Almost Ideals and Fuzzy Almost Ideals in Algebraic Structures

Sudaporn Suebsung

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

Prince of Songkla University

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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ชื่อวิทยานิพนธ์ เกือบไอดีลและเกือบไอดีลวิภัชนัยในโครงสร้างพีชคณิต
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บทคัดย่อ

กึ่งกรุป คือ คู่อันดับ $(S,\cdot)$ โดยที่ S ไม่เป็นเซตว่าง และ $\cdot$ เป็นการดำเนินการทวิภาคที่มีสมบัติการเปลี่ยนหมู่ สำหรับกึ่งกรุป $(S,\cdot)$ ที่มีอันดับบางส่วน $\leq$ เป็นกึ่งกรุปอันดับ ถ้า $x \leq y$ แล้ว $x \cdot z \leq y \cdot z$ และ $z \cdot x \leq z \cdot y$ สำหรับทุก $x, y, z \in S$ กึ่งไฮเปอร์กรุป $(H, *)$ ถูกนิยามในทำนองเดียวกันกับกึ่งกรุป แต่ต่างกันที่การดำเนินการ $*$ ของกึ่งไฮเปอร์กรุปจะเป็นฟังก์ชันที่ส่งจากเซต $H \times H$ ไปยัง $P^*(H)$ โดยที่ $P^*(H)$ คือเซตของเซตย่อยที่ไม่เป็นเซตว่างทั้งหมดของ H

ในงานวิจัยนี้ เราได้นิยามและศึกษาสมบัติบางประการของเกือบ (m,n)-ไอดีล และเกือบ (m,n)-ไอดีลวิภัชนัยในกึ่งกรุป นอกจากนี้เราได้นิยามเกือบไอดีลอันดับ เกือบไบไอดีลอันดับ เกือบควอซีไอดีลอันดับ เกือบไอดีลอันดับวิภัชนัย เกือบไบไอดีลอันดับวิภัชนัย และเกือบควอซีไอดีลอันดับวิภัชนัยในกึ่งกรุปอันดับ พร้อมทั้งศึกษาความสัมพันธ์ของไอดีลอันดับกับไอดีลอันดับวิภัชนัยเหล่านี้อีกด้วย ยิ่งไปกว่านั้นเรายังนิยามเกี่ยวกับเกือบไฮเปอร์ไอดีล เกือบไบไฮเปอร์ไอดีล และเกือบควอซีไฮเปอร์ไอดีลในกึ่งไฮเปอร์กรุป และกล่าวถึงคุณสมบัติและความสัมพันธ์น่าสนใจบางประการของไฮเปอร์ไอดีลเหล่านี้

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ABSTRACT

A semigroup is an ordered pair $(S, \cdot)$, where S is a nonempty set and $\cdot$ is an associative binary operation. The semigroup $(S, \cdot)$ with a partial order $\leq$ is an ordered semigroup if $x \leq y$, then $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for all $x, y, z \in S$. A semihypergroup $(H, *)$ can be defined in a similar way to the semigroup, but the operation $*$ of the semihypergroup is a function from $H \times H$ into $P^*(H)$, where $P^*(H)$ is a set of all nonempty subsets of H.

In this thesis, we define almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups and study some of their properties. In addition, we define ordered almost ideals, ordered almost bi-ideals, ordered almost quasi-ideals, fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups and we give the relations of them. Moreover, we define almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semihypergroups, and give some interesting properties and relations of them.

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Sudaporn Suebsung

CONTENTS

CHAPTER 1

Introduction

1.1 Background and significance

The notion of almost ideals (or A-ideals) was first introduced in semilattices by Grosek [5] in 1979. A later year, Satko and Grosek [23] generalized this notion to semigroups. They discovered minimal almost ideals and the smallest almost ideals of semigroups in [24] and [6], respectively. In 1981, Bogdanovic [1] used the concepts of almost ideals and bi-ideals in semigroups to define almost bi-ideals in semigroups. Later, Wattanatripop, Chinram and Changphas [28] defined almost quasi-ideals by using the concepts of almost ideals and quasi-ideals in semigroups, and provided some properties of almost quasi-ideals in semigroups. In [25], Solano, Suebsung and Chinram extended almost ideals in n-ary semigroups.

The theory of algebraic hyperstructures was introduced by Marty [20] in 1934. He defined hypergroups under the hyperoperation that was a function into a set, while the operation on classical algebraic structures was a function into an element. Moreover, he studied some properties of these structures and applied these structures to groups. In 1999, Hasankhani [7] began to study semihypergroups and introduced the concept of ideals in semihypergroups. Moreover, he studied the relationships between ideals and the hyper versions of Green's relations. Hila, Davvaz and Naka [8] introduced the notion of quasi-hyperideals in semihypergroups, and provided (m, n)-quasi-hyperideals, n-right hyperideals and m-left hyperideals in semihypergroups. Also, some interesting properties were investigated. In [2], Changphas and Davvaz studied hyperideals in ordered semihypergroups, and provided their properties.

The concept of fuzzy subsets was initially introduced by Zadeh [29] in 1965. This notion by Zadeh was adapted to groups by Rosenfeld [22]; he provided definitions of fuzzy subgroups and fuzzy ideals in groups. A fuzzy subset in a semigroup was introduced by Kuroki [13]. He studied various kinds of fuzzy ideals in semigroups and characterized them in [13]-[17]. In 2019, Mahboob, Davvaz and Khan [19] defined fuzzy (m, n)-ideals, fuzzy $(m, 0)$-ideals and fuzzy $(0, n)$-ideals for all positive integers m, n in semigroups. Furthermore, Kehayopulu and Tsingelis introduced the notions of fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in [11], [9], and [12], respectively. In 2018, fuzzy almost ideals and fuzzy almost quasi-ideals in semigroups were defined by Wattanatripop, Chinram and Changphas [28], using the ideas of almost ideals and almost quasi-ideals in semigroups. With this idea, they also defined fuzzy almost bi-ideals in semigroups in [27]. Recently, Gaketem generalized results in [27] to study interval valued fuzzy almost bi-ideals of semigroups in [4]. In [26], Suebsung, Wattanatripop and Chinram defined and studied some properties of almost ideals and fuzzy almost ideals of ternary semigroups.

In this thesis, we define almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups and we study some interesting properties. In addition, we define ordered almost ideals, ordered almost bi-ideals, ordered almost quasi-ideals, fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups, and give the relations of them. Moreover, we define almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semihypergroups, and give some interesting properties and relations of them.

1.2 Objectives and scope

1. To study almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups.
2. To study ordered almost ideals, ordered almost bi-ideals and ordered almost quasi-ideals in ordered semigroups.
3. To study fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups.
4. To study almost hyperideals, almost bi-hyperideals, and almost quasi-hyperideals in hypersemigroups.

1.3 Research plan

Task	2019	2020			2021	
	08-12	01-03	04-06	07-12	01-03	04-06
Literature review	*	*				
Write up the thesis proposal		*	*			
Present the thesis proposal			*			
Work on the problems				*	*	
Write up the thesis				*	*	*
Present the thesis						*

1.4 Expected benefits

Some new knowledge about almost ideals and fuzzy almost ideals in many algebraic structures.

CHAPTER 2

Preliminaries

In this chapter, we present some definitions and results, which will be used throughout this thesis.

2.1 Semigroups

In this section, we will introduce some definitions and properties of semigroups that will be used in this thesis. First of all, the definition of a semigroup can be defined as follows:

Definition 2.1.1. A ***semigroup*** is a pair $(S, \cdot)$ in which S is a nonempty set and

1. $\cdot$ is a binary operation, that is, $\cdot : S \times S \to S$,
2. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$.

For simplicity, a binary operation $\cdot$ will be identified with a multiplication of two elements, i.e., $x \cdot y$ will be identified with xy. Let A and B be nonempty subsets of a nonempty set S. A ***product*** $A \cdot B$, commonly written as AB, is the set

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

Proposition 2.1.2. Let A, B and C be nonempty subsets of a semigroup S. If $A \subseteq B$, then $AC \subseteq BC$ and $CA \subseteq CB$.

Definition 2.1.3. A nonempty subset A of a semigroup S is called a ***subsemigroup*** of S if $AA \subseteq A$.

Definition 2.1.4. Let A be a nonempty subset of a semigroup S.

1. A is called a ***left ideal*** of S if $SA \subseteq A$.

2. A is called a ***right ideal*** of S if $AS \subseteq A$.

3. A is called an ***ideal*** of S if it is both a left ideal and a right ideal of S.

Definition 2.1.5. An ideal A of a semigroup S is ***prime*** if $xy \in A$ implies $x \in A$ or $y \in A$ for all $x, y \in S$.

Definition 2.1.6. An ideal A of a semigroup S is ***semiprime*** if $x^2 \in A$ implies $x \in A$ for all $x \in S$.

The definition of (m, n)-ideals in semigroups was introduced by Lajos in [18] as follows:

Definition 2.1.7. Let m and n be non-negative integers. A subsemigroup A of a semigroup S is called an (m, n)***-ideal*** of S if $A^mSA^n \subseteq A$ (A^m is suppressed if $m = 0$).

Remark 2.1.8. Let S be a semigroup. Then the following statements hold.

(i) A left ideal of S is a $(0, 1)$-ideal of S.

(ii) A right ideal of S is a $(1, 0)$-ideal of S.

In [23], Satko and Grosek introduced the notions of left almost ideals, right almost ideals and almost ideals in semigroups as follows:

Definition 2.1.9. Let A be a nonempty subset of a semigroup S.

1. A is called a ***left almost ideal*** of S if $xA \cap A \neq \emptyset$ for all $x \in S$.

2. A is called a ***right almost ideal*** of S if $Ax \cap A \neq \emptyset$ for all $x \in S$.

3. A is called an ***almost ideal*** of S if it is both a left almost ideal and a right almost ideal of S.

Example 2.1.10. Consider a semigroup $S = \{a, b, c, d, e\}$ under the multiplication $\cdot$ defined as in the following table.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

We see that the left almost ideals of S are $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$, $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{a, b, c, d\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$, $\{a, c, d, e\}$, $\{b, c, d, e\}$ and S. And the right almost ideals of S are $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, b, c, d\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$ and S.

Thus $\{a, b\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, b, c, d\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$ and S are almost ideals of S.

Remark 2.1.11. Let S be a semigroup. Then the following statements are true.

(i) Every left ideal of S is a left almost ideal of S.

(ii) Every right ideal of S is a right almost ideal of S.

(iii) Every ideal of S is an almost ideal of S.

In 1981, Bogdanovic introduced the definition of almost bi-ideals in semigroups in [1] as follows:

Definition 2.1.12. A nonempty subset B of a semigroup S is called an ***almost bi-ideal*** of S if $BxB \cap B \neq \emptyset$ for all $x \in S$.

Example 2.1.13. Consider a semigroup $S = \{a, b, c, d\}$ with the multiplication table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

We have that $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and S are all almost bi-ideals of S.

Remark 2.1.14. Every bi-ideal of a semigroup S is an almost bi-ideal of S.

In [28], Wattanatripop, Chinram and Changphas defined almost quasi-ideals in semigroups by using the concepts of almost ideals and quasi-ideals in semigroups.

Definition 2.1.15. Let S be a semigroup. A nonempty subset Q of S is called an ***almost quasi-ideal*** of S if $(xQ \cap Qx) \cap Q \neq \emptyset$ for all $x \in S$.

Example 2.1.16. Consider a semigroup $S = \{a, b, c, d\}$ with the multiplication table:

$\cdot$	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	d	c
d	a	b	c	d

The almost quasi-ideals of S are $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$, and S.

2.2 Ordered semigroups

In this section, we recall some basic notions in ordered semigroups. Furthermore, we introduce definitions of ideals, bi-ideals and quasi-ideals (some authors call ordered ideals, ordered bi-ideals and ordered quasi-ideals) in ordered semigroups.

Definition 2.2.1. Let S be a nonempty set with a binary relation $\leq$. Then $(S, \leq)$ is called a ***partially ordered set*** if $\leq$ is a partial order on S, that is, for all $x, y, z \in S$,

1. $x \leq x$ (reflexive),
2. if $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetric),
3. if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive).

Definition 2.2.2. Let S be a set with a binary operation $\cdot$ and a binary relation $\leq$. Then $(S, \cdot, \leq)$ is called an ***ordered semigroup*** if

1. $(S, \cdot)$ is a semigroup,
2. $(S, \leq)$ is a partially ordered set,
3. for all $x, y, z \in S$, if $x \leq y$, then $xz \leq yz$ and $zx \leq zy$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For a nonempty subset A of S, we denote $(A] := \{x \in S \mid x \leq a \text{ for some } a \in A\}$.

Proposition 2.2.3. Let A and B be nonempty subsets of an ordered semigroup $(S, \cdot, \leq)$. The following properties are true.

(1) $A \subseteq (A]$.

(2) If $A \subseteq B$, then $(A] \subseteq (B]$.

(3) $(A \cap B] \subseteq (A] \cap (B]$.

(4) $(A \cup B] = (A] \cup (B]$.

Definition 2.2.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. An element $a \in S$ is called an ***ordered idempotent*** if $a \leq a^2$.

Definition 2.2.5. Let A be a nonempty subset of an ordered semigroup $(S, \cdot, \leq)$.

1. A is called a ***left ordered ideal*** of S if $SA \subseteq A$ and $(A] = A$.

2. A is called a ***right ordered ideal*** of S if $AS \subseteq A$ and $(A] = A$.

3. A is called an ***ordered ideal*** of S if A is both a left ordered ideal and a right ordered ideal of S

Example 2.2.6. Consider an ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ given below.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a, a), (a, b), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, e)\}.$$

The left ordered ideals of S are $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and S. The right ordered ideals of S are $\{a, b, c, d\}$ and S. The ordered ideals of S are $\{a, b, c, d\}$ and S.

Definition 2.2.7. Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup B of S is called an ***ordered bi-ideal*** of S if $BSB \subseteq B$ and $(B] = B$.

Example 2.2.8. From Example 2.2.6, the ordered bi-ideals of S are $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and S.

Definition 2.2.9. Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup Q of S is called an ***ordered quasi-ideal*** of S if $SQ \cap QS \subseteq Q$ and $(Q] = Q$.

Example 2.2.10. From Example 2.2.6, we have that the ordered quasi-ideals of S are $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and S.

2.3 Semihypergroups

In this section, definitions of semihypergroups and hyperideals are introduced. Moreover, some properties of them that will be used in chapter 3 are given. We begin this section with the definition of semihypergroups.

Let H be a nonempty set. A ***hyperoperation*** on H is a function $*$ from $H \times H$ into $P^*(H)$, where $P^*(H)$ is a set of all nonempty subsets of H. For subsets A and B of H and $x \in H$, we denote

$$A * B = \bigcup_{a \in A,\ b \in B} a * b, \quad x * A = \{x\} * A, \quad \text{and} \quad B * x = B * \{x\}.$$

Definition 2.3.1. Let H be a nonempty set with a hyperoperation $*$. An ordered pair $(H, *)$ is called a ***semihypergroup*** if the following assertion is satisfied:

$$(x * y) * z = x * (y * z) \text{ for all } x, y, z \in H.$$

Example 2.3.2. Let $H = \{x, y, z\}$ be a set of three elements and define a hyperoperation $*$ on H as follows:

$*$	x	y	z
x	$\{x\}$	$\{x, y\}$	$\{x, z\}$
y	$\{x\}$	$\{x, y\}$	$\{x, y\}$
z	$\{x\}$	$\{x, y\}$	$\{z\}$

Then $(H, *)$ is a semihypergroup.

Proposition 2.3.3. Let A, B and C be nonempty subsets of a semihypergroup $(H, *)$. If $A \subseteq B$, then $A * C \subseteq B * C$ and $C * A \subseteq C * B$.

Definition 2.3.4. A nonempty subset A of a semihypergroup $(H, *)$ is called a ***subsemihypergroup*** of H if $A * A \subseteq A$.

The concept of ideals in semihypergroups is defined by Hasankhani in [7]. Later, a book on semihypergroups was published by Davvaz [3], and he also defined hyperideals in semihypergroups as follows:

Definition 2.3.5. Let $(H, *)$ be a semihypergroup.

1. A nonempty subset L of H is called a ***left hyperideal*** of H if $H * L \subseteq L$.
2. A nonempty subset R of H is called a ***right hyperideal*** of H if $R * H \subseteq R$.
3. A nonempty subset I of H is called a ***hyperideal*** of H if I is a left hyperideal and a right hyperideal of H.

Example 2.3.6. Let $H = \{a, b, c, d, f\}$ be a semihypergroup under the hyperoperation $*$ defined by the following table.

$*$	a	b	c	d	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,d\}$	$\{a\}$
c	$\{a\}$	$\{a,f\}$	$\{a,c\}$	$\{a,c\}$	$\{a,f\}$
d	$\{a\}$	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
f	$\{a\}$	$\{a,f\}$	$\{a\}$	$\{a,c\}$	$\{a\}$

We can deduce that

$\{a\}, \{a, b, f\}, \{a, c, d\}$ and H are left hyperideals of H,

$\{a\}, \{a, b, d\}, \{a, c, f\}$ and H are right hyperideals of H.

Thus the hyperideals of H are $\{a\}$ and H.

Definition 2.3.7. A subsemihypergroup B of a semihypergroup $(H, *)$ is called a ***bi-hyperideal*** of H if $B * H * B \subseteq B$.

Example 2.3.8. A unit real interval numbers $H = [0, 1]$ is a semihypergroup under the hyperoperation $*$ defined by

$$x * y = [0, xy] \text{ for all } x, y \in H.$$

Let $B = [0, t]$ with $0 \leq t \leq 1$. Then B is a subsemihypergroup of H. We have $B * H * B = [0, t^2] \subseteq [0, t] = B$. Therefore, B is a bi-hyperideal of H.

Theorem 2.3.9. *Every hyperideal of a semihypergroup is a bi-hyperideal.*

The definition of quasi-hyperideals was given by Hila, Davvaz and Naka [8] as follows:

Definition 2.3.10. A nonempty subset Q of a semihypergroup $(H, *)$ is called a ***quasi-hyperideal*** of H if $(H * Q) \cap (Q * H) \subseteq Q$.

Example 2.3.11. Let $H = \{a, b, c, d\}$ be a semihypergroup under the hyperoperation $*$ defined as in the following table.

$*$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$	H
b	$\{b\}$	$\{b\}$	$\{b, d\}$	$\{b, d\}$
c	$\{c\}$	$\{c, d\}$	$\{c\}$	$\{c, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

The quasi-hyperideals of H are $\{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}$ and H.

Theorem 2.3.12. *Every quasi-hyperideal of a semihypergroup is a subsemihypergroup.*

2.4 Fuzzy subsets and fuzzy ideals

In this section, we present some definitions and results of fuzzy subsets. In addition, we introduce the definition of fuzzy ideals in semigroups and ordered semigroups, and some interesting properties of them.

In 1965, Zadeh introduced fuzzy subsets. A function f from a set S to the unit interval $[0, 1]$ is a ***fuzzy subset*** of S.

For any two fuzzy subsets f and g of a nonempty set S, the ***union*** and ***intersection*** of f and g, denoted by $f \cup g$ and $f \cap g$, are fuzzy subsets of S defined by, for all $x \in S$,

$$(f \cup g)(x) = \max\{f(x), g(x)\},$$

$$(f \cap g)(x) = \min\{f(x), g(x)\}.$$

Let $F(S)$ be a set of all fuzzy subsets of a set S. A relation on $F(S)$ is defined by, for all fuzzy subsets f and g of S,

$$f \subseteq g \Longleftrightarrow f(x) \leq g(x) \text{ for all } x \in S.$$

For a fuzzy subset f of a set S, a ***support*** of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

Let A be a subset of a set S. A ***characteristic function*** C_A is a function from S to $[0,1]$ defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A definition of fuzzy points was given by Pao-Ming and Ying-Ming [21]. Let $s \in S$ and $\alpha \in (0,1]$. A ***fuzzy point*** s_α of a set S is a fuzzy subset of S defined by

$$s_\alpha(x) = \begin{cases} \alpha & \text{if } x = s, \\ 0 & \text{if } x \neq s. \end{cases}$$

For a nonempty set S, fuzzy subsets 1 and 0 of S are defined by

$$1(x) = 1 \text{ and } 0(x) = 0 \text{ for all } x \in S.$$

Proposition 2.4.1. Let A and B be nonempty subsets of a nonempty set S. If $A \subseteq B$, then $C_A \subseteq C_B$.

Next, we give products of fuzzy subsets in semigroups and ordered semigroups. For any two fuzzy subsets f and g of a semigroup S, we define the ***product*** of f and g by, for all $x \in S$,

$$(f \circ g)(x) := \begin{cases} \sup\limits_{x=ab} \min\{f(a), g(b)\} & \text{if } x \in S^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. The ***product*** of fuzzy subsets f and g of S is defined by, for all $x \in S$,

$$(f \circ g)(x) := \begin{cases} \sup\limits_{x \leq uv} \min\{f(u), g(v)\} & \text{if } x \in (S^2], \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.4.2. Let $F(S)$ be a set of all fuzzy subsets of a semigroup S. Then $(F(S), \circ)$ is a semigroup.

Proposition 2.4.3. Let $F(S)$ be a set of all fuzzy subsets of an ordered semigroup $(S, \cdot, \leq)$. Then $(F(S), \circ, \subseteq)$ is an ordered semigroup.

Proposition 2.4.4. Let f, g and h be fuzzy subsets of a semigroup (or an ordered semigroup). Then the following statements hold.

(1) If $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$.

(2) If $f \subseteq g$, then $f \cap h \subseteq g \cap h$.

(3) If $f \subseteq g$, then $f \cup h \subseteq g \cup h$.

(4) If $f \subseteq g$, then $supp(f) \subseteq supp(g)$.

Definition 2.4.5. A fuzzy subset f of a semigroup S is called a ***fuzzy subsemigroup*** of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

Definition 2.4.6. Let f be a fuzzy subset of a semigroup S.

1. f is called a ***fuzzy left ideal*** of S if $f(xy) \geq f(y)$ for all $x, y \in S$.

2. f is called a ***fuzzy right ideal*** of S if $f(xy) \geq f(x)$ for all $x, y \in S$.

3. f is called a ***fuzzy ideal*** of S if it is both a fuzzy left ideal and a fuzzy right ideal of S, that is, $f(xy) \geq \max\{f(x), f(y)\}$ for all $x, y \in S$.

4. f is called a ***fuzzy bi-ideal*** of S if $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$.

5. f is called a ***fuzzy quasi-ideal*** of S if $(f \circ 1) \cap (1 \circ f) \subseteq f$

Mahboob, Davvaz and Khan defined fuzzy (m,n)-ideals, where m and n are any positive integers, of semigroups in [19].

Definition 2.4.7. Let S be a semigroup, and m and n be positive integers. A fuzzy subsemigroup f of S is called a ***fuzzy*** (m,n)***-ideal*** of S if

$$f(x_1x_2\cdots x_m z y_1 y_2\cdots y_n) \geq \min\{f(x_1), f(x_2), ..., f(x_m), f(y_1), f(y_2), ..., f(y_n)\}$$

for all $x_1, x_2, ..., x_m, z, y_1, y_2, ..., y_n \in S$.

Next, we will introduce the defintions of a fuzzy ordered ideal, a fuzzy ordered bi-ideal and a fuzzy ordered quasi-ideal in an ordered semigroup, which can be defined in a similar way to a semigroup, by adding add one more condition.

Definition 2.4.8. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a ***fuzzy left ordered ideal*** of S if for all $x, y \in S$,

1. $x \leq y$ implies $f(x) \geq f(y)$ and
2. $f(xy) \geq f(y)$.

A fuzzy subset f of S is called a ***fuzzy right ordered ideal*** of S if for all $x, y \in S$,

1. $x \leq y$ implies $f(x) \geq f(y)$ and
2. $f(xy) \geq f(x)$.

A fuzzy subset f of S is called a ***fuzzy ordered ideal*** of S if it is both a fuzzy left ordered ideal and a fuzzy right ordered ideal of S, that is, for all $x, y \in S$,

1. $x \leq y$ implies $f(x) \geq f(y)$ and
2. $f(xy) \geq \max\{f(x), f(y)\}$.

Definition 2.4.9. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a ***fuzzy ordered bi-ideal*** of S if for all $x, y, z \in S$,

1. $x \leq y$ implies $f(x) \geq f(y)$ and
2. $f(xyz) \geq \min\{f(x), f(z)\}$.

Definition 2.4.10. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a ***fuzzy ordered quasi-ideal*** of S provide that

1. if $x, y \in S$ such that $x \leq y$, then $f(x) \geq f(y)$, and
2. $(f \circ 1) \cap (1 \circ f) \subseteq f$.

From the definitions of fuzzy ordered bi-ideals and fuzzy ordered quasi-ideals in ordered semigroups, the fuzzy ordered bi-ideal is defined in term of the fuzzy subset f itself while the fuzzy ordered quasi-ideal in terms of the product $f \circ 1$ and $1 \circ f$. In [10], the fuzzy ordered quasi-ideal f can be defined in a similar way using only the fuzzy subset f itself as in the following theorem.

Theorem 2.4.11. *Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is a fuzzy ordered quasi-ideal of S if and only if the following conditions are satisfied.*

(1) *If $x \leq y$, then $f(x) \geq f(y)$ for all $x, y \in S$.*

(2) *If $x \leq ab$ and $x \leq cd$, then $f(x) \geq \min\{f(a), f(d)\}$ for all $x, a, b, c, d \in S$.*

In [28], Wattanatripop, Chinram and Changphas defined the notion of almost quasi-ideals of a semigroup. Moreover, they introduced the notions of fuzzy almost ideals and fuzzy almost quasi-ideals of a semigroup.

Definition 2.4.12. Let f be a nonzero fuzzy subset of a semigroup S.

1. f is called a ***fuzzy left almost ideal*** of S if $(C_s \circ f) \cap f \neq 0$ for all $s \in S$.
2. f is called a ***fuzzy right almost ideal*** of S if $(f \circ C_s) \cap f \neq 0$ for all $s \in S$.
3. f is called a ***fuzzy almost ideal*** of S if f is both a fuzzy left almost ideal and a fuzzy right almost ideal of S.
4. f is called a ***fuzzy almost quasi-ideal*** of S if $(C_s \circ f) \cap (f \circ C_s) \cap f \neq 0$ for all $s \in S$.

Theorem 2.4.13. *Let A be a nonempty subset of a semigroup S.*

(1) *A is a left almost ideal of S if and only if C_A is a fuzzy left almost ideal of S.*

(2) *A is a right almost ideal of S if and only if C_A is a fuzzy right almost ideal of S.*

(2) *A is an almost ideal of S if and only if C_A is a fuzzy almost ideal of S.*

(4) *A is an almost quasi-ideal of S if and only if C_A is a fuzzy almost quasi-ideal of S.*

Theorem 2.4.14. *Let f be a fuzzy subset of a semigroup S.*

(1) *f is a fuzzy left almost ideal of S if and only if $supp(f)$ is a left almost ideal of S.*

(2) *f is a fuzzy right almost ideal of S if and only if $supp(f)$ is a right almost ideal of S.*

(3) *f is a fuzzy almost ideal of S if and only if $supp(f)$ is an almost ideal of S.*

(4) *f is a fuzzy almost quasi-ideal of S if and only if $supp(f)$ is an almost quasi-ideal of S.*

In [27], Wattanatripop, Chinram and Changphas defined fuzzy almost bi-ideals in semigroups and give some relationship between almost bi-ideals and fuzzy almost bi-ideals of semigroups.

Definition 2.4.15. Let f be a nonzero fuzzy subset of a semigroup S. Then f is called a ***fuzzy almost bi-ideal*** of S if $(f \circ C_s \circ f) \cap f \neq 0$ for all $s \in S$.

Theorem 2.4.16. *Let B be a nonempty subset of a semigroup S. Then B is an almost bi-ideal of S if and only if C_B is a fuzzy almost bi-ideal of S.*

Theorem 2.4.17. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy almost bi-ideal of S if and only if $supp(f)$ is an almost bi-ideal of S.*

CHAPTER 3

Almost (m, n)-ideals and fuzzy almost (m, n)-ideals in semigroups

In this chapter, definitions of almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups are introduced. Moreover, we give some properties and a relation of almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups. Throughout this chapter unless stated otherwise m and n stand for non-negative integers.

3.1 Almost (m, n)-ideals in semigroups

In this section, we use the concepts of (m,n)-ideals and almost ideals in semigroups to define an almost (m,n)-ideal in a semigroup and study some properties of them. Let S be a semigroup. For $a, s \in S$ and $k \in \mathbb{N}$, we denote

1. $a^k := aaa \dots a \quad (k$ copies),
2. $a^k s a^0 := a^k s$,
3. $a^0 s a^k := s a^k$, and
4. $a^0 s a^0 := s$.

Let A be a nonempty subset of a semigroup S and $s \in S$. For $k \in \mathbb{N}$, we define

1. $A^k := AAA \ldots A \quad (k \text{ copies})$,
2. $A^k s A^0 := A^k s$,
3. $A^0 s A^k := s A^k$, and
4. $A^0 s A^0 := \{s\}$.

Firstly, we give a definition of almost (m, n)-ideals in semigroups as follows:

Definition 3.1.1. A nonempty subset A of a semigroup S is called an ***almost*** (m, n)-***ideal*** of S if $A^m s A^n \cap A \neq \emptyset$ for all $s \in S$.

Remark 3.1.2. Let S be a semigroup. The following statements hold.

(i) An almost $(0, 1)$-ideal of S is a left almost ideal of S.

(ii) An almost $(1, 0)$-ideal of S is a right almost ideal of S.

(iii) Every (m, n)-ideal of S is an almost (m, n)-ideal of S.

Proof. Clearly, (i) and (ii) are true. Let A be an (m, n)-ideal of S and let $s \in S$. Then we have $A \neq \emptyset$ and $A^m S A^n \subseteq A$, so $A^m s A^n \neq \emptyset$ and $A^m s A^n \subseteq A^m S A^n \subseteq A$. Thus $A^m s A^n \cap A = A^m s A^n \neq \emptyset$. Hence, A is an almost (m, n)-ideal of S. $\square$

Example 3.1.3. Consider a semigroup $(\mathbb{Z}_6, +)$. Let $A = \{\bar{1}, \bar{4}, \bar{5}\}$. We see that

$$(A^1 + \bar{0} + A^0) \cap A = (A + \bar{0}) \cap A = \{\bar{1}, \bar{4}, \bar{5}\} \cap A = \{\bar{1}, \bar{4}, \bar{5}\},$$
$$(A^1 + \bar{1} + A^0) \cap A = (A + \bar{1}) \cap A = \{\bar{0}, \bar{2}, \bar{5}\} \cap A = \{\bar{5}\},$$
$$(A^1 + \bar{2} + A^0) \cap A = (A + \bar{2}) \cap A = \{\bar{0}, \bar{1}, \bar{3}\} \cap A = \{\bar{1}\},$$
$$(A^1 + \bar{3} + A^0) \cap A = (A + \bar{3}) \cap A = \{\bar{1}, \bar{2}, \bar{4}\} \cap A = \{\bar{1}, \bar{4}\},$$
$$(A^1 + \bar{4} + A^0) \cap A = (A + \bar{4}) \cap A = \{\bar{2}, \bar{3}, \bar{5}\} \cap A = \{\bar{5}\},$$
$$(A^1 + \bar{5} + A^0) \cap A = (A + \bar{5}) \cap A = \{\bar{0}, \bar{3}, \bar{4}\} \cap A = \{\bar{4}\}.$$

Thus $(A^1 + \bar{s} + A^0) \cap A \neq \emptyset$ for all $\bar{s} \in \mathbb{Z}_6$. Hence, A is an almost $(1, 0)$-ideal of $\mathbb{Z}_6$ but A is not a $(1, 0)$-ideal of $\mathbb{Z}_6$ because A is not a subsemigroup of $\mathbb{Z}_6$.

From Example 3.1.3, an almost (m,n)-ideal of a semigroup S need not be an (m,n)-ideal of S. Thus we can see that the converse of Remark 3.1.2(iii) is not true in general. Next, we will explore some interesting properties of almost (m,n)-ideals in semigroups.

Proposition 3.1.4. Let A be an almost (m,n)-ideal of a semigroup S. Then every subset of S containing A is also an almost (m,n)-ideal of S.

Proof. Let B be a subset of S such that $A \subseteq B$ and let $s \in S$. Then $A^m \subseteq B^m$ and $A^n \subseteq B^n$, so $A^m sA^n \subseteq B^m sB^n$. Thus $A^m sA^n \cap A \subseteq B^m sB^n \cap B$. Since A is an almost (m,n)-ideal of S, we have $A^m sA^n \cap A \neq \emptyset$. Hence, $B^m sB^n \cap B \neq \emptyset$. Therefore, B is an almost (m,n)-ideal of S. □

Corollary 3.1.5. *The union of any two almost (m,n)-ideals of a semigroup S is an almost (m,n)-ideal of S.*

Proof. Let A_1 and A_2 be any two almost (m,n)-ideals of S. Since $A_1 \subseteq A_1 \cup A_2$ and A_1 is an almost (m,n)-ideal of S, by Proposition 3.1.4, $A_1 \cup A_2$ is an almost (m,n)-ideal of S. □

From the proof of Corollary 3.1.5, we can see that it is true if A_1 or A_2 is an almost (m,n)-ideal of S. The intersection of any two almost (m,n)-ideals of a semigroup S need not be an almost (m,n)-ideal of S as can be seen in the following example.

Example 3.1.6. Consider a semigroup $\mathbb{Z}_6$ under an addition modulo 6. Let $A_1 = \{\bar{1},\bar{4},\bar{5}\}$ and $A_2 = \{\bar{1},\bar{2},\bar{5}\}$. From Example 3.1.3, we obtain that A_1 is an almost (1,0)-ideal of $\mathbb{Z}_6$. We have

$$(A_2^1 + \bar{0} + A_2^0) \cap A_2 = (A_2 + \bar{0}) \cap A_2 = \{\bar{1},\bar{2},\bar{5}\} \cap A_2 = \{\bar{1},\bar{2},\bar{5}\},$$

$$(A_2^1 + \bar{1} + A_2^0) \cap A_2 = (A_2 + \bar{1}) \cap A_2 = \{\bar{0},\bar{2},\bar{3}\} \cap A_2 = \{\bar{2}\},$$

$$(A_2^1 + \bar{2} + A_2^0) \cap A_2 = (A_2 + \bar{2}) \cap A_2 = \{\bar{1},\bar{3},\bar{4}\} \cap A_2 = \{\bar{1}\},$$

$$(A_2^1 + \bar{3} + A_2^0) \cap A_2 = (A_2 + \bar{3}) \cap A_2 = \{\bar{2},\bar{4},\bar{5}\} \cap A_2 = \{\bar{2},\bar{5}\},$$

$$(A_2^1 + \bar{4} + A_2^0) \cap A_2 = (A_2 + \bar{4}) \cap A_2 = \{\bar{0},\bar{3},\bar{5}\} \cap A_2 = \{\bar{5}\},$$

$$(A_2^1 + \bar{5} + A_2^0) \cap A_2 = (A_2 + \bar{5}) \cap A_2 = \{\bar{0},\bar{1},\bar{4}\} \cap A_2 = \{\bar{1}\}.$$

So $(A_2^1 + \bar{s} + A_2^0) \cap A_2 \neq \emptyset$ for all $\bar{s} \in \mathbb{Z}_6$. Thus A_2 is an almost $(1,0)$-ideal of $\mathbb{Z}_6$. Consider $A_1 \cap A_2 = \{\bar{1}, \bar{5}\}$. Since

$$(\{\bar{1}, \bar{5}\}^1 + \bar{1} + \{\bar{1}, \bar{5}\}^0) \cap \{\bar{1}, \bar{5}\} = \{\bar{0}, \bar{2}\} \cap \{\bar{1}, \bar{5}\} = \emptyset,$$

$A_1 \cap A_2$ is not an almost (1,0)-ideal of S.

Theorem 3.1.7. *A semigroup S has no proper almost (m,n)-ideals if and only if for each $a \in S$, there exists $s_a \in S$ such that $(S - \{a\})^m s_a (S - \{a\})^n \subseteq \{a\}$.*

Proof. Assume that S has no proper almost (m,n)-ideals and let $a \in S$. Then $S - \{a\}$ is not an almost (m,n)-ideal of S. Then there exists $s_a \in S$ such that

$$\big[(S - \{a\})^m s_a (S - \{a\})^n\big] \cap (S - \{a\}) = \emptyset.$$

Thus $(S - \{a\})^m s_a (S - \{a\})^n \subseteq \{a\}$.

Conversely, let A be a proper subset of S. Then $A \subseteq S - \{a\}$ for some $a \in S$. By assumption, there exists $s_a \in S$ such that $(S - \{a\})^m s_a (S - \{a\})^n \subseteq \{a\}$. Thus

$$\big[(S - \{a\})^m s_a (S - \{a\})^n\big] \cap (S - \{a\}) \subseteq \{a\} \cap (S - \{a\}).$$

Since $\{a\} \cap (S - \{a\}) = \emptyset$, we have $\big[(S - \{a\})^m s_a (S - \{a\})^n\big] \cap (S - \{a\}) = \emptyset$. Since $A \subseteq S - \{a\}$, we have $A^m s_a A^n \subseteq (S - \{a\})^m s_a (S - \{a\})^n$, so

$$(A^m s_a A^n) \cap A \subseteq \big[(S - \{a\})^m s_a (S - \{a\})^n\big] \cap (S - \{a\}).$$

Thus $(A^m s_a A^n) \cap A = \emptyset$. Hence, A is not an almost (m,n)-ideal of S. Therefore, S has no proper almost (m,n)-ideals. □

Theorem 3.1.8. *Let S be a semigroup and $a \in S$. If S has no proper almost (m,n)-ideals, then at least one of the following statements is true.*

(1) $a = a^{m+n+1}$.

(2) $a = a^{(m+n)^3+1}$.

(3) $a = a^{(m+n+1)(m+n)+1}$.

Proof. Assume that S has no proper almost (m,n)-ideals. By Theorem 3.1.7, there exists $s_a \in S$ such that $(S-\{a\})^m s_a (S-\{a\})^n \subseteq \{a\}$. Suppose that $a \neq a^{m+n+1}$. Then $a^{m+n+1} \in S-\{a\}$, so

$$(a^{m+n+1})^m s_a (a^{m+n+1})^n \in (S-\{a\})^m s_a (S-\{a\})^n \subseteq \{a\}.$$

This implies that $(a^{m+n+1})^m s_a (a^{m+n+1})^n = a$.
Case 1: $s_a = a$. Then $a = (a^{m+n+1})^m a (a^{m+n+1})^n = a^{(m+n+1)(m+n)+1}$.
Case 2: $s_a \neq a$. Then $s_a \in S-\{a\}$. This implies that

$$s_a^{m+n+1} = (s_a)^m s_a (s_a)^n \in (S-\{a\})^m s_a (S-\{a\})^n \subseteq \{a\}.$$

So $s_a^{m+n+1} = a$. Since $a = (a^{m+n+1})^m s_a (a^{m+n+1})^n$, we have

$$\begin{aligned} a &= \big[(s_a^{m+n+1})^{m+n+1}\big]^m s_a \big[(s_a^{m+n+1})^{m+n+1}\big]^n \\ &= s_a^{(m+n+1)(m+n+1)(m+n)+1} \\ &= \big[(s_a^{m+n+1})^{m+n+1}\big]^{m+n} s_a \\ &= (a^{m+n+1})^{m+n} s_a \\ &= a^{(m+n)+(m+n)^2} s_a \\ &= a^{m+n} a^{(m+n)^2} s_a. \end{aligned}$$

Thus $a = a^{m+n} a^{(m+n)^2} s_a$. Since $a \neq a^{m+n+1} = a^{m+n} a$, we have $a^{(m+n)^2} s_a \neq a$. This implies that $a^{(m+n)^2} s_a \in S-\{a\}$. Thus we have

$$(a^{(m+n)^2} s_a)^m s_a (a^{(m+n)^2} s_a)^n \in (S-\{a\})^m s_a (S-\{a\})^n \subseteq \{a\},$$

so $(a^{(m+n)^2} s_a)^m s_a (a^{(m+n)^2} s_a)^n = a$. Since $s_a^{m+n+1} = a$,

$$\begin{aligned} a &= \big[(s_a^{m+n+1})^{(m+n)^2} s_a\big]^m s_a \big[(s_a^{m+n+1})^{(m+n)^2} s_a\big]^n \\ &= s_a^{(m+n+1)(m+n)^3+m+n+1} \\ &= (s_a^{m+n+1})^{(m+n)^3} s_a^{m+n+1} \\ &= a^{(m+n)^3} a \\ &= a^{(m+n)^3+1}. \end{aligned}$$

Hence, $a = a^{(m+n)^3+1}$. ∎

Corollary 3.1.9. *Let S be a semigroup and $a \in S$. If S has no proper left (or right) almost ideals, then $a = a^2$ or $a = a^3$.*

Proof. Assume that S has no proper left (or right) almost ideals. That is, S has no proper almost $(0, 1)$-ideals (or S has no proper almost $(1, 0)$-ideals). By Theorem 3.1.8, $a = a^2$ or $a = a^3$. □

3.2 Fuzzy almost (m, n)-ideals in semigroups

In this section, we give the definition and some properties of fuzzy almost (m, n)-ideals in semigroups by using the concept of almost (m, n)-ideals. Let f be a fuzzy subset and s_α be a fuzzy point of a semigroup S. For $k \in \mathbb{N}$, let

1. $f^k := f \circ f \circ \ldots \circ f \quad (k$ copies),
2. $f^k \circ s_\alpha \circ f^0 := f^k \circ s_\alpha$,
3. $f^0 \circ s_\alpha \circ f^k := s_\alpha \circ f^k$, and
4. $f^0 \circ s_\alpha \circ f^0 := s_\alpha$.

Proposition 3.2.1. Let f and g be fuzzy subsets of S. If $f \subseteq g$, then

$$f^n \subseteq g^n \text{ for all } n \in \mathbb{N}.$$

Proof. Assume that $f \subseteq g$. If $n = 1$, then we are done. Assume that $f^n \subseteq g^n$ where $n \geq 1$. We will show that $f^{n+1} \subseteq g^{n+1}$. Let $x \in S$.

If $x \notin S^2$, then $f^{n+1}(x) = (f^n \circ f)(x) = 0 \leq g^{n+1}(x)$.

If $x \in S^2$, then we have

$$\begin{aligned} f^{n+1}(x) &= (f^n \circ f)(x) \\ &= \sup_{x=ab} \min\{f^n(a), f(b)\} \\ &\leq \sup_{x=ab} \min\{g^n(a), g(b)\} \\ &= (g^n \circ g)(x) \\ &= g^{n+1}(x). \end{aligned}$$

Thus $f^{n+1}(x) \leq g^{n+1}(x)$ for all $x \in S$. Hence, we can conclude that $f^{n+1} \subseteq g^{n+1}$. Therefore, by the principle of mathematical induction, $f^n \subseteq g^n$ for all $n \in \mathbb{N}$. $\square$

Previously, we provided the definitions and some properties of fuzzy subsets and fuzzy ideals in semigroups. Nexte, we will give the definition of fuzzy almost (m, n)-ideals in semigroups as follows:

Definition 3.2.2. A fuzzy subset f of a semigroup S is called a ***fuzzy almost*** (m, n)***-ideal*** of S if $(f^m \circ s_\alpha \circ f^n) \cap f \neq 0$ for all fuzzy point s_α of S.

Remark 3.2.3. Let S be a semigroup. The following statements hold.

(i) A fuzzy almost $(0, 1)$-ideal of S is a fuzzy left almost ideal of S.

(ii) A fuzzy almost $(1, 0)$-ideal of S is a fuzzy right almost ideal of S.

Example 3.2.4. Consider a semigroup $(\mathbb{Z}_6, +)$. Let $f : \mathbb{Z}_6 \to [0, 1]$ be defined by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.2, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.5, f(\bar{5}) = 0.3.$$

We will show that f is a fuzzy almost $(1, 0)$-ideal of $\mathbb{Z}_6$ and a fuzzy almost $(1, 2)$-ideal of $\mathbb{Z}_6$. Let $\alpha \in (0, 1]$. Firstly, we want to show that f is a fuzzy almost $(1, 0)$-ideal of $\mathbb{Z}_6$. We consider the following result.

$$\begin{aligned}
(f^1 \circ \bar{0}_\alpha \circ f^0)(\bar{1}) &= (f \circ \bar{0}_\alpha)(\bar{1}) \\
&= \sup_{\bar{1} = \bar{a} + \bar{b}} \min\{f(\bar{a}), \bar{0}_\alpha(\bar{b})\} \\
&\geq \min\{f(\bar{1}), \bar{0}_\alpha(\bar{0})\} \;\; (\text{since } \bar{1} = \bar{1} + \bar{0}) \\
&= \min\{0.2, \alpha\}.
\end{aligned}$$

Since $\min\{0.2, \alpha\} \neq 0$, we have $(f^1 \circ \bar{0}_\alpha \circ f^0)(\bar{1}) \neq 0$. This implies that

$$\big[(f^1 \circ \bar{0}_\alpha \circ f^0) \cap f\big](\bar{1}) = \min\{(f^1 \circ \bar{0}_\alpha \circ f^0)(\bar{1}), f(\bar{1})\} \neq 0.$$

Hence, we can see that if there exist elements $\bar{x}, \bar{a} \in \mathbb{Z}_6$ such that $\bar{x} = \bar{a} + \bar{0}$ and $f(\bar{x}), f(\bar{a}) \neq 0$, then $\big[(f^1 \circ \bar{0}_\alpha \circ f^0) \cap f\big](\bar{x}) \neq 0$. In a similar way, we have the following:

$$\begin{aligned}
\bar{5} = \bar{4} + \bar{1} \text{ where } f(\bar{4}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \quad \big[(f^1 \circ \bar{1}_\alpha \circ f^0) \cap f\big](\bar{5}) \neq 0 \\
\bar{1} = \bar{5} + \bar{2} \text{ where } f(\bar{1}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \quad \big[(f^1 \circ \bar{2}_\alpha \circ f^0) \cap f\big](\bar{1}) \neq 0 \\
\bar{4} = \bar{1} + \bar{3} \text{ where } f(\bar{1}), f(\bar{4}) \neq 0 \quad &\Longrightarrow \quad \big[(f^1 \circ \bar{3}_\alpha \circ f^0) \cap f\big](\bar{4}) \neq 0 \\
\bar{5} = \bar{1} + \bar{4} \text{ where } f(\bar{1}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \quad \big[(f^1 \circ \bar{4}_\alpha \circ f^0) \cap f\big](\bar{5}) \neq 0 \\
\bar{4} = \bar{5} + \bar{5} \text{ where } f(\bar{4}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \quad \big[(f^1 \circ \bar{5}_\alpha \circ f^0) \cap f\big](\bar{4}) \neq 0.
\end{aligned}$$

Thus we can conclude that $(f^1 \circ \bar{s}_\alpha \circ f^0) \cap f \neq 0$ for all $\bar{s} \in \mathbb{Z}_6$. Hence, f is a fuzzy almost $(1,0)$-ideal of $\mathbb{Z}_6$. Next, we will show that f is a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$. We have

$$\begin{aligned}
(f^1 \circ \bar{0}_\alpha \circ f^2)(\bar{1}) &= \sup_{\bar{1}=\bar{a}+\bar{b}} \min\{(f \circ \bar{0}_\alpha)(\bar{a}), (f \circ f)(\bar{b})\} \\
&\geq \min\{(f \circ \bar{0}_\alpha)(\bar{1}), (f \circ f)(\bar{0})\} \\
&= \min\Big\{ \sup_{\bar{1}=\bar{x}+\bar{y}} \min\{f(\bar{x}), \bar{0}_\alpha(\bar{y})\}, \sup_{\bar{0}=\bar{u}+\bar{v}} \min\{f(\bar{u}), f(\bar{v})\}\Big\} \\
&\geq \min\Big\{ \min\{f(\bar{1}), \bar{0}_\alpha(\bar{0})\}, \min\{f(\bar{1}) + f(\bar{5})\}\Big\} \\
&= \min\{f(\bar{1}), f(\bar{5}), \alpha\} \\
&= \min\{0.2, 0.3, \alpha\}.
\end{aligned}$$

Since $\min\{0.2, 0.3, \alpha\} \neq 0$, we have that $(f^1 \circ \bar{0}_\alpha \circ f^2)(\bar{1}) \neq 0$. This implies that

$$\big[(f^1 \circ \bar{0}_\alpha \circ f^2) \cap f\big](\bar{1}) = \min\{(f^1 \circ \bar{0}_\alpha \circ f^2)(\bar{1}), f(\bar{1})\} \neq 0.$$

Thus we can see that if there are $\bar{x}, \bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_6$ such that $\bar{x} = \bar{a} + \bar{0} + (\bar{b} + \bar{c})$ and $f(\bar{x}), f(\bar{a}), f(\bar{b}), f(\bar{c}) \neq 0$, then $\big[(f^1 \circ \bar{0}_\alpha \circ f^2) \cap f\big](\bar{x}) \neq 0$. Similarly, we can see that

$$\begin{aligned}
\bar{4} = \bar{1} + \bar{1} + (\bar{1} + \bar{1}) \text{ where } f(\bar{1}), f(\bar{4}) \neq 0 \quad &\Longrightarrow \big[(f^1 \circ \bar{1}_\alpha \circ f^2) \cap f\big](\bar{4}) \neq 0 \\
\bar{5} = \bar{1} + \bar{2} + (\bar{1} + \bar{1}) \text{ where } f(\bar{1}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \big[(f^1 \circ \bar{2}_\alpha \circ f^2) \cap f\big](\bar{5}) \neq 0 \\
\bar{4} = \bar{5} + \bar{3} + (\bar{4} + \bar{4}) \text{ where } f(\bar{4}), f(\bar{5}) \neq 0 \quad &\Longrightarrow \big[(f^1 \circ \bar{3}_\alpha \circ f^2) \cap f\big](\bar{4}) \neq 0 \\
\bar{1} = \bar{1} + \bar{4} + (\bar{1} + \bar{1}) \text{ where } f(\bar{1}) \neq 0 \quad &\Longrightarrow \big[(f^1 \circ \bar{4}_\alpha \circ f^2) \cap f\big](\bar{1}) \neq 0 \\
\bar{5} = \bar{1} + \bar{5} + (\bar{4} + \bar{1}) \text{ where } f(\bar{1}), f(\bar{4}), f(\bar{5}) \neq 0 &\Longrightarrow \big[(f^1 \circ \bar{5}_\alpha \circ f^2) \cap f\big](\bar{5}) \neq 0.
\end{aligned}$$

This implies that $(f^1 \circ \bar{s}_\alpha \circ f^2) \cap f \neq 0$ for all $\bar{s} \in \mathbb{Z}_6$. Hence, f is a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$. Therefore, f is a fuzzy almost $(1,0)$-ideal and a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$.

From the definition of fuzzy almost (m, n)-ideals, we see that f is a fuzzy almost (m, n)-ideal of a semigroup S if and only if for each fuzzy point s_α of S, there exists $x \in S$ such that $[(f^m \circ s_\alpha \circ f^n) \cap f](x) \neq 0$, i.e., there is an element $x \in S$ such that

$$x = (a_1 a_2 \cdots a_m)s(b_1 b_2 \cdots b_n)$$

for some $a_1, ..., a_m, b_1, ..., b_n \in S$ and $f(x), f(a_1), ..., f(a_m), f(b_1), ..., f(b_n) \neq 0$.

Theorem 3.2.5. *Let f be a nonzero fuzzy subset of a semigroup S. If f is a fuzzy (m, n)-ideal of S, then f is a fuzzy almost (m, n)-ideal of S.*

Proof. Let f be a fuzzy (m, n)-ideal of S and let s_α be a fuzzy point of S. Since f is a nonzero fuzzy subset of S, there is an element $a \in S$ such that $f(a) \neq 0$. Since f is a fuzzy (m, n)-ideal of S, we have

$$f(a^m s a^n) \geq \min\Big\{ \underbrace{f(a), f(a), ..., f(a)}_{m \text{ copies}}, \underbrace{f(a), f(a), ..., f(a)}_{n \text{ copies}} \Big\} = f(a).$$

Thus $f(a^m s a^n) \neq 0$. Hence, $(f^m \circ s_\alpha \circ f^n \cap f)(a^m s a^n) \neq 0$. Therefore, f is a fuzzy almost (m, n)-ideal of S. □

In the previous section, we give some properties of almost (m, n)-ideals in semigroups. Next, we will illustrate these properties in fuzzy almost (m, n)-ideals by using the same idea.

Proposition 3.2.6. Let f be a fuzzy almost (m, n)-ideal of a semigroup S. Then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy almost (m, n)-ideal of S.

Proof. Let g be a fuzzy subset of S such that $f \subseteq g$ and let s_α be a fuzzy point in S. Then $f^m \subseteq g^m$ and $f^n \subseteq g^n$, so $f^m \circ s_\alpha \circ f^n \subseteq g^m \circ s_\alpha \circ g^n$. This implies that

$$(f^m \circ s_\alpha \circ f^n) \cap f \subseteq (g^m \circ s_\alpha \circ g^n) \cap g.$$

Since f is a fuzzy almost (m, n)-ideal of S, we have $(f^m \circ s_\alpha \circ f^n) \cap f \neq 0$. Hence, $(g^m \circ s_\alpha \circ g^n) \cap g \neq 0$. Therefore, g is a fuzzy almost (m, n)-ideal of S. □

Corollary 3.2.7. *The union of any two fuzzy almost (m,n)-ideals of a semigroup S is a fuzzy almost (m,n)-ideal of S.*

Proof. Let f and g be fuzzy almost (m,n)-ideals of S. Since $f \subseteq f \cup g$, by Proposition 3.2.6, $f \cup g$ is a fuzzy almost (m,n)-ideal of S. $\square$

Note that the proof of Corollary 3.2.7 is true if f or g is a fuzzy almost (m,n)-ideal of S.

Example 3.2.8. Consider a semigroup $(\mathbb{Z}_6,+)$. Let $f:\mathbb{Z}_6 \to [0,1]$ be defined by

$$f(\bar{0})=0, f(\bar{1})=0.2, f(\bar{2})=0, f(\bar{3})=0, f(\bar{4})=0.5, f(\bar{5})=0.3$$

and $g:\mathbb{Z}_6 \to [0,1]$ be defined by

$$g(\bar{0})=0, g(\bar{1})=0.8, g(\bar{2})=0.4, g(\bar{3})=0, g(\bar{4})=0, g(\bar{5})=0.3.$$

From Example 3.2.4, f is a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$. Let $\alpha \in (0,1]$. We have

$$\begin{aligned}
\bar{5} &= \bar{1}+\bar{0}+(\bar{2}+\bar{2}) \text{ where } g(\bar{1}),g(\bar{2}),g(\bar{5}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{0}_\alpha \circ g^2)\cap g\big](\bar{5}) \neq 0\\
\bar{5} &= \bar{1}+\bar{1}+(\bar{2}+\bar{1}) \text{ where } g(\bar{1}),g(\bar{2}),g(\bar{5}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{1}_\alpha \circ g^2)\cap g\big](\bar{5}) \neq 0\\
\bar{5} &= \bar{1}+\bar{2}+(\bar{1}+\bar{1}) \text{ where } g(\bar{1}),g(\bar{5}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{2}_\alpha \circ g^2)\cap g\big](\bar{5}) \neq 0\\
\bar{1} &= \bar{1}+\bar{3}+(\bar{2}+\bar{1}) \text{ where } g(\bar{1}),g(\bar{2}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{3}_\alpha \circ g^2)\cap g\big](\bar{1}) \neq 0\\
\bar{1} &= \bar{1}+\bar{4}+(\bar{1}+\bar{1}) \text{ where } g(\bar{1}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{4}_\alpha \circ g^2)\cap g\big](\bar{1}) \neq 0\\
\bar{2} &= \bar{1}+\bar{5}+(\bar{1}+\bar{1}) \text{ where } g(\bar{1}),g(\bar{2}) \neq 0 \Longrightarrow \big[(g^1\circ \bar{5}_\alpha \circ g^2)\cap g\big](\bar{2}) \neq 0.
\end{aligned}$$

Then $(g^1\circ \bar{s}_\alpha \circ g^2)\cap g \neq 0$ for all $\bar{s} \in \mathbb{Z}_6$. Thus g is a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$. Consider the intersection $f \cap g:\mathbb{Z}_6 \to [0,1]$ of f and g defined by, for all $\bar{x} \in \mathbb{Z}_6$,

$$(f\cap g)(\bar{x}) = \min\{f(\bar{x}), g(\bar{x})\},$$

that is, $(f\cap g)(\bar{0})=0,\quad (f\cap g)(\bar{1})=0.2,\quad (f\cap g)(\bar{2})=0,$
$(f\cap g)(\bar{3})=0,\quad (f\cap g)(\bar{4})=0,\quad (f\cap g)(\bar{5})=0.3.$

We can see that for all $\bar{x},\bar{a},\bar{b},\bar{c} \in \mathbb{Z}_6$ such that $f(\bar{x}), f(\bar{a}), f(\bar{b}), f(\bar{c}) \neq 0$, we have

$$\bar{x} \neq \bar{a}+\bar{1}+(\bar{b}+\bar{c}),$$

which implies that $\Big[\big[(f\cap g)^1\circ \bar{1}_\alpha \circ (f\cap g)^2\big]\cap (f\cap g)\Big](\bar{x}) = 0$ for all $\bar{x} \in \mathbb{Z}_6$. Thus $\big[(f\cap g)^1\circ \bar{1}_\alpha \circ (f\cap g)^2\big] = 0$. Hence, $f\cap g$ is not a fuzzy almost $(1,2)$-ideal of $\mathbb{Z}_6$.

Example 3.2.8 shows that, the intersection of two fuzzy almost (m,n)-ideals of a semigroup S need not be a fuzzy almost (m,n)-ideal of S.

3.3 The relations of almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups

In this section, we study the relations of almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups. Firstly, we present the following lemma.

Lemma 3.3.1. *Let A be a subset of a semigroup S. Then we have*

$$C_A^n = C_{A^n} \text{ for all } n \in \mathbb{N}.$$

Proof. Clearly, the statement is true when $n = 1$. Assume that $C_A^n = C_{A^n}$ where $n \geq 1$. Let $x \in S$.

Case 1: $x \notin A^{n+1}$. Then $C_{A^{n+1}}(x) = 0$, and $x \neq ab$ for all $a \in A^n$ and $b \in A$. Thus

$$C_A^{n+1}(x) = \big(C_A^n \circ C_A\big)(x) = (C_{A^n} \circ C_A)(x) = 0.$$

Hence, $C_A^{n+1}(x) = C_{A^{n+1}}(x) = 0$.

Case 2: $x \in A^{n+1}$. Then $C_{A^{n+1}}(x) = 1$ and $x = ab$ for some $a \in A^n, b \in A$, so $C_{A^n}(a) = 1$ and $C_A(b) = 1$. Thus we have

$$\begin{aligned} C_A^{n+1}(x) &= \big(C_A^n \circ C_A\big)(x) \\ &= (C_{A^n} \circ C_A)(x) \\ &= \sup_{x=uv} \min\{C_{A^n}(u), C_A(v)\} \\ &\geq \min\{C_{A^n}(a), C_A(b)\} \\ &= 1. \end{aligned}$$

So $C_A^{n+1}(x) = 1$. Hence, $C_A^{n+1}(x) = C_{A^{n+1}}(x)$. Therefore, $C_A^{n+1} = C_{A^{n+1}}$. By the principle of mathematical induction, $C_A^n = C_{A^n}$ for all $n \in \mathbb{N}$. □

Theorem 3.3.2. *Let A be a nonempty subset of a semigroup S. Then A is an almost (m,n)-ideal of S if and only if C_A is a fuzzy almost (m,n)-ideal of S.*

Proof. Assume that A is an almost (m,n)-ideal of S. Let s_α be a fuzzy point in S. Then $A^m s A^n \cap A \neq \emptyset$. Thus there exists $x \in A$, and $x = asb$ for some $a \in A^m$ and $b \in A^n$, which implies that $C_{A^m}(a), C_{A^n}(b), C_A(x) \neq 0$. Hence,

$$\big[(C_{A^m} \circ s_\alpha \circ C_{A^n}) \cap C_A\big](x) \neq 0.$$

By Lemma 3.3.1, we have

$$\Big[\big(C_A^m \circ s_\alpha \circ C_A^n\big) \cap C_A\Big](x) \neq 0.$$

Therefore, C_A is a fuzzy almost (m,n)-ideal of S.

Conversely, assume that C_A is a fuzzy almost (m,n)-ideal of S. Let $s \in S$. Then $\big(C_A^m \circ s_\alpha \circ C_A^n\big) \cap C_A \neq 0$ for all $\alpha \in (0,1]$, so there exists $x \in S$ such that

$$\Big[\big(C_A^m \circ s_\alpha \circ C_A^n\big) \cap C_A\Big](x) \neq 0.$$

Thus $x = asb$ for some $a, b \in S$ and $C_A(x), C_A^m(a), C_A^n(b) \neq 0$. By Lemma 3.3.1, we have $C_A^m(a) = C_{A^m}(a)$ and $C_A^n(b) = C_{A^n}(b)$, so $C_{A^m}(a) \neq 0$ and $C_{A^n}(b) \neq 0$. Then $x \in A$ and $x = asb$ where $a \in A^m$, $b \in A^n$. Thus $x \in A^m s A^n \cap A$. Hence, $A^m s A^n \cap A \neq \emptyset$. Therefore, A is an almost (m,n)-ideal of S. □

Theorem 3.3.3. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy almost (m,n)-ideal of S if and only if $supp(f)$ is an almost (m,n)-ideal of S.*

Proof. Assume that f is a fuzzy almost (m,n)-ideal of S. Let $s \in S$. Then for any $\alpha \in (0,1]$, there exists $x \in S$ such that

$$x = (a_1a_2...a_m)s(b_1b_2...b_n)$$

for some $a_1, ..., a_m, b_1, ..., b_n \in S$ and $f(x), f(a_1), ..., f(a_m), f(b_1), ..., f(b_n) \neq 0$. This implies that $x, a_1, ..., a_m, b_1, ..., b_n \in supp(f)$. So $x \in \big(supp(f)\big)^m s\big(supp(f)\big)^n$ and $x \in supp(f)$. It follows that $x \in \Big(\big(supp(f)\big)^m s\big(supp(f)\big)^n\Big) \cap supp(f)$. Hence, $supp(f)$ is an almost (m,n)-ideal of S.

Conversely, assume that $supp(f)$ is an almost (m,n)-ideal of S. Let s_α be a fuzzy point in S. Then there exists $x \in \Big(\big(supp(f)\big)^m s\big(supp(f)\big)^n\Big) \cap supp(f)$. Thus there are $a_1, ..., a_m, b_1, ..., b_n \in supp(f)$ such that

$$x = (a_1a_2...a_m)s(b_1b_2...b_n)$$

and $x \in supp(f)$. This implies that $f(x), f(a_1), ..., f(a_m), f(b_1), ..., f(b_n) \neq 0$. Hence, we have

$$[(f^m \circ s_\alpha \circ f^n) \cap f](x) \neq 0.$$

Consequently, f is a fuzzy almost (m,n)-ideal of S. □

A ***minimal*** almost (m, n)-ideal of a semigroup is an almost (m, n)-ideal which contains no other almost (m, n)-ideal. The definition of minimal fuzzy almost (m, n)-ideals in semigroups, is defined as follows:

Definition 3.3.4. Let S be a semigroup. A fuzzy almost (m, n)-ideal f of S is called ***minimal*** if for all nonzero fuzzy almost (m, n)-ideal g of S such that $g \subseteq f$, we have $supp(f) = supp(g)$.

Next, we consider the relationship between minimal almost (m, n)-ideals and minimal fuzzy almost (m, n)-ideals in semigroups by the following theorem.

Theorem 3.3.5. *Let A be a nonempty subset of a semigroup S. Then A is a minimal almost (m, n)-ideal of S if and only if C_A is a minimal fuzzy almost (m, n)-ideal of S.*

Proof. Assume that A is a minimal almost (m, n)-ideal of S. By Theorem 3.3.2, C_A is a fuzzy almost (m, n)-ideal of S. We will show that C_A is minimal. Let g be a fuzzy almost (m, n)-ideal of S such that $g \subseteq C_A$. Then $supp(g) \subseteq supp(C_A) = A$. Since g is a fuzzy almost (m, n)-ideal of S, by Theorem 3.3.3, we have $supp(g)$ is an almost (m, n)-ideal of S. Since A is a minimal almost (m, n)-ideal of S and $supp(g) \subseteq A$, it follows that $supp(g) = A = supp(C_A)$. Hence, C_A is minimal. Therefore, C_A is a minimal fuzzy almost (m, n)-ideal of S.

Conversely, assume that C_A is a minimal fuzzy almost (m, n)-ideal of S. By Theorem 3.3.2, A is an almost (m, n)-ideal of S. We want to show that A is minimal. Let G be an almost (m, n)-ideal of S such that $G \subseteq A$. Then C_G is a fuzzy almost (m, n)-ideal of S and $C_G \subseteq C_A$. Since C_A is minimal, $supp(C_G) = supp(C_A)$. Thus $G = A$. Hence, A is minimal. Therefore, A is a minimal almost (m, n)-ideal of S. □

Corollary 3.3.6. *A semigroup S has no proper almost (m, n)-ideals if and only if $supp(f) = S$ for all fuzzy almost (m, n)-ideal f of S.*

Proof. Let f be a fuzzy almost (m, n)-ideal of S. By Theorem 3.3.3, $supp(f)$ is an almost (m, n)-ideal of S. Since S has no proper almost (m, n)-ideals, $supp(f) = S$.

Conversely, let A be an almost (m, n)-ideal of S. By Theorem 3.3.2, C_A is a fuzzy almost (m, n)-ideal of S. Thus, by assumption, we have $supp(C_A) = S$, so $A = S$. Hence, S has no proper almost (m, n)-ideals. □

In a semigroup S, a ***prime*** almost (m, n)-ideal A of S is an almost (m, n)-ideal such that for all $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$. An almost (m, n)-ideal A of S is ***semiprime*** if for all $x \in S, x^2 \in A$ implies $x \in A$. The definitions of prime fuzzy almost (m, n)-ideals and semiprime fuzzy almost (m, n)-ideals of S are given below.

Definition 3.3.7. A fuzzy almost (m, n)-ideal f of a semigroup S is ***prime*** if

$$f(xy) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in S.$$

Definition 3.3.8. Let S be a semigroup. A fuzzy almost (m, n)-ideal f is ***semiprime*** if $f(x^2) \leq f(x)$ for all $x \in S$.

The relationship between prime (semiprime) almost (m, n)-ideals and prime (semiprime) fuzzy almost (m, n)-ideals in semigroups is provided in the following theorems.

Theorem 3.3.9. *A nonempty subset A of a semigroup S is a prime almost (m, n)-ideal of S if and only if C_A is a prime fuzzy almost (m, n)-ideal of S.*

Proof. Assume that A is a prime almost (m, n)-ideal of S. By Theorem 3.3.2, C_A is a fuzzy almost (m, n)-ideal of S. Let $x, y \in S$.

Case 1: $xy \in A$. Then $C_A(xy) = 1$. Since A is prime, $x \in A$ or $y \in A$, so $C_A(x) = 1$ or $C_A(y) = 1$. Thus $\max\{C_A(x), C_A(y)\} = 1 = C_A(xy)$.

Case 2: $xy \notin A$. Then $C_A(xy) = 0 \leq \max\{C_A(x), C_A(y)\}$.

Thus we conclude that $C_A(xy) \leq \max\{C_A(x), C_A(y)\}$. Hence, C_A is a prime fuzzy almost (m, n)-ideal of S.

Conversely, assume that C_A is a prime fuzzy almost (m, n)-ideal of S. Then A is an almost (m, n)-ideal of S. Let $x, y \in S$ such that $xy \in A$. This implies that $C_A(xy) = 1$. Since C_A is prime, we have $1 = C_A(xy) \leq \max\{C_A(x), C_A(y)\}$, so $\max\{C_A(x), C_A(y)\} = 1$. Thus $C_A(x) = 1$ or $C_A(y) = 1$. Hence, $x \in A$ or $y \in A$. Therefore, A is a prime almost (m, n)-ideal of S. □

Theorem 3.3.10. *A nonempty subset A of a semigroup S is a semiprime almost (m, n)-ideal of S if and only if C_A is a semiprime fuzzy almost (m, n)-ideal of S.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.3.9. □

CHAPTER 4

Ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups

In this chapter, we introduce the notions of ordered almost ideals, ordered almost bi-ideals, ordered almost quasi-ideals, fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups. Moreover, some properties and the relations of them are discussed.

4.1 Ordered almost ideals in ordered semigroups

In this section, we define ordered almost ideals, ordered almost bi-ideals and ordered almost quasi-ideals in ordered semigroups and we study some of their properties.

Definition 4.1.1. Let $(S, \cdot, \leq)$ be an ordered semigroup.

1. A nonempty subset L of S is called a ***left ordered almost ideal*** of S if

$$(sL] \cap L \neq \emptyset \quad \text{for all } s \in S.$$

2. A nonempty subset R of S is called a ***right ordered almost ideal*** of S if

$$(Rs] \cap R \neq \emptyset \quad \text{for all } s \in S.$$

3. A nonempty subset I of S is called an ***ordered almost ideal*** of S if I is a left ordered almost ideal and a right ordered almost ideal of S.

4. A nonempty subset B of S is called an ***ordered almost bi-ideal*** of S if

$$(BsB] \cap B \neq \emptyset \quad \text{for all } s \in S.$$

5. A nonempty subset Q of S is called an ***ordered almost quasi-ideal*** of S if

$$(sQ] \cap (Qs] \cap Q \neq \emptyset \quad \text{for all } s \in S.$$

Example 4.1.2. Consider an ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ below.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a,a),(a,b),(b,b),(c,a),(c,b),(c,c),(d,a),(d,b),(d,d),(e,e)\}.$$

Then every nonempty subset of S is an ordered almost bi-ideal of S except for $\{e\}$, and every nonempty subset of S except for $\{b\}, \{e\}$ and $\{b, e\}$ is an ordered almost ideal and an ordered almost quasi-ideal of S.

Remark 4.1.3. Let $(S, \cdot, \leq)$ be an ordered semigroup.

(i) Every left ordered ideal of S is a left ordered almost ideal of S.

(ii) Every right ordered ideal of S is a right ordered almost ideal of S.

(iii) Every ordered ideal of S is an ordered almost ideal of S.

(iv) Every ordered bi-ideal of S is an ordered almost bi-ideal of S.

(v) If Q is an ordered quasi-ideal of S and $sQ \cap Qs \neq \emptyset$ for all $s \in S$, then Q is an ordered almost quasi-ideal of S.

Proof. (i) Let I be a left ordered ideal of S and $s \in S$. Then $sI \subseteq SI \subseteq I$ and $(I] = I$, so $(sI] \subseteq (I] = I$. Since $I \neq \emptyset$, we have $sI \neq \emptyset$, so $(sI] \neq \emptyset$. This implies that $(sI] \cap I = (sI] \neq \emptyset$. Therefore, I is a left ordered almost ideal of S.

(ii) This can be proved in similar manner.

(iii) The proof follows (i) and (ii).

(iv) Let B be an ordered bi-ideal of S and $s \in S$. Then $BSB \subseteq B$ and $(B] = B$. Since $BsB \subseteq BSB$, we have $BsB \subseteq B$, so $(BsB] \subseteq (B]$. Thus $(BsB] \subseteq B$. Since $B \neq \emptyset$, we have that $(BsB] \neq \emptyset$. Hence, $(BsB] \cap B = (BsB] \neq \emptyset$. Therefore, B is an ordered almost bi-ideal of S.

(v) Assume that Q is an ordered quasi-ideal of S and $xQ \cap Qx \neq \emptyset$ for all $x \in S$. Let $s \in S$. Then $sQ \subseteq SQ$ and $Qs \subseteq QS$, so $sQ \cap Qs \subseteq SQ \cap QS \subseteq Q$. This implies that $(sQ \cap Qs] \subseteq (Q] = Q$. By assumption, $sQ \cap Qs \neq \emptyset$. Thus we have

$$sQ \cap Qs \subseteq (sQ \cap Qs] = (sQ \cap Qs] \cap Q \subseteq (sQ] \cap (Qs] \cap Q.$$

Thus $(sQ] \cap (Qs] \cap Q \neq \emptyset$. Hence, Q is an ordered almost quasi-ideal of S. □

Example 4.1.4. From Example 4.1.2, we can see that $\{a, b, c\}$ is an ordered almost ideal, an ordered almost bi-ideal and an ordered almost quasi-ideal of S, but it is neither an ordered ideal, an ordered bi-ideal, nor an ordered quasi-ideal of S because

$$\big(\{a, b, c\}\big] = \{a, b, c, d\} \neq \{a, b, c\}.$$

From Example 4.1.4, in general, ordered ideals (resp. ordered bi-ideals, ordered quasi-ideals) need not be an ordered almost ideal (resp. an ordered almost bi-ideal, an ordered almost quasi-ideal) in ordered semigroups. Next, we will give some interesting properties of ordered almost ideals, ordered almost bi-ideals and ordered almost quasi-ideals in ordered semigroups.

Proposition 4.1.5. Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following statements hold.

(1) If L is a left ordered almost ideal of S, then every subset of S containing L is also a left ordered almost ideal of S.

(2) If R is a right ordered almost ideal of S, then every subset of S containing R is also a right ordered almost ideal of S.

(3) If I is an ordered almost ideal of S, then every subset of S containing I is also an ordered almost ideal of S.

(4) If B is an ordered almost bi-ideal of S, then every subset of S containing B is also an ordered almost bi-ideal of S.

(5) If Q is an ordered almost quasi-ideal of S, then every subset of S containing Q is also an ordered almost quasi-ideal of S.

Proof. (1) Assume that L is a left ordered almost ideal of S. Let A be a subset of S such that $L \subseteq A$ and let $s \in S$. Then $sL \subseteq sA$, so $(sL] \subseteq (sA]$. Thus $(sL] \cap L \subseteq (sA] \cap A$. Since L is a left ordered almost ideal of S, $(sL] \cap L \neq \emptyset$. This implies that $(sA] \cap A \neq \emptyset$. Hence, A is a left ordered almost ideal of S.

(2) The proof of this statement is similar to the proof of statement (1).

(3) This statement follows from statements (1) and (2).

(4) Assume that B is an ordered almost bi-ideal of S. Let A be a subset of S such that $B \subseteq A$, and let $s \in S$. Then $BsB \subseteq AsA$, so $(BsB] \subseteq (AsA]$. Thus we have $(BsB] \cap B \subseteq (AsA] \cap A$. Since B is an ordered almost bi-ideal of S, $(BsB] \cap B \neq \emptyset$. Hence, $(AsA] \cap A \neq \emptyset$. Therefore, A is an ordered almost bi-ideal of S.

(5) Assume that Q is an ordered almost quasi-ideal of S. Let A be a subset of S such that $Q \subseteq A$ and let $s \in S$. Then $sQ \subseteq sA$ and $Qs \subseteq As$, so $(sQ] \subseteq (sA]$ and $(Qs] \subseteq (As]$. Thus $(sQ] \cap (Qs] \cap Q \subseteq (sA] \cap (As] \cap A$. Since Q is an ordered almost quasi-ideal of S, $(sQ] \cap (Qs] \cap Q \neq \emptyset$. Hence, $(sA] \cap (As] \cap A \neq \emptyset$. Therefore, A is an ordered almost quasi-ideal of S. □

The next result follows directly from Proposition 4.1.5.

Corollary 4.1.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup.*

(1) *The arbitrary union of left ordered almost ideals of S is also a left ordered almost ideal of S.*

(2) *The arbitrary union of right ordered almost ideals of S is also a right ordered almost ideal of S.*

(3) *The arbitrary union of ordered almost ideals of S is also an ordered almost ideal of S.*

(4) *The arbitrary union of ordered almost bi-ideals of S is also an ordered almost bi-ideal of S.*

(5) *The arbitrary union of ordered almost quasi-ideals of S is also an ordered almost quasi-ideal of S.*

Example 4.1.7. From Example 4.1.2, we have $\{a, d, e\}$ and $\{b, c, e\}$ are ordered almost ideals, ordered almost bi-ideals and ordered almost quasi-ideals of S. But $\{a, d, e\} \cap \{b, c, e\} = \{e\}$ is neither an ordered almost ideal, an ordered almost bi-ideal, nor an ordered almost quasi-ideal of S.

From Example 4.1.7, we can see that the arbitrary intersection of ordered almost ideals (resp. ordered almost bi-ideals, ordered almost quasi-ideals) in ordered semigroups need not be an ordered almost ideal (resp. ordered almost bi-ideal, ordered almost quasi-ideal).

Theorem 4.1.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $|S| > 1$. Then the following statements hold.*

(1) *S has no proper left ordered almost ideals if and only if for every $a \in S$, there exists an element $x_a \in S$ such that $\big(x_a(S - \{a\})\big] = \{a\}$.*

(2) *S has no proper right ordered almost ideals if and only if for every $a \in S$, there exists an element $y_a \in S$ such that $\big((S - \{a\})y_a\big] = \{a\}$.*

(3) *S has no proper ordered almost ideals if and only if for every $a \in S$, there exist elements $x_a, y_a \in S$ such that*

$$\big(x_a(S-\{a\})\big] = \{a\} \text{ or } \big((S-\{a\})y_a\big] = \{a\}.$$

(4) *S has no proper ordered almost bi-ideals if and only if for every $a \in S$, there exists an element $x_a \in S$ such that $\big((S-\{a\})x_a(S-\{a\})\big] = \{a\}$.*

(5) *S has no proper ordered almost quasi-ideals if and only if for every $a \in S$, there exists an element $x_a \in S$ such that $\big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] \subseteq \{a\}$.*

Proof. (1) Assume that S has no proper left ordered almost ideals and let $a \in S$. Then $S-\{a\}$ is not a left ordered almost ideal of S. That is, there exists $x_a \in S$ such that

$$\big(x_a(S-\{a\})\big] \cap (S-\{a\}) = \emptyset.$$

Since $\big(x_a(S-\{a\})\big] \neq \emptyset$ and $S-\{a\} \neq \emptyset$, we have $\big(x_a(S-\{a\})\big] = \{a\}$.

Conversely, let L be a proper nonempty subset of S. Then $L \subseteq S-\{a\}$ for some $a \in S$. By assumption, there exists $x_a \in S$ such that

$$\big(x_a(S-\{a\})\big] = \{a\}.$$

Since $x_aL \subseteq x_a(S-\{a\})$, we have that $(x_aL] \subseteq \big(x_a(S-\{a\})\big]$. Then

$$(x_aL] \cap L \subseteq \big(x_a(S-\{a\})\big] \cap (S-\{a\}) = \{a\} \cap (S-\{a\}).$$

Since $\{a\} \cap (S-\{a\}) = \emptyset$, we have $(x_aL] \cap L = \emptyset$. This implies that L is not a left ordered almost ideal of S. Therefore, S has no proper left ordered almost ideals.

(2) This statement can be proved in a similar manner as the statement (1).

(3) Let $a \in S$. Assume that S has no proper ordered almost ideals. Then S has no proper left ordered almost ideals or S has no proper right ordered almost ideals.

If S has no proper left ordered almost ideals, then by the statement (1), there exists an element $x_a \in S$ such that $\big(x_a(S-\{a\})\big] = \{a\}$.

If S has no proper right ordered almost ideals, then by statements (2), there exists an element $y_a \in S$ such that $\big((S-\{a\})y_a\big] = \{a\}$.

Thus we can conclude that there exist elements $x_a, y_a \in S$ such that

$$\big(x_a(S-\{a\})\big] = \{a\} \text{ or } \big((S-\{a\})y_a\big] = \{a\}.$$

Conversely, by assumption, and statements (1) and (2), S has no proper left ordered almost ideals or S has no proper right ordered almost ideals. Thus S has no proper ordered almost ideals.

(4) Assume that S has no proper ordered almost bi-ideals. Then $S-\{a\}$ is not an ordered almost bi-ideal of S for all $a \in S$. Thus for each $a \in S$, there exists $x_a \in S$ such that

$$\big((S-\{a\})x_a(S-\{a\})\big] \cap (S-\{a\}) = \emptyset,$$

so we have $\big((S-\{a\})x_a(S-\{a\})\big] = \{a\}$.

Conversely, suppose that B is a proper ordered almost bi-ideal of S. This implies that $B \subseteq S-\{t\}$ for some $t \in S$. By Proposition 4.1.5(4), $S-\{t\}$ is an ordered almost bi-ideal of S. By assumption, there exists $x_t \in S$ such that

$$\big((S-\{t\})x_t(S-\{t\})\big] = \{t\}.$$

Thus $\big((S-\{t\})x_t(S-\{t\})\big] \cap (S-\{t\}) = \emptyset$, a contradiction. Hence, B is not a proper ordered almost bi-ideal of S. Therefore, S has no proper ordered almost bi-ideals.

(5) Assume that S has no proper ordered almost quasi-ideals. Let $a \in S$. Then $S-\{a\}$ is not an ordered almost quasi-ideal of S. Thus there is $x_a \in S$ such that

$$\big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] \cap (S-\{a\}) = \emptyset.$$

Since $S-\{a\} \neq \emptyset$, we obtain that

$$\big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] = \emptyset \text{ or } \big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] = \{a\}.$$

Thus $\big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] \subseteq \{a\}$.

Conversely, suppose that Q is a proper ordered almost quasi-ideal of S. Then we have $Q \subseteq S-\{u\}$ for some $u \in S$. This implies that $S-\{u\}$ is an ordered almost quasi-ideal of S by Proposition 4.1.5(5). By assumption, there is $x_u \in S$ such that

$$\big(x_u(S-\{u\})\big] \cap \big((S-\{u\})x_u\big] \subseteq \{u\}.$$

Thus $\big(x_u(S-\{u\})\big] \cap \big((S-\{u\})x_u\big] \cap (S-\{u\}) \subseteq \{u\} \cap (S-\{u\})$. Since $\{u\} \cap (S-\{u\}) = \emptyset$, we have $\big(x_u(S-\{u\})\big] \cap \big((S-\{u\})x_u\big] \cap (S-\{u\}) = \emptyset$. This is a contradiction. Therefore, S has no proper ordered almost quasi-ideals. $\square$

Theorem 4.1.9. *Let $(S, \cdot, \leq)$ be an ordered semigroup such that $|S| > 1$ and $a \in S$. Then the following statements hold.*

(1) *If S has no proper left ordered almost ideals, then either a or a^2 is an ordered idempotent.*

(2) *If S has no proper right ordered almost ideals, then either a or a^2 is an ordered idempotent.*

(3) *If S has no proper ordered almost ideals, then either a or a^2 is an ordered idempotent.*

(4) *If S has no proper ordered almost bi-ideals, then either a or a^4 is an ordered idempotent.*

(5) *If S has no proper ordered almost quasi-ideals, then either a or a^2 is an ordered idempotent.*

Proof. (1) Assume that S has no proper left ordered almost ideals. By Theorem 4.1.8(1), there exists an element $x_a \in S$ such that

$$\big(x_a(S-\{a\})\big] = \{a\}.$$

Case 1: $a = a^2$. Then $a = a^2 \leq a^2$, so a is an ordered idempotent.

Case 2: $a \neq a^2$. This implies that $a^2 \in S-\{a\}$. Thus $x_a a^2 = a$.

Subcase 2.1: $x_a \nleq a$. That is $x_a \in S-\{a\}$. So $x_a x_a = a$. Suppose that $x_a a \nleq a$. Then $x_a a \in S-\{a\}$, it follows that $x_a(x_a a) = a$. We have $a = x_a x_a a = aa = a^2$, a contradiction. Then $x_a a \leq a$, so $a = x_a a^2 \leq a^2$. Thus a is an ordered idempotent.

Subcase 2.2: $x_a \leq a$. Then $a = x_a a^2 \leq a^3$, so $a^2 \leq a^4$. Thus a^2 is an ordered idempotent.

Thus we can conclude that a or a^2 is an ordered idempotent.

(2) The proof of this statement is similar to the statement (1).

(3) This follows from the statements (1) and (2).

(4) Assume that S has no proper ordered almost bi-ideals. By Theorem 4.1.8(4), there exists an element $x_a \in S$ such that

$$\big((S-\{a\})x_a(S-\{a\})\big] = \{a\}.$$

Case 1: $a = a^2$. Then $a = a^2 \leq a^2$, so a is an ordered idempotent.

Case 2: $a \neq a^2$. Then $a^2 \in S - \{a\}$, which implies that $a^2x_aa^2 = a$.

Subcase 2.1: $x_a \nleq a$. Then $x_a \in S - \{a\}$, so $x_a^3 = a$. Since $x_a \in S - \{a\}$ and $a^2 \in S - \{a\}$, we have that $x_a^2a^2 = x_ax_aa^2 = a$. If $x_a^2a \nleq a$, then $x_a^2a \in S - \{a\}$, so $x_a^2ax_a^3a = a$. Thus we have

$$a = x_a^2ax_a^3a = x_a^2aaa = (x_a^2a^2)a = aa = a^2,$$

a contradiction. Thus $x_a^2a \leq a$, so $a = x_a^2a^2 = (x_a^2a)a \leq aa = a^2$. This implies that a is an ordered idempotent.

Subcase 2.2: $x_a \leq a$. Then $a = a^2x_aa^2 \leq a^2aa^2 = a^5$, so $a^4 \leq a^8 = (a^4)^2$. Thus we have a^4 is an ordered idempotent.

Therefore, a or a^4 is an ordered idempotent.

(5) Assume that S has no proper ordered almost quasi-ideals. By Theorem 4.1.8(5), there is an element $x_a \in S$ such that

$$\big(x_a(S-\{a\})\big] \cap \big((S-\{a\})x_a\big] \subseteq \{a\}.$$

Case 1: $a = a^2$. Then $a = a^2 \leq a^2$, so a is an ordered idempotent.

Case 2: $a \neq a^2$. Then $a^2 \in S - \{a\}$, so $(x_aa^2] \cap (a^2x_a] \subseteq \{a\}$. Suppose for the contrary that $a \nleq x_a$. Then $x_a \in S - \{a\}$, so $x_a^2 \in \big(x_a^2\big] \cap \big(x_a^2\big] \subseteq \{a\}$. This implies that $x_a^2 = a$. Thus we have $a^2 = ax_a^2$ and $a^2 = x_a^2a$. We consider the following four cases:

If $a \leq x_aa$ and $a \leq ax_a$, then $a^2 \leq x_aa^2$ and $a^2 \leq a^2x_a$. This implies that $a^2 \in \big(x_aa^2\big]$ and $a^2 \in (a^2x_a]$, so $a^2 \in (x_aa^2] \cap (a^2x_a] \subseteq \{a\}$. Thus $a = a^2$, a contradiction.

If $a \leq x_aa$ and $a \nleq ax_a$, then $a^2 \leq x_aa^2$ and $ax_a \in S - \{a\}$. Since $a^2 \in S - \{a\}$ and $ax_a \in S - \{a\}$, we have that $(x_aa^2] \cap (ax_a^2] \subseteq \{a\}$. Since $a^2 \leq x_aa^2$ and $a^2 = ax_a^2$, we have $a^2 \in (x_aa^2] \cap (ax_a^2] \subseteq \{a\}$, so $a = a^2$, a contradiction.

If $a \not\leq x_a a$ and $a \leq ax_a$, then $x_a a \in S-\{a\}$ and $a^2 \leq a^2 x_a$. Since $x_a a \in S-\{a\}$ and $a^2 \in S-\{a\}$, we have $(x_a^2 a] \cap (a^2 x_a] \subseteq \{a\}$. Since $a^2 = (x_a)^2 a$ and $a^2 \leq a^2 x_a$, $a^2 \in (x_a^2 a] \cap (a^2 x_a] \subseteq \{a\}$, so $a = a^2$, a contradiction.

If $a \not\leq x_a a$ and $a \not\leq ax_a$, then $x_a a \in S - \{a\}$ and $ax_a \in S - \{a\}$. This implies that $(x_a^2 a] \cap (ax_a^2] \subseteq \{a\}$. Since $a^2 = x_a^2 a$ and $a^2 = ax_a^2$, we have that $a^2 \in (x_a^2 a] \cap (ax_a^2] \subseteq \{a\}$. Thus $a = a^2$, a contradiction.

Hence, $a \leq x_a$. Then $a^3 \leq x_a a^2$ and $a^3 \leq a^2 x_a$. Thus $a^3 \in (x_a a^2]$ and $a^3 \in (a^2 x_a]$. So $a^3 \in (x_a a^2] \cap (a^2 x_a] \subseteq \{a\}$. Thus $a = a^3$, so $a^2 = a^4 \leq (a^2)^2$. This implies that a^2 is an ordered idempotent. Consequently, a or a^2 is an ordered idempotent. □

4.2 Fuzzy ordered almost ideals in ordered semigroups

In this section, we define fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups. Furthermore, some interesting properties are provided. First of all, we give some basic definitions and results, which are necessary for this section.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For a fuzzy subset f of S, we define $(f] : S \longrightarrow [0, 1]$ by

$$(f](x) = \sup_{x \leq y} f(y) \text{ for all } x \in S.$$

Proposition 4.2.1. Let f, g and h be fuzzy subsets of an ordered semigroup $(S, \cdot, \leq)$. Then the following properties hold.

(1) $f \subseteq (f]$.

(2) If $f \subseteq g$, then $(f] \subseteq (g]$.

(3) If $f \subseteq g$, then $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$.

Proof. (1) Let $x \in S$. Since $x \leq x$, we have $(f](x) = \sup_{x \leq y} f(y) \geq f(x)$. Thus $f(x) \leq (f](x)$ for all $x \in S$. This implies that $f \subseteq (f]$.

(2) Assume that $f \subseteq g$. Then $f(u) \leq g(u)$ for all $u \in S$. Let $x \in S$. Thus $(f](x) = \sup_{x \leq y} f(y) \leq \sup_{x \leq y} g(y) = (g](x)$. This shows that $(f](x) \leq (g](x)$ for all $x \in S$. Hence, $(f] \subseteq (g]$.

(3) Assume that $f \subseteq g$. By Proposition 2.4.4(1), we have $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$. It follows from (2) that, $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$. $\square$

Proposition 4.2.2. Let f be a fuzzy subset of an ordered semigroup $(S, \cdot, \leq)$. Then the following statements are equivalent.

(1) For all $x, y \in S$, $x \leq y$ implies $f(x) \geq f(y)$.

(2) $(f] = f$.

Proof. First, we prove that (1) implies (2). Let $x \in S$. By assumption, we have $f(x) \geq f(y)$ for all $y \in S$ with $x \leq y$. Then $\sup\limits_{x \leq y} f(y) \leq f(x)$, which implies that $(f](x) \leq f(x)$. Hence, $(f] \subseteq f$. By Proposition 4.2.1(1), $(f] = f$.

Next, we will prove that (2) implies (1). Let $x, y \in S$ and $x \leq y$. By assumption, we have $f(x) = (f](x) = \sup\limits_{x \leq u} f(u) \geq f(y)$. Thus $f(x) \geq f(y)$. $\square$

Next, we give definitions of fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups.

Definition 4.2.3. Let f be a nonzero fuzzy subset of an ordered semigroup $(S, \cdot, \leq)$.

1. f is called a ***fuzzy left ordered almost ideal*** of S if $(s_\alpha \circ f] \cap f \neq 0$ for all fuzzy point s_α of S.
2. f is called a ***fuzzy right ordered almost ideal*** of S if $(f \circ s_\alpha] \cap f \neq 0$ for all fuzzy point s_α of S.
3. f is called a ***fuzzy ordered almost ideal*** of S if f is both a fuzzy left ordered almost ideal and a fuzzy right ordered almost ideal of S.
4. f is called a ***fuzzy ordered almost bi-ideal*** of S if $(f \circ s_\alpha \circ f] \cap f \neq 0$ for all fuzzy point s_α of S.
5. f is called a ***fuzzy ordered almost quasi-ideal*** of S if $(s_\alpha \circ f] \cap (f \circ s_\alpha] \cap f \neq 0$ for all fuzzy point s_α of S.

Lemma 4.2.4. *Let f be a nonzero fuzzy subset and s_α a fuzzy point of an ordered semigroup $(S, \cdot, \leq)$, and let $x \in S$. Then the following statements hold.*

(1) $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$ *if and only if there exists an element $b \in S$ such that*

$$x \leq sb \ \textit{and} \ f(x), f(b) \neq 0.$$

(2) $\big((f \circ s_\alpha] \cap f\big)(x) \neq 0$ *if and only if there exists an element $a \in S$ such that*

$$x \leq as \ \textit{and} \ f(x), f(a) \neq 0.$$

(3) $\big((f \circ s_\alpha \circ f] \cap f\big)(x) \neq 0$ *if and only if there exist elements $a, b \in S$ such that*

$$x \leq asb \ \textit{and} \ f(x), f(a), f(b) \neq 0.$$

(4) $\big((s_\alpha \circ f] \cap (f \circ s_\alpha] \cap f\big)(x) \neq 0$ *if and only if there exist elements $a, b \in S$ such that*

$$x \leq as, \ x \leq sb \ \textit{and} \ f(x), f(a), f(b) \neq 0.$$

Proof. (1) Assume that $\big((s_\alpha \circ f] \cap f\big)(x) \neq 0$. Then $\min\{(s_\alpha \circ f](x), f(x)\} \neq 0$, so $(s_\alpha \circ f](x) \neq 0$ and $f(x) \neq 0$. Thus we have,

$$(s_\alpha \circ f](x) = \sup_{x \leq y}(s_\alpha \circ f)(y) \neq 0.$$

Then there is $z \in S$ such that $x \leq z$ and $(s_\alpha \circ f)(z) \neq 0$. This implies that

$$(s_\alpha \circ f)(z) = \sup_{z \leq uv} \min\{s_\alpha(u), f(v)\} \neq 0.$$

So there exist $a, b \in S$ such that $z \leq ab$ and $\min\{s_\alpha(a), f(b)\} \neq 0$. Thus $s_\alpha(a) \neq 0$ (implies that $s = a$) and $f(b) \neq 0$. Hence, $x \leq z \leq ab = sb$ and $f(x), f(b) \neq 0$.

Conversely, assume that there exists an element $a \in S$ such that

$$x \leq sb \text{ and } f(x), f(b) \neq 0.$$

Then we have

$$\begin{aligned}(s_\alpha \circ f](x) &= \sup_{x\le y}(s_\alpha \circ f)(y)\\ &\ge (s_\alpha \circ f)(x) \ \ (\text{since } x \le x)\\ &= \sup_{x\le uv}\min\{s_\alpha(u), f(v)\}\\ &\ge \min\{s_\alpha(s), f(b)\} \ \ (\text{since } x \le sb)\\ &= \min\{\alpha, f(b)\}\\ &\ne 0.\end{aligned}$$

Thus $(s_\alpha \circ f](x) \ne 0$. Since $(s_\alpha \circ f](x) \ne 0$ and $f(x) \ne 0$, it follows that $\min\{(s_\alpha \circ f](x), f(x)\} \ne 0$. Hence, $\big((s_\alpha \circ f] \cap f\big)(x) \ne 0$.

(2) The proof of this statement is similar to the statement (1).

(3) Assume that $\big((f \circ s_\alpha \circ f] \cap f\big)(x) \ne 0$. Then $\min\{(f \circ s_\alpha \circ f](x), f(x)\} \ne 0$, so $(f \circ s_\alpha \circ f](x) \ne 0$ and $f(x) \ne 0$. Thus we have,

$$(f \circ s_\alpha \circ f](x) = \sup_{x\le y}(f \circ s_\alpha \circ f)(y) \ne 0.$$

Then there is $z \in S$ such that $x \le z$ and $(f \circ s_\alpha \circ f)(z) \ne 0$. This implies that

$$(f \circ s_\alpha \circ f)(z) = \sup_{z\le uv}\min\{f(u), (s_\alpha \circ f)(v)\} \ne 0.$$

So there exist $a, b \in S$ such that $z \le ab$ and $\min\{f(a), (s_\alpha \circ f)(b)\} \ne 0$. Thus $f(a) \ne 0$ and $(s_\alpha \circ f)(b) \ne 0$. Since $(s_\alpha \circ f)(b) \ne 0$, we get

$$\sup_{b\le u'v'}\min\{s_\alpha(u'), f(v')\} \ne 0.$$

Thus there exist $a', b' \in S$ such that $b \le a'b'$ and $\min\{s_\alpha(a'), f(b')\} \ne 0$, so $a' = s$ and $f(b') \ne 0$. Hence, $x \le z \le ab \le a(a'b')$. Therefore, $x \le asb'$ and $f(x), f(a), f(b') \ne 0$.

Conversely, assume that there exist elements $a, b \in S$ such that

$$x \le asb \text{ and } f(x), f(a), f(b) \ne 0.$$

Then we have

$$
\begin{aligned}
(f \circ s_\alpha \circ f](x) &= \sup_{x \le y} (f \circ s_\alpha \circ f)(y) \\
&\ge (f \circ s_\alpha \circ f)(x) \ \ (\text{since } x \le x) \\
&= \sup_{x \le uv} \min\{(f \circ s_\alpha)(u), f(v)\} \\
&\ge \min\{(f \circ s_\alpha)(as), f(b)\} \ \ (\text{since } x \le (as)b) \\
&= \min\Big\{ \sup_{as \le z_1 z_2} \min\{f(z_1), s_\alpha(z_2)\}, f(b) \Big\} \\
&\ge \min\big\{ \min\{f(a), s_\alpha(s)\}, f(b) \big\} \ \ (\text{since } as \le as) \\
&= \min\{f(a), \alpha, f(b)\} \\
&\ne 0.
\end{aligned}
$$

Thus $(f \circ s_\alpha \circ f](x) \ne 0$. Since $(f \circ s_\alpha \circ f](x) \ne 0$ and $f(x) \ne 0$, we have that $\min\{(f \circ s_\alpha \circ f](x), f(x)\} \ne 0$. Hence, $\big((f \circ s_\alpha \circ f] \cap f\big)(x) \ne 0$.

(4) Assume that $\big((s_\alpha \circ f] \cap (f \circ s_\alpha] \cap f\big)(x) \ne 0$. Then $(s_\alpha \circ f](x) \ne 0$, $(f \circ s_\alpha](x) \ne 0$ and $f(x) \ne 0$. This implies that

$$\big((s_\alpha \circ f] \cap f\big)(x) \ne 0 \text{ and } \big((f \circ s_\alpha] \cap f\big)(x) \ne 0.$$

By statements (1) and (2), there exist elements $a, b \in S$ such that $x \le sb$, $x \le as$ and $f(x), f(a), f(b) \ne 0$.

Conversely, assume that there exist elements $a, b \in S$ such that $x \le as, x \le sb$ and $f(x), f(a), f(b) \ne 0$. By the converse of statements (1) and (2), we get

$$\big((s_\alpha \circ f] \cap f\big)(x) \ne 0 \text{ and } \big((f \circ s_\alpha] \cap f\big)(x) \ne 0,$$

so $(s_\alpha \circ f](x) \ne 0, (f \circ s_\alpha](x) \ne 0$ and $f(x) \ne 0$. Thus $\big((s_\alpha \circ f] \cap (f \circ s_\alpha] \cap f\big)(x) \ne 0$. □

The following theorem follows directly from Definition 4.2.3 and Lemma 4.2.4.

Theorem 4.2.5. *Let f be a fuzzy subset of an ordered semigroup $(S, \cdot, \leq)$. Then the following statements hold.*

(1) *f is a fuzzy left ordered almost ideal of S if and only if for each fuzzy point s_α, there exist $x, b \in S$ such that*

$$x \leq sb \text{ and } f(x), f(b) \neq 0.$$

(2) *f is a fuzzy right ordered almost ideal of S if and only if for each fuzzy point s_α, there exist $x, a \in S$ such that*

$$x \leq as \text{ and } f(x), f(a) \neq 0.$$

(3) *f is a fuzzy ordered almost ideal of S if and only if for each fuzzy point s_α, there exist $x, y, a, b \in S$ such that*

$$x \leq as,\ y \leq sb \text{ and } f(x), f(y), f(a), f(b) \neq 0.$$

(4) *f is a fuzzy ordered almost bi-ideal of S if and only if for each fuzzy point s_α, there exist $x, a, b \in S$ such that*

$$x \leq asb \text{ and } f(x), f(a), f(b) \neq 0.$$

(5) *f is a fuzzy ordered almost quasi-ideal of S if and only if for each fuzzy point s_α, there exist $x, a, b \in S$ such that*

$$x \leq as, x \leq sb \text{ and } f(x), f(a), f(b) \neq 0.$$

Example 4.2.6. Consider an ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ given below.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a,a),(a,b),(b,b),(c,a),(c,b),(c,c),(d,a),(d,b),(d,d),(e,e\}.$$

Define a function $f : S \longrightarrow [0,1]$ by

$$f(a) = 0, f(b) = 0.3, f(c) = 0, f(d) = 0.1 \text{ and } f(e) = 0.2.$$

Let $\alpha \in (0,1]$. Then we have

- $b \leq ab$ and $f(b) \neq 0 \Longrightarrow \big[(a_\alpha \circ f] \cap f\big](b) \neq 0,$
  $d \leq ba$ and $f(b), f(d) \neq 0 \Longrightarrow \big[(f \circ a_\alpha] \cap f\big](d) \neq 0,$
- $b \leq bb$ and$f(b) \neq 0 \Longrightarrow \big[(b_\alpha \circ f] \cap f\big](b) \neq 0$ and $\big[(f \circ b_\alpha] \cap f\big](b) \neq 0,$
- $b \leq cb$ and $f(b) \neq 0 \Longrightarrow \big[(c_\alpha \circ f] \cap f\big](b) \neq 0,$
  $d \leq bc$ and $f(b), f(d) \neq 0 \Longrightarrow \big[(f \circ c_\alpha] \cap f\big](d) \neq 0,$
- $b \leq db$ and $f(b) \neq 0 \Longrightarrow \big[(d_\alpha \circ f] \cap f\big](b) \neq 0,$
  $d \leq bd$ and $f(b), f(d) \neq 0 \Longrightarrow \big[(f \circ d_\alpha] \cap f\big](d) \neq 0,$
- $e \leq ee$ and $f(e) \neq 0 \Longrightarrow \big[(e_\alpha \circ f] \cap f\big](e) \neq 0$ and $\big[(f \circ e_\alpha] \cap f\big](e) \neq 0.$

This implies that $(s_\alpha \circ f] \cap f \neq 0$ and $(f \circ s_\alpha] \cap f \neq 0$ for all $s \in S$. Thus f is a fuzzy ordered almost ideal of S. Also, we have

$$\begin{aligned}
b \leq bab \text{ where } f(b) \neq 0 &\Longrightarrow \big[(f \circ a_\alpha \circ f] \cap f\big](b) \neq 0,\\
b \leq bbb \text{ where } f(b) \neq 0 &\Longrightarrow \big[(f \circ b_\alpha \circ f] \cap f\big](b) \neq 0,\\
b \leq bcb \text{ where } f(b) \neq 0 &\Longrightarrow \big[(f \circ c_\alpha \circ f] \cap f\big](b) \neq 0,\\
b \leq bdb \text{ where } f(b) \neq 0 &\Longrightarrow \big[(f \circ d_\alpha \circ f] \cap f\big](b) \neq 0,\\
e \leq eee \text{ where } f(e) \neq 0 &\Longrightarrow \big[(f \circ e_\alpha \circ f] \cap f\big](e) \neq 0.
\end{aligned}$$

Then $(f \circ s_\alpha \circ f) \cap f \neq 0$ for all $s \in S$, which implies that f is a fuzzy ordered almost bi-ideal of S. Finally, we show that f is a fuzzy ordered almost quasi-ideal of S as follows:

$$\begin{aligned}
d \leq ab \text{ and } d \leq ba \text{ where } f(b), f(d) \neq 0 &\Longrightarrow \big[(a_\alpha \circ f] \cap (f \circ a_\alpha] \cap f\big](d) \neq 0,\\
b \leq bb \text{ where } f(b) \neq 0 &\Longrightarrow \big[(b_\alpha \circ f] \cap (f \circ b_\alpha] \cap f\big](b) \neq 0,\\
d \leq cb \text{ and } d \leq dc \text{ where } f(b), f(d) \neq 0 &\Longrightarrow \big[(c_\alpha \circ f] \cap (f \circ c_\alpha] \cap f\big](d) \neq 0,\\
d \leq db \text{ and } d \leq dd \text{ where } f(b), f(d) \neq 0 &\Longrightarrow \big[(d_\alpha \circ f] \cap (f \circ d_\alpha] \cap f\big](d) \neq 0,
\end{aligned}$$

$d \leq eb$ and $d \leq de$ where $f(b), f(d) \neq 0 \implies \big[(e_\alpha \circ f] \cap (f \circ e_\alpha] \cap f\big](d) \neq 0$. We can conclude that $(s_\alpha \circ f] \cap (f \circ s_\alpha] \cap f \neq 0$ for all $s \in S$, so f is a fuzzy ordered almost quasi-ideal of S. Therefore, the fuzzy subset f in this example is a fuzzy ordered almost ideal, fuzzy ordered almost bi-ideal and fuzzy ordered almost quasi-ideal of S.

Remark 4.2.7. Let f be a nonzero fuzzy subset of an ordered semigroup $(S, \cdot, \leq)$. The following statements hold.

(i) Every fuzzy left ordered ideal of S is a fuzzy left ordered almost ideal of S.

(ii) Every fuzzy right ordered ideal of S is a fuzzy right ordered almost ideal of S.

(iii) Every fuzzy ordered ideal of S is a fuzzy ordered almost ideal of S.

(iv) Every fuzzy ordered bi-ideal of S is a fuzzy ordered almost bi-ideal of S.

(v) If S is commutative, then every fuzzy ordered quasi-ideal of S is a fuzzy ordered almost quasi-ideal of S.

Proof. (i) Assume that f is a fuzzy left ordered ideal of S. Let s_α be a fuzzy point of S. Since f is a nonzero fuzzy subset of S, there exists an element $a \in S$ such that $f(a) \neq 0$. Since f is a fuzzy left ordered ideal of S, $f(sa) \geq f(a)$, so $f(sa) \neq 0$. Let $x = sa$. Then $x \leq sa$, $f(x) \neq 0$ and $f(a) \neq 0$. By Theorem 4.2.5(1), f is a fuzzy left ordered almost ideal of S.

(ii) The proof of this statement is similar to the statement (i).

(iii) This statement follows from (i) and (ii).

(iv) Assume that f is a fuzzy ordered bi-ideal of S. Let s_α be a fuzzy point of S. Since f is a nonzero fuzzy subset of S, there exists an element $a \in S$ such that $f(a) \neq 0$. Since f is a fuzzy ordered bi-ideal of S,

$$f(asa) \geq \min\{f(a), f(a)\} = f(a),$$

which implies that $f(asa) \neq 0$. Let $x = asa$. Thus we have $x \leq asa$ and $f(x), f(a) \neq 0$. By Theorem 4.2.5(4), f is a fuzzy ordered almost bi-ideal of S.

(v) Assume that f is a fuzzy ordered quasi-ideal of S. Let s_α be a fuzzy point of S. Since f is a nonzero fuzzy subset of S, there exists $a \in S$ such that $f(a) \neq 0$. Let $x = sa$. Since S is commutative, $x \leq as$ and $x \leq sa$. By Theorem 2.4.11,

$$f(x) \geq \min\{f(a), f(a)\} = f(a),$$

so $f(x) \neq 0$. By Theorem 4.2.5(5), f is a fuzzy ordered almost quasi-ideal of S. $\square$

The following example shows that the converse of the Remark 4.2.7 is not true.

Example 4.2.8. From Example 4.2.6, we have that f is a fuzzy ordered almost ideal, fuzzy ordered almost bi-ideal and fuzzy ordered almost quasi-ideal of S. However, f is neither a fuzzy ordered ideal nor a fuzzy ordered bi-ideal nor a fuzzy ordered quasi-ideal of S because $a \leq b$ but $f(a) = 0 \leq 0.3 = f(b)$.

Next, we give some properties of fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups by using the same concepts of ordered almost ideals, ordered almost bi-ideals and ordered almost quasi-ideals, respectively.

Proposition 4.2.9. Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following statements are true.

(1) If f is a fuzzy left ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy left ordered almost ideal of S.

(2) If f is a fuzzy right ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy right ordered almost ideal of S.

(3) If f is a fuzzy ordered almost ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy ordered almost ideal of S.

(4) If f is a fuzzy ordered almost bi-ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy ordered almost bi-ideal of S.

(5) If f is a fuzzy ordered almost quasi-ideal of S, then every fuzzy subset g of S such that $f \subseteq g$ is also a fuzzy ordered almost quasi-ideal of S.

Proof. Assume that f is a fuzzy left ordered almost ideal of S. Let g be a fuzzy subset of S such that $f \subseteq g$ and let s_α be a fuzzy point of S. Then $s_\alpha \circ f \subseteq s_\alpha \circ g$ and so $(s_\alpha \circ f] \subseteq (s_\alpha \circ g]$. Thus $(s_\alpha \circ f] \cap f \subseteq (s_\alpha \circ g] \cap g$. Since f is a fuzzy left ordered almost ideal of S, $(s_\alpha \circ f] \cap f \neq \emptyset$. Hence, $(s_\alpha \circ g] \cap g \neq \emptyset$. Therefore, g is a fuzzy left ordered almost ideal of S. This verifies (1). The proof of (2), (3), (4), and (5) are similar to the proof of (1). □

Corollary 4.2.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then we have the following statements hold.*

(1) *The arbitrary union of fuzzy left ordered almost ideals of S is also a fuzzy left ordered almost ideal of S.*

(2) *The arbitrary union of fuzzy right ordered almost ideals of S is also a fuzzy right ordered almost ideal of S.*

(3) *The arbitrary union of fuzzy ordered almost ideals of S is also a fuzzy ordered almost ideal of S.*

(4) *The arbitrary union of fuzzy ordered almost bi-ideals of S is also a fuzzy ordered almost bi-ideal of S.*

(5) *The arbitrary union of fuzzy ordered almost quasi-ideals of S is also a fuzzy ordered almost quasi-ideal of S.*

From Corollary 4.2.10, we have the case of the arbitrary union is true, but the case of the arbitrary intersection need not be true. The following example gives the answer.

Example 4.2.11. Consider an ordered semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition modulo 6 and the order $\leq := \{(\bar{a}, \bar{a}) \mid \bar{a} \in \mathbb{Z}_6\}$. Define functions $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$f(\bar{0}) = 0, f(\bar{1}) = 0.3, f(\bar{2}) = 0, f(\bar{3}) = 0, f(\bar{4}) = 0.2, f(\bar{5}) = 0.1,$$

and $g : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$g(\bar{0}) = 0, g(\bar{1}) = 0.3, g(\bar{2}) = 0.1, g(\bar{3}) = 0, g(\bar{4}) = 0, g(\bar{5}) = 0.3.$$

Then f and g are fuzzy left ordered almost ideals, fuzzy right ordered almost ideals, fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals, and fuzzy ordered almost quasi-ideals of $\mathbb{Z}_6$. Next, we consider a function $f \cap g : \mathbb{Z}_6 \longrightarrow [0, 1]$ defined by $(f \cap g)(\bar{x}) = \min\{f(\bar{x}), g(\bar{x})\}$ for all $\bar{x} \in \mathbb{Z}_6$, that is

$$(f \cap g)(\bar{0}) = 0, \;\; (f \cap g)(\bar{1}) = 0.3, \;\; (f \cap g)(\bar{2}) = 0,$$
$$(f \cap g)(\bar{3}) = 0, \;\; (f \cap g)(\bar{4}) = 0, \;\; (f \cap g)(\bar{5}) = 0.1.$$

Then $f \cap g$ is neither a fuzzy left ordered almost ideal, a fuzzy right ordered almost ideal, a fuzzy ordered almost ideal, a fuzzy ordered almost bi-ideal, nor a fuzzy ordered almost quasi-ideal of $\mathbb{Z}_6$ because for $\alpha \in (0, 1]$,

$$\big(\bar{1}_\alpha \circ (f \cap g)\big] \cap (f \cap g) = 0,$$
$$\big((f \cap g) \circ \bar{1}_\alpha\big] \cap (f \cap g) = 0,$$
$$\big((f \cap g) \circ \bar{0}_\alpha \circ (f \cap g)\big] \cap (f \cap g) = 0,$$
$$\big(\bar{1}_\alpha \circ (f \cap g)\big] \cap \big((f \cap g) \circ \bar{1}_\alpha\big] \cap (f \cap g) = 0.$$

4.3 The relations of ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups

In this section, some connections of ordered almost ideals and fuzzy ordered almost ideals in ordered semigroups are discussed.

4.3.1 The relations of ordered almost ideals

Firstly, we discuss about the relations of various almost ideals as follows:

Theorem 4.3.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Every ordered almost ideal of S is an ordered almost bi-ideal of S.*

Proof. Let I be an ordered almost ideal of S and let $s \in S$. Since $I \neq \emptyset$, there exists an element $a \in I$. Then $asI \subseteq IsI$, so $(asI] \subseteq (IsI]$. Thus we have

$$(asI] \cap I \subseteq (IsI] \cap I.$$

Since I is an ordered almost ideal of S and $as \in S$, we have that $\big((as)I\big] \cap I \neq \emptyset$. This implies that $(IsI] \cap I \neq \emptyset$. Hence, I is an ordered almost bi-ideal of S. □

From the proof of Theorem 4.3.1, we can see that if I is a left ordered almost ideal or a right ordered almost ideal of S, then I is an ordered almost bi-ideal of S. The following corollary follows directly from Remark 4.1.3(iii) and Theorem 4.3.1.

Corollary 4.3.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then every ordered ideal of S is an ordered almost bi-ideal of S.*

Theorem 4.3.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Every ordered almost quasi-ideal of S is an ordered almost ideal of S.*

Proof. Let Q be an ordered almost quasi-ideal of S and let $s \in S$. Then we have $(sQ] \cap (Qs] \cap Q \neq \emptyset$. Since $(sQ] \cap (Qs] \subseteq (sQ]$ and $(sQ] \cap (Qs] \subseteq (Qs]$,

$$(sQ] \cap (Qs] \cap Q \subseteq (sQ] \cap Q \text{ and } (sQ] \cap (Qs] \cap Q \subseteq (Qs] \cap Q.$$

Thus $(sQ] \cap Q \neq \emptyset$ and $(Qs] \cap Q \neq \emptyset$. Hence, Q is a left ordered almost ideal and a right ordered almost ideal of S. Therefore, Q is an ordered almost ideal of S. □

As a consequence of Theorem 4.3.1 and Theorem 4.3.3, the relationship between ordered almost bi-ideals and ordered almost quasi-ideals in ordered semigroups is given as follows:

Corollary 4.3.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Every ordered almost quasi-ideal of S is an ordered almost bi-ideal of S.*

Example 4.3.5. From Example 4.1.2, the set of all ordered almost ideals of S and the set of all ordered almost quasi-ideals of S coincide. While $\{b\}$ and $\{b, e\}$ are ordered almost bi-ideals of S, they are neither ordered almost ideals nor ordered almost quasi-ideals of S.

From Example 4.3.5, we can conclude that the converses of Theorem 4.3.1 and Corollary 4.3.4 are not true. The converse of Theorem 4.3.3 is not true either, which can be seen in the following example.

Example 4.3.6. Consider an ordered semigroup consisting of five elements $S = \{a, b, c, d, e\}$, where the product $\cdot$ and the order relation $\leq$ are given below.

$\cdot$	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	e
c	a	c	b	d	e
d	d	d	d	a	a
e	d	d	d	a	a

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e\}.$$

Let $I = \{b, d, e\}$. Then I is an ordered almost ideal of S, but it is not an ordered almost quasi-ideal of S because $(eI] \cap (Ie] \cap I = \{a, d\} \cap \{a, e\} \cap I = \{a\} \cap I = \emptyset$.

The converses of Theorems 4.3.1, 4.3.3, and 4.3.4 may be not true in general (see in Example 4.3.5 and 4.3.6). The question is, when are the converses of these theorems true? The theorems below give the answer.

Theorem 4.3.7. *Let $(S, \cdot, \leq)$ be a commutative ordered semigroup. Then a subsemigroup of S is an ordered almost ideal of S if and only if it is an ordered almost bi-ideal of S.*

Proof. Let A be a subsemigroup of S. Assume that A is an ordered almost ideal of S. It follows from Theorem 4.3.1 that A is an ordered almost bi-ideal of S.

Conversely, assume that A is an ordered almost bi-ideal of S. Let $s \in S$. Then $(AsA] \cap A \neq \emptyset$. Since S is commutative and A is a subsemigroup of S,

$$(AsA] \cap A = (sAA] \cap A \subseteq (sA] \cap A,$$
$$(AsA] \cap A = (AAs] \cap A \subseteq (As] \cap A.$$

Thus $(sA] \cap A \neq \emptyset$ and $(As] \cap A \neq \emptyset$. Hence, A is a left ordered almost ideal and a right ordered almost ideal of S. Therefore, A is an ordered almost ideal of S. $\square$

Theorem 4.3.8. *Let $(S, \cdot, \leq)$ be a commutative ordered semigroup. Then a nonempty subset of S is an ordered almost quasi-ideal of S if and only if it is an ordered almost ideal of S.*

Proof. By Theorem 4.3.3, an ordered almost quasi-ideal of S is an ordered almost ideal of S. Next, assume that I is an ordered almost ideal of S. We will show that I is an ordered almost quasi-ideal of S. Let $s \in S$. Since S is commutative, $(sI] = (Is]$. Then we have $(sI] \cap I = (sI] \cap (Is] \cap I$. Since I is an ordered almost ideal of S, $(sI] \cap I \neq \emptyset$. Thus $(sI] \cap (Is] \cap I \neq \emptyset$. Hence, I is an ordered almost quasi-ideal of S. □

The next result follows directly from Theorem 4.3.7 and Theorem 4.3.8.

Corollary 4.3.9. *Let $(S, \cdot, \leq)$ be a commutative ordered semigroup. A subsemigroup of S is an ordered almost quasi-ideal of S if and only if it is an ordered almost bi-ideal of S.*

4.3.2 The relations of fuzzy ordered almost ideals

In the previous subsection, we studied the relations of ordered almost ideals and ordered almost bi-ideals, ordered almost ideals and ordered almost quasi-ideals, and ordered almost bi-ideals and ordered almost quasi-ideals in ordered semigroups. Next, we provide the relations of various fuzzy ordered almost ideals in ordered semigroups.

Theorem 4.3.10. *Every fuzzy ordered ideal of an ordered semigroup $(S, \cdot, \leq)$ is a fuzzy ordered almost bi-ideal of S.*

Proof. Let f be a fuzzy ordered ideal of S and let s_α be a fuzzy point of S. Since f is a nonzero fuzzy subset of S, $f(a) \neq 0$ for some $a \in S$. Let $x = asa$. Since f is a fuzzy ordered ideal of S, we have that

$$f(x) = f(asa) = f\big((as)a\big) \geq f(a).$$

Thus $f(x) \neq 0$. By Theorem 4.2.5(4), f is a fuzzy ordered almost bi-ideal of S. □

From the proof of Theorem 4.3.10, we can see that if a fuzzy ordered ideal is replaced by fuzzy left ordered ideal or fuzzy right ordered ideal, then the statement still hold. The following example shows that the converse of Theorem 4.3.10 is not true.

Example 4.3.11. Consider an ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ below.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a, a), (a, b), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, e\}.$$

Define a function $g : S \longrightarrow [0, 1]$ by

$$g(a) = 0, g(b) = 0.5, g(c) = 0, g(d) = 0 \text{ and } g(e) = 0.1.$$

Then for each fuzzy point s_α of S, we have $b \leq bsb$ and $g(b) \neq 0$. Thus g is a fuzzy ordered almost bi-ideal of S. But g is not a fuzzy ordered ideal of S because $a \leq b$ while $f(a) \leq f(b)$.

Theorem 4.3.12. *Let f be a fuzzy subsemigroup of an ordered semigroup $(S, \cdot, \leq)$. If f is a fuzzy ordered almost ideal of S, then it is a fuzzy ordered almost bi-ideal of S.*

Proof. Assume that f is a fuzzy ordered almost ideal of S. Let s_α be a fuzzy point of S. By Theorem 4.2.5(3), there exist elements $x, b \in S$ such that

$$x \leq sb \text{ and } f(x), f(b) \neq 0.$$

Thus $bx \leq bsb$ and $\min\{f(b), f(x)\} \neq 0$. Since f is a fuzzy subsemigroup of S,

$$f(bx) \geq \min\{f(b), f(x)\},$$

which implies that $f(bx) \neq 0$. Hence, we can conclude that $bx \leq bsb$, $f(bx) \neq 0$ and $f(b) \neq 0$. By Theorem 4.2.5(4), f is a fuzzy ordered almost bi-ideals of S. $\square$

Theorem 4.3.13. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Every fuzzy ordered almost quasi-ideal of S is a fuzzy ordered almost ideal of S.*

Proof. Assume that f is a fuzzy ordered almost quasi-ideal of S. Let s_α be a fuzzy point of S. By Theorem 4.2.5(5), there are $x, a, b \in S$ such that

$$x \leq as,\ x \leq sb \text{ and } f(x), f(a), f(b) \neq 0.$$

By Theorem 4.2.5(1) and (2), f is a fuzzy left ordered almost ideal and a fuzzy right ordered almost ideal of S. Therefore, f is a fuzzy ordered almost ideal of S. $\square$

We have the following result by combining Theorem 4.3.12 and Theorem 4.3.13.

Corollary 4.3.14. *If a fuzzy subsemigroup f of an ordered semigroup $(S, \cdot, \leq)$ is a fuzzy ordered almost quasi-ideal of S, then f is a fuzzy ordered almost bi-ideal of S.*

From the proof of Theorems 4.3.12-4.3.14, we see that if f is a fuzzy left ordered almost ideal or a fuzzy right ordered almost ideal of S, then it is enough to make these theorem true.

Theorem 4.3.15. *Let f be a fuzzy subsemigroup of an ordered semigroup $(S, \cdot, \leq)$. Then the following statements hold.*

(1) *f is a fuzzy ordered almost quasi-ideal of S if and only if f is a fuzzy ordered almost ideal of S.*

(2) *If S is commutative, then f is a fuzzy ordered almost ideal of S if and only if f is a fuzzy ordered almost bi-ideal of S.*

(3) *If S is commutative, then f is a fuzzy ordered almost quasi-ideal of S if and only if f is a fuzzy ordered almost bi-ideal of S.*

Proof. (1) Assume that f is a fuzzy ordered almost quasi-ideal of S. By Theorem 4.3.13, f is a fuzzy ordered almost ideal of S.

Conversely, assume that f is a fuzzy ordered almost ideal of S. Let s_α be a fuzzy point of S. By Theorem 4.2.5(3), there exist elements $x, y, a, b \in S$ such that

$$x \leq as, y \leq sb \text{ and } f(x), f(y), f(a), f(b) \neq 0.$$

Thus $yx \leq yas$ and $yx \leq sbx$. Since f is a fuzzy subsemigroup of S,

$$\begin{aligned} f(yx) &\geq \min\{f(y), f(x)\}, \\ f(ya) &\geq \min\{f(y), f(a)\}, \\ f(bx) &\geq \min\{f(b), f(x)\}. \end{aligned}$$

Since $\min\{f(y), f(x)\} \neq 0$, $\min\{f(y), f(a)\} \neq 0$ and $\min\{f(b), f(x)\} \neq 0$, $f(yx), f(ya), f(bx) \neq 0$. By Theorem 4.2.5$(5)$, f is a fuzzy ordered almost quasi-ideal of S.

(2) Let S be commutative. By Theorem 4.3.12, a fuzzy ordered almost ideal of S is a fuzzy ordered almost bi-ideal of S. Assume that f is a fuzzy ordered almost bi-ideal of S. Let s_α be a fuzzy point of S. By Theorem 4.2.5(4), there exist $x, a, b \in S$ such that

$$x \leq asb \text{ and } f(x), f(a), f(b) \neq 0.$$

Then $\min\{f(a), f(b)\} \neq 0$. Since S is commutative, $asb = s(ab) = (ab)s$. Since f is a fuzzy subsemigroup of S,

$$f(ab) \geq \min\{f(a), f(b)\},$$

so $f(ab) \neq 0$. Hence, we can conclude that

$$x \leq s(ab), x \leq (ab)s \text{ and } f(x), f(ab) \neq 0.$$

By Theorem 4.2.5(3), f is a fuzzy ordered almost ideal of S.

(3) This follows from (1) and (2). $\square$

4.3.3 The relations of ordered almost ideals and fuzzy ordered almost ideals

In this subsection, the relations of ordered almost ideals and fuzzy ordered almost ideals, ordered almost bi-ideals and fuzzy ordered almost bi-ideals, and ordered almost quasi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups are provided.

Theorem 4.3.16. *Let A be a nonempty subset of an ordered semigroup $(S, \cdot, \leq)$. Then the following statements hold.*

(1) *A is a left ordered almost ideal of S if and only if C_A is a fuzzy left ordered almost ideal of S.*

(2) *A is a right ordered almost ideal of S if and only if C_A is a fuzzy right ordered almost ideal of S.*

(3) *A is an ordered almost ideal of S if and only if C_A is a fuzzy ordered almost ideal of S.*

(4) *A is an ordered almost bi-ideal of S if and only if C_A is a fuzzy ordered almost bi-ideal of S.*

(5) *A is an ordered almost quasi-ideal of S if and only if C_A is a fuzzy ordered almost quasi-ideal of S.*

Proof. Assume A is a left ordered almost ideal of S. Let s_α be a fuzzy point of S. Then we have $(sA] \cap A \neq \emptyset$. Thus there exists $x \in S$ such that $x \in A$ and $x \in (sA]$, so $C_A(x) = 1$ and $x \leq sa$ for some $a \in A$. This implies that

$$x \leq sa \text{ and } C_A(x), C_A(a) \neq 0.$$

By Theorem 4.2.5(1), C_A is a fuzzy left ordered almost ideal of S.

Conversely, assume that C_A is a fuzzy left ordered almost ideal of S. Let $s \in S$. By Theorem 4.2.5(1), there are elements $x, a \in S$ such that

$$x \leq sa \text{ and } C_A(x), C_A(a) \neq 0,$$

which implies that $x, a \in A$. Thus $x \leq sa \in sA$, so $x \in (sA]$. Hence $x \in (sA] \cap A$. Therefore, A is a left ordered almost ideal of S. The same argument can be applied to prove statements (2)-(5). $\square$

Theorem 4.3.17. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following statements hold.*

(1) *f is a fuzzy left ordered almost ideal of S if and only if $supp(f)$ is a left ordered almost ideal of S.*

(2) *f is a fuzzy right ordered almost ideal of S if and only if $supp(f)$ is a right ordered almost ideal of S.*

(3) *f is a fuzzy ordered almost ideal of S if and only if $supp(f)$ is an ordered almost ideal of S.*

(4) *f is a fuzzy ordered almost bi-ideal of S if and only if $supp(f)$ is an ordered almost bi-ideal of S.*

(5) *f is a fuzzy ordered almost quasi-ideal of S if and only if $supp(f)$ is an ordered almost quasi-ideal of S.*

Proof. Assume that f is a fuzzy left ordered almost ideal of S. Let $s \in S$. By Theorem 4.2.5(1), there exist elements $x, b \in S$ such that

$$x \leq sb \text{ and } f(x), f(b) \neq 0,$$

so $x \in supp(f)$ and $b \in supp(f)$. Thus we have

$$x \leq sb \in s(supp(f)) \text{ and } x \in supp(f),$$

which implies $x \in \big(s(supp(f))\big] \cap supp(f)$. Hence, $\big(s(supp(f))\big] \cap supp(f) \neq \emptyset$. Therefore, $supp(f)$ is a left ordered almost ideal of S.

Conversely, assume that $supp(f)$ is a left ordered almost ideal of S. By Theorem 4.3.16(1), $C_{supp(f)}$ is a fuzzy left ordered almost ideal of S. Let s_α be a fuzzy point of S. By Theorem 4.2.5(1), there exist elements $x, b \in S$ such that

$$x \leq sb \text{ and } C_{supp(f)}(x), C_{supp(f)}(b) \neq 0.$$

Thus $x \in supp(f)$ and $b \in supp(f)$, so $f(x) \neq 0$ and $f(b) \neq 0$. Hence, f is a fuzzy left ordered almost ideal of S. The statements (2), (3), (4) and (5) can be proved using the semilar manner. □

In the section 3.3, we consider minimal, prime, and semiprime almost (m, n)-ideals and minimal, prime, and semiprime fuzzy almost (m, n)-ideals in semigroups. Next, we discuss such structures in ordered semigroups. The proof of the following theorems are similar to the proof of Theorems 3.3.5, 3.3.6, 3.3.9 and 3.3.10, respectively.

Theorem 4.3.18. *Let A be a nonempty subset of an ordered semigroup $(S, \cdot, \leq)$.*

(1) *A is a minimal left ordered almost ideal of S if and only if C_A is a minimal fuzzy left ordered almost ideal of S.*

(2) *A is a minimal right ordered almost ideal of S if and only if C_A is a minimal fuzzy right ordered almost ideal of S.*

(3) *A is a minimal ordered almost ideal of S if and only if C_A is a minimal fuzzy ordered almost ideal of S.*

(4) *A is a minimal ordered almost bi-ideal of S if and only if C_A is a minimal fuzzy ordered almost bi-ideal of S.*

(5) *A is a minimal ordered almost quasi-ideal of S if and only if C_A is a minimal fuzzy ordered almost quasi-ideal of S.*

Corollary 4.3.19. *Let $(S, \cdot, \leq)$ be an ordered semigroup.*

(1) *S has no proper left ordered almost ideals if and only if $supp(f) = S$ for all fuzzy left ordered almost ideal f of S.*

(2) *S has no proper right ordered almost ideals if and only if $supp(f) = S$ for all fuzzy right ordered almost ideal f of S.*

(3) *S has no proper ordered almost ideals if and only if $supp(f) = S$ for all fuzzy ordered almost ideal f of S.*

(4) *S has no proper ordered almost bi-ideals if and only if $supp(f) = S$ for all fuzzy ordered almost bi-ideal f of S.*

(5) *S has no proper ordered almost quasi-ideals if and only if $supp(f) = S$ for all fuzzy ordered almost quasi-ideal f of S.*

Theorem 4.3.20. *Let A be a nonempty subset of an ordered semigroup $(S, \cdot, \leq)$.*

(1) *A is a prime left ordered almost ideal of S if and only if C_A is a prime fuzzy left ordered almost ideal of S.*

(2) *A is a prime right ordered almost ideal of S if and only if C_A is a prime fuzzy right ordered almost ideal of S.*

(3) *A is a prime ordered almost ideal of S if and only if C_A is a prime fuzzy ordered almost ideal of S.*

(4) *A is a prime ordered almost bi-ideal of S if and only if C_A is a prime fuzzy ordered almost bi-ideal of S.*

(5) *A is a prime ordered almost quasi-ideal of S if and only if C_A is a prime fuzzy ordered almost quasi-ideal of S.*

Theorem 4.3.21. *Let A be a nonempty subset of an ordered semigroup $(S, \cdot, \leq)$.*

(1) *A is a semiprime left ordered almost ideal of S if and only if C_A is a semiprime fuzzy left ordered almost ideal of S.*

(2) *A is a semiprime right ordered almost ideal of S if and only if C_A is a semiprime fuzzy right ordered almost ideal of S.*

(3) *A is a semiprime ordered almost ideal of S if and only if C_A is a semiprime fuzzy ordered almost ideal of S.*

(4) *A is a semiprime ordered almost bi-ideal of S if and only if C_A is a semiprime fuzzy ordered almost bi-ideal of S.*

(5) *A is a semiprime ordered almost quasi-ideal of S if and only if C_A is a semiprime fuzzy ordered almost quasi-ideal of S.*

CHAPTER 5

Almost hyperideals in semihypergroups

In this chapter, we define almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semihypergroups, and give some properties of them. Moreover, the relationships among them are established.

5.1 Almost hyperideals in semihypergroups

In this section, we introduce definitions of almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semihypergroups by using the concept of almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semigroups, respectively. In addition, we present some interesting properties of them.

Definition 5.1.1. Let $(H, *)$ be a semihypergroup.

1. A nonempty subset L of H is called a ***left almost hyperideal*** of H if

$$s * L \cap L \neq \emptyset \text{ for all } s \in H.$$

2. A nonempty subset R of H is called a ***right almost hyperideal*** of H if

$$R * s \cap R \neq \emptyset \text{ for all } s \in H.$$

3. A nonempty subset I of H is called an ***almost hyperideal*** of H if I is a left almost hyperideal and a right almost hyperideal of H.

4. A nonempty subset B of H is called an ***almost bi-hyperideal*** of H if

$$(B * s * B) \cap B \neq \emptyset \text{ for all } s \in H.$$

5. A nonempty subset Q of H is called an ***almost quasi-hyperideal*** of H if

$$(s * Q \cap Q * s) \cap Q \neq \emptyset \text{ for all } s \in H.$$

Example 5.1.2. Let H be a semihypergroup of three elements $\{x, y, z\}$ with the following hyperoperation.

$\cdot$	x	y	z
x	$\{x\}$	$\{x, y\}$	$\{x, z\}$
y	$\{x\}$	$\{x, y\}$	$\{x, y\}$
z	$\{x\}$	$\{x, y\}$	$\{z\}$

Then every nonempty subset of H is an almost bi-hyperideal of H. The almost hyperideals and almost quasi-hyperideals of H are $\{x\}, \{x, y\}, \{x, z\}$ and H.

Remark 5.1.3. Let $(H, *)$ be a semihypergroup. The following statements are true.

(i) Every left hyperideal of H is a left almost hyperideal of H.

(ii) Every right hyperideal of H is a right almost hyperideal of H.

(iii) Every hyperideal of H is an almost hyperideal of H.

(iv) Every bi-hyperideal of H is an almost bi-hyperideal of H.

(v) If Q is a quasi-hyperideal of H and $s * Q \cap Q * s \neq \emptyset$ for all $s \in H$, then Q is an almost quasi-hyperideal of H.

Proof. (i) Assume that L is a left hyperideal of H and let $s \in H$. Then we have $s * L \subseteq S * L \subseteq L$. Since $L \neq \emptyset$, it follows that $s * L \neq \emptyset$. This implies that

$$s * L \cap L = s * L \neq \emptyset.$$

Hence, L is a left almost ideal of H.

(ii) This can be proved in a similar manner as the satement (i).

(iii) This follows from (i) and (ii).

(iv) Let B be a bi-hyperideal of H and $s \in H$. Then we have $B * s * B \neq \emptyset$ and $B * H * B \subseteq B$. Thus $B * s * B \subseteq B * H * B \subseteq B$. This implies that

$$(B * s * B) \cap B = B * s * B \neq \emptyset.$$

Hence, B is an almost bi-hyperideal of H.

(v) Let $s \in H$. Then we have $s * Q \subseteq S * Q$ and $Q * s \subseteq Q * S$, so

$$(s * Q) \cap (Q * s) \subseteq (S * Q) \cap (Q * S) \subseteq Q.$$

By assumption, $(s * Q) \cap (Q * s) \neq \emptyset$. This implies that

$$\big[(s * Q) \cap (Q * s)\big] \cap Q = (s * Q) \cap (Q * s) \neq \emptyset.$$

Hence, $(s*Q \cap Q*s) \cap Q \neq \emptyset$. Therefore, Q is an almost quasi-hyperideal of H. $\square$

Example 5.1.4. From Example 5.1.2, we have $\{x, y\}$ is an almost hyperideal of H. However, it is not a hyperideal of H because

$$\{x, y\} * H = \{x, y, z\} \not\subseteq \{x, y\}.$$

Example 5.1.5. From Example 5.1.2, we obtain that $\{y, z\}$ is an almost bi-hyperideal of H, but it is not a bi-hyperideal of H because

$$\{y, z\} * H * \{y, z\} = \{x, y, z\} \not\subseteq \{y, z\}.$$

Example 5.1.6. From Example 5.1.2, we can see that $\{x, z\}$ is an almost quasi-hyperideal of H. However, $\{x, z\}$ is not a quasi-hyperideal of H because

$$\big(H * \{x, z\}\big) \cap \big(\{x, z\} * H\big) = \{x, y, z\} \not\subseteq \{x, z\}.$$

Examples 5.1.4, 5.1.5 and 5.1.6 show that the converse of Remark 5.1.3 is not true in general. In the previous chapters, we provided some interesting properties of almost ideals in many algebraic structures. Next, we discuss these properties in semihypergroups.

Proposition 5.1.7. Let $(H, *)$ be a semihypergroup. Then the following statements hold.

(1) If L is a left almost hyperideal of H, then every subset of H containing L is a left almost hyperideal of H.

(2) If R is a right almost hyperideal of H, then every subset of H containing R is a right almost hyperideal of H.

(3) If I is an almost hyperideal of H, then every subset of H containing I is an almost hyperideal of H.

(4) If B is an almost bi-hyperideal of H, then every subset of H containing B is an almost bi-hyperideal of H.

(5) If Q is an almost quasi-hyperideal of H, then every subset of H containing Q is an almost quasi-hyperideal of H.

Proof. (1) Let L be a left almost hyperideal of H, and let $s \in H$. Then $s*L \cap L \neq \emptyset$. Assume that A is a subset of H such that $L \subseteq A$. Then $s * L \subseteq s * A$, which implies that $s * L \cap L \subseteq s * A \cap A$. Thus $s * A \cap A \neq \emptyset$. Hence, A is a left almost hyperideal of H.

(2) This proof is similar to the proof of (1).

(3) This follows from (1) and (2).

(4) Let B be an almost bi-hyperideal of S, and let $s \in S$. Then $(B*s*B) \cap B \neq \emptyset$. Assume that A is a subset of H such that $B \subseteq A$. Thus $B * s * B \subseteq A * s * A$, so $(B * s * B) \cap B \subseteq (A * s * A) \cap A$. This implies that $(A * s * A) \cap A \neq \emptyset$. Therefore, A is an almost bi-hyperideal of H.

(5) Let Q be an almost quasi-hyperideal of S, and let $s \in S$. Then we have $(s * Q \cap Q * s) \cap Q \neq \emptyset$. Assume that A is a subset of H and $Q \subseteq A$. Thus $s * Q \subseteq s * A$ and $Q * s \subseteq A * s$, so

$$s * Q \cap Q * s \subseteq s * A \cap A * s.$$

This implies that $(s*Q\cap Q*s)\cap Q \subseteq (s*A\cap A*s)\cap A$, so $(s*A\cap A*s)\cap A \neq \emptyset$. Therefore, A is an almost quasi-hyperideal of H. □

The following corollary follows directly from Proposition 5.1.7.

Corollary 5.1.8. *Let $(H, *)$ be a semihypergroup.*

(1) *The arbitrary union of left almost hyperideals of H is also a left almost hyperideal of H.*

(2) *The arbitrary union of right almost hyperideals of H is also a right almost hyperideal of H.*

(3) *The arbitrary union of almost hyperideals of H is also an almost hyperideal of H.*

(4) *The arbitrary union of almost bi-hyperideals of H is also an almost bi-hyperideal of H.*

(5) *The arbitrary union of almost quasi-hyperideals of H is also an almost quasi-hyperideal of H.*

Example 5.1.9. Let $H = \{a, b, c, d, e\}$ be a semihypergroup under the hyperoperation $*$ defined as in the following table.

$*$	a	b	c	d	e
a	$\{b,c\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
c	$\{a\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
d	$\{a\}$	$\{b,c\}$	$\{b,d\}$	$\{d,e\}$	$\{d,e\}$
e	$\{a\}$	$\{b,c\}$	$\{c\}$	$\{d,e\}$	$\{e\}$

Then we have $\{a,b,e\}$ and $\{a,c,d\}$ are left almost hyperideals, right almost hyperideals, almost hyperideals, almost bi-hyperideals, and almost quasi-hyperideals of H. But, the intersection of them is neither a left almost hyperideal, a right almost hyperideal, an almost hyperideal, an almost bi-hyperideal, nor an almost quasi-hyperideal of H because

$$a*\{a\}\cap\{a\}=\{a\}*a\cap\{a\}=\{b,c\}\cap\{a\}=\emptyset,$$
$$\big(\{a\}*b*\{a\}\big)\cap\{a\}=\{b,c\}\cap\{a\}=\emptyset,$$
$$\big(a*\{a\}\big)\cap\big(\{a\}*a\big)\cap\{a\}=\{b,c\}\cap\{a\}=\emptyset.$$

From Corollary 5.1.8, the union of almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals in semihypergroups is an almost hyperideal, an almost bi-hyperideal and an almost quasi-hyperideal, respectively. But, this is not the case for intersections of these structures as seen in Example 5.1.9.

Theorem 5.1.10. *Let $(H,*)$ be a semihypergroup and $|H|>1$. Then the following statements are true.*

(1) *H has no proper left almost hyperideals if and only if for any $a\in H$, there exists an element $h_a\in H$ such that $h_a*(H-\{a\})=\{a\}$.*

(2) *H has no proper right almost hyperideals if and only if for any $a\in H$, there exists an element $h_a\in H$ such that $(H-\{a\})*h_a=\{a\}$.*

(3) *H has no proper almost hyperideals if and only if for any $a\in H$, there exist elements $h_a,k_a\in H$ such that*

$$h_a*(H-\{a\})=\{a\} \text{ or } (H-\{a\})*k_a=\{a\}.$$

(4) *H has no proper almost bi-hyperideals if and only if for any $a\in H$, there exists an element $h_a\in H$ such that $(H-\{a\})*h_a*(H-\{a\})=\{a\}$.*

(5) *H has no proper almost quasi-hyperideals if and only if for any $a\in H$, there is an element $h_a\in H$ such that $h_a*(H-\{a\})\cap(H-\{a\})*h_a\subseteq\{a\}$.*

Proof. Assume that H has no proper left almost hyperideals. Let $a \in H$. Then $H - \{a\}$ is not a left almost hyperideal of H. Then there exists $h_a \in H$ such that

$$h_a * (H - \{a\}) \cap (H - \{a\}) = \emptyset,$$

which implies that $h_a * (H - \{a\}) = \{a\}$.

Conversely, let A be a proper subset of H. Then $A \subseteq H - \{u\}$ for some $u \in H$. By assumption, there exists an element $h_a \in H$ such that

$$h_a * (H - \{u\}) \cap (H - \{u\}) = \{a\} \cap (H - \{a\}) = \emptyset.$$

Since $A \subseteq H - \{u\}$, we have $h_a * A \cap A \subseteq h_a * (H - \{u\}) \cap (H - \{u\})$. Thus $h_a * A \cap A = \emptyset$. Hence, A is not a left almost hyperideal of H. Therefore, H has no proper left almost hyperideals. This is the proof of the statement (1). The proof of statements (2), (3), (4) and (5) are similar to that of the statement (1). □

Theorem 5.1.11. *Let $(H, \cdot)$ be a semihypergroup where $|H| > 1$, and let $a \in S$. Then the following statements hold.*

(1) *If H has no proper left almost hyperideals, then either $a \in a^2$ or $a \in a^3$.*

(2) *If H has no proper right almost hyperideals, then either $a \in a^2$ or $a \in a^3$.*

(3) *If H has no proper almost hyperideals, then either $a \in a^2$ or $a \in a^3$.*

(4) *If H has no proper almost bi-hyperideals, then either $a \in a^2$ or $a \in a^5$.*

(5) *If H has no proper almost quasi-hyperideals, then either $a \in a^2$ or $a \in a^3$.*

Proof. (1) Let $a \in H$. Assume that H has no proper left almost hyperideals. By Theorem 5.1.10(1), there exists an element $h_a \in H$ such that

$$h_a * (H - \{a\}) = \{a\}.$$

Assume that $a \notin a^2$. Then $a^2 \subseteq H - \{a\}$, so we have $h_a * a^2 \subseteq h_a * (H - \{a\}) = \{a\}$. Thus $h_a * a^2 = \{a\}$. Suppose that $h_a \neq a$. Then $h_a \in H - \{a\}$, so $h_a^2 = \{a\}$.

If $a \in h_a * a$, then $a^2 \subseteq h_a * a^2 = \{a\}$, a contradiction.

If $a \notin h_a * a$, then $h_a * a \subseteq H - \{a\}$, so $h_a^2 * a = h_a * h_a * a = \{a\}$. Thus we have $\{a\} = h_a^2 * a = a * a = a^2$, a contradiction.

Hence, $h_a = a$. Then we have $a^3 = a * a^2 = h_a * a^2 = \{a\}$. Thus $a \in a^3$.

(2) This proof is similar to the proof of the statement (1).

(3) This proof follows from the statements (1) and (2).

(4) Let $a \in H$. Assume that H has no proper almost bi-hyperideals. By Theorem 5.1.10(4), there exists an element $h_a \in H$ such that

$$(H - \{a\}) * h_a * (H - \{a\}) = \{a\}.$$

Assume that $a \notin a^2$. Then $a^2 \subseteq H - \{a\}$, which implies that $a^2 * h_a * a^2 = \{a\}$. Suppose that $h_a \neq a$. Then $h_a \in H - \{a\}$, so $h_a^3 = \{a\}$. Since $h_a \in H - \{a\}$ and $a^2 \subseteq H - \{a\}$, we have that $h_a^2 * a^2 = \{a\}$.

If $a \in h_a^2 * a$, then $a^2 \subseteq h_a^2 * a^2 = \{a\}$, a contradiction.

If $a \notin h_a^2 * a$, then $h_a^2 * a \subseteq H - \{a\}$, so $h_a^2 * a * h_a^3 * a = \{a\}$. Thus

$$\{a\} = h_a^2 * a * h_a^3 * a = h_a^2 * a * a * a = h_a^2 * a^2 * a = a \cdot a = a^2.$$

This implies that $a^2 = \{a\}$, a contradiction.

Thus $h_a = a$. Hence, $\{a\} = a^2 * h_a * a^2 = a^2 * a * a^2 = a^5$. Therefore, $a \in a^5$.

(5) Let $a \in H$. Assume that H has no proper almost quasi-hyperideals. By Theorem 5.1.10(5), there exists an element $h_a \in H$ such that

$$h_a * (H - \{a\}) \cap (H - \{a\}) * h_a \subseteq \{a\}.$$

Assume that $a \notin a^2$. This implies that $a^2 \subseteq H - \{a\}$. Then we have

$$(h_a * a^2) \cap (a^2 * h_a) \subseteq \{a\}.$$

Suppose that $h_a \neq a$. Then $h_a \in H - \{a\}$, so $h_a^2 = h_a * h_a \cap h_a * h_a \subseteq \{a\}$. Thus we have $h_a^2 = \{a\}$. We consider the following four cases:

Case 1: $a \in h_a * a$ and $a \in a * h_a$. Then $a^2 \subseteq h_a * a^2$ and $a^2 \subseteq a^2 * h_a$, which implies that $a^2 \subseteq (h_a * a^2) \cap (a^2 * h_a) \subseteq \{a\}$. This is a contradiction.

Case 2: $a \in h_a * a$ and $a \notin a * h_a$. Then $a^2 \subseteq h_a * a^2$ and $a * h_a \subseteq H - \{a\}$. Since $a^2 \subseteq H - \{a\}$ and $a * h_a \subseteq H - \{a\}$,

$$h_a * a^2 \cap a * h_a^2 \subseteq \{a\}.$$

Since $h_a^2 = \{a\}$, we have $a^2 = a * h_a^2$. Thus $a^2 \subseteq \left(h_a * a^2\right) \cap \left(a * h_a^2\right) \subseteq \{a\}$. This is a contradiction.

Case 3: $a \notin h_a * a$ and $a \in a * h_a$. This implies that $h_a * a \subseteq H - \{a\}$ and $a^2 \subseteq a^2 * h_a$. Since $h_a * a \subseteq H - \{a\}$ and $a^2 \subseteq H - \{a\}$,

$$h_a^2 * a \cap a^2 * h_a \subseteq \{a\}.$$

Since $h_a^2 = \{a\}$, we have $a^2 = h_a^2 * a$. Thus $a^2 \subseteq h_a^2 * a \cap a^2 * h_a \subseteq \{a\}$. This is a contradiction.

Case 4: $a \notin h_a * a$ and $a \notin a * h_a$. Then $h_a * a \subseteq H - \{a\}$ and $a * h_a \subseteq H - \{a\}$. Thus $h_a^2 * a \cap a * h_a^2 \subseteq \{a\}$. Since $h_a^2 = \{a\}$, we have

$$a^2 = a * a \cap a * a = h_a^2 * a \cap a * h_a^2 \subseteq \{a\},$$

so $a^2 = \{a\}$. This is a contradiction.

Hence, $h_a = a$. Since $h_a * a^2 \cap a^2 * h_a \subseteq \{a\}$, we have

$$a^3 = a * a^2 \cap a^2 * a = h_a * a^2 \cap a^2 * h_a \subseteq \{a\},$$

so $\{a\} = a^3$. Therefore, we can conclude that $a \in a^3$. □

5.2 The relations of almost hyperideals in semihypergroups

In this section, we provide some connections of almost hyperideals and almost bi-hyperideals, almost hyperideals and almost quasi-hyperideals, and almost bi-hyperideals and almost quasi-hyperideals in semihypergroups.

Theorem 5.2.1. *Let $(H, *)$ be a semihypergroup. Then every almost hyperideal of H is an almost bi-hyperideal of S.*

Proof. Let I be an almost hyperideal of H and $s \in H$. Since $I \neq \emptyset$, there is an element $a \in I$. Then $a * s * I \subseteq I * s * I$, so $(a * s * I) \cap I \subseteq (I * s * I) \cap I$. Since I is an almost hyperideal of H, $(a * s * I) \cap I \neq \emptyset$. Hence, $(I * s * I) \cap I \neq \emptyset$. Therefore, I is an almost bi-hyperideal of H. □

Combining Remark 5.1.3(iii) and Theorem 5.2.1, we have the following result.

Corollary 5.2.2. *Every hyperideal of a semihypergroup $(H, *)$ is an almost bi-hyperideal of H.*

Theorem 5.2.3. *Let $(H, *)$ be a semihypergroup. Then every almost quasi-hyperideal of H is an almost hyperideal of H.*

Proof. Assume that Q is an almost quasi-hyperideal of H. Let $s \in H$. Then we have $(s * Q \cap Q * s) \cap Q \neq \emptyset$. Since $s * Q \cap Q * s \subseteq s * Q$ and $s * Q \cap Q * s \subseteq Q * s$,

$$(s * Q \cap Q * s) \cap Q \subseteq s * Q \cap Q \text{ and } (s * Q \cap Q * s) \cap Q \subseteq Q * s \cap Q.$$

Thus $s * Q \cap Q \neq \emptyset$ and $Q * s \cap Q \neq \emptyset$. Hence, Q is a left almost hyperideal and a right almost hyperideal of H. Therefore, Q is an almost hyperideal of H. □

If we combine Theorem 5.2.1 and Theorem 5.2.3, then the result of the relationship between almost bi-hyperideals and almost quasi-hyperideals in semihypergroups is obtained as in the following corollary.

Corollary 5.2.4. *Every almost quasi-hyperideal of a semihypergroup $(H, *)$ is an almost bi-hyperideal of H.*

Example 5.2.5. From Example 5.1.2, we see that an almost hyperideal and an almost quasi-hyperideal of H are almost bi-hyperideals of H. Moreover, this shows that the converses of Theorem 5.2.1 and Corollary 5.2.4 are not true because $\{y, z\}$ is an almost bi-hyperideal of H but it is neither an almost hyperideal nor an almost quasi-hyperideal of H.

The following example shows that the converse of Theorem 5.2.3 is not true.

Example 5.2.6. Let $H = \{a, b, c, d, e\}$ be a semihypergroup under the hyperoperation $*$ below.

$\cdot$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{d\}$	$\{d\}$
b	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
c	$\{a\}$	$\{c\}$	$\{b\}$	$\{d\}$	$\{e\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{a\}$	$\{a\}$
e	$\{d\}$	$\{d\}$	$\{d\}$	$\{a\}$	$\{a\}$

Then $I = \{b, d, e\}$ is an almost hyperideal of H, but it is not an almost quasi-hyperideal of H because

$$(e * I) \cap (I * e) \cap I = \{a, d\} \cap \{a, e\} \cap I = \{a\} \cap I = \emptyset.$$

The coverses of Theorems 5.2.1, 5.2.3 and 5.2.4 can be true when they have the same conditions as in ordered semigroups. The proofs of the following theorems are the same as the proofs of Theorems 4.3.7, 4.3.8, and 4.3.9, respectively.

Theorem 5.2.7. *If $(H, *)$ is a commutative semihypergroup and A is a subsemihypergroup of H, then A is an almost hyperideal of H if and only if it is an almost bi-hyperideal of H.*

Theorem 5.2.8. *Let $(H, *)$ be a commutative semihypergroup. Then A is an almost quasi-hyperideals of H if and only if A is an almost hyperideals of H.*

Corollary 5.2.9. *Let $(H, *)$ be a commutative semihypergroup. A subsemihypergroup of H is an almost quasi-ideal of H if and only if it is an almost bi-ideal of H.*

CHAPTER 6

Conclusions and suggestions

6.1 Conclusions

In this thesis, we study a variety of almost ideals in three structures: semigroups, ordered semigroups and semihypergroups. Additionally, various fuzzy almost ideals in semigroups and ordered semigroups are studied.

In a semigroup, we introduce the defintions of almost (m, n)-ideals and fuzzy almost (m, n)-ideals, and provide their properties. The relations of almost (m, n)-ideals and fuzzy almost (m, n)-ideals is given by using a characteristic function and a support of fuzzy subsets. Moreover, we give the relations of minimal, prime and semiprime almost (m, n)-ideals, and minimal, prime and semiprime fuzzy almost (m, n)-ideals.

In an ordered semigroup, we define ordered almost ideals, ordered almost bi-ideals, ordered almost quasi-ideals, fuzzy ordered almost ideals, fuzzy ordered almost bi-ideals and fuzzy ordered almost quasi-ideals. Moreover, we provide the relations of all kinds of ordered almost ideals and fuzzy ordered almost ideals. For example, an ordered almost ideal is an ordered almost bi-ideal, an ordered almost quasi-ideal is an ordered almost ideal, an ordered almost quasi-ideal is an ordered almost bi-ideal, a fuzzy ordered almost ideal is a fuzzy ordered almost bi-ideal if it is a fuzzy subsemigroup, and a fuzzy ordered almost quasi-ideal is a fuzzy ordered almost ideal. For the relations of ordered almost ideals and fuzzy ordered almost ideals, ordered almost bi-ideals and fuzzy ordered almost bi-ideals, and ordered almost quasi-ideals and fuzzy ordered almost quasi-ideals in ordered semigroups, the same idea as in semigroups can be applied.

In a semihypergroup, we define almost hyperideals, almost bi-hyperideals and almost quasi-hyperideals by using the same notions as almost ideals, almost bi-ideals and almost quasi-ideals in semigroups, respectively. For properties and relations of all kinds of almost hyperideals in semihypergroups, we can do the same with ordered semigroups.

In this study, we can see that in these three structures, almost ideals have the same definitions and properties. The definitions and properties of fuzzy almost ideals in semigroups and ordered semigroups are the same.

6.2 Suggestions

1. Study almost ideals and fuzzy almost ideals in other algebraic structures.
2. Study fuzzy almost ideals in semihypergroups.

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