Invariant Subspace Method for Fractional Telegraph Equations

Somavatey Meas

A Thesis Submitted in Partial Fulfillment of the Requirements for the
Degree of Master of Science in Mathematics
Prince of Songkla University
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This is to certify that the work here submitted is the result of the candidate's own investigations. Due acknowledgement has been made of any assistance received.

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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ABSTRACT

In this thesis, we use the invariant subspace method to find the solutions of three classes of fractional telegraph equations, i.e., space-, time-, and space and time-fractional telegraph equations, in which fractional derivatives are considered in the Caputo sense. In this method, we first classify all possible invariant subspaces with respect to the differential operator. By assuming the solution to be a linear combination of functions in the appropriate invariant subspace, the fractional telegraph equation is reduced to a system of fractional ordinary differential equations. Finally, solving the system of fractional ordinary differential equations yields the solution of fractional telegraph equation.

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Miss Somavatey Meas

Contents

Chapter 1

Introduction

1.1 Background and significance

Diffusion is one of the most ubiquitous phenomena that has been observed in many branches of science and engineering. In general, most diffusion processes are studied under the assumption that the diffusion is normal-the mean square displacement of a randomly walking particle grows linearly with time. In addition, in the process, the particle can wait between successive jumps and the jump size distribution must have finite moments. However, in some cases, these conditions are not met, for example, anomalous diffusion, which is characterized by power laws with exponents not equal to one [1, 2]. Mathematically, anomalous diffusion is usually described by fractional partial differential equations, in which the integer order derivatives are replaced by fractional order derivatives in time.

The telegraph equation is a simple example of a diffusion-like process, which has characteristics of both wave motion and diffusion. For this reason, it has been used to describe in various fields of applied science and engineering, for instance, the diffusion of light in turbid [3, 4], distribution of organisms [5], population dynamics [6] and hyperbolic heat transfer [7, 8].

In the previous works, Momani [9] used the Adomian decomposition method to derive the analytical and approximate solutions of the space- and time-fractional telegraph equations. By using the separation of variables method, Chen et al. [10] solved the time-fractional telegraph equation with certain non-homogeneous boundary conditions. Srivastava et al. [11] used the reduced differential transformation method to find the approximate solutions of the time-fractional telegraph equations. Kumar [12] derived the analytical and approximate solutions of the space-fractional telegraph equation by using the homotopy analysis and Laplace transform methods. Das et al. [13] obtained the approximate solutions of time-fractional telegraph equation by applying the homotopy analysis method.

The invariant subspace method was initially proposed by Galaktinov and Svirshchevskii [14] for solving non-linear partial differential equations. Later on, it was extended by many authors [15, 16, 17, 18] to construct exact solutions for fractional partial differential equations.

In this thesis, we apply the invariant subspace method to find exact solutions of three classes of fractional telegraph equations as follows

1. The space-fractional telegraph equation of the form

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu + f(x,t), \quad x > 0, t > 0,$$

   where a, b are constants, f is a function of x and t, and $0 < \alpha \leq 1$ is the order of the space-fractional derivative.

2. The time-fractional telegraph equation of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\beta}u}{\partial t^{2\beta}} + a\frac{\partial^{\beta}u}{\partial t^{\beta}} + bu + f(x,t),$$

   where $0 < \beta \leq 1$ is the order of the time-fractional derivative.

3. The space and time-fractional telegraph equation of the form

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^{2\beta}u}{\partial t^{2\beta}} + a\frac{\partial^{\beta}u}{\partial t^{\beta}} + bu + f(x,t),$$

   where $0 < \alpha \leq 1,\ 0 < \beta \leq 1$ are the order of the space- and the time-fractional derivatives, respectively.

In order to solve these problems by using the invariant subspace method, first, we choose the differential operator and classify all possible invariant subspaces. By choosing an appropriate invariant subspace, the solution can be assumed as a linear combination of the elements in it. Then the fractional telegraph equation can be reduced to a system of fractional ordinary differential equations. Finally, solving this system by using the Laplace transform method, we obtain the solution of fractional telegraph equation.

1.2 Objective of study

The objective of this thesis is to show how the invariant subspace method provides exact solutions for space-, time-, and space and time-fractional telegraph equations.

1.3 Expected advantage of this study

We will apply the invariant subspace method to find exact solutions of three classes of fractional telegraph equations, i.e., space-, time-, and space and time-fractional telegraph equations.

Chapter 2

Preliminaries

In this chapter, we introduce some basic definitions of fractional integrals and derivatives and some useful properties.

2.1 Definitions and Properties

Definition 2.1.1. Suppose that α and t are positive real numbers. The Riemann-Liouville fractional integral is defined by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx,$$

where

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt,$$

is the Gamma function.

Definition 2.1.2. Riemann-Liouville fractional derivative of order $\alpha > 0$ of the function f is defined by

$$D^{\alpha} f(t) = \begin{cases} \frac{d^n}{dt^n} J^{n-\alpha} f(t), & n-1 < \alpha < n, \;\; n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases}$$

Definition 2.1.3. Caputo fractional derivative of order $\alpha > 0$ of the function f is defined by

$$D_*^{\alpha} f(t) = \begin{cases} J^{n-\alpha} \frac{d^n}{dt^n} f(t), & n-1 < \alpha < n, \;\; n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases}$$

Example 2.1.4. Let $n = 1$, $0 < \alpha < 1$, $c \in \mathbb{R}$.

(1) The Caputo fractional derivative of constant is

$$D_*^\alpha c = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{dc}{dx}}{(t-x)^{\alpha+1-1}} dx = 0.$$

(2) The Riemann-Liouville fractional derivative of constant is

$$\begin{aligned} D^\alpha c &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{c}{(t-x)^\alpha} dx \\ &= -\frac{c}{\Gamma(1-\alpha)} \frac{d}{dt} \left[-\frac{t^{1-\alpha}}{1-\alpha} \right] \\ &= \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}. \end{aligned}$$

Example 2.1.5. Let $n = 1$, $0 < \alpha < 1$, $f(t) = t$.

(1) The Caputo fractional derivative of the function $f(t) = t$ is

$$\begin{aligned} D_*^\alpha t &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{dx}{dx}}{(t-x)^\alpha} dx \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-x)^\alpha} dx \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} t^{1-\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}. \end{aligned}$$

(2) The Riemann-Liouville fractional derivative of the function $f(t) = t$ is

$$D^\alpha t = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \underbrace{\int_0^t \frac{x}{(t-x)^\alpha} dx}_{A}$$

Let $u = x \Rightarrow du = dx$, $dv = \dfrac{1}{(t-x)^\alpha} dx \Rightarrow v = -\dfrac{1}{1-\alpha}(t-x)^{1-\alpha}$

$$A = -\frac{x}{1-\alpha}(t-x)^{1-\alpha} \Big|_0^t + \frac{1}{1-\alpha} \int_0^t (t-x)^{1-\alpha} dx = \frac{t^{2-\alpha}}{(1-\alpha)(2-\alpha)}.$$

We obtain

$$D^\alpha t = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\frac{t^{2-\alpha}}{(1-\alpha)(2-\alpha)} \right] = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}.$$

In general case, the Riemann-Liouville and Caputo fractional derivative of the power function can be shown as follows:

Proposition 2.1.6. *The Riemann-Liouville fractional derivative of power function satisfies*

$$D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \;\; n-1<\alpha<n, \;\; \beta>-1, \;\; \beta\in\mathbb{R}.$$

Proposition 2.1.7. *The Caputo fractional derivative of the power function satisfies*

$$D_*^{\alpha}t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & n-1<\alpha<n, \;\; \beta>n-1, \;\; \beta\in\mathbb{R} \\ 0, & n-1<\alpha<n, \;\; \beta\leq n-1, \;\; \beta\in\mathbb{N}. \end{cases}$$

Definition 2.1.8. [18, 19] Two-parameter function of Mittag-Leffler type is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \;\; \alpha,\beta,z\in\mathbb{C}, \;\; Re(\alpha)>0, Re(\beta)>0, \tag{2.1}$$

e.g.

- $E_{1,1}(z) = e^z$
- $E_{1,2}(z) = \dfrac{e^z-1}{z}$
- $E_{2,1}(z^2) = \cosh(z)$
- $E_{2,2}(z^2) = \dfrac{\sinh(z)}{z}$
- $E_{2,1}(-z^2) = \cos(z)$
- $E_{2,2}(-z^2) = \dfrac{\sin z}{z}$
- $z^2E_{2,3}(z^2) = E_{2,1}(z^2) - 1$
- $(-2z)E_{1,2}(-2z) = E_{1,1}(-2z) - 1.$

The n-th order derivative of $E_{\alpha,\beta}(z)$ is given by

$$E_{\alpha,\beta}^{(n)}(z) = \frac{d^n}{dz^n}E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{(k+n)!z^k}{k!\Gamma(\alpha k+\alpha n+\beta)}, \;\; n=0,1,2,\cdots. \tag{2.2}$$

Derivative of Mittag-Leffler function is given by

$$\frac{d^{\alpha}}{dz^{\alpha}}\Big[E_{\alpha,1}(az^{\alpha})\Big] = aE_{\alpha,1}(az^{\alpha}), \;\; Re(\alpha)>0, a\in\mathbb{R}.$$

Proposition 2.1.9. *Let $n-1<\alpha\leq n, n\in\mathbb{N}$. The Laplace transform of the Caputo derivative of order α is defined as*

$$\mathcal{L}\left\{\frac{d^{\alpha}f}{dx^{\alpha}}; s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0), \tag{2.3}$$

where $F(s)$ is the Laplace transform of f.
Let $\alpha, \beta, \lambda \in \mathbb{R},\ \alpha, \beta > 0,\ n \in \mathbb{N}$. Then the Laplace transform of the two-parameter function of Mittag-Leffler type (2.2) is given by

$$\mathcal{L}\{z^{\alpha n+\beta-1}E^{(n)}_{\alpha,\beta}(\pm\lambda z^{\alpha}); s\} = \frac{n!s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{n+1}},\ \ Re(s) > |\lambda|^{1/\alpha}, \tag{2.4}$$

when $n = 0$, we have

$$\mathcal{L}\{z^{\beta-1}E_{\alpha,\beta}(\pm\lambda z^{\alpha}); s\} = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp \lambda}. \tag{2.5}$$

Example 2.1.10. Let $n = 1$. Find the solution of this problem

$$\begin{cases} \dfrac{d^{\alpha}y}{dx^{\alpha}} & = y(x), \\ y(0) & = 1. \end{cases} \tag{2.6}$$

Applying the Laplace transform yields

$$s^{\alpha}Y(s) - s^{\alpha-1}y(0) = Y(s)$$
$$Y(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1}$$

By using (2.5), we have

$$Y(s) = \mathcal{L}\big\{E_{\alpha,1}(x^{\alpha})\big\}.$$

Taking inverse Laplace transform, we get

$$y(x) = E_{\alpha,1}(x^{\alpha}),$$

which is the solution of this problem.
If $\alpha = 1$, then the solution of ordinary differential equation is

$$y(x) = E_{1,1}(x) = e^{x}.$$

2.2 Invariant Subspace Method

Consider evolution partial differential equation of the form

$$u_t = F[u], \tag{2.7}$$

where F is non-linear differential operator of order k, that is,

$$F[u] = F(x, u, u_x, ..., \partial^k_x u),\ \ \partial^k_x u = \frac{\partial^k u}{\partial x^k}.$$

Let W_n be a finite dimensional linear space spanned by linearly independent functions $f_1(x), f_2(x), ..., f_n(x)$, that is,

$$\begin{aligned} W_n &= L\{f_1(x), f_2(x), ..., f_n(x)\} \\ &= \left\{ \sum_{i=1}^{n} c_i f_i(x) \mid \ c_i \in \mathbb{R}, \ \ i = 1, 2, ..., n \right\}. \end{aligned}$$

Definition 2.2.1. A finite dimensional linear space W_n is said to be invariant with respect to a differential operator F if $F[W_n] \subseteq W_n$, that is, $F[u] \in W_n$ for all $u \in W_n$.

As a means to solve the equation (2.7), we suppose that W_n is an invariant subspace with respect to a given operator F if $F[W_n] \subseteq W_n$ and then the operator F is said to preserve or admit W_n which means:

$$F[u] = F\left[\sum_{i=1}^{n} c_i(t) f_i(x)\right] = \sum_{i=1}^{n} \Psi_i\big(c_1(t), c_2(t), ..., c_n(t)\big) f_i(x), \tag{2.8}$$

where $\{\Psi_i\}$ are the expansion coefficients of $F[u] \in W_n$ on the basis $\{f_i\}$.
We assume the solution of equation (2.7) to be a combination of functions in W_n, that is,

$$u(x,t) = \sum_{i=1}^{n} c_i(t) f_i(x), \tag{2.9}$$

where $f_i(x) \in W_n, \ \ i = 1, 2, ..., n.$
Since W_n is invariant subspace under the operator F, we obtain equation (2.8).
By substituting equation (2.8) and (2.9) into (2.7), we get

$$\sum_{i=1}^{n} c_i'(t) f_i(x) = \sum_{i=1}^{n} \Psi_i\big(c_1(t), c_2(t), ..., c_n(t)\big) f_i(x)$$

$$\sum_{i=1}^{n} \left[c_i'(t) - \Psi_i\big(c_1(t), c_2(t), ..., c_n(t)\big) \right] f_i(x) = 0.$$

Since $f_1(x), f_2(x), ..., f_n(x)$ are linearly independent functions, we obtain a system of ordinary differential equations

$$c_i'(t) = \Psi_i(c_1(t), c_2(t), ..., c_n(t)), \ \ i = 1, 2, ..., n.$$

Finally, by solving this system, we obtain the desired solution (2.9).

Example 2.2.2. (Galaktionov and Svirshchevskii [14]) Consider a non-linear diffusion equation

$$v_t = \big(v^{\sigma} v_x\big)_x - v^{1-\sigma}, \;\; \sigma > 0. \tag{2.10}$$

By using the transformation $u = v^{\sigma} \Rightarrow v = u^{\frac{1}{\sigma}}$ and $v_t = \frac{1}{\sigma} u^{\frac{1}{\sigma}-1} u_t$, we get

$$\begin{aligned}\big(v^{\sigma} v_x\big)_x - v^{1-\sigma} &= \left(\frac{1}{\sigma} u u^{\frac{1}{\sigma}-1} u_x\right)_x - u^{\frac{1}{\sigma}-1} \\ &= \frac{1}{\sigma}\left[\frac{1}{\sigma} u^{\frac{1}{\sigma}-1} u_x^2 + u_{xx} u^{\frac{1}{\sigma}}\right] - u^{\frac{1}{\sigma}-1}.\end{aligned}$$

Substituting these terms into (2.10), the equation (2.10) turns to be

$$u_t = u u_{xx} + \frac{1}{\sigma}(u_x)^2 - \sigma. \tag{2.11}$$

We choose the operator

$$F[u] = u u_{xx} + \frac{1}{\sigma}(u_x)^2 - \sigma.$$

The subspace $W_2 = L\{1, x^2\}$ is invariant under F because

$$\begin{aligned}F[c_1 + c_2 x^2] &= (c_1 + c_2 x^2)\frac{d^2}{dx^2}\Big[c_1 + c_2 x^2\Big] + \frac{1}{\sigma}\left[\frac{d}{dx}\big(c_1 + c_2 x^2\big)\right]^2 - \sigma \\ &= \big(2c_1 c_2 - \sigma\big) + 2\left(1 + \frac{2}{\sigma}\right) c_2^2 x^2 \in W_2.\end{aligned}$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2, that is,

$$u(x,t) = c_1(t) + c_2(t) x^2.$$

Substituting $u(x,t)$ into the equation (2.11) , we obtain

$$c_1'(t) + c_2'(t) x^2 = 2c_1(t)c_2(t) - \sigma + 2\left(1 + \frac{2}{\sigma}\right) c_2^2(t) x^2$$

$$\Big[c_1'(t) - 2c_1(t)c_2(t) + \sigma\Big] + \left[c_2'(t) - 2\left(1 + \frac{2}{\sigma}\right) c_2^2(t)\right] x^2 = 0.$$

Since 1 and x^2 are linearly independent functions, we obtain a system of ordinary differential equations

$$c_1'(t) = 2c_1(t)c_2(t) - \sigma, \tag{2.12}$$

$$c_2'(t) = 2\left(1 + \frac{2}{\sigma}\right) c_2^2(t). \tag{2.13}$$

Taking integral to both sides of equation (2.13) yields

$$\int \frac{1}{c_2^2(t)} c_2'(t)dt = \int 2\left(1+\frac{2}{\sigma}\right)dt$$

$$-\frac{1}{c_2(t)} = 2\left(1+\frac{2}{\sigma}\right)t$$

$$c_2(t) = -\frac{\sigma}{2(\sigma+2)t}.$$

Substituting $c_2(t)$ into equation (2.12), we get

$$c_1'(t) + \frac{\sigma}{(\sigma+2)t}c_1(t) = -\sigma, \tag{2.14}$$

which is the first order linear differential equation of the form

$$y' + p(t)y = q(t),$$

where $p(t) = \dfrac{\sigma}{(\sigma+2)t}$ and $q(t) = -\sigma$.
Then the general solution is

$$c_1(t) = y(t) = e^{-P(t)} \int q(t)e^{P(t)}dt,$$

where

$$P(t) = \int p(t)dt = \int \frac{\sigma}{(\sigma+2)t}dt = \frac{\sigma}{(\sigma+2)} \ln t.$$

So, we obtain

$$\begin{aligned} c_1(t) &= e^{-\frac{\sigma}{(\sigma+2)}\ln t}\left[-\sigma \int e^{\frac{\sigma}{(\sigma+2)}\ln t} dt\right] \\ &= t^{-\frac{\sigma}{\sigma+2}}\left[-\sigma \int t^{\frac{\sigma}{\sigma+2}} dt\right] \\ &= t^{-\frac{\sigma}{\sigma+2}}\left[-\frac{\sigma(\sigma+2)}{2\sigma+2}t^{\frac{2\sigma+2}{\sigma+2}} + B\right] \\ &= Bt^{-\frac{\sigma}{\sigma+2}} - \frac{\sigma(\sigma+2)}{2(\sigma+1)}t, \end{aligned}$$

where B is an arbitrary constant.
Hence, the solution of the equation (2.11) is

$$u(x,t) = Bt^{-\frac{\sigma}{\sigma+2}} - \frac{\sigma(\sigma+2)}{2(\sigma+1)}t - \frac{\sigma}{2(\sigma+2)t}x^2.$$

Therefore, the solution of the equation (2.10) is

$$v(x,t) = \left[Bt^{-\frac{\sigma}{\sigma+2}} - \frac{\sigma(\sigma+2)}{2(\sigma+1)}t - \frac{\sigma}{2(\sigma+2)t}x^2\right]^{\frac{1}{\sigma}}.$$

Chapter 3

Explicit solution of fractional telegraph equations

In this chapter, we show how the invariant subspace can be extended to three classes of fractional telegraph equations, i.e., space-, time-, and space- and time telegraph equations.

3.1 The space-fractional telegraph equations

Consider the space-fractional telegraph equation with $0 < \alpha \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \;\; x > 0, \;\; t > 0, \tag{3.1}$$

where $\dfrac{\partial^{2\alpha}}{\partial x^{2\alpha}}$ is a space-fractional derivative in the Caputo sense. Now, we denote the left side of equation (3.1) by

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u. \tag{3.2}$$

The following theorem shows an exact solution to the space-fractional telegraph equation (3.1) by using the invariant subspace method.

Theorem 3.1.1. *The space-fractional telegraph equation (3.1) admits a solution of the form*

$$u(x,t) = c_1(x) + c_2(x)t + \cdots + c_{n+1}(x)t^n,$$

where $c_i(x),\ i = 1, ..., n+1$ *are solutions of the following system of fractional* *ordinary differential equations*

$$\begin{cases} \dfrac{d^{2\alpha}c_1(x)}{dx^{\alpha}} & = 2c_3(x) + c_2(x) + c_1(x), \\ \dfrac{d^{2\alpha}c_2(x)}{dx^{\alpha}} & = 6c_4(x) + 2c_3(x) + c_2(x), \\ & \vdots \\ \dfrac{d^{2\alpha}c_{n+1}(x)}{dx^{\alpha}} & = c_{n+1}(x). \end{cases} \tag{3.3}$$

Proof. The operator $F[.]$ defined by (3.2) is invariant under $W_n = L\{1, t, \cdots, t^n\}$ because

$$F(c_1 + c_2 t + \cdots + c_{n+1}t^n) = (2c_3 + c_2 + c_1) + (6c_4 + 2c_3 + c_2)t + \cdots + c_{n+1}t^n \in W_n.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_n, that is,

$$u(x,t) = c_1(x) + c_2(x)t + \cdots + c_{n+1}(x)t^n. \tag{3.4}$$

Then we have

$$F[u(x,t)] = \big[2c_3(x) + c_2(x) + c_1(x)\big] + \big[6c_4(x) + 2c_3(x) + c_2(x)\big]t + \cdots + c_{n+1}(x)t^n. \tag{3.5}$$

Taking the fractional derivative of order 2α with respect to x in both sides of equation (3.4), we obtain

$$\frac{d^{2\alpha}u(x,t)}{dx^{2\alpha}} = \frac{d^{2\alpha}c_1(x)}{dx^{2\alpha}} + \frac{d^{2\alpha}c_2(x)}{dx^{2\alpha}}t + \cdots + \frac{d^{2\alpha}c_{n+1}(x)}{dx^{2\alpha}}t^n. \tag{3.6}$$

Substituting equation (3.6) and (3.5) into the equation (3.1), we get

$$\left[\frac{d^{2\alpha}c_1(x)}{dx^{2\alpha}} - 2c_3(x) - c_2(x) - c_1(x)\right] + t\left[\frac{d^{2\alpha}c_2(x)}{dx^{2\alpha}} - 6c_4(x) - 2c_3(x) - c_2(x)\right] + \cdots + t^n\left[\frac{d^{2\alpha}c_{n+1}(x)}{dx^{2\alpha}} - c_{n+1}(x)\right] = 0.$$

Since $1, t, \cdots, t^n$ are linearly independent functions, we obtain a system of fractional ordinary differential equations (3.3). □

Remark 3.1.2. Under the operator (3.2), there are several invariant subspaces which can be proved in a similar way. In below, we classify some invariant subspaces with respect to the operator (3.2).

1. The subspace $W_2 = L\{1, e^{at}\}$, $a \neq 0$ is invariant under F because

$$F(c_1 + c_2 e^{at}) = c_1 + (a^2 c_2 + a c_2 + c_2) e^{at} \in W_2.$$

2. The subspace $W_3^1 = L\{1, \sin(at), \cos(at)\}$, $a \neq 0$ is invariant under F because

$$\begin{aligned} F[c_1 + c_2 \sin(at) + c_3 \cos(at)] = c_1 &+ [c_2 - a c_3 - a^2 c_2] \sin(at) \\ &+ [c_3 + a c_2 - a^2 c_3] \cos(at) \in W_3^1. \end{aligned}$$

3. The subspace $W_3^2 = L\{1, \sinh(at), \cosh(at)\}$, $a \neq 0$ is invariant under F because

$$\begin{aligned} F[c_1 + c_2 \sinh(at) + c_3 \cosh(at)] = c_1 &+ [c_2 + a c_3 + a^2 c_2] \sinh(at) \\ &+ [c_3 + a c_2 + a^2 c_3] \cosh(at) \in W_3^2. \end{aligned}$$

4. The subspace $W_3^3 = L\{1, e^{at}, te^{at}\}$, $a \neq 0$ is invariant under F because

$$\begin{aligned} F[c_1 + c_2 e^{at} + c_3 t e^{at}] = c_1 &+ [(1 + a + a^2) c_2 + (1 + 2a) c_3] e^{at} \\ &+ c_3 (1 + a + a^2) t e^{at} \in W_3^3. \end{aligned}$$

5. The subspace $W_3^4 = L\{1, e^{at} \cos bt, e^{at} \sin bt\}$, $a, b \neq 0$ is invariant under F because

$$\begin{aligned} &F[c_1 + c_2 e^{at} \cos bt + c_3 e^{at} \sin bt] \\ &\qquad = c_1 + [c_2 + (a c_2 + b c_3) + (a^2 c_2 + b^2 c_2)] e^{at} \cos bt \\ &\qquad + [c_3 + (a c_3 - b c_2) - (a b c_2 + b^2 c_3) \\ &\qquad + (a^2 c_3 - a b c_2)] e^{at} \sin bt \in W_3^4. \end{aligned}$$

The advantage of these different invariant subspaces is that, by choosing an appropriate invariant subspace, we can solve the space-fractional telegraph equation subject to different boundary conditions.

Next, we will apply the invariant subspace method to solve some examples as follows:

Example 3.1.3. (Momani [9]) Consider the space-fractional telegraph equation with $0 < \alpha \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad x > 0, \quad t > 0, \tag{3.7}$$

subject to the boundary conditions

$$u(0,t) = e^{-t}, \frac{\partial u(0,t)}{\partial x} = e^{-t}.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u,$$

we choose the invariant subspace

$$W_2 = L\{1, e^{-t}\}.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2, that is,

$$u(x,t) = a(x) + b(x)e^{-t}.$$

By using the boundary conditions

- $u(0,t) = e^{-t}$, that

$$a(0) + b(0)e^{-t} = e^{-t} \Rightarrow a(0) = 0, \quad b(0) = 1,$$

- $\dfrac{\partial}{\partial x} u(0,t) = e^{-t}$, that

$$a'(0) + b'(0)e^{-t} = e^{-t} \Rightarrow a'(0) = 0, \quad b'(0) = 1.$$

Substituting $u(x,t)$ into the equation (3.7), we get

$$\frac{d^{2\alpha}}{dx^{2\alpha}} a(x) + e^{-t} \frac{d^{2\alpha}}{dx^{2\alpha}} b(x) = a(x) + b(x)e^{-t}$$

$$\left[\frac{d^{2\alpha}}{dx^{2\alpha}} a(x) - a(x)\right] + e^{-t}\left[\frac{d^{2\alpha}}{dx^{2\alpha}} b(x) - b(x)\right] = 0.$$

Since 1 and e^{-t} are linearly independent functions, we obtain a system of space-fractional ordinary differential equations

$$\frac{d^{2\alpha}}{dx^{2\alpha}} a(x) = a(x), \quad a(0) = a'(0) = 0, \tag{3.8}$$

$$\frac{d^{2\alpha}}{dx^{2\alpha}} b(x) = b(x), \quad b(0) = b'(0) = 1. \tag{3.9}$$

Applying the Laplace transform to both sides of equation (3.8), we obtain

$$\begin{aligned}\mathcal{L}\Big\{\frac{d^{2\alpha}a(x)}{dx^{2\alpha}};s\Big\} &= \mathcal{L}\big\{a(x);s\big\}\\ s^{2\alpha}A(s)-s^{2\alpha-1}a(0)-s^{2\alpha-2}a'(0) &= A(s)\\ A(s) &= 0,\end{aligned}$$

where $A(s)$ is the Laplace transform of $a(x)$.
Taking inverse Laplace transform yields

$$a(x)=0.$$

Applying the Laplace transform to both sides of equation (3.9), we obtain

$$\begin{aligned}\mathcal{L}\Big\{\frac{d^{2\alpha}b(x)}{dx^{2\alpha}};s\Big\} &= \mathcal{L}\big\{b(x);s\big\}\\ s^{2\alpha}B(s)-s^{2\alpha-1}b(0)-s^{2\alpha-2}b'(0) &= B(s)\\ B(s) &= \frac{s^{2\alpha-1}}{s^{2\alpha}-1}+\frac{s^{2\alpha-2}}{s^{2\alpha}-1},\end{aligned}$$

where $B(s)$ is the Laplace transform of $b(x)$.
By using (2.5), we have

$$B(s)=\mathcal{L}\{E_{2\alpha,1}(x^{2\alpha});s\}+\mathcal{L}\{xE_{2\alpha,2}(x^{2\alpha});s\}.$$

Taking inverse Laplace transform, we get

$$b(x)=E_{2\alpha,1}(x^{2\alpha})+xE_{2\alpha,2}(x^{2\alpha}).$$

Therefore, the exact solution of equation (3.7) is

$$u(x,t)=e^{-t}\big[E_{2\alpha,1}(x^{2\alpha})+xE_{2\alpha,2}(x^{2\alpha})\big],$$

which is the same solution obtained by the Adomian decomposition method by Momani [9].
In particular, if $\alpha=1$, then the exact solution of classical telegraph equation is

$$u(x,t)=e^{-t}\big[E_{2,1}(x^2)+xE_{2,2}(x^2)\big]=e^{-t}\big[\cosh x+\sinh x\big]=e^{x-t}.$$

Example 3.1.4. (Momani [9]) Consider the non-homogeneous space-fractional telegraph equation with $0 < \alpha \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u - x^2 - t + 1, \;\; x > 0, \;\; t > 0, \tag{3.10}$$

subject to the boundary conditions

$$u(0,t) = t, \frac{\partial u(0,t)}{\partial x} = 0.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u,$$

we choose the invariant subspace

$$W_2^1 = L\{1, t\}.$$

Now, we assume the solution $u(x,t)$ as a linear combination of functions in the invariant subspace W_2^1, that is,

$$u(x,t) = a(x) + b(x)t.$$

Using the boundary conditions

- $u(0,t) = t \Rightarrow a(0) + b(0)t = t \Rightarrow a(0) = 0, \;\; b(0) = 1,$
- $\frac{\partial u(0,t)}{\partial x} = 0 \Rightarrow a'(0) + b'(0)t = 0 \Rightarrow a'(0) = 0, \;\; b'(0) = 0.$

Substituting $u(x,t)$ into the equation (3.10), we get

$$\frac{d^{2\alpha}}{dx^{2\alpha}} a(x) + t \frac{d^{2\alpha}}{dx^{2\alpha}} b(x) + x^2 + t - 1 = b(x) + a(x) + b(x)t$$

$$\left[\frac{d^{2\alpha}}{dx^{2\alpha}} a(x) - a(x) - b(x) + x^2 - 1 \right] + t \left[\frac{d^{2\alpha}}{dx^{2\alpha}} b(x) - b(x) + 1 \right] = 0.$$

Since 1 and t are linearly independent functions, we obtain a system of space-fractional ordinary differential equations

$$\frac{d^{2\alpha} a(x)}{dx^{2\alpha}} = a(x) + b(x) - x^2 + 1, \;\; a(0) = a'(0) = 0, \tag{3.11}$$

$$\frac{d^{2\alpha} b(x)}{dx^{2\alpha}} = b(x) - 1, \;\; b(0) = 1, \;\; b'(0) = 0. \tag{3.12}$$

Applying the Laplace transform to both sides of equation (3.12), we get

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\alpha}b(x)}{dx^{2\alpha}};s\right\} &= \mathcal{L}\big\{b(x);s\big\} - \mathcal{L}\{1\} \\
s^{2\alpha}B(s) - s^{2\alpha-1}b(0) - s^{2\alpha-2}b'(0) &= B(s) - \frac{1}{s} \\
B(s) &= \frac{s^{2\alpha-1}}{s^{2\alpha}-1} - \frac{1}{s(s^{2\alpha}-1)} \\
&= \frac{s^{2\alpha-1}}{s^{2\alpha}-1} - \left[\frac{s^{2\alpha-1}}{s^{2\alpha}-1} - \frac{1}{s}\right] \\
&= \frac{1}{s} = \mathcal{L}\{1\}.
\end{aligned}$$

Taking inverse Laplace transform yields

$$b(x) = 1.$$

Substituting $b(x)$ into the equation (3.11) and applying the Laplace transform to both sides, we obtain

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\alpha}a(x)}{dx^{2\alpha}};s\right\} &= \mathcal{L}\big\{a(x);s\big\} - \mathcal{L}\{x^2;s\} + \mathcal{L}\{2\} \\
s^{2\alpha}A(s) - s^{2\alpha-1}a(0) - s^{2\alpha-2}a'(0) &= A(s) - \frac{2}{s^3} + \frac{2}{s} \\
A(s) &= \frac{2}{s(s^{2\alpha}-1)} - \frac{2}{s^3(s^{2\alpha}-1)} \\
&= 2\left[\frac{s^{2\alpha-1}}{s^{2\alpha}-1} - \frac{1}{s} - \frac{s^{2\alpha-3}}{s^{2\alpha}-1} + \frac{1}{s^3}\right]
\end{aligned}$$

By using (2.5), we have

$$A(s) = 2\mathcal{L}\{E_{2\alpha,1}(x^{2\alpha});s\} - 2\mathcal{L}\{1\} - 2\mathcal{L}\{x^2E_{2\alpha,3}(x^{2\alpha});s\} + \mathcal{L}\{x^2\}.$$

Taking inverse Laplace transform yields

$$a(x) = 2E_{2\alpha,1}(x^{2\alpha}) - 2 - 2x^2E_{2\alpha,3}(x^{2\alpha}) + x^2.$$

Therefore, the exact solution of equation (3.10) is

$$u(x,t) = 2E_{2\alpha,1}(x^{2\alpha}) - 2 - 2x^2E_{2\alpha,3}(x^{2\alpha}) + x^2 + t,$$

which is the same solution obtained by the Adomian decomposition method by Momani [9].
In particular, if $\alpha = 1$, then the exact solution of classical telegraph equation is

$$u(x,t) = 2E_{2,1}(x^2) - 2 - 2\big[E_{2,1}(x^2) - 1\big] + x^2 + t = x^2 + t.$$

Example 3.1.5. Consider the space-fractional telegraph equation with $0 < \alpha \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u, \quad x > 0, \quad t > 0, \tag{3.13}$$

subject to the boundary conditions

$$u(0,t) = \sin t, \quad u_x(0,t) = \cos t.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u,$$

we choose the invariant subspace

$$W_3 = L\{1, \sin t, \cos t\}.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_3, that is,

$$u(x,t) = a(x) + b(x)\sin t + c(x)\cos t.$$

By using the boundary conditions $u(0,t) = \sin t$, that

$$a(0) + b(0)\sin t + c(0)\cos t = \sin t \Rightarrow a(0) = 0, b(0) = 1, c(0) = 0,$$

and $u_x(0,t) = \cos t$, that

$$a'(0) + b'(0)\sin t + c'(0)\cos t = \cos t \Rightarrow a'(0) = b'(0) = 0, c'(0) = 1.$$

Substituting $u(x,t)$ into the equation (3.13), we get

$$\frac{d^{2\alpha}}{dx^{2\alpha}}a(x) + \sin t\frac{d^{2\alpha}}{dx^{2\alpha}}b(x) + \cos t\frac{d^{2\alpha}}{dx^{2\alpha}}c(x) = a(x) - c(x)\sin t + b(x)\cos t$$

$$\left[\frac{d^{2\alpha}}{dx^{2\alpha}}a(x) - a(x)\right] + \sin t\left[\frac{d^{2\alpha}}{dx^{2\alpha}}b(x) + c(x)\right] + \cos t\left[\frac{d^{2\alpha}}{dx^{2\alpha}}c(x) - b(x)\right] = 0.$$

Since 1, $\sin t$ and $\cos t$ are linearly independent functions, we obtain a system of space-fractional ordinary differential equations

$$\frac{d^{2\alpha}a(x)}{dx^{2\alpha}} = a(x), \quad a(0) = a'(0) = 0, \tag{3.14}$$

$$\frac{d^{2\alpha}b(x)}{dx^{2\alpha}} = -c(x), \quad b(0) = 1, \quad b'(0) = 0, \tag{3.15}$$

$$\frac{d^{2\alpha}c(x)}{dx^{2\alpha}} = b(x), \quad c(0) = 0, \quad c'(0) = 1. \tag{3.16}$$

Applying the Laplace transform to both sides of equation (3.14), we get

$$\begin{aligned}\mathcal{L}\left\{\frac{d^{2\alpha}a(x)}{dx^{2\alpha}};s\right\} &= \mathcal{L}\{a(x);s\}\\ s^{2\alpha}A(s)-s^{2\alpha-1}a(0)-s^{2\alpha-2}a'(0) &= A(s)\\ A(s) &= 0.\end{aligned}$$

Taking inverse Laplace transform yields

$$a(x)=0.$$

Now, we transform equation (3.15) and (3.16) by setting

$$\vec{z}(x)=\begin{bmatrix}b(x)\\ c(x)\end{bmatrix}.$$

Then

$$\frac{d^{2\alpha}}{dx^{2\alpha}}\vec{z}(x)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}b(x)\\ c(x)\end{bmatrix}=A\vec{z}(x),$$

where $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $\vec{z}(0)=\begin{bmatrix}1\\ 0\end{bmatrix}$, $\vec{z}'(0)=\begin{bmatrix}0\\ 1\end{bmatrix}$.

Applying Laplace transform to both sides, then we obtain

$$\begin{aligned}s^{2\alpha}\vec{Z}(s)-s^{2\alpha-1}\vec{z}(0)-s^{2\alpha-2}\vec{z}'(0) &= A\vec{Z}(s)\\ (s^{2\alpha}I-A)\vec{Z}(s) &= s^{2\alpha-1}\vec{z}(0)+s^{2\alpha-2}\vec{z}'(0)\\ \begin{bmatrix}s^{2\alpha} & 1\\ -1 & s^{2\alpha}\end{bmatrix}\vec{Z}(s) &= \begin{bmatrix}s^{2\alpha-1}\\ 0\end{bmatrix}+\begin{bmatrix}0\\ s^{2\alpha-2}\end{bmatrix}=\begin{bmatrix}s^{2\alpha-1}\\ s^{2\alpha-2}\end{bmatrix}\\ \vec{Z}(s) &= \begin{bmatrix}s^{2\alpha} & 1\\ -1 & s^{2\alpha}\end{bmatrix}^{-1}\begin{bmatrix}s^{2\alpha-1}\\ s^{2\alpha-2}\end{bmatrix}\\ &= \frac{1}{s^{4\alpha}+1}\begin{bmatrix}s^{2\alpha} & -1\\ 1 & s^{2\alpha}\end{bmatrix}\begin{bmatrix}s^{2\alpha-1}\\ s^{2\alpha-2}\end{bmatrix}\\ \begin{bmatrix}B(s)\\ C(s)\end{bmatrix} &= \begin{bmatrix}\dfrac{s^{4\alpha-1}-s^{2\alpha-2}}{s^{4\alpha}+1}\\ \dfrac{s^{4\alpha-2}+s^{2\alpha-1}}{s^{4\alpha}+1}\end{bmatrix}.\end{aligned}$$

Then we get

$$B(s)=\frac{s^{4\alpha-1}}{s^{4\alpha}+1}-\frac{s^{2\alpha-2}}{s^{4\alpha}+1}.$$

By using (2.5), we have

$$B(s) = \mathcal{L}\{E_{4\alpha,1}(-x^{4\alpha})\} - \mathcal{L}\{x^{2\alpha+1}E_{4\alpha,2\alpha+2}(-x^{4\alpha})\}$$

Taking inverse Laplace transform yields

$$b(x) = E_{4\alpha,1}(-x^{4\alpha}) - x^{2\alpha+1}E_{4\alpha,2\alpha+2}(-x^{4\alpha}).$$

And

$$C(s) = \frac{s^{4\alpha-2}}{s^{4\alpha}+1} + \frac{s^{2\alpha-1}}{s^{4\alpha}+1}.$$

By using (2.5), we have

$$C(s) = \mathcal{L}\{xE_{4\alpha,2}(-x^{4\alpha})\} + \mathcal{L}\{x^{2\alpha}E_{4\alpha,2\alpha+1}(-x^{4\alpha})\}.$$

Taking inverse Laplace transform yields

$$c(x) = xE_{4\alpha,2}(-x^{4\alpha}) + x^{2\alpha}E_{4\alpha,2\alpha+1}(-x^{4\alpha}).$$

Therefore, the solution of equation (3.13) is

$$\begin{aligned} u(x,t) = &\left[E_{4\alpha,1}(-x^{4\alpha}) - x^{2\alpha+1}E_{4\alpha,2\alpha+2}(-x^{4\alpha})\right]\sin t \\ &+ \left[xE_{4\alpha,2}(-x^{4\alpha}) + x^{2\alpha}E_{4\alpha,2\alpha+1}(-x^{4\alpha})\right]\cos t. \end{aligned}$$

If $\alpha = 1$, then the solution of classical telegraph equation is

$$u(x,t) = \left[E_{4,1}(-x^4) - x^3E_{4,4}(-x^4)\right]\sin t + \left[xE_{4,2}(-x^4) + x^2E_{4,3}(-x^4)\right]\cos t.$$

3.2 The time-fractional telegraph equations

Consider the time-fractional telegraph equation with $0 < \beta \leq 1$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + \frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \quad x > 0, \quad t > 0, \tag{3.17}$$

where $\frac{\partial^{2\beta}}{\partial t^{2\beta}}$ and $\frac{\partial^{\beta}}{\partial t^{\beta}}$ are time-fractional derivatives in the Caputo sense. Now, we set the differential operator

$$F[u] = \frac{\partial^2 u}{\partial x^2} - u. \tag{3.18}$$

To obtain an exact solution of time-fractional telegraph equation (3.17) by applying the invariant subspace method is stated in the following theorem.

Theorem 3.2.1. *The time-fractional telegraph equation (3.17) admits a solution of the form*

$$u(x,t) = c_1(t) + c_2(t)e^{ax} + c_3(t)xe^{ax},$$

where $c_1(t), c_2(t)$, and $c_3(t)$ are solutions of the following system of fractional ordinary differential equations

$$\begin{cases} \dfrac{d^{2\beta}c_1(t)}{dt^{2\beta}} + \dfrac{d^{\beta}c_1(t)}{dt^{\beta}} & = -c_1(t), \\ \dfrac{d^{2\beta}c_2(t)}{dt^{2\beta}} + \dfrac{d^{\beta}c_2(t)}{dt^{\beta}} & = a^2c_2(t) + 2ac_3(t) - c_2(t), \\ \dfrac{d^{2\beta}c_3(t)}{dt^{2\beta}} + \dfrac{d^{\beta}c_3(t)}{dt^{\beta}} & = a^2c_3(t) - c_3(t). \end{cases} \tag{3.19}$$

Proof. Under the operator $F[.]$ defined by (3.18), we choose the invariant subspace $W_3^3 = L\{1, e^{ax}, xe^{ax}\}, a \neq 0$ because

$$F[c_1 + c_2e^{ax} + c_3xe^{ax}] = -c_1 + (a^2c_2 + 2ac_3 - c_2)e^{ax} + (a^2c_3 - c_3)xe^{ax} \in W_3^3.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_3^3, that is,

$$u(x,t) = c_1(t) + c_2(t)e^{ax} + c_3(t)xe^{ax}. \tag{3.20}$$

Then we have

$$F[u(x,t)] = -c_1(t) + \big(a^2c_2(t) + 2ac_3(t) - c_2(t)\big)e^{ax} + \big(a^2c_3(t) - c_3(t)\big)xe^{ax}. \tag{3.21}$$

Applying the fractional derivative of order 2α and α with respect to t in both sides of equation (3.20), we sum them together, we obtain

$$\frac{d^{2\beta}u(x,t)}{dt^{2\beta}} + \frac{d^{\beta}u(x,t)}{dt^{\beta}} = \frac{d^{2\beta}c_1(t)}{dt^{2\beta}} + \frac{d^{\beta}c_1(t)}{dt^{\beta}} + \frac{d^{2\beta}c_2(t)}{dt^{2\beta}}e^{ax} + \frac{d^{\beta}c_2(t)}{dt^{\beta}}e^{ax} + \frac{d^{2\beta}c_3(t)}{dt^{2\beta}}xe^{ax} + \frac{d^{\beta}c_3(t)}{dt^{\beta}}xe^{ax}. \tag{3.22}$$

Substituting (3.22) and (3.21) in (3.17), we get

$$\left[\frac{d^{2\beta}c_1(t)}{dt^{2\beta}} + \frac{d^{\beta}c_1(t)}{dt^{\beta}} + c_1(t)\right] + e^{ax}\left[\frac{d^{2\beta}c_2(t)}{dt^{2\beta}} + \frac{d^{\beta}c_2(t)}{dt^{\beta}} - a^2c_2(t) - 2ac_3(t) + c_2(t)\right] + xe^{ax}\left[\frac{d^{2\beta}c_3(t)}{dt^{2\beta}} + \frac{d^{\beta}c_3(t)}{dt^{\beta}} - a^2c_3(t) + c_3(t)\right] = 0.$$

Since 1, e^{ax}, and xe^{ax} are linearly independent functions, we get a system of fractional ordinary differential equations (3.19). □

Remark 3.2.2. We would like to mention that, in a similar way, there are several invariant subspaces under the operator (3.18) can be proved this theorem. In the following, we classify all possible invariant subspaces with respect to the differential operator (3.18)

1. The subspace $W_n = L\{1, x, ..., x^n\}$ is invariant under F because

$$\begin{aligned} F(c_1 + c_2 x + ... + c_{n+1} x^n) &= (2c_3 - c_1) + (6c_4 - c_2)x \\ &\quad - ... - c_{n+1} x^n \in W_n. \end{aligned}$$

2. The subspace $W_2 = L\{1, e^{ax}\}$, $a \neq 0$ is invariant under F because

$$F(c_1 + c_2 e^{ax}) = -c_1 + (a^2 c_2 - c_2) e^{ax} \in W_2.$$

3. The subspace $W_3^1 = L\{1, \sin(ax), \cos(ax)\}$, $a \neq 0$ is invariant under F because

$$\begin{aligned} F[c_1 + c_2 \sin(ax) + c_3 \cos(ax)] &= -c_1 - [c_2 + a^2 c_2] \sin(ax) \\ &\quad - [c_3 + a^2 c_3] \cos(ax) \in W_3^1. \end{aligned}$$

4. The subspace $W_3^2 = L\{1, \sinh(ax), \cosh(ax)\}$, $a \neq 0$ is invariant under F because

$$\begin{aligned} F[c_1 + c_2 \sinh(ax) + c_3 \cosh(ax)] &= -c_1 + [a^2 c_2 - c_2] \sinh(ax) \\ &\quad + [a^2 c_3 - c_3] \cosh(ax) \in W_3^2. \end{aligned}$$

5. The subspace $W_3^4 = L\{1, e^{ax} \cos bx, e^{ax} \sin bx\}$, $a, b \neq 0$ is invariant under F because

$$\begin{aligned} &F[c_1 + c_2 e^{ax} \cos bx + c_3 e^{ax} \sin bx] \\ &\qquad = -c_1 + [a^2 c_3 - b^2 c_2 + 2abc_3 - c_2] e^{ax} \cos bx \\ &\qquad + [a^2 c_3 - 2abc_3 - b^2 c_3 - c_3] e^{ax} \sin bx \in W_3^4. \end{aligned}$$

The benefit of these different invariant subspaces is that, by choosing an appropriate invariant subspace with respect to the initial conditions, we are able to solve the time-fractional telegraph equation.

Next, we solve some examples which are stated in [11, 13] by using the invariant subspace method.

Example 3.2.3. (Srivastava et al. [11]) Consider the time-fractional telegraph equation with $0 < \beta \leq 1$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \quad x > 0, \quad t > 0, \tag{3.23}$$

subject to the initial conditions

$$u(x,0) = e^x, \frac{\partial u(x,0)}{\partial t} = -2e^x.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial x^2} - u,$$

we choose the invariant subspace

$$W_2 = L\{1, e^x\}.$$

Now, we assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2, that is,

$$u(x,t) = a(t) + b(t)e^x.$$

It follows the initial conditions

- $u(x,0) = e^x \Rightarrow a(0) + b(0)e^x = e^x \Rightarrow a(0) = 0, b(0) = 1,$
- $\dfrac{\partial u(x,0)}{\partial t} = -2e^x \Rightarrow a'(0) + b'(0)e^x = -2e^x \Rightarrow a'(0) = 0, b'(0) = -2.$

Substituting $u(x,t)$ into the equation (3.23), we obtain

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t) + e^x\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) + 2e^x\frac{d^{\beta}}{dt^{\beta}}b(t) = -a(t)$$

$$\left[\frac{d^{2\beta}}{dt^{2\beta}}a(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) + a(t)\right] + e^x\left[\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}b(t)\right] = 0.$$

Since 1 and e^x are linearly independent functions, we obtain a system of time-fractional ordinary differential equations

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) = -a(t), \quad a(0) = a'(0) = 0, \tag{3.24}$$

$$\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}b(t) = 0, \quad b(0) = 1, b'(0) = -2. \tag{3.25}$$

Applying the Laplace transform to both sides of equation (3.24), we obtain

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\beta}}{dt^{2\beta}}a(t);s\right\}+2\mathcal{L}\left\{\frac{d^{\beta}}{dt^{\beta}}a(t);s\right\}&=-\mathcal{L}\{a(t);s\}\\
s^{2\beta}A(s)-s^{2\beta-1}a(0)-s^{2\beta-2}a'(0)+2s^{\beta}A(s)-2s^{\beta-1}a(0)&=-A(s)\\
A(s)&=0.
\end{aligned}$$

Taking inverse Laplace transform yields

$$a(t)=0.$$

Applying the Laplace transform to both sides of equation (3.25), we get

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\beta}}{dt^{2\beta}}b(t);s\right\}&=-2\mathcal{L}\left\{\frac{d^{\beta}}{dt^{\beta}}b(t);s\right\}\\
s^{2\beta}B(s)-s^{2\beta-1}b(0)-s^{2\beta-2}b'(0)&=-2s^{\beta}B(s)+2s^{\beta-1}b(0)\\
s^{2\beta}B(s)+2s^{\beta}B(s)&=2s^{\beta-1}+s^{2\beta-1}-2s^{2\beta-2}\\
B(s)&=2\frac{s^{\beta-1}}{s^{2\beta}+2s^{\beta}}+\frac{s^{2\beta-1}}{s^{2\beta}+2s^{\beta}}-2\frac{s^{2\beta-2}}{s^{2\beta}+2s^{\beta}}\\
&=2\frac{s^{\beta}(s^{-1})}{s^{\beta}(s^{\beta}+2)}+\frac{s^{\beta}(s^{\beta-1})}{s^{\beta}(s^{\beta}+2)}-2\frac{s^{\beta}(s^{\beta-2})}{s^{\beta}(s^{\beta}+2)}\\
&=2\frac{1}{s(s^{\beta}+2)}+\frac{s^{\beta-1}}{s^{\beta}+2}-2\frac{s^{\beta-2}}{s^{\beta}+2}\\
&=\frac{1}{s}-\frac{s^{\beta-1}}{s^{\beta}+2}+\frac{s^{\beta-1}}{s^{\beta}+2}-2\frac{s^{\beta-2}}{s^{\beta}+2}\\
&=\frac{1}{s}-2\frac{s^{\beta-2}}{s^{\beta}+2}.
\end{aligned}$$

By using (2.5), we have

$$B(s)=\mathcal{L}\{1\}-2\mathcal{L}\{tE_{\beta,2}(-2t^{\beta})\}.$$

Taking inverse Laplace transform yields

$$b(t)=1-2tE_{\beta,2}(-2t^{\beta}).$$

Therefore, the exact solution of equation (3.23) is

$$u(x,t)=e^{x}\big[1-2tE_{\beta,2}(-2t^{\beta})\big],$$

which is the same solution obtained by the reduced differential transform method by Srivastava et al. [11].

In particular, if $\beta=1$, then the exact solution of classical telegraph equation is

$$u(x,t)=e^{x}\big[1-2tE_{1,2}(-2t)\big]=e^{x}\big[1+E_{1,1}(-2t)-1\big]=e^{x-2t}.$$

Example 3.2.4. (Srivastava et al. [11]) Consider the following time-fractional telegraph equation with $0 < \beta \leq 1$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \;\; x > 0, \;\; t > 0, \tag{3.26}$$

subject to the initial conditions

$$u(x,0) = \sinh x, \frac{\partial u(x,0)}{\partial t} = -2\sinh x.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial x^2} - u,$$

we choose the invariant subspace

$$W_2^1 = L\{1, \sinh x\}.$$

We assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2^1, that is,

$$u(x,t) = a(t) + b(t)\sinh x.$$

By using the initial conditions $u(x,0) = \sinh x$, that

$$a(0) + b(0)\sinh x = \sinh x \Rightarrow a(0) = 0, \;\; b(0) = 1,$$

and $\dfrac{\partial u(x,0)}{\partial t} = -2\sinh x$, that

$$a'(0) + b'(0)\sinh x = -2\sinh x \Rightarrow a'(0) = 0, \;\; b'(0) = -2.$$

Substituting $u(x,t)$ into the equation (3.26), we obtain

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t) + \sinh x\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) + 2\sinh x\frac{d^{\beta}}{dt^{\beta}}b(t) = -a(t)$$

$$\left[\frac{d^{2\beta}}{dt^{2\beta}}a(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) + a(t)\right] + \sinh x\left[\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}b(t)\right] = 0.$$

Since 1 and $\sinh x$ are linearly independent, we obtain a system of time-fractional ordinary differential equations

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t) + 2\frac{d^{\beta}}{dt^{\beta}}a(t) = -a(t), \;\; a(0) = a'(0) = 0, \tag{3.27}$$

$$\frac{d^{2\beta}}{dt^{2\beta}}b(t) + 2\frac{d^{\beta}}{dt^{\beta}}b(t) = 0, \;\; b(0) = 1, b'(0) = -2, \tag{3.28}$$

where a system of fractional ordinary differential equations has already found in a example (3.2.3), that

$$a(t) = 0, \;\; b(t) = 1 - 2tE_{\beta,2}(-2t^{\beta}).$$

Therefore, the exact solution of equation (3.26) is

$$u(x,t) = \left[1 - 2tE_{\beta,2}(-2t^{\beta})\right] \sinh x,$$

which is the same solution obtained by the reduced differential transform method by Srivastava et al. [11].
In particular, if $\beta = 1$, then the exact solution of classical telegraph equation is

$$u(x,t) = e^{-2t} \sinh x.$$

Example 3.2.5. Consider the time-fractional telegraph equation with $0 < \beta \leq 1$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \;\; x > 0, \;\; t > 0, \tag{3.29}$$

subject to the initial conditions

$$u(x,0) = \cosh x, \frac{\partial u(x,0)}{\partial t} = -2\cosh x.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial x^2} - u,$$

we choose the invariant subspace

$$W_2^2 = L\{1, \cosh x\}.$$

Now we assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2^2, that is,

$$u(x,t) = a(t) + b(t)\cosh x.$$

Using the initial conditions

- $u(x,0) = \cosh x \Rightarrow a(0) + b(0)\cosh x = \cosh x \Rightarrow a(0) = 0, b(0) = 1,$
- $\dfrac{\partial u(x,0)}{\partial t} = -2\cosh \Rightarrow a'(0) + b'(0)\cosh x = -2\cosh x$

$$\Rightarrow a'(0) = 0, b'(0) = -2.$$

Substituting $u(x,t)$ into the equation (3.29), we obtain

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t)+\cosh x\frac{d^{2\beta}}{dt^{2\beta}}b(t)+2\frac{d^{\beta}}{dt^{\beta}}a(t)+2\cosh x\frac{d^{\beta}}{dt^{\beta}}b(t)=-a(t)$$

$$\left[\frac{d^{2\beta}}{dt^{2\beta}}a(t)+2\frac{d^{\beta}}{dt^{\beta}}a(t)+a(t)\right]+\cosh x\left[\frac{d^{2\beta}}{dt^{2\beta}}b(t)+2\frac{d^{\beta}}{dt^{\beta}}b(t)\right]=0.$$

Since 1 and $\cosh x$ are linearly independent, we obtain a system of time-fractional ordinary differential equations

$$\frac{d^{2\beta}}{dt^{2\beta}}a(t)+2\frac{d^{\beta}}{dt^{\beta}}a(t)=-a(t),\ \ a(0)=a'(0)=0, \tag{3.30}$$

$$\frac{d^{2\beta}}{dt^{2\beta}}b(t)+2\frac{d^{\beta}}{dt^{\beta}}b(t)=0,\ \ b(0)=1,b'(0)=-2, \tag{3.31}$$

where a system of fractional ordinary differential equations has found in a example (3.2.4), we have

$$a(t)=0,\ \ b(t)=1-2tE_{\beta,2}(-2t^{\beta}).$$

Therefore, the exact solution of equation (3.29) is

$$u(x,t)=\left[1-2tE_{\beta,2}(-2t^{\beta})\right]\cosh x.$$

In particular, if $\beta=1$, then the exact solution of classical telegraph equation is

$$u(x,t)=e^{-2t}\cosh x.$$

Example 3.2.6. Consider the time-fractional telegraph equation with $0<\beta\le 1$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^{2\beta}u}{\partial t^{2\beta}}+2\frac{\partial^{\beta}u}{\partial t^{\beta}}+u,\ \ x>0,\ \ t>0, \tag{3.32}$$

subject to the initial conditions

$$u(x,0)=\cos x,\frac{\partial u(x,0)}{\partial t}=\sin x.$$

Under the operator

$$F[u]=\frac{\partial^2 u}{\partial x^2}-u,$$

we choose the invariant subspace

$$W_3=L\big\{1,\sin x,\cos x\big\}.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_3, that is,

$$u(x,t)=a(t)+b(t)\sin x+c(t)\cos x.$$

By using the initial conditions

- $u(x,0) = \cos x \Rightarrow a(0) + b(0)\sin x + c(0)\cos x = \cos x$

$$\Rightarrow a(0) = b(0) = 0, c(0) = 1,$$

- $\dfrac{\partial}{\partial t}u(x,0) = \sin x \Rightarrow a'(0) + b'(0)\sin x + c'(0)\cos x = \sin x$

$$\Rightarrow a'(0) = 0, b'(0) = 1, c'(0) = 0.$$

Substituting $u(x,t)$ into the equation (3.32), we get

$$\frac{d^{2\beta}a(t)}{dt^{2\beta}} + \sin x\frac{d^{2\beta}b(t)}{dt^{2\beta}} + \cos x\frac{d^{2\beta}c(t)}{dt^{2\beta}} + 2\frac{d^{\beta}a(t)}{dt^{\beta}} + 2\sin x\frac{d^{\beta}b(t)}{dt^{\beta}} + 2\cos x\frac{d^{\beta}c(t)}{dt^{\beta}}$$
$$= -a(t) - 2b(t)\sin x - 2c(t)\cos x$$
$$\left[\frac{d^{2\beta}a(t)}{dt^{2\beta}} + 2\frac{d^{\beta}a(t)}{dt^{\beta}} + a(t)\right] + \sin x\left[\frac{d^{2\beta}b(t)}{dt^{2\beta}} + 2\frac{d^{\beta}b(t)}{dt^{\beta}} + 2b(t)\right]$$
$$+ \cos x\left[\frac{d^{2\beta}c(t)}{dt^{2\beta}} + 2\frac{d^{\beta}c(t)}{dt^{\beta}} + 2c(t)\right] = 0.$$

Since 1, $\sin x$ and $\cos x$ are linearly independent functions, we get a system of fractional ordinary differential equations

$$\frac{d^{2\beta}a(t)}{dt^{2\beta}} + 2\frac{d^{\beta}a(t)}{dt^{\beta}} = -a(t), \;\; a(0) = a'(0) = 0, \tag{3.33}$$
$$\frac{d^{2\beta}b(t)}{dt^{2\beta}} + 2\frac{d^{\beta}b(t)}{dt^{\beta}} = -2b(t), \;\; b(0) = 0, \;\; b'(0) = 1, \tag{3.34}$$
$$\frac{d^{2\beta}c(t)}{dt^{2\beta}} + 2\frac{d^{\beta}c(t)}{dt^{\beta}} = -2c(t), \;\; c(0) = 1, \;\; c'(0) = 0. \tag{3.35}$$

Applying the Laplace transform to both sides of equation (3.33), we obtain

$$\mathcal{L}\left\{\frac{d^{2\beta}a(t)}{dt^{2\beta}}; s\right\} + 2\mathcal{L}\left\{\frac{d^{\beta}a(t)}{dt^{\beta}}; s\right\} = -\mathcal{L}\{a(t); s\}$$
$$s^{2\beta}A(s) - s^{2\beta-1}a(0) - s^{2\beta-2}a'(0) + 2s^{\beta}A(s) - 2s^{\beta-1}a(0) = -A(s)$$
$$A(s) = 0.$$

Taking inverse Laplace transform yields

$$a(x) = 0.$$

Applying the Laplace transform to both sides of equation (3.34), we get

$$\mathcal{L}\left\{\frac{d^{2\beta}b(t)}{dt^{2\beta}}; s\right\} + 2\mathcal{L}\left\{\frac{d^{\beta}b(t)}{dt^{\beta}}; s\right\} = -2\mathcal{L}\{b(t); s\}$$
$$s^{2\beta}B(s) - s^{2\beta-1}b(0) - s^{2\beta-2}b'(0) + 2s^{\beta}B(s) - 2s^{\beta-1}b(0) = -2B(s)$$
$$B(s)\left[s^{2\beta} + 2s^{\beta} + 2\right] = s^{2\beta-2}$$

$$\begin{aligned} B(s) &= \frac{s^{2\beta-2}}{s^{2\beta}+2s^{\beta}+2} \\ &= \frac{s^{2\beta-2}}{(s^{\beta}+1)^2+1} \\ &= \frac{s^{2\beta-2}}{(s^{\beta}+1)^2}\left[\frac{1}{1+\frac{1}{(s^{\beta}+1)^2}}\right]. \end{aligned}$$

We have

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n.$$

Then

$$\frac{1}{1+\dfrac{1}{(s^{\beta}+1)^2}} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(s^{\beta}+1)^{2n}}.$$

Hence

$$B(s) = \sum_{n=0}^{\infty}(-1)^n \frac{s^{2\beta-2}}{(s^{\beta}+1)^{2n+2}}.$$

By using

$$\mathcal{L}\{z^{\alpha n+\beta-1}E_{\alpha,\beta}^{(n)}(\pm\lambda z^{\alpha}); s\} = \frac{n!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{n+1}},$$

we get

$$B(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathcal{L}\big\{t^{2n\beta+1}E_{\beta,2-\beta}^{(2n+1)}(-t^{\beta})\big\}.$$

Applying the inverse Laplace transform to both sides, we get

$$b(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n\beta+1}E_{\beta,2-\beta}^{(2n+1)}(-t^{\beta}).$$

Applying the Laplace transform to both sides of equation (3.35), we obtain

$$\begin{aligned} \mathcal{L}\left\{\frac{d^{2\beta}c(t)}{dt^{2\beta}}; s\right\} + 2\mathcal{L}\left\{\frac{d^{\beta}c(t)}{dt^{\beta}}; s\right\} &= -2\mathcal{L}\big\{c(t); s\big\} \\ s^{2\beta}C(s) - s^{2\beta-1}c(0) - s^{2\beta-2}c'(0) + 2s^{\beta}C(s) - 2s^{\beta-1}c(0) &= -2C(s) \\ C(s)\big[s^{2\beta}+2s^{\beta}+2\big] &= s^{2\beta-1}+2s^{\beta-1} \end{aligned}$$

$$\begin{aligned} C(s) &= \frac{s^{2\beta-1}+2s^{\beta-1}}{s^{2\beta}+2s^{\beta}+2} \\ &= \frac{s^{\beta-1}(s^{\beta}+1)+s^{\beta-1}}{(s^{\beta}+1)^2+1} \\ &= \frac{s^{\beta-1}(s^{\beta}+1)}{(s^{\beta}+1)^2+1}+\frac{s^{\beta-1}}{(s^{\beta}+1)^2+1} \\ &= \frac{s^{\beta-1}(s^{\beta}+1)}{(s^{\beta}+1)^2}\left[\frac{1}{1+\frac{1}{(s^{\beta}+1)^2}}\right]+\frac{s^{\beta-1}}{(s^{\beta}+1)^2}\left[\frac{1}{1+\frac{1}{(s^{\beta}+1)^2}}\right] \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{s^{\beta-1}}{(s^{\beta}+1)^{2n+1}}+\sum_{n=0}^{\infty}(-1)^n\frac{s^{\beta-1}}{(s^{\beta}+1)^{2n+2}}. \end{aligned}$$

We have

$$\mathcal{L}\{z^{\alpha n+\beta-1}E^{(n)}_{\alpha,\beta}(\pm\lambda z^{\alpha});s\} = \frac{n!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{n+1}}.$$

Thus

$$C(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\mathcal{L}\left\{t^{2\beta n}E^{(2n)}_{\beta,1}(-t^{\beta})\right\}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathcal{L}\left\{t^{(2n+1)\beta}E^{(2n+1)}_{\beta,1}(-t^{\beta})\right\}.$$

Taking the inverse Laplace transform yields

$$c(t) = \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2\beta n}}{(2n)!}E^{(2n)}_{\beta,1}(-t^{\beta})+\frac{t^{(2n+1)\beta}}{(2n+1)!}E^{(2n+1)}_{\beta,1}(-t^{\beta})\right].$$

Therefore, the solution of equation (3.32) is

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2n\beta+1}}{(2n+1)!}E^{(2n+1)}_{\beta,2-\beta}(-t^{\beta})\right]\sin x \\ &+\sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2\beta n}}{(2n)!}E^{(2n)}_{\beta,1}(-t^{\alpha})+\frac{t^{(2n+1)\beta}}{(2n+1)!}E^{(2n+1)}_{\beta,1}(-t^{\beta})\right]\cos x. \end{aligned}$$

If $\beta = 1$, then the solution of classical telegraph equation is

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2n+1}}{(2n+1)!}E^{(2n+1)}_{1,1}(-t)\right]\sin x \\ &+\sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2n}}{(2n)!}E^{(2n)}_{1,1}(-t)+\frac{t^{2n+1}}{(2n+1)!}E^{(2n+1)}_{1,1}(-t)\right]\cos x \\ &= -e^{-t}\sin(t)\sin x+e^{-t}\cos t\cos x-e^{-t}\sin t\cos x \\ &= e^{-t}\cos(t+x)-e^{-t}\sin t\cos x. \end{aligned}$$

Example 3.2.7. (Das et al. [13]) Consider time-fractional telegraph equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^\mu u}{\partial t^\mu} + \frac{\partial^{\mu-1} u}{\partial t^{\mu-1}} + u + \frac{t^n}{n!}\sinh x, \;\; 1 < \mu < 2, \tag{3.36}$$

subject to initial condition

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0.$$

Under the operator

$$F[u] = \frac{\partial^2 u}{\partial x^2} - u,$$

we choose the invariant subspace

$$W_2 = L\{1, \sinh x\}.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2, that is,

$$u(x,t) = a(t) + b(t)\sinh x.$$

It follows the initial condition $u(x,0) = 0$, we have

$$a(0) + b(0)\sinh x = 0 \Rightarrow a(0) = b(0) = 0.$$

And another initial condition $\dfrac{\partial u(x,0)}{\partial t} = 0$, we have

$$a'(0) + b'(0)\sinh x = 0 \Rightarrow a'(0) = b'(0) = 0.$$

Substituting $u(x,t)$ into the equation (3.36), we obtain

$$\frac{d^\mu a(t)}{dt^\mu} + \sinh x \frac{d^\mu b(t)}{dt^\mu} + \frac{d^{\mu-1} a(t)}{dt^{\mu-1}} + \sinh x \frac{d^{\mu-1} b(t)}{dt^{\mu-1}} = -a(t) + \frac{t^n}{n!}\sinh x$$

$$\left[\frac{d^\mu a(t)}{dt^\mu} + \frac{d^{\mu-1} a(t)}{dt^{\mu-1}} + a(t)\right] + \sinh x \left[\frac{d^\mu b(t)}{dt^\mu} + \frac{d^{\mu-1} b(t)}{dt^{\mu-1}} - \frac{t^n}{n!}\right] = 0.$$

Since 1 and $\sinh x$ are linearly independent functions, we get a system of fractional ordinary differential equations

$$\frac{d^\mu a(t)}{dt^\mu} + \frac{d^{\mu-1} a(t)}{dt^{\mu-1}} = -a(t), \;\; a(0) = a'(0) = 0, \tag{3.37}$$

$$\frac{d^\mu b(t)}{dt^\mu} + \frac{d^{\mu-1} b(t)}{dt^{\mu-1}} = \frac{t^n}{n!}, \;\; b(0) = b'(0) = 0. \tag{3.38}$$

Applying Laplace transform to both sides of equation (3.37), we obtain

$$\begin{aligned}\mathcal{L}\left\{\frac{d^{\mu}a(t)}{dt^{\mu}};s\right\}+\mathcal{L}\left\{\frac{d^{\mu-1}a(t)}{dt^{\mu-1}};s\right\}&=-\mathcal{L}\{a(t);s\}\\ s^{\mu}A(s)-s^{\mu-1}a(0)-s^{\mu-2}a'(0)+s^{\mu-1}A(s)-s^{\mu-2}a(0)&=-A(s)\\ A(s)\big[s^{\mu}+s^{\mu-1}+1\big]&=0\\ A(s)&=0.\end{aligned}$$

Applying inverse Laplace transform yields

$$a(t)=0.$$

Taking Laplace transform to both sides of equation (3.38), we get

$$\begin{aligned}\mathcal{L}\left\{\frac{d^{\mu}b(t)}{dt^{\mu}};s\right\}+\mathcal{L}\left\{\frac{d^{\mu-1}b(t)}{dt^{\mu-1}};s\right\}&=\mathcal{L}\left\{\frac{t^{n}}{n!};s\right\}\\ s^{\mu}B(s)-s^{\mu-1}b(0)-s^{\mu-2}b'(0)+s^{\mu-1}B(s)+s^{\mu-2}b(0)&=\frac{1}{n!}\left(\frac{n!}{s^{n+1}}\right)\\ B(s)\big[s^{\mu}+s^{\mu-1}\big]&=\frac{1}{s^{n+1}}\\ B(s)&=\frac{1}{s^{n+1}\big(s^{\mu}+s^{\mu-1}\big)}\\ &=\frac{1}{s^{n+1}s^{\mu-1}(s+1)}\\ &=\frac{s^{-(n+\mu)}}{s+1}.\end{aligned}$$

By using

$$\mathcal{L}\{z^{\beta-1}E_{\alpha,\beta}(\pm\lambda z^{\alpha});s\}=\frac{s^{\alpha-\beta}}{s^{\alpha}\mp\lambda},$$

we have

$$B(s)=\mathcal{L}\{t^{n+\mu}E_{1,n+\mu+1}(-t)\}$$

Taking inverse Laplace transform yields

$$b(t)=t^{n+\mu}E_{1,n+\mu+1}(-t).$$

Therefore, the solution of equation (3.36) is

$$u(x,t)=\big[t^{n+\mu}E_{1,n+\mu+1}(-t)\big]\sinh x,$$

which is the same solution obtained by the homotopy analysis method by Das et al. [13].

If $\mu = 2$, then the solution of classical telegraph equation is

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}(-1)^k \frac{t^{k+n+2}}{(k+n+2)!}\sinh x \\ &= \sum_{m=n+2}^{\infty}(-1)^{m-(n+2)}\frac{t^m}{m!}\sinh x, \quad k = m-(n+2) \\ &= (-1)^{-(n+2)}\sum_{m=n+2}^{\infty}(-1)^m\frac{t^m}{m!}\sinh x \\ &= (-1)^{-(n+2)}\left[e^{-t} - \sum_{m=0}^{n+1}\frac{(-t)^m}{m!}\right]\sinh x. \end{aligned}$$

3.3 The space and time-fractional telegraph equations

Consider the space and time-fractional telegraph equation with $0 < \alpha, \beta \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \quad x > 0, \quad t > 0, \tag{3.39}$$

where $\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}$ and $\frac{\partial^{2\beta}}{\partial t^{2\beta}}$ are space-fractional and time-fractional derivatives in the Caputo sense, respectively.

If $\alpha = 1$, then equation (3.39) becomes to time-fractional telegraph equation of the form

$$\frac{\partial^{2} u}{\partial x^{2}} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u.$$

If $\beta = 1$, then equation (3.39) becomes to space-fractional telegraph equation of the form

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^{2} u}{\partial t^{2}} + 2\frac{\partial u}{\partial t} + u.$$

As a means to find the solution of the space and time-fractional telegraph equation (3.39) by using the invariant subspace method, we need to choose the operator

$$F[u] = \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - u. \tag{3.40}$$

The way to obtain an exact solution of equation (3.39) by using the invariant subspace method will be shown in the following theorem.

Theorem 3.3.1. *The space and time-fractional telegraph equation (3.39) admits a solution of the form*

$$u(x,t) = c_1(t) + c_2(t)x^{2\alpha},$$

where $c_1(t)$, $c_2(t)$ are solutions of the following system of fractional ordinary differential equations

$$\begin{cases} \dfrac{d^{2\beta}c_1(t)}{dt^{2\beta}} + 2\dfrac{d^{\beta}c_1(t)}{dt^{\beta}} &= c_2(t)\Gamma(2\alpha+1) - c_1(t), \\ \dfrac{d^{2\beta}c_2(t)}{dt^{2\beta}} + 2\dfrac{d^{\beta}c_2(t)}{dt^{\beta}} &= -c_2(t). \end{cases} \tag{3.41}$$

Proof. The subspace $W_2 = L\{1, x^{2\alpha}\}$ is invariant under the differential operator $F[.]$ defined by (3.40) because

$$F\big[c_1 + c_2x^{2\alpha}\big] = \big[c_2\Gamma(2\alpha+1) - c_1\big] - c_2x^{2\alpha} \in W_2.$$

Assume the solution $u(x,t)$ as a linear combination of the elements in the invariant subspace W_2, that is,

$$u(x,t) = c_1(t) + c_2(t)x^{2\alpha}. \tag{3.42}$$

Then we have

$$F[u(x,t)] = \big[c_2(t)\Gamma(2\alpha+1) - c_1(t)\big] - c_2(t)x^{2\alpha}. \tag{3.43}$$

Taking the fractional derivative of order 2β and β with respect to t in both sides of equation (3.42), we sum them together, we obtain

$$\frac{d^{2\beta}u(x,t)}{dt^{2\beta}} + 2\frac{d^{\beta}u(x,t)}{dt^{\beta}} = \frac{d^{2\beta}c_1(t)}{dt^{2\beta}} + x^{2\alpha}\frac{d^{2\beta}c_2(t)}{dt^{2\beta}} + 2\frac{d^{\beta}c_1(t)}{dt^{\beta}} + 2x^{2\alpha}\frac{d^{\beta}c_2(t)}{dt^{\beta}}. \tag{3.44}$$

Substituting equation (3.44) and (3.43) in equation (3.39), we get

$$\left[\frac{d^{2\beta}c_1(t)}{dt^{2\beta}} + 2\frac{d^{\beta}c_1(t)}{dt^{\beta}} - c_2(t)\Gamma(2\alpha+1) + c_1(t)\right] + x^{2\alpha}\left[\frac{d^{2\beta}c_2(t)}{dt^{2\beta}} + 2\frac{d^{\beta}c_2(t)}{dt^{\beta}} + c_2(t)\right] = 0.$$

Since 1 and $x^{2\alpha}$ are linearly independent functions, we get a system of fractional ordinary differential equations (3.41). □

Remark 3.3.2. Hence, this theorem can be stated in a similar way when we choose other invariant subspaces with respect to the operator (3.40). Under the operator (3.40), we classify all possibilities of invariant subspaces as follows

1. The subspace $W_2^1 = L\{1, E_{2\alpha}(ax^{2\alpha})\},\ \ a \neq 0$ is invariant under F because

$$\begin{aligned} F\big[c_1 + c_2E_{2\alpha}(ax^{2\alpha})\big] &= \frac{d^{2\alpha}}{dx^{2\alpha}}\Big[c_1 + c_2E_{2\alpha}(ax^{2\alpha})\Big] - \Big[c_1 + c_2E_{2\alpha}(ax^{2\alpha})\Big] \\ &= ac_2E_{2\alpha}(ax^{2\alpha}) - c_1 - c_2E_{2\alpha}(ax^{2\alpha}) \\ &= -c_1 + \big[ac_2 - c_2\big]E_{2\alpha}(ax^{2\alpha}) \in W_2^1. \end{aligned}$$

2. The subspace $W_2^2 = L\{1, E_{2\alpha}(x^{2\alpha})\}$ is invariant under F because

$$F\big[c_1 + c_2 E_{2\alpha}(x^{2\alpha})\big] = \frac{d^{2\alpha}}{dx^{2\alpha}}\Big[c_1 + c_2 E_{2\alpha}(x^{2\alpha})\Big] - \Big[c_1 + c_2 E_{2\alpha}(x^{2\alpha})\Big]$$
$$= c_2 E_{2\alpha}(x^{2\alpha}) - c_1 - c_2 E_{2\alpha}(x^{2\alpha}) = -c_1 \in W_2^2.$$

3. The subspace $W_n = L\{1, x^{2\alpha}, \cdots, x^{(2n)\alpha}\}$ is invariant under F because

$$F\big[c_1 + c_2 x^{2\alpha} + \cdots + c_{n+1} x^{(2n)\alpha}\big] = c_2 \Gamma(2\alpha + 1) - c_1$$
$$- \cdots - c_{n+1} x^{(2n)\alpha} \in W_n.$$

The usefulness of these distinct invariant subspaces is that, by choosing an appropriate invariant subspace, we can solve the space and time-fractional telegraph equation with respect to distinct initial conditions.

In the following example is the same as the time-fractional telegraph equation when the space-fractional order derivative closes to one.

Example 3.3.3. Consider the following space and time-fractional telegraph equation with $0 < \alpha, \beta \leq 1$

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^{2\beta} u}{\partial t^{2\beta}} + 2\frac{\partial^{\beta} u}{\partial t^{\beta}} + u, \quad x > 0, \quad t > 0, \tag{3.45}$$

subject to the initial conditions

$$u(x, 0) = E_{2\alpha,1}(x^{2\alpha}), u_t(x, 0) = -2E_{2\alpha,1}(x^{2\alpha}).$$

Under the operator

$$F[u] = \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - u,$$

we choose the invariant subspace

$$W_2^2 = L\{1, E_{2\alpha,1}(x^{2\alpha})\}.$$

Assume the solution $u(x, t)$ as a linear combination of the elements in the invariant subspace W_2^2, that is,

$$u(x, t) = a(t) + b(t)E_{2\alpha,1}(x^{2\alpha}).$$

By using the initial conditions

- $u(x, 0) = x^{2\alpha}$, we have

$$a(0) + b(0)E_{2\alpha,1}(x^{2\alpha}) = E_{2\alpha,1}(x^{2\alpha}) \Rightarrow a(0) = 0, b(0) = 1,$$

- $u_t(x,0) = -2E_{2\alpha,1}(x^{2\alpha})$, we have

$$a'(0) + b'(0)E_{2\alpha,1}(x^{2\alpha}) = -2E_{2\alpha,1}(x^{2\alpha}) \Rightarrow a'(0) = 0, b'(0) = -2.$$

Substituting $u(x,t)$ into the equation (3.45), we obtain

$$\frac{d^{2\beta}}{dt^{2\beta}}\left[a(t) + b(t)E_{2\alpha,1}(x^{2\alpha})\right] + 2\frac{d^{\beta}}{dt^{\beta}}\left[a(t) + b(t)E_{2\alpha,1}(x^{2\alpha})\right] = -a(t)$$
$$\left[\frac{d^{2\beta}a(t)}{dt^{2\beta}} + 2\frac{d^{\beta}a(t)}{dt^{\beta}} + a(t)\right] + E_{2\alpha,1}(x^{2\alpha})\left[\frac{d^{2\beta}b(t)}{dt^{2\beta}} + 2\frac{d^{\beta}b(t)}{dt^{\beta}}\right] = 0.$$

Since 1 and $E_{2\alpha,1}(x^{2\alpha})$ are linearly independent functions, we get a system of fractional ordinary differential equations.

$$\frac{d^{2\beta}a(t)}{dt^{2\beta}} + 2\frac{d^{\beta}a(t)}{dt^{\beta}} = -a(t), \;\; a(0) = 0, a'(0) = 0, \tag{3.46}$$
$$\frac{d^{2\beta}b(t)}{dt^{2\beta}} + 2\frac{d^{\beta}b(t)}{dt^{\beta}} = 0, \;\; b(0) = 1, b'(0) = -2. \tag{3.47}$$

Applying Laplace transform to both sides of equation (3.47), we get

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\beta}b(t)}{dt^{2\beta}}; s\right\} &= -2\mathcal{L}\left\{\frac{d^{\beta}b(t)}{dt^{\beta}}; s\right\} \\
s^{2\beta}B(s) - s^{2\beta-1}b(0) - s^{2\beta-2}b'(0) &= -2s^{\beta}B(s) + 2s^{\beta-1}b(0) \\
B(s)\left[s^{2\beta} + 2s^{\beta}\right] &= s^{2\beta-1} - 2s^{2\beta-2} + 2s^{\beta-1} \\
B(s) &= \frac{s^{2\beta-1} - 2s^{2\beta-2} + 2s^{\beta-1}}{s^{2\beta} + 2s^{\beta}} \\
&= \frac{s^{\beta-1} - 2s^{\beta-2} + 2s^{-1}}{s^{\beta} + 2} \\
&= \frac{s^{\beta-1}}{s^{\beta}+2} - 2\frac{s^{\beta-2}}{s^{\beta}+2} + \frac{2}{s(s^{\beta}+2)} \\
&= \frac{s^{\beta-1}}{s^{\beta}+2} - 2\frac{s^{\beta-2}}{s^{\beta}+2} + \frac{1}{s} - \frac{s^{\beta-1}}{s^{\beta}+2} \\
&= \frac{1}{s} - 2\frac{s^{\beta-2}}{s^{\beta}+2}.
\end{aligned}$$

By using (2.5), we have

$$B(s) = \mathcal{L}\{1\} - 2\mathcal{L}\{tE_{\beta,2}(-2t^{\beta})\}.$$

Taking inverse Laplace transform yields

$$b(t) = 1 - 2tE_{\beta,2}(-2t^{\beta}).$$

Applying Laplace transform to both sides of equation (3.46), we obtain

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^{2\beta}a(t)}{dt^{2\beta}};s\right\} + 2\mathcal{L}\left\{\frac{d^{\beta}a(t)}{dt^{\beta}};s\right\} &= -\mathcal{L}\{a(t)\} \\
s^{2\beta}A(s) - s^{2\beta-1}a(0) - s^{2\beta-2}a'(0) + 2s^{\beta}A(s) - 2s^{\beta-1}a(0) &= -A(s) \\
A(s)\left[s^{2\beta} - 2s^{\beta} + 1\right] &= 0 \\
A(s) &= 0.
\end{aligned}$$

Taking inverse Laplace transform yields

$$a(t) = 0.$$

Therefore, the solution of equation (3.45) is

$$u(x,t) = \left[1 - 2tE_{\beta,2}(-2t^{\beta})\right]E_{2\alpha,1}(x^{2\alpha}).$$

If $\alpha = 1$, then the solution of equation (3.45) is

$$u(x,t) = \left[1 - 2tE_{\beta,2}(-2t^{\beta})\right]\cosh x,$$

which is the same as solution of time-fractional telegraph equation in the previous example (3.2.5).

Chapter 4

Conclusions

The objective of this thesis was to construct exact solutions of three classes of fractional telegraph equations, i.e., space-, time-, and space and time-fractional telegraph equations. The invariant subspace method by Galaktinov and Svirshchevskii [14] was mainly used in our study.
The following are the summarized results we have obtained:

1. In theorem 3.1.1, we have constructed an exact solution of space-fractional telegraph equation by using the invariant subspace method under the invariant subspace in time $W_n = L\{1, t, ..., t^n\}$.

2. In remark 3.1.2, we have given some invariant subspace in time.

3. In example 3.1.3-3.1.5, we have applied the invariant subspace method to derive solutions to space-fractional telegraph equation with different boundary conditions.

4. In theorem 3.2.1, we have derived an exact solution of time-fractional telegraph equation by using the invariant subspce method along with the invariant subspace $W_3^3 = L\{1, e^{ax}, xe^{ax}\}$.

5. In remark 3.2.2, we have listed other invariant subspace in space.

6. In example 3.2.3-3.2.6, we have derived explicit solutions of time-fractional telegraph equation with different initial conditions.

7. In theorem 3.3.1, we have combined both space- and time-fractional derivatives in the telegraph equation and shown an exact solution by using the invariant subspace method under the invariant subspace in space $W_2 = L\{1, x^{2\alpha}\}$.

8. In remark 3.3.2, we have given other invariant subspace in space.

9. In example 3.3.3, we have modified an example of time-fractional telegraph equation by replacing the integer order in space with fractional order and derived the solution. In particular, we have shown that the obtained solution closes to the solution in time-fractional telegraph equation when the space-fractional order derivative closes to one.

Bibliography

[1] Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. Physics reports. 2000 Dec 1;339(1):1-77.

[2] Chen W, Sun H, Zhang X, Koroak D. Anomalous diffusion modeling by fractal and fractional derivatives. Computers and Mathematics with Applications. 2010 Mar 1;59(5):1754-8.

[3] Ishimaru A. Diffusion of a pulse in densely distributed scatterers. JOSA. 1978 Aug 1;68(8):1045-50.

[4] Ishimaru A. Diffusion of light in turbid material. Applied optics. 1989 Jun 15;28(12):2210-5.

[5] Holmes EE. Are diffusion models too simple? A comparison with telegraph models of invasion. The American Naturalist. 1993 Nov 1;142(5):779-95.

[6] Van Gorder RA, Vajravelu K. Analytical and numerical solutions of the density dependent Nagumo telegraph equation. Nonlinear Analysis: Real World Applications. 2010 Oct 1;11(5):3923-9.

[7] Baumeister KJ, Hamill TD. Discussion:Hyperbolic Heat-Conduction EquationA Solution for the Semi-Infinite Body Problem(Baumeister, KJ, and Hamill, TD, 1969, ASME J. Heat Transfer, 91, pp. 543548). Journal of Heat Transfer. 1971 Feb 1;93(1):126-7.

[8] Barletta A, Zanchini E. A thermal potential formulation of hyperbolic heat conduction. Journal of heat transfer. 1999 Feb 1;121(1):166-9.

[9] S. Momani, Analytic and approximate solutions of the space-and time-fractional telegraph equations, Appl. Math. Comput. 170 (2005) 1126-34.

[10] J. Chen, F. Liu and V. Anh, Analytical solution for the time-fractional telegraph equation by the method of separating variables, J. Math. Anal. Appl. 338 (2008) 1364-77.

[11] V.K. Srivastava, M. K. Awasthi and M. Tamsir, RDTM solutions of Caputo time fractional-order hyperbolic telegraph equation, AIP Advances. 3 032142(2013).

[12] S. Kumar, A new analytical modelling for fractional telegraph equation via Laplace transform, Applied Mathematical Modelling. 38(2014) 3154-3163.

[13] S. Das, K. Vishal, and PK. Gupta, Yildirim A. An approximate analytical solution of time-fractional telegraph equation. Applied Mathematics and Computation. 2011 May 15;217(18):7405-11.

[14] V. Galaktionov and S. Svirshchevskii, Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics, Chapman and Hall/CRC, Boca Raton, 2007.

[15] R. K. Gazizov and A. A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method, Computors and Mathematics with Applications. 66 (2013) 576-584.

[16] R. Sahadevan and T. Bakkyaraj, Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations, Fractional Calculus and Applied Analysis. 18 (2015) 146-162.

[17] R. Sahadevan and P. Prakash, Exact solution of certain time fractional nonlinear partial differential equations, Nonlinear Dynamics. 85 (2016) 659-673.

[18] S. Choudhary and V. Daftardar-Gejji, Invariant subspace method: a tool for solving fractional partial differential equations, Fractional Calculus and Applied Analysis. 20 (2017) 477-93.

[19] M. Ishteva, Properties and applications of the Caputo fractional operator. Department of Mathematics, University of Karlsruhe, Karlsruhe. Feb. (2005).

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