sections 5.1 and 5.2. We show interesting properties of these ideals and fuzzy ideals. Moreover, we show the relationships between these ideals and their fuzzifications. Furthermore, we study new types of almost ideals and fuzzy ideals in section 5.3 and 5.4. The relationships between almost ideals and fuzzy ideals are provided.

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