Green's Relations and Congruences for Γ-semigroups

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ชื่อวิทยานิพนธ์	ความสัมพันธ์ของกรีนและสมภาคสำหรับแกมมากึ่งกรุป
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บทคัดย่อ

กำหนดให้ S เป็นแกมมากึ่งกรุป และ α เป็นสมาชิกที่ถูกกำหนดใน Γ นิยาม $ab = a\alpha b$ สำหรับทุก $a, b \in S$ เห็นได้ชัดว่า S เป็นกึ่งกรุป และเราแทนกึ่งกรุปนี้ด้วยสัญลักษณ์ S_α ความสัมพันธ์ของกรีน $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ และ $\mathcal{D}$ บนแกมมากึ่งกรุป S ถูกนิยามโดย เอ็น เค ซาฮา ในปี ค.ศ. 1987 คลาส $\mathcal{L}$, คลาส $\mathcal{R}$, คลาส $\mathcal{H}$ และ คลาส $\mathcal{D}$ ที่บรรจุสมาชิก a ของแกมมากึ่งกรุป S ถูกแทนด้วยสัญลักษณ์ L_a, R_a, H_a และ D_a ตามลำดับ

เราศึกษาความสัมพันธ์ของกรีนสำหรับแกมมากึ่งกรุปและให้สมบัติบางประการที่น่าสนใจ ดังตัวอย่าง เราพิสูจน์ว่า ถ้า a และ b เป็นสมาชิกในแกมมากึ่งกรุป S โดยที่ $a\mathcal{D}b$ แล้ว $|L_a| = |L_b|$, $|R_a| = |R_b|$ และ $|H_a| = |H_b|$ อีกทั้งเราพบว่า ถ้า a เป็นสมาชิกในแกมมากึ่งกรุป S และ $\alpha \in \Gamma$ แล้ว $H_a\alpha H_a \cap H_a = \emptyset$ หรือ $H_a\alpha H_a = H_a$ ในกรณีที่ $H_a\alpha H_a = H_a$ แล้ว H_a เป็นกึ่งกรุปของ S_α

ยิ่งไปกว่านั้นเราศึกษาสมภาคสำหรับแกมมากึ่งกรุปและสร้างความสัมพันธ์ระหว่างสมภาค และเซตผลหารของสมภาคเหล่านั้นบนความสัมพันธ์ของกรีน เรานิยามสมภาค ρ_r และ ρ_l บนแกมมากึ่งกรุป S ดังนี้

$$\rho_r = \{(a,b) \in S \times S \mid a\gamma t = b\gamma t \text{ สำหรับทุก } t \in S \text{ และสำหรับทุก } \gamma \in \Gamma\},$$
$$\rho_l = \{(a,b) \in S \times S \mid t\gamma a = t\gamma b \text{ สำหรับทุก } t \in S \text{ และสำหรับทุก } \gamma \in \Gamma\}$$

ถ้า S เป็นแกมมากึ่งกรุปปกติ เราพบว่า ρ_r และ ρ_l เป็นสมภาคขวาลดรูปและสมภาคซ้ายลดรูปที่เล็กที่สุด ตามลำดับ

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ABSTRACT

Let S be a Γ-semigroup and α a fixed element in Γ. Define $ab = a\alpha b$ for all $a, b \in S$. Then S is a semigroup and we denote this semigroup by S_α. Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a Γ-semigroups S were defined by N. K. Saha in the year 1987. The $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class and $\mathcal{D}$-class containing the element a of a Γ-semigroup S will be written as L_a, R_a, H_a and D_a, respectively.

We study Green's relations for Γ-semigroups and give some interesting properties. For example, we prove that if a and b are elements in a Γ-semigroup S such that $a\mathcal{D}b$, then $|L_a| = |L_b|$, $|R_a| = |R_b|$ and $|H_a| = |H_b|$. We also observe that if a is an element in a Γ-semigroup S and $\alpha \in \Gamma$, then $H_a\alpha H_a \cap H_a = \emptyset$ or $H_a\alpha H_a = H_a$. Moreover, if $H_a\alpha H_a = H_a$, then H_a is a subsemigroup of S_α.

Furthermore, we study congruences for Γ-semigroups and give some connections between congruences and their quotient sets on Green's relations. We also define two congruences ρ_r and ρ_l on a Γ-semigroup S as follows:

$$\rho_r = \{(a, b) \in S \times S \mid a\gamma t = b\gamma t \text{ for all } t \in S \text{ and } \gamma \in \Gamma\};$$
$$\rho_l = \{(a, b) \in S \times S \mid t\gamma a = t\gamma b \text{ for all } t \in S \text{ and } \gamma \in \Gamma\}.$$

If S is a regular Γ-semigroup, we obtain that ρ_r and ρ_l are the minimum right and left reductive congruences on S, respectively.

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Prathana Siammai

CONTENTS

CHAPTER 1

Introduction and Preliminaries

The notion of Γ-semigroups was introduced by M. K. Sen in the year 1981. Since Γ-semigroups generalize semigroups, many classical notions of semigroups have been extended to Γ-semigroups. For example, in the year 1987, N. K. Saha introduced Green's relations for Γ-semigroups analogous to Green's relations for semigroups. In fact, any semigroup S can be considered to be a Γ-semigroup, by define $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. On the other hand, let S be a Γ-semigroup and α a fixed element in Γ. We define $ab = a\alpha b$ for all $a, b \in S$, then we can show that S is a semigroup and we denote this semigroup by S_α.

In this thesis, we study Green's relations and congruences for Γ-semigroups, and also give characterizations for reductive congruences and reductive Γ-semigroups. Moreover, we give some connections between Green's relations and simple Γ-semigroups.

1.1 Semigroups

We will use the notation and terminology of Howie (1976) to introduce the notion of semigroups.

Definition 1.1. Let S be a nonempty set and $*$ a binary operation on S. Then $(S, *)$ is called a *semigroup* if $*$ is associative, i.e.,

$$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in S.$$

We give some examples of semigroups.

Example 1.1. $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}, \times)$ and $(\mathbb{R}, \times)$ are semigroups where $+$ is the usual addition and $\times$ is the usual multiplication.

Example 1.2. $(\mathbb{Z}, -)$ is not a semigroup since for $a, b, c \in \mathbb{Z}$ such that $c \neq 0$, we have

$$a - (b - c) = a - b + c \neq a - b - c = (a - b) - c.$$

Example 1.3. Let X be a nonempty set and $T(X)$ the set of all mappings from X into X. Define a composition of mappings in $T(X)$ by

$$(x)(\alpha \circ \beta) = ((x)\alpha)\beta \quad \text{for all } x \in X.$$

Thus for $\alpha, \beta \in T(X)$, we have $\alpha \circ \beta \in T_x$. Clearly,

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \quad \text{for all } \alpha, \beta, \gamma \in T(X).$$

So $\circ$ is associative. Hence $(T(X), \circ)$ is a semigroup. The semigroup $T(X)$ is called a *full transformation semigroup* on X.

Definition 1.2. Let S be a semigroup. A nonempty subset T of S is called a *subsemigroup* of S if it is closed under the binary operation of S, i.e. if

$$ab \in T \quad \text{for all } a, b \in T.$$

Example 1.4. $(\mathbb{N}, +)$ is a subsemigroup of $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \times)$ is a subsemigroup of $(\mathbb{R}, \times)$ where $+$ is the usual addition and $\times$ is the usual multiplication.

Definition 1.3. Let S be a semigroup. An element $a \in S$ is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if every element of S is regular.

Example 1.5. Consider the semigroup $(T(X), \circ)$ in Example 1.3. To show that $(T(X), \circ)$ is a regular semigroup, let $\alpha \in T(X)$. Then for each $x \in ran\ (\alpha)$ there exists $a_x \in X$ such that $(a_x)\alpha = x$.

Define $\beta : X \to X$ by

$$(x)\beta = \begin{cases} a_x & \text{if} \quad x \in ran\ (\alpha), \\ x & \text{if} \quad x \notin ran\ (\alpha). \end{cases}$$

Claim that $\alpha = \alpha \circ \beta \circ \alpha$. We must show that $dom\ (\alpha) = dom\ (\alpha \circ \beta \circ \alpha)$ and $(a)(\alpha \circ \beta \circ \alpha) = (a)\alpha$ for all $a \in dom\ (\alpha)$. Obviously, $dom\ (\alpha \circ \beta \circ \alpha) \subseteq dom\ (\alpha)$. Let $y \in dom\ (\alpha)$. We have $(y)(\alpha \circ \beta \circ \alpha) = (((y)\alpha)\beta)\alpha = (a_{(y)\alpha})\alpha = (y)\alpha$, from which it follows that $dom\ (\alpha) \subseteq dom\ (\alpha \circ \beta \circ \alpha)$ and $(y)(\alpha \circ \beta \circ \alpha) = (y)\alpha$ for all $y \in dom\ (\alpha)$. Hence $\alpha = \alpha \circ \beta \circ \alpha$, as claimed.

We have $(T(X), \circ)$ is a regular semigroup since for $\alpha \in T(X)$, there exists $\beta \in T(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

Definition 1.4. Let S be a semigroup and $a \in S$. An element $a' \in S$ is called an *inverse* of a if

$$a = aa'a \text{ and } a' = a'aa'.$$

The set of all inverses of a is denoted by $V(a)$.

Definition 1.5. Let S be a semigroup. An element $e \in S$ is said to be an *idempotent* if $e^2 = e$. The set of all idempotents is denoted by $E(S)$. A semigroup S is called a *band* if $S = E(S)$.

Definition 1.6. Let A be a nonempty set. A *relation* on A we mean an arbitrary subset of $A \times A$.

Definition 1.7. Let A be a nonempty set and ρ a relation on A. Then

ρ is called *reflexive* if $(a, a) \in \rho$ for all $a \in A$;

ρ is called *symmetric* if for $a, b \in A$, $(a, b) \in \rho \Rightarrow (b, a) \in \rho$;

ρ is called *transitive* if for $a, b, c \in A$, $(a, b) \in \rho$ and $(b, c) \in \rho \Rightarrow (a, c) \in \rho$.

Definition 1.8. Let A be a nonempty set. A relation ρ on A is called an *equivalence relation* on A if it is reflexive, symmetric and transitive.

Example 1.6. Let ρ be a relation on $\mathbb{Z}$ defined by

$$a\rho b \Leftrightarrow 4|\ a - b \quad \text{for all } a, b \in \mathbb{Z}.$$

We have ρ is an equivalence relation on $\mathbb{Z}$ since

ρ is reflexive: for $a \in \mathbb{Z}, 4|\ a-a \Rightarrow (a,a) \in \rho$;

ρ is symmetric: for $a, b \in \mathbb{Z}$,

$$\begin{aligned} (a,b) \in \rho &\Rightarrow 4|\ a-b \\ &\Rightarrow 4x = a-b \qquad \text{for some } x \in \mathbb{Z} \\ &\Rightarrow 4(-x) = b-a \quad \text{because } -x \in \mathbb{Z} \\ &\Rightarrow 4|\ b-a \\ &\Rightarrow (b,a) \in \rho; \end{aligned}$$

ρ is transitive: for $a, b, c \in \mathbb{Z}$,

$$\begin{aligned} (a,b) \in \rho \text{ and } (b,c) \in \rho &\Rightarrow 4|\ a-b \text{ and } 4|\ b-c \\ &\Rightarrow 4x = a-b \text{ and } 4y = b-c \quad \text{for some } x, y \in \mathbb{Z} \\ &\Rightarrow 4x+4y = (a-b)+(b-c) \\ &\Rightarrow 4(x+y) = a-c \qquad\qquad \text{because } x+y \in \mathbb{Z} \\ &\Rightarrow 4|\ a-c \\ &\Rightarrow (a,c) \in \rho. \end{aligned}$$

Definition 1.9. The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S were introduced by J. A. Green (1951) as the following rules:

(i) $a\mathcal{L}b$ if and only if $S^1a = S^1b$, where $S^1a = Sa \cup \{a\}$;

(ii) $a\mathcal{R}b$ if and only if $aS^1 = bS^1$, where $aS^1 = aS \cup \{a\}$;

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$;

(iv) $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

We have the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a semigroup S are equivalence relations. The equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ are called *Green's relations*. An alternative characterization is given in the following remark:

Remark 1.1. Let a, b be elements of a semigroup S. We have

(i) $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$;

(ii) $a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$ and $by = a$;

(iii) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$;

(iv) $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$;

where

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

The following theorem shows that the relations $\mathcal{L}$ and $\mathcal{R}$ commute.

Theorem 1.1. $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Again, we will use the notation and terminology of Howie (1976) to introduce a congruence for semigroups as follows:

Definition 1.10. Let S be a semigroup . An equivalence relation ρ on S is called a *right congruence* on S if

$$(a, b) \in \rho \Rightarrow (at, bt) \in \rho \quad \text{for all } a, b, t \in S,$$

and a *left congruence* on S if

$$(a, b) \in \rho \Rightarrow (ta, tb) \in \rho \quad \text{for all } a, b, t \in S.$$

An equivalence relation ρ on S is called a *congruence* on S if it is both a right and a left congruence on S.

Example 1.7. Let ρ be an equivalence relation on a semigroup $(\mathbb{Z}, +)$ defined by

$$a\rho b \Leftrightarrow 4|\ a - b \quad \text{for all } a, b \in \mathbb{Z}.$$

We have ρ is a right congruence on $\mathbb{Z}$ since for $a, b, t \in \mathbb{Z}$,

$$\begin{aligned}(a,b) \in \rho \Rightarrow\ & 4|\ a-b \\ \Rightarrow\ & 4x = a-b \quad \text{for some } x \in \mathbb{Z} \\ \Rightarrow\ & 4x = (a+t)-(b+t) \\ \Rightarrow\ & 4|\ (a+t)-(b+t) \\ \Rightarrow\ & (a+t, b+t) \in \rho.\end{aligned}$$

A similar argument shows that ρ is a left congruence on $\mathbb{Z}$. Hence ρ is a congruence on $\mathbb{Z}$.

Example 1.8. Let ρ be an equivalence relation on a semigroup $(\mathbb{Z}, \times)$ defined by

$$a\rho b \Leftrightarrow 4|\ a-b \quad \text{for all } a, b \in \mathbb{Z}.$$

We have ρ is a right congruence on $\mathbb{Z}$ since for $a, b, t \in \mathbb{Z}$,

$$\begin{aligned}(a,b) \in \rho \Rightarrow\ & 4|\ a-b \\ \Rightarrow\ & 4x = a-b \quad \text{for some } x \in \mathbb{Z} \\ \Rightarrow\ & 4xt = (a-b)t \\ \Rightarrow\ & 4(xt) = at-bt \quad \text{because } xt \in \mathbb{Z} \\ \Rightarrow\ & 4|\ at-bt \\ \Rightarrow\ & (at, bt) \in \rho.\end{aligned}$$

A similar argument shows that ρ is a left congruence on $\mathbb{Z}$. Hence ρ is a congruence on $\mathbb{Z}$.

In 1955, G. Thierrin introduced the notions of a reductive congruence and a reductive semigroup (Fattahi and Vishki, 2004: 262) as the following.

Definition 1.11. Let S be a semigroup. A congruence ρ is called *right reductive* on S if

$$(at, bt) \in \rho \Rightarrow (a,b) \in \rho \qquad \text{for all } a, b, t \in S,$$

and *left reductive* on S if

$$(ta, tb) \in \rho \Rightarrow (a, b) \in \rho \quad \text{for all } a, b, t \in S.$$

A congruence ρ on S is called *reductive* on S if it is both right and left reductive on S.

Example 1.9. Consider the congruence ρ on $(\mathbb{Z}, +)$ in Example 1.7. We have ρ is right reductive on $\mathbb{Z}$ since for $a, b, t \in \mathbb{Z}$,

$$\begin{aligned}(a+t, b+t) \in \rho &\Rightarrow 4 \mid (a+t) - (b+t) \\ &\Rightarrow 4x = (a+t) - (b+t) \quad \text{for some } x \in \mathbb{Z} \\ &\Rightarrow 4x = a - b \\ &\Rightarrow 4 \mid a - b \\ &\Rightarrow (a, b) \in \rho.\end{aligned}$$

A similar argument shows that ρ is left reductive on $\mathbb{Z}$. Hence ρ is reductive on $\mathbb{Z}$.

Example 1.10. Consider the congruence ρ on $(\mathbb{Z}, \times)$ in Example 1.8. We have ρ is not right and left reductive on $\mathbb{Z}$.

Definition 1.12. A semigroup S is called *right (resp. left) reductive* if equality on S is a right (resp. left) reductive congruence. In other words, S is called *right reductive* if

$$at = bt \Rightarrow a = b \quad \text{for all } a, b, t \in S,$$

and *left reductive* if

$$ta = tb \Rightarrow a = b \quad \text{for all } a, b, t \in S.$$

A semigroup S is called *reductive* if it is both right and left reductive.

Example 1.11. Consider the semigroup $(\mathbb{Z}, +)$. We have $(\mathbb{Z}, +)$ is a reductive semigroup since for $a, b, t \in \mathbb{Z}$,

$$a + t = b + t \Rightarrow a = b,$$

and

$$t + a = t + b \Rightarrow a = b.$$

Example 1.12. Consider the semigroup $(\mathbb{Z}, \times)$. We have $(\mathbb{Z}, \times)$ is not a right and a left reductive semigroup.

Definition 1.13. Let A be a set of all right (resp. left) reductive congruence on a semigroup S. A congruence ρ on S is called the *minimum right (resp. left) reductive* if $\rho \subseteq \rho'$ for all $\rho' \in A$.

1.2 Γ-semigroups

We first recall some definitions and examples from Sen and Saha (1986), and Saha (1987).

Definition 1.14. Let S and Γ be nonempty sets. Then S is called a Γ-*semigroup* if it satisfies $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

We give some examples of Γ-semigroups.

Example 1.13. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n|\ n \in \mathbb{N}\}$. Define $a\alpha b = a + \alpha + b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $+$ is the usual addition. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.14. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n|\ n \in \mathbb{N}\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.15. Let $\mathbb{R}$ be the set of all real numbers and $\Gamma = \left\{\frac{1}{n}\middle|\ n \in \mathbb{N}\right\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{R}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have $\mathbb{R}$ is a Γ-semigroup.

Example 1.16. Let S be a set of all negative rational numbers and $\Gamma = \left\{ -\frac{1}{p} \middle| \ p \text{ is prime} \right\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in S$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have S is a Γ-semigroup.

Example 1.17. Let $S = \{4z + 3 |\ z \in \mathbb{Z}\}$ and $\Gamma = \{4n + 1 |\ n \in \mathbb{N}\}$. Define $a\alpha b = a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ where $+$ is the usual addition. We have S is a Γ-semigroup.

Example 1.18. For nonempty sets X and Y, let $T(X, Y)$ denote the set of all mappings from X to Y. Let Γ be a nonempty subset of $T(Y, X)$. Define a mapping $T(X, Y) \times \Gamma \times T(X, Y) \to T(X, Y)$ by $\alpha\gamma\beta = \alpha \circ \gamma \circ \beta$ for all $\alpha, \beta \in T(X, Y)$ and $\gamma \in \Gamma$ where $\circ$ is the composition of functions. We have $T(X, Y)$ is a Γ-semigroup.

Definition 1.15. Let S be a Γ-semigroup. A nonempty subset T of S is said to be a Γ-subsemigroup of S if $T\Gamma T \subseteq T$ where $T\Gamma T = \{a\alpha b \mid a, b \in T \text{ and } \alpha \in \Gamma\}$.

Example 1.19. Consider the Γ-semigroups in Example 1.13. Let $\mathbb{N}$ be the set of all natural numbers. We have $\mathbb{N}$ is a Γ-subsemigroup of $\mathbb{Z}$ since $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{N}\Gamma\mathbb{N} \subseteq \mathbb{N}$.

Example 1.20. Consider the Γ-semigroup S in Example 1.17. Let $T = \{4n - 1 |\ n \in \mathbb{N}\}$. We have T is a Γ-subsemigroup of S since $T \subseteq S$ and $T\Gamma T \subseteq T$.

Definition 1.16. Let S be a Γ-semigroup. An element $a \in S$ is said to be *regular* if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. The Γ-semigroup S is said to be a *regular* Γ*-semigroup* if every element of S is regular.

Example 1.21. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.13. We have $\mathbb{Z}$ is a regular Γ-semigroup since for $a \in \mathbb{Z}$, if $\alpha = m$ and $\beta = n$ for some $m, n \in \mathbb{N}$, then there exists $x = -m - n - a \in \mathbb{Z}$ such that $a\alpha x\beta a = a + m + x + n + a = a + m + (-m - n - a) + n + a = a$.

Example 1.22. Consider the Γ-semigroup S in Example 1.16. Without loss of generality, let $a = \frac{m}{n} \in S$ where $m > 0$ and $n < 0$.

If $m = 1$, then $a = \dfrac{1}{n}$. Now $\dfrac{1}{n} = \dfrac{1}{n} \times \Big(-\dfrac{1}{p_1}\Big) \times \dfrac{np_1p_2}{1} \times \Big(-\dfrac{1}{p_2}\Big) \times \dfrac{1}{n}$ where p_1 and p_2 are prime. Thus taking $x = \dfrac{np_1p_2}{1}, \alpha = -\dfrac{1}{p_1}$ and $\beta = -\dfrac{1}{p_2}$. Then $a = a\alpha x\beta a$.

If $m \neq 1$, then $m = p_1p_2 \dots p_k$ where p_i's are prime. Now $\dfrac{p_1p_2 \dots p_k}{n} = \dfrac{p_1p_2 \dots p_k}{n} \times \Big(-\dfrac{1}{p_1}\Big) \times \dfrac{n}{p_2p_3 \dots p_{k-1}} \times \Big(-\dfrac{1}{p_k}\Big) \times \dfrac{p_1p_2 \dots p_k}{n}$. Thus taking $x = \dfrac{n}{p_2p_3 \dots p_{k-1}}, \alpha = -\dfrac{1}{p_1}$ and $\beta = -\dfrac{1}{p_k}$. Then $a = a\alpha x\beta a$.

Hence S is a regular Γ-semigroup.

Definition 1.17. Let S be a Γ-semigroup and $a \in S$. Let $b \in S$ and $\alpha, \beta \in \Gamma$. An element b of S is called an *(α, β)-inverse* of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$.

Definition 1.18. Let S be a Γ-semigroup and $\alpha \in \Gamma$. An element $e \in S$ is said to be an *α-idempotent* if $e\alpha e = e$. The set of all α-idempotents is denoted by $E_\alpha(S)$. We denote $\bigcup\limits_{\alpha\in\Gamma} E_\alpha(S)$ by $E(S)$. Any element of $E(S)$ is called an *idempotent element* of S. A Γ-semigroup S is called an *idempotent Γ-semigroup* if $S = E(S)$.

Definition 1.19. Let S be a Γ-semigroup. A nonempty subset A of S is called a *left ideal* of S if $S\Gamma A \subseteq A$, a *right ideal* of S if $A\Gamma S \subseteq A$, and an *ideal* of S if it is both a left and a right ideal of S.

Example 1.23. Consider the Γ-semigroups $\mathbb{Z}$ in Example 1.14. Let $A = \{0\} \subseteq \mathbb{Z}$. We have A is a left and a right ideal of $\mathbb{Z}$ since $\mathbb{Z}\Gamma A \subseteq A$ and $A\Gamma\mathbb{Z} \subseteq A$, respectively. Therefore A is an ideal of $\mathbb{Z}$.

Definition 1.20. Let S be a Γ-semigroup. A Γ-semigroup S is called *left simple* if S is the unique left ideal of S, *right simple* if S is the unique right ideal of S, and *simple* if S is the unique ideal of S.

Example 1.24. Consider the Γ-semigroups $\mathbb{Z}$ in Example 1.13. We have $\mathbb{Z}$ is left and right simple since $\mathbb{Z}$ is the unique left and right ideal of $\mathbb{Z}$, respectively. Also Γ-semigroup $\mathbb{Z}$ is simple since $\mathbb{Z}$ is the unique ideal of $\mathbb{Z}$.

Example 1.25. Consider the Γ-semigroups $\mathbb{Z}$ in Example 1.14. By Example 1.23, we see that $\mathbb{Z}$ is not left and right simple. Also $\mathbb{Z}$ is not simple.

CHAPTER 2

Green's relations for Γ-semigroups

Green's relations for semigroups were introduced by J. A. Green in the year 1951. They are equivalence relations that characterize the element of a semigroup in term of the principal ideals they generate. They have played a fundamental role in the development of semigroup theory. In this chapter, we study Green's relations for Γ-semigroups and also investigate some interesting properties of these relations.

We demonstrate this chapter in four sections. In the first section, we introduce the notions of Green's relations for Γ-semigroups and present some results which will be used in the next sections. Next, we point out the structure of $\mathcal{D}$-classes and generalize Green's Theorem for semigroups to Green's Theorem for Γ-semigroups. In the third section, we focus regular $\mathcal{D}$-classes. In the last section, ideals of Γ-semigroups and simple Γ-semigroups are studied.

2.1 The equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$

Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on a Γ-semigroup S were defined by N. K. Saha (1987) as the following rules:

(i) $a\mathcal{L}b$ if and only if $S^1\Gamma a = S^1\Gamma b$, where $S^1\Gamma a = S\Gamma a \cup \{a\}$;

(ii) $a\mathcal{R}b$ if and only if $a\Gamma S^1 = b\Gamma S^1$, where $a\Gamma S^1 = a\Gamma S \cup \{a\}$;

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$;

(iv) $\mathcal{D} = \mathcal{L} o \mathcal{R}$.

An alternative characterization, making the aspect of these relations more explicit, is given in the following remark:

Remark 2.1. Let a, b be elements of a Γ-semigroup S. We have

(i) $a\mathcal{L}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$;

(ii) $a\mathcal{R}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$;

(iii) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$;

(iv) $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$.

Another immediate properties of $\mathcal{L}, \mathcal{R}, \mathcal{H}$, and $\mathcal{D}$ are as the following:

Proposition 2.1. *Let S be a Γ-semigroup. We have $\mathcal{L}, \mathcal{R}, \mathcal{H}$, and $\mathcal{D}$ are equivalence relations on S.*

Proof. Let $a \in S$. Obviously, we have $(a, a) \in \mathcal{L}$. Hence $\mathcal{L}$ is reflexive.

Let $a, b \in S$. If $(a, b) \in \mathcal{L}$, then $S^1\Gamma a = S^1\Gamma b$. It is easy to see that $(b, a) \in \mathcal{L}$. Hence $\mathcal{L}$ is symmetric.

Let $a, b, c \in S$. If $(a, b) \in \mathcal{L}$ and $(b, c) \in \mathcal{L}$, then $S^1\Gamma a = S^1\Gamma b$ and $S^1\Gamma b = S^1\Gamma c$. We have $S^1\Gamma a = S^1\Gamma c$. It follows that $(a, c) \in \mathcal{L}$. Hence $\mathcal{L}$ is transitive.

Therefore $\mathcal{L}$ is an equivalence relation on S. The relation $\mathcal{R}$ is proved in an analogous way.

Since the intersection of two equivalence relations is again an equivalence relation, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ is an equivalence relation on S. □

For the proof of $\mathcal{D}$, we must use the following theorem:

Theorem 2.2. $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Let $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$.

Case 1. $a = c$. Then $a\mathcal{R}b$. Since $a\mathcal{R}b$ and $b\mathcal{L}b$, $(a, b) \in \mathcal{R} \circ \mathcal{L}$.

Case 2. $b = c$. Then $a\mathcal{L}b$. Since $a\mathcal{R}a$ and $a\mathcal{L}b$, $(a, b) \in \mathcal{R} \circ \mathcal{L}$.

Case 3. $a \neq c$ and $b \neq c$. Since $a\mathcal{L}c$ and $c\mathcal{R}b$, there exist $x, y, u, v \in S$ and $\alpha, \beta, \eta, \mu \in \Gamma$ such that

$$x\alpha a = c, \quad y\beta c = a, \quad c\eta u = b, \quad b\mu v = c.$$

Let $d = y\beta c\eta u$. Then

$$a\eta u = y\beta c\eta u = d$$

and

$$d\mu v = y\beta c\eta u\mu v = y\beta b\mu v = y\beta c = a,$$

from which it follows $a\mathcal{R}d$. Also,

$$y\beta b = y\beta c\eta u = d$$

and

$$x\alpha d = x\alpha y\beta c\eta u = x\alpha a\eta u = c\eta u = b,$$

so $d\mathcal{L}b$. We deduce that $(a,b) \in \mathcal{R} \circ \mathcal{L}$. Therefore $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$.

Similarly, we can prove that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. $\square$

Now we show that $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ is an equivalence relation on S.

Let $a \in S$. Since $(a,a) \in \mathcal{L}$ and $(a,a) \in \mathcal{R}$, $(a,a) \in \mathcal{L} \circ \mathcal{R}$. Hence $\mathcal{L} \circ \mathcal{R}$ is reflexive.

Let $a, b \in S$ such that $(a,b) \in \mathcal{L} \circ \mathcal{R}$. It follows that $a\mathcal{L} \circ \mathcal{R}b$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. We have $b\mathcal{R}c$ and $c\mathcal{L}a$. Thus $b\mathcal{R} \circ \mathcal{L}a$. Since $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, $b\mathcal{L} \circ \mathcal{R}a$. It follows that $(b,a) \in \mathcal{L} \circ \mathcal{R}$. Hence $\mathcal{L} \circ \mathcal{R}$ is symmetric.

Let $a, b, c \in S$ such that $(a,b) \in \mathcal{L} \circ \mathcal{R}$ and $(b,c) \in \mathcal{L} \circ \mathcal{R}$. Then there exist $x, y \in S$ such that $a\mathcal{L}x$, $x\mathcal{R}b$, $b\mathcal{L}y$ and $y\mathcal{R}c$. Since $x\mathcal{R}b$ and $b\mathcal{L}y$, $x\mathcal{R} \circ \mathcal{L}y$. Again from $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, thus $x\mathcal{L} \circ \mathcal{R}y$. Then there exists $z \in S$ such that $x\mathcal{L}z$ and $z\mathcal{R}y$. Since $a\mathcal{L}x$ and $x\mathcal{L}z$, $a\mathcal{L}z$. Since $z\mathcal{R}y$ and $y\mathcal{R}c$, $z\mathcal{R}c$. since $a\mathcal{L}z$ and $z\mathcal{R}c$, $a\mathcal{L} \circ \mathcal{R}c$. It follows that $(a,c) \in \mathcal{L} \circ \mathcal{R}$. Hence $\mathcal{L} \circ \mathcal{R}$ is transitive.

Therefore $\mathcal{L} \circ \mathcal{R} = \mathcal{D}$ is an equivalence relation on S.

Lemma 2.3. *Let S be a Γ-semigroup. The following statements hold.*

(i) $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D}$, $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D}$.

(ii) The relation $\mathcal{L}$ is a right congruence on S and the relation $\mathcal{R}$ is a left congruence on S.

Proof. (i) Let $a, b \in S$. We have

$$\begin{aligned}(a,b) \in \mathcal{H} &\Rightarrow (a,b) \in \mathcal{L} \text{ and } (a,b) \in \mathcal{R} \\ &\Rightarrow (a,b) \in \mathcal{L} \\ &\Rightarrow (a,b) \in \mathcal{L} \text{ and } (b,b) \in \mathcal{R} \\ &\Rightarrow (a,b) \in \mathcal{D}.\end{aligned}$$

We have $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D}$. Similarly, $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D}$.

We can see the proof of (ii) in chapter 3. □

The $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element a of a Γ-semigroup S will be written as L_a (resp. R_a, H_a, D_a).

Proposition 2.4. *If a and b are elements of a Γ-semigroup S such that $L_a \cap R_b \neq \emptyset$, then $L_a \cap R_b = H_x$ for all $x \in L_a \cap R_b$.*

Proof. Let $x \in L_a \cap R_b$. Then $L_x = L_a$ and $R_x = R_b$. We have $L_a \cap R_b = L_x \cap R_x = H_x$. □

Lemma 2.5. *If a is a regular element of a Γ-semigroup S, then $S^1\Gamma a = S\Gamma a$, $a\Gamma S^1 = a\Gamma S$ and $S^1\Gamma a\Gamma S^1 = S\Gamma a\Gamma S$.*

Proof. Let $a \in S$. Then $a = a\alpha x\beta a$ for some $x \in S$ and $\alpha, \beta \in \Gamma$. We have

$$S^1\Gamma a = S^1\Gamma a\alpha x\beta a = (S^1\Gamma a\alpha x)\beta a \subseteq S\Gamma a \subseteq S^1\Gamma a,$$

from which it follows $S^1\Gamma a = S\Gamma a$. Similarly, we have $a\Gamma S^1 = a\Gamma S$. Thus

$$S^1\Gamma a\Gamma S^1 = (S^1\Gamma a)\Gamma S^1 = S\Gamma a\Gamma S^1 = S\Gamma(a\Gamma S^1) = S\Gamma(a\Gamma S) = S\Gamma a\Gamma S.$$

□

The following theorem will be used variously in this chapter.

Theorem 2.6. *Let S be a Γ-semigroup, $\alpha \in \Gamma$ and e an α-idempotent. Then* *(i)* $a\alpha e = a$ *for all* $a \in L_e$*;*

(ii) $e\alpha a = a$ *for all* $a \in R_e$*;*

(iii) $a\alpha e = a = e\alpha a$ *for all* $a \in H_e$*;*

(iv) For all $a \in S, |H_a \cap E_\alpha(S)| \leq 1$.

Proof. (i) Let $a \in L_e$. Then $a\mathcal{L}e$. It follows that $S^1\Gamma a = S^1\Gamma e$. Then $a = e$ or there exist $x \in S$ and $\gamma \in \Gamma$ such that $a = x\gamma e$. If $a = e$, then $a\alpha e = e\alpha e = e = a$. If $a = x\gamma e$, then $a\alpha e = (x\gamma e)\alpha e = x\gamma(e\alpha e) = x\gamma e = a$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii).

(iv) Let $e, f \in H_a \cap E_\alpha$. Then $e\mathcal{H}f$. So $e\mathcal{L}f$ and $e\mathcal{R}f$. Then $f \in L_e$ and $e \in R_f$. By (i) and (ii), respectively, we have $f\alpha e = f$ and $f\alpha e = e$. Therefore $e = f$. It follows that $|H_a \cap E_\alpha(S)| \leq 1$. □

Lemma 2.7. *If* a *and* x *are elements of a* Γ*-semigroup* S *and* $\alpha, \beta \in \Gamma$ *such that* $a = a\alpha x\beta a$*, then* $S^1\Gamma a = S^1\Gamma x\beta a$ *and* $a\Gamma S^1 = a\alpha x\Gamma S^1$*.*

Proof. Since $S^1\Gamma a = S^1\Gamma a\alpha x\beta a = (S^1\Gamma a)\alpha x\beta a \subseteq S^1\Gamma x\beta a = (S^1\Gamma x)\beta a \subseteq S^1\Gamma a$, $S^1\Gamma a = S^1\Gamma x\beta a$. Similarly, $a\Gamma S^1 = a\alpha x\Gamma S^1$. □

Theorem 2.8. *Let* S *be a* Γ*-semigroup and* $\alpha, \beta \in \Gamma$*. The following statements hold.*

(i) If $a, x \in S$ *such that* $a = a\alpha x\beta a$*, then* $a\mathcal{L}x\beta a$ *and* $a\mathcal{R}a\alpha x$*.*

(ii) For $a \in S$*, if* a' *is an* (α, β)*-inverse of* a*, then* $R_a \cap L_{a'}$ *and* $R_{a'} \cap L_a$ *contain a* β*-idempotent* $a\alpha a'$ *and an* α*-idempotent* $a'\beta a$*, respectively.*

Proof. (i) It follows from Lemma 2.7 and definitions of $\mathcal{L}$ and $\mathcal{R}$ on S.

(ii) Let $a \in S$ and a' be an (α, β)-inverse of a. Then $a = a\alpha a'\beta a$ and $a' = a'\beta a\alpha a'$. By $a = a\alpha a'\beta a$ and (i), we have $a'\beta a \in L_a$ and $a\alpha a' \in R_a$. By $a' = a'\beta a\alpha a'$ and (i), we have $a\alpha a' \in L_{a'}$ and $a'\beta a \in R_{a'}$. Thus $a\alpha a' \in R_a \cap L_{a'}$ and $a'\beta a \in R_{a'} \cap L_a$. Since $a = a\alpha a'\beta a, a'\beta a = a'\beta a\alpha a'\beta a$ and $a\alpha a' = a\alpha a'\beta a\alpha a'$. Therefore $a'\beta a$ is an α-idempotent and $a\alpha a'$ is a β-idempotent. □

Immediately by Proposition 2.4, Theorem 2.6 (iv) and Theorem 2.8 (ii), we have the following theorem:

Theorem 2.9. *Let a be an element of a Γ-semigroup S and $\alpha, \beta \in \Gamma$. If b and c are (α, β)-inverses of a such that $b\mathcal{H}c$, then $b = c$.*

Proof. Let $a \in S$. Assume that b and c are (α, β)-inverses of a. By Theorem 2.8 (ii), we have $a\alpha b \in R_a \cap L_b$, $b\beta a \in R_b \cap L_a$, $a\alpha c \in R_a \cap L_c$ and $c\beta a \in R_c \cap L_a$. Since $b\mathcal{H}c$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $L_b = L_c$ and $R_b = R_c$. Thus

$$a\alpha b, a\alpha c \in R_a \cap L_b, b\beta a, c\beta a \in R_b \cap L_a.$$

By Proposition 2.4, we have $H_{a\alpha b} = H_{a\alpha c}$ and $H_{b\beta a} = H_{c\beta a}$. Since $a\alpha b, a\alpha c \in E_\beta$ and $b\beta a$, $c\beta a \in E_\alpha$, by Theorem 2.6 (iv), we have $a\alpha b = a\alpha c$ and $b\beta a = c\beta a$. It follows that

$$b = b\beta a\alpha b = b\beta(a\alpha b) = b\beta(a\alpha c) = b\beta a\alpha c = (b\beta a)\alpha c = c\beta a\alpha c = c.$$

$\square$

Lemma 2.10. *If a is an element of a Γ-semigroup S such that $S^1\Gamma a = S^1\Gamma e$ or $a\Gamma S^1 = e\Gamma S^1$ for some $e \in E_\alpha(S)$, then a is a regular element of S.*

Proof. Suppose that $S^1\Gamma a = S^1\Gamma e$. Then $a = e$ or there exist $x, y \in S$ and $\beta, \eta \in \Gamma$ such that $a = x\beta e, e = y\eta a$. If $a = e$, then $a = e = e\alpha e\alpha e$. If $a = x\beta e$ and $e = y\eta a$, then $a = x\beta e = (x\beta e)\alpha e = a\alpha y\eta a$. Thus a is a regular element of S. Similarly, if $a\Gamma S^1 = e\Gamma S^1$, then a is a regular element of S. $\square$

Theorem 2.11. *Let S be a Γ-semigroup and $\alpha \in \Gamma$. If $e \in E_\alpha(S)$ and*

$$G_e = \{x \in S | \ x\alpha e = e\alpha x = x \text{ and } x\alpha y = y\alpha x = e \text{ for some } y \in S\},$$

then G_e is a subgroup of S_α where e is an identity and

$$G_e = \{x \in S | \ x \in e\alpha S \cap S\alpha e \text{ and } e \in x\alpha S \cap S\alpha x\}.$$

Proof. Obviously, G_e is a subsemigroup of S_α where e is an identity. We will show that G_e is a subgroup of S_α where e is an identity. Let $x \in G_e$. Then

$e\alpha x = x\alpha e = x$ and $x\alpha y = y\alpha x = e$ for some $y \in S$. Since $e\alpha y\alpha e \in S$,

$$e\alpha(e\alpha y\alpha e) = (e\alpha y\alpha e)\alpha e = e\alpha y\alpha e$$

and

$$x\alpha(e\alpha y\alpha e) = e = (e\alpha y\alpha e)\alpha x.$$

Thus $e\alpha y\alpha e \in G_e$. Therefore G_e is a subgroup of S_α where e is an identity.

Let $H = \{x \in S|\ x \in e\alpha S \cap S\alpha e$ and $e \in x\alpha S \cap S\alpha x\}$. We will show that $G_e = H$. Clearly, $G_e \subseteq H$. Let $x \in H$. By the definition of H, there exist $a, b, c, d \in S$ such that

$$x = e\alpha a = b\alpha e \text{ and } e = x\alpha c = d\alpha x.$$

Then

$$x = e\alpha a = e\alpha(e\alpha a) = e\alpha x$$

and

$$x = b\alpha e = (b\alpha e)\alpha e = x\alpha e.$$

Thus

$$x\alpha(e\alpha c\alpha e) = (x\alpha e)\alpha c\alpha e = x\alpha c\alpha e = e\alpha e = e,$$

$$(e\alpha d\alpha e)\alpha x = e\alpha d\alpha(e\alpha x) = e\alpha d\alpha x = e\alpha e = e$$

and

$$e\alpha c\alpha e = e\alpha e\alpha c\alpha e = e\alpha(d\alpha x)\alpha c\alpha e = e\alpha d\alpha(x\alpha c)\alpha e = e\alpha d\alpha e\alpha e = e\alpha d\alpha e.$$

We can conclude that

$$e\alpha x = x\alpha e = x \text{ and } x\alpha(e\alpha c\alpha e) = (e\alpha c\alpha e)\alpha x = e.$$

Thus $x \in G_e$. Hence $H \subseteq G_e$. Therefore $G_e = H$, as desired. □

Theorem 2.12. *If S is a Γ-semigroup and $\alpha \in \Gamma$, then $G_e \subseteq H_e$ for all $e \in E_\alpha(S)$.*

Proof. Let $x \in G_e$. Then $x\alpha e = e\alpha x = x$ and $x\alpha y = y\alpha x = e$ for some $y \in S$. Since $x = x\alpha e$ and $e = y\alpha x$, $x\mathcal{L}e$. Since $x = e\alpha x$ and $e = x\alpha y$, $x\mathcal{R}e$. Thus $x \in L_e \cap R_e = H_e$. Hence $G_e \subseteq H_e$. □

Theorem 2.13. *Let S be a regular Γ-semigroup and $a, b \in S$. The following statements are equivalent.*

(i) $a \in S\Gamma b\Gamma S$.

(ii) There exists $c \in S$ such that $a\mathcal{R}c$ and $c \in S\Gamma b$.

Proof. (i)⇒(ii). Assume that $a \in S\Gamma b\Gamma S$. Then $a = x\alpha b\beta y$ for some $x, y \in S$ and $\alpha, \beta \in \Gamma$. Since S is regular, there exist $z \in S$ and $\eta, \mu \in \Gamma$ such that $a = a\eta z\mu a$. Let $c = (a\eta z\mu x)\alpha b$. Thus $c \in S\Gamma b$. Since $c = a\eta(z\mu x\alpha b)$ and $a = a\eta z\mu a = a\eta z\mu(x\alpha b\beta y) = (a\eta z\mu x\alpha b)\beta y = c\beta y$, $a\mathcal{R}c$.

(ii)⇒(i). Assume that there exists $c \in S$ such that $a\mathcal{R}c$ and $c \in S\Gamma b$. Since S is regular and $a\mathcal{R}c$, $a \in a\Gamma S = c\Gamma S \subseteq S\Gamma b\Gamma S$. □

2.2 The structure of $\mathcal{D}$-classes

Each $\mathcal{D}$-class in a Γ-semigroup is a union of $\mathcal{L}$-classes and also a union of $\mathcal{R}$-classes. The intersection of $\mathcal{L}$-classes and $\mathcal{R}$-classes is empty or is an $\mathcal{H}$-class. In fact, by the definition of $\mathcal{D}$,

$$a\mathcal{D}b \Leftrightarrow R_a \cap L_b \neq \emptyset \Leftrightarrow L_a \cap R_b \neq \emptyset.$$

It is convenient to visualize a $\mathcal{D}-$class as what Clifford and Preston (Howie, 1972: 42) have called an *egg-box* in which each row represents an $\mathcal{R}-$class, each column an $\mathcal{L}-$class and each cell an $\mathcal{H}-$class. We can see that as follows:

a		b
c		
		d, e

From this, we have $R_a = R_b, R_d = R_e, L_a = L_c, L_b = L_d = L_e, H_d = H_e$ and $D_a = D_b = D_c = D_d = D_e$.

Proposition 2.14. *Let a be an element in a Γ-semigroup S. We have $L_a \subseteq D_a$ and $R_a \subseteq D_a$.*

Proof. It follows from the definition of Green's relation $\mathcal{D}$. □

Theorem 2.15. *Let a and b be elements in a Γ-semigroup S. We have $L_a \cap R_b \neq \emptyset$ if and only if $D_a = D_b$.*

Proof. Suppose that $L_a \cap R_b \neq \emptyset$. Let $x \in L_a \cap R_b$. Then $x \in L_a \subseteq D_a$ and $x \in R_b \subseteq D_b$. Thus $D_a \cap D_b \neq \emptyset$. Hence $D_a = D_b$.

Conversely, suppose that $D_a = D_b$. We have $a\mathcal{D}b$. Since $\mathcal{D} = \mathcal{L}o\mathcal{R}$, there exists $x \in S$ such that $a\mathcal{L}x$ and $x\mathcal{R}b$. Hence $x \in L_a \cap R_b$. □

The following lemma is similar to Green's Lemma for semigroups.

Lemma 2.16. *(Green's Lemma for Γ-semigroups) Let a and b be elements in a Γ-semigroup S such that $a\mathcal{R}b$. Then $a = b$ or there exist $s, s' \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha s = b$ and $b\beta s' = a$.*

If $a = b$, define $\varphi : L_a \to S$ and $\psi : L_b \to S$ by

$$\varphi = \psi = 1_{L_a} = 1_{L_b}$$

where 1_{L_a} and 1_{L_b} are identity maps on L_a and L_b, respectively.

If $a \neq b$, define $\varphi : L_a \to S$ and $\psi : L_b \to S$ by

$$(x)\varphi = x\alpha s \quad \text{if} \quad x \in L_a,$$

and

$$(y)\psi = y\beta s' \quad \text{if} \quad y \in L_b.$$

We have the following statements hold.

(i) $(L_a)\varphi = L_b$ and $(L_b)\psi = L_a$.

(ii) $\varphi\psi = 1_{L_a}$ and $\psi\varphi = 1_{L_b}$.

(iii) If $x \in L_a$, then $((x)\varphi)\mathcal{R}x$ and if $y \in L_b$, then $((y)\psi)\mathcal{R}y$.

Proof. (i) If $a = b$, then obviously $(L_a)\varphi = L_b$.

Assume $a \neq b$. Let $x \in L_a$. Then $x\mathcal{L}a$. Thus $x\alpha s\mathcal{L}a\alpha s$. So $(x)\varphi = x\alpha s \in L_{a\alpha s} = L_b$. Hence $(L_a)\varphi \subseteq L_b$. Next, let $y \in L_b$. We have $y\mathcal{L}b$. Then $y = b$ or there exist $t, t' \in S$ and $\nu, \eta \in \Gamma$ such that $t\nu y = b$ and $t'\eta b = y$. If $y = b$, then $y = b = a\alpha s = (a)\varphi \in (L_a)\varphi$. If $y \neq b$, we have $a = b\beta s' = t\nu y\beta s' = t\nu t'\eta b\beta s' = t\nu t'\eta a$. Then $a\mathcal{L}t'\eta a$. Thus $y = t'\eta b = t'\eta a\alpha s = (t'\eta a)\varphi \in (L_a)\varphi$. Hence $L_b \subseteq (L_a)\varphi$. Therefore $(L_a)\varphi = L_b$.

Similarly, $(L_b)\psi = L_a$.

(ii) If $a = b$, then obviously $\varphi\psi = 1_{L_a}$.

Assume $a \neq b$. Let $x \in L_a$. Then $x = a$ or there exist $t \in S$ and $\nu \in \Gamma$ such that $x = t\nu a$. If $x = a$, then $(x)\varphi\psi = x\alpha s\beta s' = a\alpha s\beta s' = b\beta s' = a = x$. If $x = t\nu a$, then $(x)\varphi\psi = x\alpha s\beta s' = t\nu a\alpha s\beta s' = t\nu b\beta s' = t\nu a = x$. Therefore $\varphi\psi = 1_{L_a}$.

Similarly, $\psi\varphi = 1_{L_b}$.

(iii) Let $x \in L_a$.

Case 1. $a = b$. Then obviously $((x)\varphi)\mathcal{R}x$.

Case 2. $a \neq b$. Then $(x)\varphi = x\alpha s$ and $x = (x)\varphi\psi = ((x)\varphi)\beta s'$. We have $((x)\varphi)\mathcal{R}x$.

Similarly, if $y \in L_b$, we have $((y)\psi)\mathcal{R}y$. $\square$

The left-right dual of Lemma 2.16 is proved in an analogous way.

Let a and b be elements in a Γ-semigroup S such that $a\mathcal{D}b$. By Theorem 2.15, there exists $c \in L_a \cap R_b$. Thus $L_a = L_c$ and $R_c = R_b$. Since $c\mathcal{R}b$, by Lemma 2.16, we have $|L_c| = |L_b|$. Hence $|L_a| = |L_b|$. Similarly, if $a\mathcal{L}c$, by the dual of Lemma 2.16, we have $|R_a| = |R_b|$. We have the following corollary:

Corollary 2.17. *If a and b are elements in a Γ-semigroup S such that $a\mathcal{D}b$, then $|L_a| = |L_b|$ and $|R_a| = |R_b|$.*

Lemma 2.18. *Let a and b be elements in a Γ-semigroup S such that $a\mathcal{R}b$. If $s \in S$ such that $a\alpha s = b$ for some $\alpha \in \Gamma$, then $H_x\alpha s = H_{x\alpha s}$ for all $x \in L_a$.*

Proof. Let $x \in L_a$. By Lemma 2.16 (i) and (iii), we have $L_a\alpha s = L_b$ and $x\mathcal{R}(x\alpha s)$. Thus $x\alpha s \in L_b$. Hence $L_{x\alpha s} = L_b$.

To prove $H_x\alpha s \subseteq H_{x\alpha s}$, let $y \in H_x$. Then $y \in L_x = L_a$. By Lemma 2.16 (iii), we have $y\mathcal{R}(y\alpha s)$. Thus $y\alpha s \in R_y = R_x = R_{x\alpha s}$. Hence $H_x\alpha s \subseteq R_{x\alpha s}$. Since $H_x \subseteq L_x = L_a, H_x\alpha s \subseteq L_a\alpha s = L_b = L_{x\alpha s}$. Therefore $H_x\alpha s \subseteq R_{x\alpha s} \cap L_{x\alpha s} = H_{x\alpha s}$, this imples $H_x\alpha s \subseteq H_{x\alpha s}$.

Conversely, let $z \in H_{x\alpha s}$. Then $z \in L_{x\alpha s} = L_b$. Since $L_a\alpha s = L_b$, there exists $w \in L_a$ such that $w\alpha s = z$. By Lemma 2.16 (iii), we have $w\mathcal{R}(w\alpha s)$. Then $w\mathcal{R}z$. We have $w \in R_z = R_{x\alpha s}$. Thus $w \in L_a \cap R_{x\alpha s} = L_x \cap R_x = H_x$. Hence $z = w\alpha s \in H_x\alpha s$. Therefore $H_{x\alpha s} \subseteq H_x\alpha s$.

We can conclude that $H_x\alpha s = H_{x\alpha s}$ for all $x \in L_a$. $\square$

The left-right dual of Lemma 2.18 is proved in an analogous way.

Theorem 2.19. *Let a and c be elements in a Γ-semigroup S such that $a\mathcal{D}c$. Let $b \in S$ such that $a\mathcal{R}b$ and $b\mathcal{L}c$. Then $a = b$ or there exist $s, s' \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha s = b, b\beta s' = a$ and $b = c$ or there exist $t, t' \in S$ and $\nu, \eta \in \Gamma$ such that $t\nu b = c, t'\eta c = b$.*

If $a = b$ and $b = c$, define $\varphi : H_a \to S$ and $\psi : H_c \to S$ by

$$\varphi = \psi = 1_{H_a} = 1_{H_c}$$

where 1_{H_a} and 1_{H_c} are identity maps on H_a and H_c, respectively.

If $a = b$ and $b \neq c$, define $\varphi : H_a \to S$ and $\psi : H_c \to S$ by

$$(x)\varphi = t\nu x \quad \text{if} \quad x \in H_a,$$

and

$$(y)\psi = t'\eta y \quad \text{if} \quad y \in H_c.$$

If $a \neq b$ and $b = c$, define $\varphi : H_a \to S$ and $\psi : H_c \to S$ by

$$(x)\varphi = x\alpha s \quad \text{if} \quad x \in H_a,$$

and

$$(y)\psi = y\beta s' \quad \text{if} \quad y \in H_c.$$

If $a \neq b$ and $b \neq c$, define $\varphi : H_a \to S$ and $\psi : H_c \to S$ by

$$(x)\varphi = t\nu x\alpha s \quad \textit{if} \quad x \in H_a,$$

and

$$(y)\psi = t'\eta y\beta s' \quad \textit{if} \quad y \in H_c.$$

We have the following statements hold.

(i) $(H_a)\varphi = H_c$ and $(H_c)\psi = H_a$.

(ii) $\varphi\psi = 1_{H_a}$ and $\psi\varphi = 1_{H_c}$.

(iii) $|H_a| = |H_c|$.

Proof. (i) and (ii). Let $x \in H_a$.

Case 1. $a = b$ and $b = c$. Then $(x)\varphi = x \in H_a = H_b = H_c$.

Case 2. $a = b$ and $t\nu b = c, t'\eta c = b$. Then $(x)\varphi = t\nu x$ and $H_a = H_b$. Since $b\mathcal{L}c$ and $t\nu b = c$, by the dual of Lemma 2.18, $t\nu H_b = H_c$. Thus $(x)\varphi = t\nu x \in t\nu H_a = t\nu H_b = H_c$.

Case 3. $a\alpha s = b, b\beta s' = a$ and $b = c$. Then $(x)\varphi = x\alpha s$. Since $a\mathcal{R}b$ and $a\alpha s = b$, by Lemma 2.18, $H_a\alpha s = H_b$. Thus $(x)\varphi = x\alpha s \in H_a\alpha s = H_b = H_c$.

Case 4. $a\alpha s = b, b\beta s' = a$ and $t\nu b = c, t'\eta c = b$. Then $(x)\varphi = t\nu x\alpha s$. Since $a\mathcal{R}b$ and $a\alpha s = b$, by Lemma 2.18, $H_a\alpha s = H_b$. Since $b\mathcal{L}c$ and $t\nu b = c$, by the dual of Lemma 2.18, $t\nu H_b = H_c$. Thus $(x)\varphi = t\nu x\alpha s \in t\nu H_a\alpha s = t\nu(H_a\alpha s) = t\nu H_b = H_c$.

Hence $(H_a)\varphi \subseteq H_c$. Similarly, $(H_c)\psi \subseteq H_a$.

Next, we show that $\varphi\psi = 1_{H_a}$.

Case 1. $a = b$ and $b = c$. Then $(x)\varphi\psi = x$.

Case 2. $a = b$ and $t\nu b = c, t'\eta c = b$. Since $x \in H_a$, $x \in R_a = R_b$. Since $b\mathcal{L}c$ and $x \in R_b$, by the dual of Lemma 2.16 (ii), we have $t'\eta t\nu x = x$. Thus $(x)\varphi\psi = t'\eta t\nu x = x$.

Case 3. $a\alpha s = b, b\beta s' = a$ and $b = c$. Since $x \in H_a$, $x \in L_a$. Since $a\mathcal{R}b$ and $x \in L_a$, by Lemma 2.16 (ii), we have $x\alpha s\beta s' = x$. Thus $(x)\varphi\psi = x\alpha s\beta s' = x$.

Case 4. $a\alpha s = b, b\beta s' = a$ and $t\nu b = c, t'\eta c = b$. Since $x \in H_a$, $x \in L_a$ and $x \in R_a$. Since $a\mathcal{R}b$, $x \in R_a = R_b$. Since $a\mathcal{R}b$ and $x \in L_a$, by Lemma 2.16 (ii), we have $x\alpha s\beta s' = x$. Since $b\mathcal{L}c$ and $x \in R_b$, by the dual of Lemma 2.16 (ii), we have $t'\eta t\nu x = x$. Thus $(x)\varphi\psi = t'\eta t\nu x\alpha s\beta s' = x$.

Hence $\varphi\psi = 1_{H_a}$. Similarly, $\psi\varphi = 1_{H_c}$.

We have φ maps H_a onto H_c and ψ maps H_c onto H_a . Hence $(H_a)\varphi = H_c$ and $(H_c)\psi = H_a$.

(iii) By (ii), we have φ is one to one. Thus $|H_a| = |(H_a)\varphi|$. By (i), we have $(H_a)\varphi = H_c$, so $|H_a| = |H_c|$. □

The following theorem holds.

Theorem 2.20. *Let a and b be elements in a Γ-semigroup S. Then $a\alpha b \in R_a \cap L_b$ if and only if $(R_b \cap L_a) \cap E_\alpha(S) \neq \emptyset$. Moreover, if $(R_b \cap L_a) \cap E_\alpha(S) \neq \emptyset$, then*

$$a\alpha H_b = H_a\alpha b = H_a\alpha H_b = R_a \cap L_b.$$

Proof. Suppose that $a\alpha b \in R_a \cap L_b$. We have $a\mathcal{R}(a\alpha b)$. Then $a = a\alpha b$ or there exist $c \in S$ and $\beta \in \Gamma$ such that $a = (a\alpha b)\beta c$. If $a = a\alpha b$, we have $L_a = L_{a\alpha b}$ By Lemma 2.16 (ii), we have

$$x\alpha b = x \quad \text{for all } x \in L_a. \tag{2.1}$$

Since $a\alpha b\mathcal{L}b$, $L_{a\alpha b} = L_b$. Thus $L_a = L_{a\alpha b} = L_b$. Since $b \in L_b = L_a$, by (2.1), we have $b\alpha b = b \in E_\alpha(S)$. Thus $b \in (R_b \cap L_a) \cap E_\alpha(S)$. If $a = (a\alpha b)\beta c$, By Lemma 2.16 (i), we have $L_{a\alpha b}\beta c = L_a$. By Lemma 2.16 (ii), we have

$$x\alpha b\beta c = x \quad \text{for all } x \in L_a. \tag{2.2}$$

Since $a\alpha b \in L_b$, $b \in L_{a\alpha b}$. Hence $b\beta c \in L_{a\alpha b}\beta c = L_a$. By (2.2), we have $(b\beta c)\alpha(b\beta c) = b\beta c \in E_\alpha(S)$. Since $b \in L_{a\alpha b}$, by Lemma 2.16 (iii), $(b\beta c)\mathcal{R}b$. We have $b\beta c \in R_b$. Thus $b\beta c \in (R_b \cap L_a) \cap E_\alpha(S)$.

Conversely, Let $e \in R_b \cap L_a \cap E_\alpha(S)$. Since $b \in R_e$, by Theorem

2.6 (ii), we have $e\alpha b = b$. Since $e\mathcal{R}b$ and $e\alpha b = b$, By Lemma 2.16 (i) and (iii), $L_e\alpha b = L_b$ and

$$(x\alpha b)\mathcal{R}x \quad \text{for all } x \in L_e. \tag{2.3}$$

Since $e \in L_a$, $a \in L_e$. By (2.3), we have $a\alpha b \in R_a$. Since $L_e\alpha b = L_b$ and $a \in L_e$, $a\alpha b \in L_b$. Thus $a\alpha b \in R_a \cap L_b$.

Let $x \in H_a$ and $y \in H_b$. Then $L_x = L_a$, $R_x = R_a$, $L_y = L_b$ and $R_y = R_b$. We have that $e \in (R_y \cap L_x) \cap E_\alpha(S)$. By the converse of this theorem, we have $x\alpha y \in R_x \cap L_y = R_a \cap L_b$, that is, $H_a\alpha H_b \subseteq R_a \cap L_b$. Since $a\alpha b \in H_a\alpha H_b \subseteq R_a \cap L_b$, $a\mathcal{R}(a\alpha b)$ and $b\mathcal{L}(a\alpha b)$. Since $a\mathcal{R}(a\alpha b)$, by Lemma 2.18, $H_a\alpha b = H_{a\alpha b}$. Since $b\mathcal{L}(a\alpha b)$, by the dual of Lemma 2.18, $a\alpha H_b = H_{a\alpha b}$. Thus

$$a\alpha H_b \subseteq H_a\alpha H_b \subseteq R_a \cap L_b = R_{a\alpha b} \cap L_{a\alpha b} = H_{a\alpha b} = a\alpha H_b$$

and

$$H_a\alpha b \subseteq H_a\alpha H_b \subseteq R_a \cap L_b = R_{a\alpha b} \cap L_{a\alpha b} = H_{a\alpha b} = H_a\alpha b.$$

Hence $a\alpha H_b = H_a\alpha b = H_a\alpha H_b = H_{a\alpha b} = R_a \cap L_b$. □

The following theorem is similar to Green's Theorem for semigroups.

Theorem 2.21. *(Green's Theorem for Γ-semigroups) Let a be an element in a Γ-semigroup S and $\alpha \in \Gamma$. Then $H_a\alpha H_a \cap H_a = \emptyset$ or $H_a\alpha H_a = H_a$. If $H_a\alpha H_a = H_a$, then H_a is a subsemigroup of S_α.*

Proof. Suppose that $H_a\alpha H_a \cap H_a \neq \emptyset$. There exists $x \in H_a$ such that $x = y\alpha z$ for some $y, z \in H_a$. Thus $x \in L_a \cap R_a$, by Theorem 2.20, we have $H_a\alpha H_a = L_a \cap R_a = H_a$. □

Applying Theorem 2.21, we obtain the following corollary.

Corollary 2.22. *Let e be an $\alpha-$idempotent of a Γ-semigroup S where $\alpha \in \Gamma$. If $H_e\alpha H_e = H_e$, then H_e is a subsemigroup of S_α.*

For this example, we characterize the egg-box of some Γ-semigroups.

Example 2.1. Consider the Γ-semigroup $T(X,Y)$ in Example 1.18. Let $X = \{a,b\}, Y = \{x,y,z\}$ and $\Gamma = \{\theta\}$ where $\theta = \begin{pmatrix} x & y & z \\ a & b & b \end{pmatrix} \in T(Y,X)$.

The Γ-semigroup $(T(X,Y),\theta)$ has three $\mathcal{D}-$classes. The three $\mathcal{D}-$classes D_1, D_2 and D_3 can be enumerated in the egg box fashion as follows:

D_1	$\begin{pmatrix} a & b \\ x & x \end{pmatrix}$	$\begin{pmatrix} a & b \\ y & y \end{pmatrix}$	$\begin{pmatrix} a & b \\ z & z \end{pmatrix}$
D_2	$\begin{pmatrix} a & b \\ x & y \end{pmatrix}$ $\begin{pmatrix} a & b \\ y & x \end{pmatrix}$	$\begin{pmatrix} a & b \\ x & z \end{pmatrix}$ $\begin{pmatrix} a & b \\ z & x \end{pmatrix}$	
D_3	$\begin{pmatrix} a & b \\ y & z \end{pmatrix}$		
	$\begin{pmatrix} a & b \\ z & y \end{pmatrix}$		

We have D_1 and D_2 are regular but D_3 is not regular.

Consider $H = \{\begin{pmatrix} a & b \\ x & y \end{pmatrix} \begin{pmatrix} a & b \\ y & x \end{pmatrix}\}$. We have $H\theta H = H$. Then H is a subsemigroup of $(T(X,Y))_\theta$.

Example 2.2. Consider the Γ-semigroup $T(X,Y)$ in Example 1.18. Let $X = \{a,b\}, Y = \{x,y\}$ and $\Gamma = \{\theta_1,\theta_2\}$ where $\theta_1 = \begin{pmatrix} x & y \\ a & b \end{pmatrix} \in T(Y,X)$ and $\theta_2 = \begin{pmatrix} x & y \\ a & a \end{pmatrix} \in T(Y,X)$.

The Γ-semigroup $(T(X,Y),\theta)$ has two $\mathcal{D}-$classes. The two $\mathcal{D}-$classes D_1, and D_2 can be enumerated in the egg box fashion as follows:

D_1	$\begin{pmatrix} a & b \\ x & x \end{pmatrix}$	$\begin{pmatrix} a & b \\ y & y \end{pmatrix}$
D_2	$\begin{pmatrix} a & b \\ x & y \end{pmatrix}$ $\begin{pmatrix} a & b \\ y & x \end{pmatrix}$	

We have D_1 and D_2 are regular.

Consider $H = \{\begin{pmatrix} a & b \\ x & y \end{pmatrix} \begin{pmatrix} a & b \\ y & x \end{pmatrix}\}$. We have $H\theta_1 H = H$. Then H is a subsemigroup of $(T(X,Y))_{\theta_1}$. However, we have $H\theta_2 H \cap H = \emptyset$.

2.3 Regular $\mathcal{D}$-classes

In the year 1936, the concept of regularity was introduced by Von Neumann in ring theory (Howie, 1972: 44). In this section, we consider some interesting properties in regular $\mathcal{D}$-classes.

Our first theorem, we have the regularity is a property of $\mathcal{D}$-classes rather than of element as the following theorem:

Theorem 2.23. *If a is a regular element of a Γ-semigroup S, then every element of D_a is regular.*

Proof. Since a is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Let $b \in D_a$. So $a\mathcal{D}b$. Then $a\mathcal{L}c$ and $c\mathcal{R}b$ for some $c \in S$. Since $a\mathcal{L}c$, $a = c$ or there exist $u, v \in S$ and $\gamma, \mu \in \Gamma$ such that

$$u\gamma a = c \text{ and } v\mu c = a.$$

Since $c\mathcal{R}b$, $b = c$ or there exist $z, t \in S$ and $\eta, \theta \in \Gamma$ such that

$$c\eta z = b \text{ and } b\theta t = c.$$

Case 1. $a = c$ and $c = b$. Then $a = b$, so b is regular.

Case 2. $a = c$ and $c\eta z = b$ and $b\theta t = c$. Then

$$b\theta(t\alpha x)\beta b = c\alpha x\beta c\eta z = a\alpha x\beta a\eta z = a\eta z = c\eta z = b.$$

Case 3. $u\gamma a = c$ and $v\mu c = a$, and $b = c$. Then

$$b\alpha(x\beta v)\mu b = c\alpha x\beta v\mu b = u\gamma a\alpha x\beta a = u\gamma a = c = b.$$

Case 4. $u\gamma a = c$ and $v\mu c = a$, and $c\eta z = b$ and $b\theta t = c$. Then

$$b\theta(t\alpha x\beta v)\mu b = c\alpha x\beta v\mu c\eta z = u\gamma a\alpha x\beta a\eta z = u\gamma a\eta z = c\eta z = b.$$

Therefore b is a regular element. □

Let D be a $\mathcal{D}$-class. Then either every element of D is regular or no element of D is regular. We call the $\mathcal{D}$-class *regular* if all its elements are regular.

Since idempotents are regular, a $\mathcal{D}$-class containing an idempotent is regular. Conversely, we can show that a regular $\mathcal{D}$-class must contain at least one idempotent as follows:

Theorem 2.24. *In a regular $\mathcal{D}$-class, each $\mathcal{L}$-class and each $\mathcal{R}$-class contains at least one idempotent.*

Proof. Let a be an element of a regular $\mathcal{D}$-class D in a Γ-semigroup S. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Then $x\beta a = x\beta(a\alpha x\beta a) = (x\beta a)\alpha(x\beta a)$. Thus $x\beta a$ is an α-idempotent. Since $a = a\alpha(x\beta a)$, $a\mathcal{L}x\beta a$. Similarly, $a\alpha x$ is a β-idempotent and $a\mathcal{R}a\alpha x$. □

Theorem 2.25. *Let a be an element of a regular $\mathcal{D}$-class D in a Γ-semigroup S. Then*

(i) If a' is an (α, β)-inverse of a, then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$, contain a β-idempotent $a\alpha a'$ and an α-idempotent $a'\beta a$, respectively;

(ii) If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain a β-idempotent e and an α-idempotent f, respectively, then H_b contains an (α, β)-inverse a^* of a such that $a\alpha a^* = e$ and $a^*\beta a = f$;

(iii) No $\mathcal{H}$-class contains more than one (α, β)-inverse of a for all ordered pair $(\alpha, \beta) \in \Gamma \times \Gamma$.

Proof. (i) Let a' be an (α, β)-inverse of a. Then $a = a\alpha a'\beta a$ and $a' = a'\beta a\alpha a'$. By Theorem 2.8 (i), we have

$$a \;\mathcal{L}\; a'\beta a, \qquad a\alpha a' \;\mathcal{R}\; a, \qquad a' \;\mathcal{L}\; a\alpha a', \qquad a'\beta a \;\mathcal{R}\; a'.$$

Thus $a'\mathcal{D}a$, from which it follows that $a' \in D$. By Theorem 2.8 (ii), we have $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$ contain a β-idempotent $a\alpha a'$ and an α-idempotent $a'\beta a$, respectively.

(ii) Since $a\mathcal{R}e$, by Theorem 2.6 (ii), $e\beta a = a$. Similarly, from $a\mathcal{L}f$ we deduce that $a\alpha f = a$ by Theorem 2.6 (i). Again from $a\mathcal{R}e$ it follows that $a = e$ or there exist $x \in S$ and $\gamma \in \Gamma$ such that $a\gamma x = e$.

Case 1 . $a = e$. Let $a^* = f\beta e$. Then

$$a\alpha a^*\beta a = a\alpha(f\beta e)\beta a = (a\alpha f)\beta(e\beta a) = a\beta a = e\beta a = a$$

and

$$a^*\beta a\alpha a^* = (f\beta e)\beta a\alpha(f\beta e) = f\beta(e\beta a)\alpha f\beta e = f\beta(a\alpha f)\beta e = f\beta(a\beta e) = f\beta e = a^*.$$

Then a^* is an (α, β)-inverse of a. Moreover

$$a\alpha a^* = a\alpha f\beta e = a\beta e = e\beta e = e.$$

Further, since $a\mathcal{L}f$, $a = f$ or $f = y\theta a$ for some $y \in S$ and $\theta \in \Gamma$. If $a = f$, then $a^*\beta a = f\beta e\beta a = e\beta e\beta e = e = f$. If $f = y\theta a$, then $a^*\beta a = f\beta e\beta a = y\theta a\beta e\beta e = y\theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

Case 2 . $a\gamma x = e$. Let $a^* = f\gamma x\beta e$. Then

$$a\alpha a^*\beta a = a\alpha(f\gamma x\beta e)\beta a = (a\alpha f)\gamma x\beta(e\beta a) = a\gamma x\beta a = e\beta a = a$$

and

$a^*\beta a\alpha a^* = (f\gamma x\beta e)\beta a\alpha(f\gamma x\beta e) = f\gamma x\beta(e\beta a)\alpha f\gamma x\beta e = f\gamma x\beta(a\alpha f)\gamma x\beta e = f\gamma x\beta(a\gamma x)\beta e = f\gamma x\beta e\beta e = f\gamma x\beta e = a^*$.

Then a^* is an (α, β)-inverse of a. Moreover

$$a\alpha a^* = a\alpha f\gamma x\beta e = a\gamma x\beta e = e\beta e = e.$$

Since $a\mathcal{L}f$, $a = f$ or there exist $y \in S$ and $\theta \in \Gamma$ such that $f = y\theta a$. If $a = f$, then $a^*\beta a = f\gamma x\beta e\beta a = a\gamma x\beta e\beta a = e\beta e\beta a = e\beta a = a = f$. If $f = y\theta a$, then $a^*\beta a = f\gamma x\beta e\beta a = y\theta(a\gamma x)\beta e\beta a = y\theta(e\beta e)\beta a = y\theta(e\beta a) = y\theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

(iii) Suppose that a' and a^* are both (α, β)-inverses of a inside the single $\mathcal{H}$-class H_b. Since $a\alpha a'$ and $a\alpha a^*$ are β-idempotents in the $\mathcal{H}$-class $R_a \cap L_b$, $a\alpha a' = a\alpha a^*$ by Theorem 2.6(iv). Similarly, $a'\beta a = a^*\beta a$ because both are α-idempotents in the $\mathcal{H}$-class $L_a \cap R_b$. Then $a' = a'\beta a\alpha a' = a^*\beta a\alpha a^* = a^*$. □

2.4 Ideals of Γ-semigroups and simple Γ-semigroups

In this section, we give characterizations for ideals and simple Γ-semigroups. Moreover, we consider some connections between Green's relations, ideals and simple Γ-semigroups.

Theorem 2.26. *Let S be a Γ-semigroup and A a nonempty subset of S. The following statements hold.*

(i) $S\Gamma A$ is a left ideal of S.

(ii) $A\Gamma S$ is a right ideal of S.

(iii) $S\Gamma A\Gamma S$ is an ideal of S.

Proof. (i) Let $x \in S, \gamma \in \Gamma$ and $y \in S\Gamma A$. Then $y = z\alpha a$ for some $z \in S, \alpha \in \Gamma$ and $a \in A$. We have $x\gamma y = x\gamma(z\alpha a) = (x\gamma z)\alpha a \in S\Gamma A$. It follows that $S\Gamma(S\Gamma A) \subseteq S\Gamma A$. Hence $S\Gamma A$ is a left ideal of S.

(ii) It is similar to (i).

(iii) Let $x \in S, \gamma \in \Gamma$ and $y \in S\Gamma A\Gamma S$. Then $y = w\alpha a\beta z$ for some $w, z \in S, \alpha, \beta \in \Gamma$ and $a \in A$. We have $x\gamma y = x\gamma(w\alpha a\beta z) = (x\gamma w)\alpha a\beta z \in S\Gamma A\Gamma S$ and $y\gamma x = (w\alpha a\beta z)\gamma x = w\alpha a\beta(z\gamma x) \in S\Gamma A\Gamma S$. It follows that $S\Gamma(S\Gamma A\Gamma S) \subseteq S\Gamma A\Gamma S$ and $(S\Gamma A\Gamma S)\Gamma S \subseteq S\Gamma A\Gamma S$. Therefore $S\Gamma A\Gamma S$ is an ideal of S. □

Theorem 2.27. *Let S be a Γ-semigroup. The following statements hold.*

(i) A Γ-semigroup S is left simple if and only if $S\Gamma x = S$ for all $x \in S$.

(ii) A Γ-semigroup S is right simple if and only if $x\Gamma S = S$ for all $x \in S$.

(iii) A Γ-semigroup S is simple if and only if $S\Gamma x\Gamma S = S$ for all $x \in S$.

Proof. (i) Assume that S is a left simple Γ-semigroup. Let $x \in S$. By Theorem 2.26 (i), we have $S\Gamma x$ is a left ideal of S. Since S is left simple, $S\Gamma x = S$.

Conversely, assume that $S\Gamma x = S$ for all $x \in S$. Let L be any left ideal of S. Then $L \subseteq S$. Let $y \in L$. We have $S = S\Gamma y \subseteq S\Gamma L \subseteq L$. Hence $L = S$. Therefore S is a left simple Γ-semigroup.

(ii) It is similar to (i).

(iii) It is similar to (i). $\square$

Theorem 2.28. *Let S be a Γ-semigroup. The following statements hold.*

(i) S is a left simple Γ-semigroup if and only if $\mathcal{L} = S \times S$.

(ii) S is a right simple Γ-semigroup if and only if $\mathcal{R} = S \times S$.

Proof. (i) Assume that S is left simple. Let $a, b \in S$. Similar to Theorem 2.26 (i), we can show that $S^1\Gamma a$ and $S^1\Gamma b$ are left ideals of S. Thus $S^1\Gamma a = S^1\Gamma b$ since S is left simple. Hence $a\mathcal{L}b$. It follows that $\mathcal{L} = S \times S$.

Conversely, assume that $\mathcal{L} = S \times S$. Let A be any left ideal of S. Obviously, $A \subseteq S$. Let $x \in S$ and $a \in A$. Then $(x, a) \in S \times S = \mathcal{L}$. Thus $x\mathcal{L}a$. It follows that $S^1\Gamma x = S^1\Gamma a$. We have $x \in S^1\Gamma x = S^1\Gamma a \subseteq A$ since $a \in A$. Hence $S = A$. Therefore S is a left simple Γ-semigroup.

(ii) It is similar to (i). $\square$

Theorem 2.29. *If S is a regular Γ-semigroup such that every left and right ideal of S is an ideal of S, then $\mathcal{L} = \mathcal{R}$.*

Proof. Let $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. We have $S\Gamma a = S\Gamma a\alpha x\beta a = S\Gamma a\alpha(x\beta a) \subseteq S\Gamma a\Gamma S$ and $a\Gamma S = a\alpha x\beta a\Gamma S = (a\alpha x)\beta a\Gamma S \subseteq S\Gamma a\Gamma S$. Since $S\Gamma a$ and $a\Gamma S$ are left ideal and right ideal of S, respectively, $S\Gamma a$ and $a\Gamma S$ are ideals of S. Thus $S\Gamma a\Gamma S \subseteq S\Gamma a$ and $S\Gamma a\Gamma S \subseteq a\Gamma S$. Hence $S\Gamma a = S\Gamma a\Gamma S = a\Gamma S$. Since a is a regular element of S, by Lemma 2.5,

$S^1\Gamma a = S\Gamma a$ and $a\Gamma S^1 = a\Gamma S$. Therefore $S^1\Gamma a = a\Gamma S^1$. We can conclude that $S^1\Gamma x = x\Gamma S^1$ for all $x \in S$. By definition of $\mathcal{L}$ and $\mathcal{R}$, we have $\mathcal{L} = \mathcal{R}$. $\square$

CHAPTER 3

Congruences for Γ-semigroups

Congruences have been widely studied in semigroup theory. They have played an important role in the concept of reductive semigroups, introduced by G. Thierrin in the year 1955. Subsequently, regular reductive semigroups were studied by A. Fattahi and H. R. E. Vishki (2004).

In this chapter, we recall from Example 1.13 and Example 1.14 that $\mathbb{Z}$ under $\Gamma = \{n|\ n \in \mathbb{N}\}$ with the usual addition and multiplication are Γ-semigroups, respectively. We separate into three sections. In the first section, we introduce the notion of congruences for Γ-semigroups. Next, we give a characterization for quotient Γ-semigroups and also we present some of its properties. In the last section, reductive Γ-semigroups are considered.

3.1 Congruences for Γ-semigroups

Let S be a Γ-semigroup. An equivalence relation ρ on S is called a *right congruence* on S if

$$(a, b) \in \rho \Rightarrow (a\gamma t, b\gamma t) \in \rho \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma,$$

and a *left congruence* on S if

$$(a, b) \in \rho \Rightarrow (t\gamma a, t\gamma b) \in \rho \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma.$$

An equivalence relation ρ on S is called a *congruence* on S if it is both a right and left congruence on S.

We give some examples of congruences for Γ-semigroups.

Example 3.1. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.13. Let ρ be an equivalence relation on a Γ-semigroup $\mathbb{Z}$ defined by

$$a\rho b \Leftrightarrow 4|\ a-b \quad \text{for all } a,b\in\mathbb{Z}.$$

We have that ρ is a right congruence on $\mathbb{Z}$ since for $a,b,t\in\mathbb{Z}$ and $\gamma\in\Gamma$,

$$\begin{aligned}
(a,b)\in\rho &\Rightarrow 4|\ a-b\\
&\Rightarrow 4x=a-b \quad \text{for some } x\in\mathbb{Z}\\
&\Rightarrow 4x=(a+\gamma+t)-(b+\gamma+t)\\
&\Rightarrow 4|\ (a+\gamma+t)-(b+\gamma+t)\\
&\Rightarrow (a+\gamma+t,b+\gamma+t)\in\rho\\
&\Rightarrow (a\gamma t,b\gamma t)\in\rho.
\end{aligned}$$

A similar argument shows that ρ is a left congruence on $\mathbb{Z}$. Hence ρ is a congruence on $\mathbb{Z}$.

Example 3.2. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.14. Let ρ be an equivalence relation on a Γ-semigroup $\mathbb{Z}$ defined by

$$a\rho b \Leftrightarrow 4|\ a-b \quad \text{for all } a,b\in\mathbb{Z}.$$

We have that ρ is a right congruence on $\mathbb{Z}$ since for $a,b,t\in\mathbb{Z}$ and $\gamma\in\Gamma$,

$$\begin{aligned}
(a,b)\in\rho &\Rightarrow 4|\ a-b\\
&\Rightarrow 4x=a-b \quad \text{for some } x\in\mathbb{Z}\\
&\Rightarrow 4x(\gamma t)=(a-b)(\gamma t)\\
&\Rightarrow 4(x\gamma t)=(a\gamma t-b\gamma t) \quad \text{because } x\gamma t\in\mathbb{Z}\\
&\Rightarrow 4|\ (a\gamma t-b\gamma t)\\
&\Rightarrow (a\gamma t,b\gamma t)\in\rho.
\end{aligned}$$

A similar argument shows that ρ is a left congruence on $\mathbb{Z}$. Hence ρ is a congruence on $\mathbb{Z}$.

Now we are ready to show that the relation $\mathcal{L}$ is a right congruence on S.

Let $a, b, t \in S$ and $\gamma \in \Gamma$. Assume that $(a, b) \in \mathcal{L}$. We have $S^1\Gamma a = S^1\Gamma b$. Then $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$. If $a = b$, then $a\gamma t = b\gamma t$. We have $(a\gamma t, b\gamma t) \in \mathcal{L}$ since $\mathcal{L}$ is reflexive. If $a = x\alpha b$ and $b = y\beta a$, we have

$$\begin{aligned}
a = x\alpha b \text{ and } b = y\beta a &\Rightarrow a\gamma t = (x\alpha b)\gamma t \text{ and } b\gamma t = (y\beta a)\gamma t \\
&\Rightarrow a\gamma t = x\alpha(b\gamma t) \text{ and } b\gamma t = y\beta(a\gamma t) \\
&\Rightarrow a\gamma t \in S\Gamma b\gamma t \text{ and } b\gamma t \in S\Gamma a\gamma t \\
&\Rightarrow S\Gamma a\gamma t \subseteq S\Gamma b\gamma t \text{ and } S\Gamma b\gamma t \subseteq S\Gamma a\gamma t \\
&\Rightarrow S\Gamma a\gamma t = S\Gamma b\gamma t.
\end{aligned}$$

Since $a\gamma t = (x\alpha b)\gamma t = (x\alpha y)\beta a\gamma t \in S\Gamma a\gamma t$, $S^1\Gamma a\gamma t = S\Gamma a\gamma t$. Similarly, $S^1\Gamma b\gamma t = S\Gamma b\gamma t$. Thus $S^1\Gamma a\gamma t = S^1\Gamma b\gamma t$. Hence $(a\gamma t, b\gamma t) \in \mathcal{L}$.

Therefore $\mathcal{L}$ is a right congruence on S.

A similarly argument shows that the relation $\mathcal{R}$ is a left congruence on S.

Theorem 3.1. *Let S be a Γ-semigroup and ρ an equivalence relation on S. Then*

$$\rho \textit{ is a congruence on } S \textit{ if and only if}$$
$$a\rho b \textit{ and } c\rho d \Leftrightarrow (a\gamma c)\rho(b\gamma d) \textit{ for all } a, b, c, d \in S \textit{ and } \gamma \in \Gamma.$$

Proof. Assume that ρ is a congruences on S. Let $a, b, c, d \in S$ such that $a\rho b$, $c\rho d$ and $\gamma \in \Gamma$. Since ρ is a right congruence on S and $a\rho b$, $(a\gamma c)\rho(b\gamma c)$. Since ρ is a left congruence on S and $c\rho d$, $(b\gamma c)\rho(b\gamma d)$. We have $(a\gamma c)\rho(b\gamma d)$ since ρ is transitive.

Conversely, assume that $a\rho b$ and $c\rho d \Leftrightarrow (a\gamma c)\rho(b\gamma d)$ for all $a, b, c, d \in S$ and $\gamma \in \Gamma$. Let $x, y, z \in S$ such that $x\rho y$ and $\alpha \in \Gamma$. Since ρ is reflexive, $z\rho z$. By assumption, we have $(x\alpha z)\rho(y\alpha z)$ and $(z\alpha x)\rho(z\alpha y)$. Thus ρ is a congruence on S. □

3.2 Quotient Γ-semigroups

Let S be a Γ-semigroup and ρ a congruence on S. For $a\rho, b\rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a\rho)\gamma(b\rho) = (a\gamma b)\rho$. This is well-defined, since for all $a, a', b, b' \in S$ and $\gamma \in \Gamma$,

$$\begin{aligned} a\rho = a'\rho \text{ and } b\rho = b'\rho &\Rightarrow (a, a'), (b, b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b), (a'\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b)\rho = (a'\gamma b')\rho. \end{aligned}$$

Let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. We have

$$(a\rho\gamma b\rho)\mu c\rho = ((a\gamma b)\rho)\mu c\rho = ((a\gamma b)\mu c)\rho = (a\gamma(b\mu c))\rho = a\rho\gamma(b\mu c)\rho = a\rho\gamma(b\rho\mu c\rho).$$

Then the quotient set S/ρ is a Γ-semigroup. The Γ-semigroup S/ρ is called a *quotient* Γ*-semigroup* of S by ρ.

Theorem 3.2. *Let S be a Γ-semigroup and ρ a congruence on S. Then*

(i) If $\rho \subseteq \mathcal{L}$, then for all $a, b \in S$, $a\mathcal{L}b$ if and only if $a\rho\ \mathcal{L}\ b\rho$ in S/ρ;

(ii) If $\rho \subseteq \mathcal{R}$, then for all $a, b \in S$, $a\mathcal{R}b$ if and only if $a\rho\ \mathcal{R}\ b\rho$ in S/ρ;

(iii) If $\rho \subseteq \mathcal{H}$, then for all $a, b \in S$, $a\mathcal{H}b$ if and only if $a\rho\ \mathcal{H}\ b\rho$ in S/ρ.

Proof. (i) Let $a, b \in S$ such that $a\mathcal{L}b$. Then $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.

Case 1. $a = b$. Then $a\rho = b\rho$.

Case 2. $a = x\alpha b$ and $b = y\beta a$. Then $a\rho = (x\alpha b)\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\beta a)\rho = (y\rho)\beta(a\rho)$. Therefore $a\rho\mathcal{L}b\rho$.

Conversely, let $a, b \in S$. Assume that $a\rho\mathcal{L}b\rho$. Then $a\rho = b\rho$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\rho)\beta(a\rho)$.

Case 1. $a\rho = b\rho$. Then $(a, b) \in \rho$. Since $\rho \subseteq \mathcal{L}$, $(a, b) \in \mathcal{L}$. So $a\mathcal{L}b$.

Case 2. $a\rho = (x\rho)\alpha(b\rho)$ and $b\rho = (y\rho)\beta(a\rho)$. Then $a\rho = (x\alpha b)\rho$ and $b\rho = (y\beta a)\rho$. Then $(a, x\alpha b) \in \rho$ and $(b, y\beta a) \in \rho$. Since $\rho \subseteq \mathcal{L}, (a, x\alpha b) \in \mathcal{L}$ and $(b, y\beta a) \in \mathcal{L}$. Then $a \in S^1\Gamma(x\alpha b)$ and $b \in S^1\Gamma(y\beta a)$. Thus $S^1\Gamma a = S^1\Gamma b$. Hence $a\mathcal{L}b$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii). $\square$

3.3 Reductive Γ-semigroups

In the year 1955, the notion of reductive semigroups was introduced by G. Thierrin. Subsequently, A. Fattahi and H. R. E. Vishki (2004) have given a characterization for regular reductive semigroups.

In this section, we introduced the notion of reductive Γ-semigroups and also present some connections between Green's relations and reductive Γ-semigroups.

Let S be a Γ-semigroup. A congruence ρ on S is called *right reductive* on S if

$$(a\gamma t, b\gamma t) \in \rho \Rightarrow (a, b) \in \rho \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma,$$

and *left reductive* on S if

$$(t\gamma a, t\gamma b) \in \rho \Rightarrow (a, b) \in \rho \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma.$$

A congruence ρ on S is called *reductive* on S if it is both a right and left reductive on S.

We give some examples of reductive congruences.

Example 3.3. Consider the congruence ρ in Example 3.1. We have ρ is right reductive on $\mathbb{Z}$ since for $a, b, t \in \mathbb{Z}$ and $\gamma \in \Gamma$,

$$\begin{aligned}
(a\gamma t, b\gamma t) \in \rho &\Rightarrow (a+\gamma+t, b+\gamma+t) \in \rho \\
&\Rightarrow 4|\ (a+\gamma+t)-(b+\gamma+t) \\
&\Rightarrow 4x = (a+\gamma+t)-(b+\gamma+t) \quad \text{for some } x \in \mathbb{Z} \\
&\Rightarrow 4x = a-b \\
&\Rightarrow 4|\ a-b \\
&\Rightarrow (a,b) \in \rho.
\end{aligned}$$

A similar argument shows that ρ is left reductive on $\mathbb{Z}$. Hence ρ is reductive on $\mathbb{Z}$.

Example 3.4. Consider the congruence ρ in Example 3.2. We have ρ is not right and left reductive on $\mathbb{Z}$.

A Γ-semigroup S is called *right reductive* if

$$a\gamma t = b\gamma t \Rightarrow a = b \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma,$$

and *left reductive* if

$$t\gamma a = t\gamma b \Rightarrow a = b \quad \text{for all } a, b, t \in S \text{ and } \gamma \in \Gamma.$$

A Γ-semigroup is called *reductive* if it is both right and left reductive.

We give some examples of reductive Γ-semigroups.

Example 3.5. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.13. We have $\mathbb{Z}$ is a reductive Γ-semigroup since for $a, b, t \in \mathbb{Z}$ and $\gamma \in \Gamma$,

$$a+\gamma+t = b+\gamma+t \Rightarrow a = b,$$

and

$$t+\gamma+a = t+\gamma+b \Rightarrow a = b.$$

Example 3.6. Consider the Γ-semigroup $\mathbb{Z}$ in Example 1.14. We have $\mathbb{Z}$ is not a right and a left reductive Γ-semigroup.

Theorem 3.3. *Let S be a Γ-semigroup and ρ a congruence on S. The following statements are true.*

(i) ρ is a right reductive congruence on S if and only if S/ρ is a right reductive Γ-semigroup.

(ii) ρ is a left reductive congruence on S if and only if S/ρ is a left reductive Γ-semigroup.

(iii) ρ is a reductive congruence on S if and only if S/ρ is a reductive Γ-semigroup.

Proof. (i) Let ρ be a right reductive congruence on S. Let $a\rho, b\rho \in S/\rho$ such that $(a\rho)\gamma(t\rho) = (b\rho)\gamma(t\rho)$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a\gamma t, b\gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Since ρ is right reductive, $(a, b) \in \rho$. Hence $a\rho = b\rho$.

Conversely, suppose that S/ρ is a right reductive Γ-semigroup. Let $a, b \in S$ such that $(a\gamma t, b\gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a\gamma t)\rho = (b\gamma t)\rho$ for all $t \in S$ and $\gamma \in \Gamma$. Thus $(a\rho)\gamma(t\rho) = (b\rho)\gamma(t\rho)$ for all $t \in S$ and $\gamma \in \Gamma$. Since S/ρ is a right reductive Γ-semigroup, $a\rho = b\rho$. Therefore $(a, b) \in \rho$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii). $\square$

Proposition 3.4. *Define the equivalence relations ρ_r and ρ_l on a Γ-semigroup S as follows:*

$$\rho_r = \{(a, b) \in S \times S \mid a\gamma t = b\gamma t \textit{ for all } t \in S \textit{ and } \gamma \in \Gamma\};$$

$$\rho_l = \{(a, b) \in S \times S \mid t\gamma a = t\gamma b \textit{ for all } t \in S \textit{ and } \gamma \in \Gamma\}.$$

Then ρ_r and ρ_l are congruences on S.

Proof. Let $a, b \in S$ such that $(a, b) \in \rho_r$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since ρ_r is reflexive, $(a\gamma t, b\gamma t) \in \rho_r$. Thus ρ_r is a right congruence on S.

Next, we show that ρ_r is a left congruence on S. Let $a, b \in S$ such that $(a, b) \in \rho_r$. Then $a\beta c = b\beta c$ for all $c \in S$ and $\beta \in \Gamma$. We have

$t\gamma(a\beta c) = t\gamma(b\beta c)$ for all $c, t \in S$ and $\gamma \in \Gamma$. It follows that $(t\gamma a)\beta c = (t\gamma b)\beta c$ for all $c \in S$ and $\beta \in \Gamma$. Thus $(t\gamma a, t\gamma b) \in \rho_r$. Hence ρ_r is a left congruence on S.

A similar argument shows that ρ_l is a left congruence on S. $\square$

The three following theorems hold.

Theorem 3.5. *Let S be a Γ-semigroup. Then*

(i) S is a right reductive Γ-semigroup if and only if $\rho_r = 1_S$;

(ii) S is a left reductive Γ-semigroup if and only if $\rho_l = 1_S$.

Proof. (i) Assume S is a right reductive Γ-semigroup. Let $a, b \in S$ such that $a\rho_r b$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since S is right reductive, $a = b$.

Conversely, suppose $\rho_r = 1_S$. Let $a, b \in S$ such that $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a, b) \in \rho_r$. Since $\rho_r = 1_S, a = b$. Hence S is a right reductive Γ-semigroup.

(ii) It is similar to (i). $\square$

Theorem 3.6. *Let S be a regular Γ-semigroup. Then*

(i) $\rho_r \subseteq \mathcal{R}$;

(ii) $\rho_l \subseteq \mathcal{L}$.

Proof. (i) Let $(a, b) \in \rho_r$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. So $a\Gamma S = b\Gamma S$. Since $a \in a\Gamma S$ and $b \in b\Gamma S$ because S is regular, $a\Gamma S^1 = b\Gamma S^1$. Therefore $(a, b) \in \mathcal{R}$. Thus $\rho_r \subseteq \mathcal{R}$.

(ii) It is similar to (i). $\square$

If A is a set of all right (resp. left) reductive congruence on a Γ-semigroup S. A congruence ρ on S is called the *minimum right (resp. left) reductive* if $\rho \subseteq \rho'$ for all $\rho' \in A$.

Theorem 3.7. *Let S be a regular Γ-semigroup. Then*

(i) ρ_r is the minimum right reductive congruence on S;

(ii) ρ_l is the minimum left reductive congruence on S.

Proof. (i) Let $a, b \in S$. Assume that $(a\gamma t, b\gamma t) \in \rho_r$ for all $t \in S$ and $\gamma \in \Gamma$. Then $a\gamma t\beta t' = b\gamma t\beta t'$ for all $t, t' \in S$ and $\gamma, \beta \in \Gamma$. Thus $a\alpha t'' = b\alpha t''$ for all $t'' \in S$ and $\alpha \in \Gamma$ because S is regular. So $(a, b) \in \rho_r$. Therefore ρ_r is a right reductive congruence on S.

Next, let ρ be any right reductive congruence on S. Let $(a, b) \in \rho_r$. Then $a\gamma t = b\gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since ρ is reflexive, $(a\gamma t, b\gamma t) \in \rho$. Therefore $(a, b) \in \rho$ because ρ is right reductive.

(ii) It is similar to (i). $\square$

REFERENCES

Fattahi, A., and Vishki H. R. E. 2004. A note on reductive semigroups. Southeast Asian Bulletin of Mathematics. 28: 261-264.

Green, J. A. 1951. On the structure of semigroups. Annuals of Mathematics. 54: 163-172.

Howie, J. M. 1976. An introduction to semigroup theory. Academic Press: London.

Saha, N. K. 1987. On Γ-semigroup II. Bulletin of Calcutta Mathematical Society. 79: 331-335.

Sen, M. K., and Saha, N. K. 1986. On Γ-semigroup I. Bulletin of Calcutta Mathematical Society. 78: 180-186.

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