Some Properties of Ordered Γ-semigroups

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Thesis Title Some Properties of Ordered Γ-semigroups

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บทคัดย่อ

กำหนดให้ S และ Γ เป็นเซตไม่ว่างและ $\leq$ เป็นความสัมพันธ์บน S เราเรียก $(S, \Gamma, \leq)$ ว่า แกมมากึ่งกรุปอันดับ ถ้า (S, Γ) เป็นแกมมากึ่งกรุปและ $(S, \leq)$ เป็นเซตอันดับบางส่วนโดยที่ ถ้า $a \leq b$ แล้ว $a\gamma c \leq b\gamma c$ และ $c\gamma a \leq c\gamma b$ สำหรับทุก $a, b, c \in S$ และสำหรับทุก $\gamma \in \Gamma$

ในการทำวิจัยนี้ เราได้ให้ทฤษฎีบทสมสัณฐานของแกมมากึ่งกรุปและแกมมากึ่งกรุปอันดับ ยิ่งไปกว่านั้น เราได้ให้ความสัมพันธ์บางประการระหว่างอันดับเทียมและแกมมากึ่งกรุปอันดับ

นอกจากนี้ เราศึกษาไบไอดีล ไบไอดีลเล็กสุดเฉพาะกลุ่ม 0-ไบไอดีลเล็กสุดเฉพาะกลุ่ม และไบไอดีลใหญ่สุดเฉพาะกลุ่มในแกมมากึ่งกรุปอันดับ

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ABSTRACT

Let S and Γ be nonempty sets and $\leq$ a relation on S. Then $(S, \Gamma, \leq)$ is called an ordered Γ-semigroup if (S, Γ) is a semigroup and $(S, \leq)$ is a partially ordered set such that

$$a \leq b \ \Rightarrow \ a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b \text{ for all } a, b, c \in S \text{ and } \gamma \in \Gamma.$$

In this thesis, we give isomorphism theorems of Γ-semigroups and ordered Γ-semigroups. Moreover, we give some connections between pseudoorder and ordered Γ-semigroups.

Furthermore, we study bi-ideals, minimal bi-ideals, 0-minimal bi-ideals and maximal bi-ideals in ordered Γ-semigroups.

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Kittisak Tinpun

CONTENTS

CHAPTER 1

Introduction and Preliminaries

The notion of Γ-semigroups was introduced by M. K. Sen in the year 1981 (Sen, 1981). Γ-semigroups generalize semigroups. Many classical notions of semigroups have been extended to Γ-semigroups. In fact, any semigroup S can be considered to be a Γ-semigroup, by define $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. On the other hand, let S be a Γ-semigroup and α a fixed element in Γ. We define $ab = a\alpha b$ for all $a, b \in S$, then we can see that S is a semigroup.

In this thesis, we give isomorphism theorems of Γ-semigroups and ordered Γ-semigroups, and also give some properties of ordered Γ-semigroups. Moreover, we give some connections between pseudoorder and ordered Γ-semigroups.

Furthermore, we study bi-ideals, minimal bi-ideals, 0-minimal bi-ideals and maximal bi-ideals in ordered Γ-semigroups.

1.1 Semigroups

We will use the notation and terminology of Howie (Howie, 1976) to introduce the notion of a semigroup as follows :

Definition 1.1. Let S be a nonempty set and $*$ a binary operation on S. $(S, *)$ is called a *semigroup* if $*$ is associative, i.e.,

$$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in S.$$

Example 1.1. $(\mathbb{N}, +)$ and $(\mathbb{R}, \times)$ are semigroups.

Example 1.2. $(\mathbb{Z}, -)$ is not a semigroup since for $a, b, c \in \mathbb{Z}$ such that $c \neq 0$, we have

$$a - (b - c) = a - b + c \neq a - b - c = (a - b) - c.$$

Definition 1.2. Let S be a semigroup. A nonempty subset T of S is called a *subsemigroup* of S if T is closed under the binary operation of S, that is, $ab \in T$ for all $a, b \in T$.

Definition 1.3. Let A be a nonempty set. An arbitrary subset of $A \times A$ is called a *relation* on A.

Definition 1.4. Let S be a semigroup. A relation ρ on S is called an *equivalence relation* on S if

(1) $a\rho a$ for all $a \in S$ (reflexive) ;

(2) $a\rho b$ implies $b\rho a$ for all $a, b \in S$ (symmetric) ;

(3) $a\rho b$ and $b\rho c$ imply $a\rho c$ for all $a, b, c \in S$ (transitive).

We will use the notation and terminology of Howie (Howie, 1976) to introduce congruences and isomorphism theorems for semigroups as follows :

Definition 1.5. Let S be a semigroup. An equivalence relation ρ on S is called a *right congruence* on S if

$$(a, b) \in \rho \text{ implies } (at, bt) \in \rho \quad \text{for all } a, b, t \in S,$$

and an equivalence relation ρ on S is called a *left congruence* on S if

$$(a, b) \in \rho \text{ implies } (ta, tb) \in \rho \quad \text{for all } a, b, t \in S.$$

An equivalence relation ρ on S is called a *congruence* on S if ρ is both a right and left congruence on S.

Example 1.3. Let ρ be a relation on a semigroup $(\mathbb{N}, +)$ defined by

$$a\rho b \Leftrightarrow 4|a - b \quad \text{for all } a, b \in \mathbb{N}.$$

We have ρ is a right congruence on $\mathbb{N}$ since for $a, b, t \in \mathbb{N}$,

$$\begin{aligned} (a, b) \in \rho &\Rightarrow 4|a - b \\ &\Rightarrow 4x = a - b \quad \text{for some } x \in \mathbb{N} \\ &\Rightarrow 4x = (a + t) - (b + t) \\ &\Rightarrow 4|(a + t) - (b + t) \\ &\Rightarrow (a + t, b + t) \in \rho. \end{aligned}$$

A similar argument shows that ρ is a left congruence on $\mathbb{N}$. Hence ρ is a congruence on $\mathbb{N}$.

Definition 1.6. Let S be a semigroup and ρ a congruence on S. Then we have

$$S/\rho = \{a\rho \mid a \in S\}.$$

Theorem 1.1. *Let S be a semigroup and ρ a congruence on S. For $a\rho, b\rho \in S/\rho$, let $(a\rho)(b\rho) = (ab)\rho$. Then S/ρ is a semigroup.*

Definition 1.7. Let S be a semigroup. A subsemigroup A of S is called a *left (resp. right) ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$). A is called an *ideal* of S if A is both a left and right ideal of S.

Example 1.4. Let $\mathbb{Z}$ be the set of all integers and $M_2(\mathbb{Z})$, the set of all 2×2 matrices over $\mathbb{Z}$. We have known that $M_2(\mathbb{Z})$ is a semigroup under the usual multiplication. Let

$$L = \{\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in \mathbb{Z}\} \text{ and } R = \{\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{Z}\}.$$

Then L is a left ideal of $M_2(\mathbb{Z})$ and R is a right ideal of $M_2(\mathbb{Z})$.

Definition 1.8. Let S be a semigroup. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$.

Example 1.5. Let $S = [0, 1]$. Then S is a semigroup under usual multiplication. Let $B = [0, \frac{1}{2}]$. Then B is a subsemigroup of S. We have that $BSB \subseteq B = [0, \frac{1}{4}] \subseteq B$. Therefore B is a bi-ideal of S.

Example 1.6. Let $\mathbb{N}$ be the set of all possitive integers. Then $\mathbb{N}$ is a semigroup under the usual multiplication. Let $B = 2\mathbb{N}$. Thus $B\mathbb{N}B = 4\mathbb{N} \subseteq 2\mathbb{N} = B$. Hence B is a bi-ideal of $\mathbb{N}$.

Definition 1.9. A semigroup S is said to be *left (resp. right) simple* if S does not contain proper left (resp. right) ideals of S.

Theorem 1.2. *Let S be a semigroup. S is a left (resp. right) simple semigroup if and only if $Sa = S$ (resp. $aS = S$) for every $a \in S$.*

Theorem 1.3. *Let S be a semigroup. The following statements are equivalent:*

(1) S *is a group.*

(2) S *has the conditions*

(a) $\exists e \in S\ \forall a \in S,\ ea = a$;

(b) $\forall a \in S\ \exists b \in S,\ ba = e$.

(3) S *has the conditions*

(a) $\exists e \in S\ \forall a \in S,\ ae = a$;

(b) $\forall a \in S\ \exists b \in S,\ ab = e$.

Theorem 1.4. *Let S be a semigroup. The following statements are equivalent:*

(1) S *is a group.*

(2) S *is a left and right simple semigroup.*

(3) $Sa = S = aS$ *for every* $a \in S$.

Definition 1.10. A semigroup S is called *t-simple* if S does not contain proper bi-ideals of S.

Definition 1.11. A semigroup S with zero is called 0-*t-simple* if $S^2 \neq \{0\}$ and S does not contain nonzero proper bi-ideals of S.

Definition 1.12. Let S and T be semigroups. The mapping $\phi : S \to T$ is called a *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$.

Example 1.7. Let $\mathbb{R}$ be a semigroup of the set of all real numbers under the usual addition. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(a) = 2a$ for all $a \in \mathbb{R}$. Let $a, b \in \mathbb{R}$. We have

$$\begin{aligned}\phi(a+b) &= 2(a+b)\\ &= 2a+2b\\ &= \phi(a)+\phi(b).\end{aligned}$$

Hence ϕ is a homomorphism.

Definition 1.13. Let S and T be semigroups. The mapping $\phi : S \to T$ is called an *isomorphism* if ϕ is a homomorphism, 1-1 and onto.

Theorem 1.5. *The following statements are true.*

(1) If ρ is a congruence on a semigroup S, then S/ρ is a semigroup and the mapping $\rho^{\#} : S \to S/\rho$ defined by

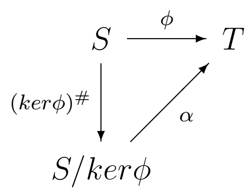
$$\rho^{\#}(x) = x\rho \text{ for all } x \in S$$

is a homomorphism.

(2) Let S and T be semigroups. If $\phi : S \to T$ is a homomorphism, then the relation

$$ker\phi = \phi^{-1} \circ \phi = \{(x, y) \in S \times S \mid \phi(x) = \phi(y)\}$$

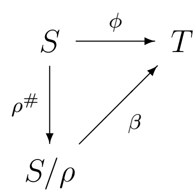
is a congruence on S and there is a monomorphism $\alpha : S/ker\phi \to T$ such that $ran\alpha = ran\phi$ and the diagram



commutes.

Theorem 1.6. *(First Isomorphism Theorem) Let S and T be semigroups. If $\phi : S \to T$ is a homomorphism, then $S/ker\phi \cong ran\phi$.*

Theorem 1.7. *Let ρ be a congruence on a semigroup S. If $\phi : S \to T$ is a homomorphism such that $\rho \subseteq ker\phi$, then there is a unique homomorphism $\beta : S/\rho \to T$ such that $ran\beta = ran\phi$ and the diagram*



commutes.

Let ρ and σ be congruences on a semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on S/ρ by

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

The following theorem holds.

Theorem 1.8. *(Third Isomorphism Theorem) Let ρ and σ be congruences on a semigroups S such that $\rho \subseteq \sigma$. The following statements hold.*

(1) σ/ρ is a congruence on S/ρ.

(2) $(S/\rho)(\sigma/\rho) \cong S/\sigma$.

1.2 Γ-semigroups

In 1981, M. K. Sen (Sen, 1981) introduced the definition of a Γ-semigroup as follows :

Definition 1.14. Let S and Γ be nonempty sets. S is called a Γ-*semigroup* if

(1) $a\alpha b \in S$, for all $a, b \in S$ and $\alpha \in \Gamma$;

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Now, we give some examples of Γ-semigroups.

Example 1.8. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n \mid n \in \mathbb{N}\}$. Define $a\alpha b = a + \alpha + b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $+$ is the usual addition. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.9. Let $\mathbb{Z}$ be the set of all integers and $\Gamma = \{n \mid n \in \mathbb{N}\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{Z}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have $\mathbb{Z}$ is a Γ-semigroup.

Example 1.10. Let $\mathbb{R}$ be the set of all real numbers and $\Gamma = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in \mathbb{R}$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication . We have $\mathbb{R}$ is a Γ-semigroup since for $a, b, c \in \mathbb{R}$ and $\alpha, \beta \in \Gamma$

$$a\alpha b = a \times \alpha \times b \in \mathbb{R} \text{ and } (a\alpha b)\beta c = a\alpha(b\beta c).$$

Example 1.11. Let S be a set of all negative rational numbers and $\Gamma = \{\frac{-1}{p} \mid p \text{ is prime}\}$. Define $a\alpha b = a \times \alpha \times b$ for all $a, b \in S$ and $\alpha \in \Gamma$ where $\times$ is the usual multiplication. We have S is a Γ-semigroup since for $a, b, c \in S$ and $\alpha, \beta \in \Gamma$

$$a\alpha b = a \times \alpha \times b \in S \text{ and } (a\alpha b)\beta c = a\alpha(b\beta c).$$

Definition 1.15. Let (S, Γ) be a Γ-semigroup and M a nonempty subset of S. Then M is called a *sub*Γ*-semigroup* of S if $a\gamma b \in M$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Example 1.12. Let $S = [0, 1]$ and $\Gamma = \{\frac{1}{n} \mid n \text{ is a positive integer}\}$. Then S is a Γ-semigroup under the usual multiplication. Next, let $M = [0, \frac{1}{2}]$. We have that M is a nonempty subset of S and $a\gamma b \in M$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a subΓ-semigroup of S.

Example 1.13. Consider the Γ-semigroup $(\mathbb{Z}, \Gamma)$ in Example 1.6. Let $\mathbb{N}$ be the set of all positive integers. We have $\mathbb{N}$ is a subΓ-semigroup of $\mathbb{Z}$ since $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{N}\Gamma\mathbb{N} \subseteq \mathbb{N}$.

Definition 1.16. Let (S, Γ) be a Γ-semigroup. A subΓ-semigroup A of S is called a *left (resp. right) ideal* of S if $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A is called an *ideal* of S if A is both a left and right ideal of S.

Definition 1.17. Let (S, Γ) be a Γ-semigroup and A a subΓ-semigroup of S. Then A is called a *bi-ideal* of S if $A\Gamma S\Gamma A \subseteq A$.

1.3 Ordered semigroups

In 1995, N. Kehayopulu and M. Tsingelis (Kehayopulu and Tsingelis, 1995) have studied ordered semigroups and given isomorphism theorems of ordered semigroups as follows :

Definition 1.18. Let S be a nonempty set and $\leq$ a relation on S. We call $\leq$ is an *order* on S if

(1) $\forall a \in S$, $a \leq a$ (reflexive) ;

(2) $\forall a, b \in S$ if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetric) ;

(3) $\forall a, b, c \in S$ if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive).

Example 1.14. We have $\leq$ is an order on $\mathbb{R}$.

(1) (reflexive) Let $a \in \mathbb{R}$. By property of $\leq$ on $\mathbb{R}$, we have $a \leq a$.

(2) (anti-symmetric) Let $a, b \in \mathbb{R}$, $a \leq b$ and $b \leq a$. By property of $\leq$ on $\mathbb{R}$, we have $a = b$.

(3) (transitive) Let $a, b, c \in \mathbb{R}$, $a \leq b$ and $b \leq c$. By property of $\leq$ on $\mathbb{R}$, we have $a \leq c$.

Example 1.15. Let X be any set. We have $\subseteq$ is an order on $P(X)$.

(1) (reflexive) Let $A \in P(X)$. Clearly, $A \subseteq A$.

(2) (anti-symmetric) Let $A, B \in P(X)$ such that $A \subseteq B$ and $B \subseteq A$. By property of $\subseteq$ on $P(X)$, we have $A = B$.

(3) (transitive) Let $A, B, C \in P(X)$ such that $A \subseteq B$ and $B \subseteq C$. By property of $\subseteq$ on $P(X)$, we have $A \subseteq C$.

Definition 1.19. If $\leq$ is an order on a nonempty set S, then $(S, \leq)$ is called a *partially ordered set.*

Example 1.16. By Example 1.11 and 1.12, we have $(\mathbb{R}, \leq)$ and $(P(X), \subseteq)$ are partially ordered sets.

Definition 1.20. Let S be a nonempty set, $\bullet$ a binary operation on S and $\leq$ a relation on S. We call $(S, \bullet, \leq)$ is an *ordered semigroup* if

(1) $(S, \bullet)$ is a semigroup ;

(2) $(S, \leq)$ is a partially ordered set ;

(3) for all $a, b, c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$.

Example 1.17. $(\mathbb{N}, +, \leq)$ is an ordered semigroup since

(1) $(\mathbb{N}, +)$ is a semigroup ;

(2) $(\mathbb{N}, \leq)$ is a partially ordered set ;

(3) Let $a, b, c \in \mathbb{N}$ such that $a \leq b$. Then $a+c \leq b+c$ and $c+a \leq c+b$.

Example 1.18. $(P(X), \cup, \subseteq)$ is an ordered semigroup since

(1) $(P(X), \cup)$ is a semigroup ;

(2) $(P(X), \subseteq)$ is a partially ordered set ;

(3) Let $A, B, C \in P(X)$ such that $A \subseteq B$. We show that $A \cup C \subseteq B \cup C$. Let $a \in A \cup C$. Then $a \in A$ and $a \in C$. Since $A \subseteq B$, we have $a \in B$ and

$a \in C$. Thus $a \in B \cup C$. Therefore $A \cup C \subseteq B \cup C$. Since $P(x)$ is commutative under $\cup$, $C \cup A \subseteq C \cup B$.

Definition 1.21. Let $(S, \bullet, \leq)$ be an ordered semigroup. A relation ρ on S is called a *pseudoorder* on S if

(1) $\leq \subseteq \rho$;

(2) for all $a, b \in S$, $(a, b) \in \rho$ and $(b, c) \in \rho$ imply $(a, c) \in \rho$;

(3) for all $a, b, c \in S$, $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$.

Definition 1.22. Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $f : S \to T$ a mapping from S to T. f is said to be *isotone* if $x, y \in S$, $x \leq_S y$ implies $f(x) \leq_T f(y)$. A mapping f is said to be *reverse isotone* if $x, y \in S$, $f(x) \leq_T f(y)$ implies $x \leq_S y$.

Remark 1.1. Each reverse isotone mapping is 1-1.

Definition 1.23. Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $f : S \to T$ a mapping from S to T. f is called a *homomorphism* if

(1) f is isotone ;

(2) $f(x \bullet y) = f(x) * f(y)$ for all $x, y \in S$.

Definition 1.24. Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups. A mapping $f : S \to T$ is called an *isomorphism* if f is homomorphism, onto and reverse isotone.

If ρ is a pseudoorder on S, let $\bar{\rho}$ be a relation on S defined by

$$\bar{\rho} = \{(a, b) \in S \times S \mid (a, b) \in \rho \text{ and } (b, a) \in \rho\}.$$

Proposition 1.9. *Let $(S, \bullet, \leq)$ be an ordered semigroup and ρ a pseudoorder on S. Then $\bar{\rho}$ is a congruence on S.*

Let S be an ordered semigroup and ρ a pseudoorder on S. By the Proposition 1.9, we have that $\bar{\rho}$ is a congruence on S. Then the set $S/\bar{\rho} = \{a\bar{\rho} \mid a \in S\}$ with multiplication $(a\bar{\rho}) \bullet (b\bar{\rho}) = (ab)\bar{\rho}$ is a semigroup and an order $\preceq_\rho$ defined by

$$\preceq_\rho = \{(a\bar{\rho}, b\bar{\rho}) \mid \exists x \in a\bar{\rho}, \exists y \in b\bar{\rho}, (x, y) \in \rho\}.$$

Proposition 1.10. *Let S be an ordered semigroup and ρ a pseudoorder on S. The following statements hold.*

(1) For $a, b \in S$, $a\bar{\rho} \preceq_\rho b\bar{\rho}$ if and only if $(a, b) \in \rho$.

(2) $\preceq_\rho$ is an order on $S/\bar{\rho}$.

Proposition 1.11. *Let $(S, \bullet, \leq)$ be an ordered semigroup and ρ a pseudoorder on S. Then $S/\bar{\rho}$ is an ordered semigroup.*

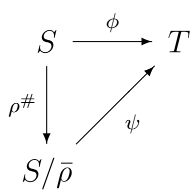
Let $\rho^{\#}$ be a homomorphism of S onto $S/\bar{\rho}$ defined by $\rho^{\#} : S \to S/\bar{\rho}$ such that $\rho^{\#}(a) = a\bar{\rho}$ for all $a \in S$.

Proposition 1.12. *Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. Define the relation $\tilde{\phi}$ on S by*

$$\tilde{\phi} = \{(a, b) \in S \times S \mid \phi(a) \leq_T \phi(b)\}.$$

Then $\tilde{\phi}$ is a pseudoorder on S.

Theorem 1.13. *Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. If ρ is a pseudoorder on S such that $\rho \subseteq \tilde{\phi}$, then the mapping $\psi : S/\bar{\rho} \to T$ defined by $\psi(a\bar{\rho}) = \phi(a)$ is a unique homomorphism of $S/\bar{\rho}$ into T such that $ran\psi = ran\phi$ and the diagram*



commutes (i.e., $\psi \circ \rho^{\#} = \phi$).

Let $(S, \bullet, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. Define $ker\phi = \{(a, b) \in S \times S \mid \phi(a) = \phi(b)\}$. It is easy to see that $ker\phi$ is a congruence on S. We have

$$\begin{aligned}(a,b)\in ker\phi &\Leftrightarrow \phi(a)=\phi(b)\\ &\Leftrightarrow \phi(a)\leq_T \phi(b) \text{ and } \phi(b)\leq_T \phi(a)\\ &\Leftrightarrow (a,b)\in\tilde{\phi} \text{ and } (b,a)\in\tilde{\phi}\\ &\Leftrightarrow (a,b)\in\bar{\tilde{\phi}}.\end{aligned}$$

So $ker\phi=\bar{\tilde{\phi}}$. The following theorem holds.

Theorem 1.14. *(First Isomorphism Theorem) Let $(S,\bullet,\leq_S)$ and $(T,*,\leq_T)$ be ordered semigroups and $\phi:S\to T$ a homomorphism. Then $S/ker\phi\cong ran\phi$.*

Theorem 1.15. *(Third Isomorphism Theorem) Let $(S,\bullet,\leq_S)$ be an ordered semigroup, ρ and σ pseudoorders on S such that $\rho\subseteq\sigma$. The following statements hold.*

(1) σ/ρ is a pseudoorder on $S/\bar{\rho}$.

(2) $(S/\bar{\rho})/(\overline{\sigma/\rho})\cong S/\bar{\sigma}$.

Definition 1.25. Let S be an ordered semigroup, T a nonempty subset of S and H a nonempty subset of T. Then we denote

$$(H]_T=\{t\in T\mid \exists h\in H,\ t\leq h\}.$$

If $T=S$, then we alway write $(H]$ instead of $(H]_S$

Example 1.19. Consider an ordered semigroup $(\mathbb{N},\bullet,\leq)$ and $T=\{3,6,7,8,9\}$, $H_1=\{6,10\}$ and $H_2=\{8\}$. Then we have

$(H_1]_T=\{3,6,7,8,9\}$;

$(H_1]=(H_1]_{\mathbb{N}}=\{1,2,3,4,5,6,7,8,9,10\}$;

$(H_2]_T=\{3,6,7,8\}$;

$(H_2]=(H_2]_{\mathbb{N}}=\{1,2,3,4,5,6,7,8\}$.

Definition 1.26. Let S be an ordered semigroup and T a nonempty subset of S. Then T is called a *subsemigroup* of S if

(1) $xy\in T$ for all $x,y\in T$;

(2) $(T]\subseteq T$.

Example 1.20. Let $X = \{4, 5, 6\}$, we have

$$P(X) = \{\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}\}.$$

(1) Let $T_1 = \{\emptyset, \{4\}\}$. We have $\emptyset \cup \emptyset = \emptyset$, $\emptyset \cup \{4\} = \{4\}$, $\{4\} \cup \{4\} = \{4\}$. Then $(T_1] = (\emptyset, \{4\}] = \{\emptyset, \{4\}\} \subseteq T$. Hence T_1 is an subsemigroup of a ordered semigroup $(P(X), \cup, \subseteq)$.

(2) Let $T_2 = \{\{4, 6\}\}$. We have $\{4, 6\} \cup \{4, 6\} = \{4, 6\}$. Then $(T_2] = (\{4, 6\}] = \{\emptyset, \{4\}, \{6\}, \{4, 6\}\} \nsubseteq T_2$. Hence T_2 is a subsemigroup of a semigroup $(P(X), \cup)$ but T_2 is not a subsemigroup of an ordered semigroup $(P(X), \cup, \subseteq)$.

Theorem 1.16. *Let S be an ordered semigroup and T a nonempty subset of S. Then T is a subsemigroup if and only if $(x] \subseteq T$ and $(xy] \subseteq T$ for all $x, y \in T$.*

Theorem 1.17. *Let S be an ordered semigroup. The following statements hold.*

(1) $A \subseteq (A]$ for every $A \subseteq S$.

(2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(3) $(A](B] \subseteq (AB]$ for every $A, B \subseteq S$.

(4) $((A]] = (A]$ for every $A \subseteq S$.

(5) If A and B are ideals of S, then $(AB]$ and $A \cup B$ are ideals of S.

Definition 1.27. Let $(S, \bullet, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left (resp. right) ideal* of S if

(1) $SA \subseteq A$ (resp. $AS \subseteq A$) ;

(2) $(A] \subseteq A$.

A is called an *ideal* of S if A is both a left and right ideal of S.

Definition 1.28. Let $(S, \bullet, \leq)$ be an ordered semigroup. A subsemigroup A of S is called a *bi-ideal* of S if

(1) $ASA \subseteq A$;

(2) $(A] \subseteq A$.

Definition 1.29. Let $(S, \bullet, \leq)$ be an ordered semigroup. A left (resp. right) ideal or a bi-ideal A of S is said to be *proper* if $A \neq S$.

Definition 1.30. Let $(S, \bullet, \leq)$ be an ordered semigroup. S is said to be *left (resp. right) simple* if S does not contain proper left (resp. right) ideals.

Remark 1.2. Equivalent definition is as follow : for every left (resp. right) ideal A of S, we have $A = S$.

Definition 1.31. Let $(S, \bullet, \leq)$ be an ordered semigroup. S is called *regular* if, for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

Remark 1.3. Equivalent definitions are as follow :

(1) $A \subseteq (ASA]$ for every $A \subseteq S$ or

(2) $a \in (aSa]$ for every $a \in S$.

Theorem 1.18. *Let S be an ordered semigroup and A a nonempty subset of S. The following statements hold.*

(1) $(Sa]$ is a left ideal of S for every $a \in S$.

(2) $(aS]$ is a right ideal of S for every $a \in S$.

(3) $(SaS]$ is an ideal of S for evry $a \in S$.

Theorem 1.19. *An ordered semigroup S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

CHAPTER 2

Isomorphism theorems

In 1995, N. Kehayopulu and M. Tsingelis (Kehayopulu and Tsingelis, 1995) have given two isomorphism theorems for ordered semigroups. Pseudo-order played an important role in concepts of congruences and quotient of ordered semigroups.

In this chapter, we separate into two sections. In the first section, we give some properties of isomorphisms for Γ-semigroups. In the last section, isomorphisms for ordered Γ-semigroups are considered.

2.1 Isomorphism theorems of Γ-semigroups

First, we give the definition of congruences of Γ-semigroups.

Definition 2.1. Let S be a Γ-semigroup. An equivalence relation ρ on S is called a *right [resp. left] congruence* on S if for each $a, b \in S$, $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ [resp. $(c\gamma a, c\gamma b) \in \rho$] for all $c \in S$ and $\gamma \in \Gamma$. An equivalence relation ρ on S is called a *congruence* on S if ρ is both a right and left congruence on S.

Theorem 2.1. *Let S be a Γ-semigroup and ρ a congruence on S. For $a\rho, b\rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a\rho)\gamma(b\rho) = (a\gamma b)\rho$. Then the quotient set S/ρ is a Γ-semigroup.*

Proof. First, we will show that a binary operation is well-defined.
Let $a, a', b, b' \in S$ and $\gamma \in \Gamma$. Consider

$$\begin{aligned} a\rho = a'\rho \text{ and } b\rho = b'\rho &\Rightarrow (a, a'), (b, b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b), (a'\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b, a'\gamma b') \in \rho \\ &\Rightarrow (a\gamma b)\rho = (a'\gamma b')\rho. \end{aligned}$$

Next, let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. We have

$$(a\rho\gamma b\rho)\mu c\rho = ((a\gamma b)\rho)\mu c\rho = ((a\gamma b)\mu c)\rho = (a\gamma(b\mu c))\rho = a\rho\gamma(b\mu c)\rho = a\rho\gamma(b\rho\mu c\rho).$$

Then the quotient set S/ρ is a Γ-semigroup. $\square$

Definition 2.2. Let S and T be Γ-semigroups under same Γ. The mapping $\phi : S \to T$ is called a Γ*-semigroup homomorphism* or *homomorphism* if $\phi(x\gamma y) = \phi(x)\gamma\phi(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Let S and T be Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Let

$$ker\phi = \phi^{-1} \circ \phi = \{(x, y) \in S \times S \mid \phi(x) = \phi(y)\}.$$

It is easy to see that $ker\phi$ is a congruence on S.

Then the following theorem holds.

Theorem 2.2. *Let S and T be Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Then there is a monomorphism $\varphi : S/ker\phi \to T$ such that $ran\varphi = ran\phi$ and the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\phi} & T \\ {\scriptstyle (ker\phi)^{\#}}\downarrow & \nearrow{\scriptstyle \varphi} & \\ S/ker\phi & & \end{array}$$

commutes (i.e., $\varphi \circ (ker\phi)^{\#} = \phi$) where the mapping $(ker\phi)^{\#} : S \to S/ker\phi$ defined by $(ker\phi)^{\#}(a) = aker\phi$ for all $a \in S$.

Proof. Define $\varphi : S/ker\phi \to T$ by

$$\varphi(aker\phi) = \phi(a) \text{ for all } a \in S.$$

We have

$$aker\phi = bker\phi \;\Leftrightarrow\; (a, b) \in ker\phi \;\Leftrightarrow\; \phi(a) = \phi(b).$$

Then φ is well-defined and 1-1. Next, we will show that φ is a homomorphism. Let $a, b \in S$ and $\gamma \in \Gamma$. Consider

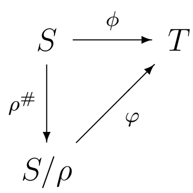
$\varphi((aker\phi)\gamma(bker\phi)) = \varphi((a\gamma b)ker\phi) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(aker\phi)\gamma\varphi(bker\phi)$.
Then φ is a homomorphism.
It is easy to see that ranϕ=ranφ. We have $\varphi \circ (ker\phi)^{\#} = \phi$ since

$$(\varphi \circ (ker\phi)^{\#})(a) = \varphi((ker\phi)^{\#}(a)) = \varphi(aker\phi) = \phi(a) \text{ for all } a \in S.$$

Hence the theorem is proved. □

Corollary 2.3. *(First Isomophism Theorem) Let S and T be Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Then $S/ker\phi \cong ran\phi$.*

Theorem 2.4. *Let S and T be Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. If ρ is a congruence on S such that $\rho \subseteq ker\phi$, then there is a unique homomorphism $\varphi : S/\rho \to T$ such that ranφ=ranϕ and the diagram*



commute (i.e., $\varphi \circ \rho^{\#} = \phi$) where the mapping $\rho^{\#} : S \to S/\rho$ defined by $\rho^{\#}(a) = a\rho$ for all $a \in S$.

Proof. Defin $\varphi : S/\rho \to T$ by

$$\varphi(a\rho) = \phi(a) \text{ for all } a \in S.$$

We have for all $a, b \in S$.

$$a\rho = b\rho \;\Rightarrow\; (a, b) \in \rho \;\Rightarrow\; (a, b) \in ker\phi \;\Rightarrow\; \phi(a) = \phi(b).$$

Then φ is well-defined. Let $a, b \in S$ and $\gamma \in \Gamma$. Consider

$$\varphi((a\rho)\gamma(b\rho)) = \varphi((a\gamma b)\rho) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(a\rho)\gamma\varphi(b\rho).$$

Hence φ is a homomorphism.
It is easy to see that ranφ=ranϕ. For each $a \in S$, we have

$$(\varphi \circ \rho^{\#})(a) = \varphi(\rho^{\#}(a)) = \varphi(a\rho) = \phi(a).$$

Thus $\varphi \circ \rho^{\#} = \phi$. Next, let $\psi : S/\rho \to T$ be any homomorphism satisfying $\psi \circ \rho^{\#} = \phi$. Then for all $a \in S$,

$$\psi(a\rho) = \psi(\rho^{\#}(a)) = \psi \circ \rho^{\#}(a) = \phi(a) = \varphi(a\rho).$$

Therefore $\psi = \varphi$. $\square$

Let ρ and σ be congruences on a Γ-semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on S/ρ by

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

We show that σ/ρ is well-defined. Let $x\rho, a\rho, y\rho, b\rho \in S/\rho$ such that $x\rho = a\rho$ and $y\rho = b\rho$. So $(x, a), (y, b) \in \rho$. Since $\rho \subseteq \sigma$, $(x, a), (y, b) \in \sigma$. This implies $(x, y) \in \sigma \ \Leftrightarrow \ (a, b) \in \sigma$.

The following theorem holds.

Theorem 2.5. *(Third Isomorphism Theorem) Let ρ and σ be congruences on a Γ-semigroup S with $\rho \subseteq \sigma$ and*

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

Then

(1) σ/ρ is a congruence on S/ρ ;

(2) $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

Proof. (1) Let $a \in S$. Then $(a, a) \in \sigma$, so $(a\rho, a\rho) \in \sigma/\rho$.

Next, let $a, b \in S$ such that $(a\rho, b\rho) \in \sigma/\rho$. Then $(a, b) \in \sigma$. Since σ is symmetric, $(b, a) \in \sigma$. Hence $(b\rho, a\rho) \in \sigma/\rho$.

Next, let $a, b, c \in S$ such that $(a\rho, b\rho), (b\rho, c\rho) \in \sigma/\rho$. So $(a, b), (b, c) \in \sigma$. Since σ is transitive, $(a, c) \in \sigma$. Therefore $(a\rho, c\rho) \in \sigma/\rho$.

Finally, let $a, b, c \in S$ and $\gamma \in \Gamma$. Assume $(a\rho, b\rho) \in \sigma/\rho$. Then $(a, b) \in \sigma$. Since σ is a congruence on S, $(a\gamma c, b\gamma c) \in \sigma$. So $((a\gamma c)\rho, (b\gamma c)\rho) \in \sigma/\rho$. Then $((a\rho)\gamma(c\rho), (b\rho)\gamma(c\rho)) \in \sigma/\rho$. Similarly, $((c\rho)\gamma(a\rho), (c\rho)\gamma(b\rho)) \in \sigma/\rho$.

Therefore σ/ρ is a congruence on S/ρ.

(2) Define $\varphi : (S/\rho)/(\sigma/\rho) \to S/\sigma$ by $\varphi((a\rho)(\sigma/\rho)) = a\sigma$ for all $a \in S$. Clearly, φ is onto. For all $a, b \in S$, we have

$$(a\rho)(\sigma/\rho) = (b\rho)(\sigma/\rho) \;\Leftrightarrow\; (a\rho, b\rho) \in \sigma/\rho \;\Leftrightarrow\; (a,b) \in \sigma \;\Leftrightarrow\; a\sigma = b\sigma.$$

Therefore φ is well-defined and 1-1.

Next, we will show that φ is a homomorphism. Let $a, b \in S$ and $\gamma \in \Gamma$. We have

$$\begin{aligned}\varphi((a\rho)(\sigma/\rho)\gamma(b\rho)(\sigma/\rho)) &= \varphi((a\rho\gamma b\rho)(\sigma/\rho)) \\ &= \varphi((a\gamma b)\rho)(\sigma/\rho)) \\ &= (a\gamma b)\sigma \\ &= a\sigma\gamma b\sigma = \varphi((a\rho)(\sigma/\rho))\gamma\varphi((b\rho)(\sigma/\rho)).\end{aligned}$$

Then φ is an isomorphism. Therefore $(S/\rho)/(\sigma/\rho) \cong S/\sigma$. $\square$

2.2 Isomorphism theorems of ordered Γ-semigroups

Definition 2.3. Let S and Γ be nonempty sets and $\leq$ a relation on S. We call $(S, \Gamma, \leq)$ is an *ordered* Γ-*semigroup* if

(1) (S, Γ) is a Γ-semigroup ;

(2) $(S, \leq)$ is a partially ordered set ;

(3) $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

Let S be a Γ-semigroup and ρ a congruence on S, in Section 2.1, we have that S/ρ is a Γ-semigroup. The following question in natural : If $(S, \Gamma, \leq)$ is an ordered Γ-semigroup and ρ is a congruence on S, then is the set S/ρ an ordered Γ-semigroup? A probable order on S/ρ could be the relation $\preceq_\rho$ on S/ρ defined by means of the order $\leq$ on S, that is,

$$a\rho \preceq_\rho b\rho \;\Leftrightarrow\; \text{there exists } x \in a\rho \text{ and } y \in a\rho \text{ such that } (x, y) \in \leq.$$

But this relation is not an order, in general. We prove it in the following example.

Example 2.1. We consider the ordered Γ-semigroup $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$ defined by the multiplication table and the order $\leq$ below :

α	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

β	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

and $\leq = \{(a,a),(a,d),(b,b),(c,c),(c,e),(d,d),(e,e)\}$.

Let $x, y, z \in S$ and $\gamma, \mu \in \Gamma$. Then we have

$$(x\gamma y)\mu a = a = x\gamma(y\mu a), (x\gamma y)\mu c = c = x\gamma(y\mu c)$$
$$(x\gamma y)\mu d = d = x\gamma(y\mu d), (x\gamma y)\mu e = e = x\gamma(y\mu e)$$
$$(x\gamma y)\alpha b = e = x\gamma(y\alpha b)$$
$$(x\gamma y)\beta b = e = x\gamma(y\beta b) \text{ if } y \neq b$$
$$(x\gamma b)\beta b = e = x\gamma(y\beta b) \text{ if } x \neq b$$
$$(b\alpha b)\beta b = e = b\alpha(b\beta b), (b\beta b)\beta b = b = b\beta(b\beta b)$$

Hence S is a Γ-semigroup. Since

$$x\gamma a \leq x\gamma d, a\gamma x = d\gamma x, x\gamma c \leq x\gamma e, c\gamma x = e\gamma x \text{ for all } x \in S \text{ and } \gamma \in \Gamma,$$

S is an ordered Γ-semigroup.

Let ρ be the congruence on S defined as follows :

$$\rho = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,e),(e,a),(c,d),(d,c)\}.$$

Let $\preceq_\rho$ be an order on S/ρ defined by means of the order $\leq$ on S, that is,

$$a\rho \preceq_\rho b\rho \;\Leftrightarrow\; \text{there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } (x,y) \in \leq.$$

We have $a\rho = \{a, e\}$, $b\rho = \{b\}$ and $c\rho = \{c, d\}$. Also we have $a\rho \preceq_\rho c\rho$ and $c\rho \preceq_\rho a\rho$ but $a\rho \neq c\rho$.

The following question arise : Is there a congruence ρ on an ordered Γ-semigroup S for which S/ρ is an ordered Γ-semigroup? This lead us to the concept of pseudoorders.

Definition 2.4. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A relation ρ is called a *pseudoorder* if

(1) $\leq \subseteq \rho$;

(2) for all $a, b, c \in S$, $(a, b) \in \rho$ and $(b, c) \in \rho$ imply $(a, c) \in \rho$;

(3) for all $a, b \in S$, $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$ for every $c \in S$ and $\gamma \in \Gamma$.

If ρ is a pseudoorder on an ordered Γ-semigroup S, let $\bar{\rho}$ be a relation on S defined by

$$\bar{\rho} = \rho \cap \rho^{-1} = \{(a, b) \in S \times S \mid (a, b) \in \rho \text{ and } (b, a) \in \rho\}.$$

Theorem 2.6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and ρ a pseudoorder on S. Then $\bar{\rho}$ is a congruence on S.*

Proof. Let $a \in S$. Since $(a, a) \in \leq$ and $\leq \subseteq \rho$, $(a, a) \in \rho$. Then $(a, a) \in \bar{\rho}$.

Next, let $a, b \in S$ such that $(a, b) \in \bar{\rho}$. Thus $(a, b) \in \rho$ and $(b, a) \in \rho$. This implies that $(b, a) \in \bar{\rho}$.

Next, we show that ρ is transitive. Let $a, b, c \in S$ such that $(a, b), (b, c) \in \bar{\rho}$. Then $(a, b), (b, a), (b, c), (c, b) \in \rho$. Thus $(a, c), (c, a) \in \rho$. Hence $(a, c) \in \bar{\rho}$.

Finally, let $a, b \in S$ such that $(a, b) \in \bar{\rho}$. Then $(a, b), (b, a) \in \rho$. Thus $(c\gamma a, c\gamma b), (a\gamma c, b\gamma c), (c\gamma b, c\gamma a), (b\gamma c, a\gamma c) \in \rho$ for all $c \in S$ and $\gamma \in \Gamma$. Therefore $(a\gamma c, b\gamma c), (c\gamma a, c\gamma b) \in \bar{\rho}$ for all $c \in S$ and $\gamma \in \Gamma$. □

Let S be an ordered Γ-semigroup and ρ a pseudoorder on S. By Theorem 2.6, we have that $\bar{\rho}$ is a congruence on S. Then $S/\bar{\rho}$ is a Γ-semigroup.

Next, for each $a\bar{\rho}, b\bar{\rho} \in S/\bar{\rho}$, define the order $\preceq_{\bar{\rho}}$ on $S/\bar{\rho}$.

$$a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho} \;\Leftrightarrow\; \text{there exist } x \in a\bar{\rho} \text{ and } y \in b\bar{\rho} \text{ such that } (x, y) \in \rho.$$

Theorem 2.7. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and ρ a pseudoorder on S. The following statements hold.*

(1) For $a, b \in S$, $a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho}$ if and only if $(a, b) \in \rho$.

(2) $\preceq_{\bar{\rho}}$ is an order on $S/\bar{\rho}$.

Proof. (1) If $(a, b) \in \rho$, then $a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho}$.

Conversly, assume that $a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho}$. Then there exist $x \in a\bar{\rho}$ and $y \in b\bar{\rho}$ such that $(x, y) \in \rho$. Since $(x, a) \in \bar{\rho}$ and $(y, b) \in \bar{\rho}$, we have $(x, a), (a, x), (b, y)$ and $(y, b) \in \rho$. Since $(a, x), (x, y)$ and $(y, b) \in \rho$, we have $(a, b) \in \rho$.

(2) Let $a, b, c \in S$. Since $(a, a) \in \leq_S \subseteq \rho$, $a\bar{\rho} \preceq_{\bar{\rho}} a\bar{\rho}$. Assume that $a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho}$ and $b\bar{\rho} \preceq_{\bar{\rho}} a\bar{\rho}$. By (1), $(a, b) \in \rho$ and $(b, a) \in \rho$. Then $(a, b) \in \bar{\rho}$. So $a\bar{\rho} = b\bar{\rho}$.

Finally, assume that $a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho}$ and $b\bar{\rho} \preceq_{\bar{\rho}} c\bar{\rho}$. By (1),$(a, b) \in \rho$ and $(b, c) \in \rho$. Therefore $(a, c) \in \rho$. By (1), $a\bar{\rho} \preceq_{\bar{\rho}} c\bar{\rho}$. Hence $\preceq_{\bar{\rho}}$ is an order on $S/\bar{\rho}$. □

The following theorem holds.

Theorem 2.8. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and ρ a pseudoorder on S. Then $S/\bar{\rho}$ is an ordered Γ-semigroup.*

Proof. By Theorem 2.6, we have $\bar{\rho}$ is a congruence on S. Then $S/\bar{\rho}$ is a Γ-semigroup. By Theorem 2.7, we have $\preceq_{\bar{\rho}}$ is an ordered on $S/\bar{\rho}$.

Let $x, y \in S$ such that $x\bar{\rho} \preceq_{\bar{\rho}} y\bar{\rho}$. Then there exist $a \in x\bar{\rho}$ and $b \in y\bar{\rho}$ such that $(a, b) \in \rho$. Thus $(x, a) \in \bar{\rho}$ and $(y, b) \in \bar{\rho}$. Then $(x, a), (a, x), (y, b)$ and $(b, y) \in \rho$. So $(x, y) \in \rho$. Let $c \in S$ and $\gamma \in \Gamma$. Therefore

$$(x\gamma c, a\gamma c), (a\gamma c, x\gamma c), (y\gamma c, b\gamma c) \text{ and } (b\gamma c, y\gamma c) \in \rho.$$

Then $(x\gamma c, a\gamma c), (y\gamma c, b\gamma c) \in \bar{\rho}$. So $(x\gamma c)\bar{\rho} = (a\gamma c)\bar{\rho}$, $(y\gamma c)\bar{\rho} = (b\gamma c)\bar{\rho}$. Similarly, $(c\gamma x)\bar{\rho} = (c\gamma a)\bar{\rho}$ and $(c\gamma y)\bar{\rho} = (c\gamma b)\bar{\rho}$. Since $(x, y) \in \rho$, we have $(x\gamma c, y\gamma c), (c\gamma x, c\gamma y) \in \rho$. Hence $(x\gamma c)\bar{\rho} \preceq_{\bar{\rho}} (y\gamma c)\bar{\rho}$ and $(c\gamma x)\bar{\rho} \preceq_{\bar{\rho}} (c\gamma y)\bar{\rho}$. Therefore $(x\bar{\rho})\gamma(c\bar{\rho}) \preceq_{\bar{\rho}} (y\bar{\rho})\gamma(c\bar{\rho})$ and $(c\bar{\rho})\gamma(x\bar{\rho}) \preceq_{\bar{\rho}} (c\bar{\rho})\gamma(y\bar{\rho})$. Hence $S/\bar{\rho}$ is an ordered Γ-semigroup. □

Definition 2.5. Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $f : S \to T$ a mapping from S into T. f is called *isotone* if $x \leq_S y$ implies $f(x) \leq_T f(y)$ for all $x, y \in S$. f is called *reverse isotone* if $f(x) \leq_T f(y)$ implies $x \leq_S y$ for all $x, y \in S$.

Definition 2.6. Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $f : S \to T$ a mapping from S into T. f is called an *ordered Γ-semigroup homomorphism* or *homomorphism* if

(1) f is isotone ;

(2) $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Theorem 2.9. *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $f : S \to T$ a mapping from S into T. If f is a reverse isotone mapping, then f is 1-1.*

Proof. Let $x, y \in S$ such that $f(x) = f(y)$. Since $f(x) \leq_T f(y)$, $x \leq_S y$.

Similarly, since $f(y) \leq_T f(x)$, $y \leq_S x$. Then $x = y$. □

Definition 2.7. Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $f : S \to T$ a mapping from S into T. f is called an *isomorphism* if f is a homomorphism, onto and reverse isotone.

Theorem 2.10. *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Define the relation $\tilde{\phi}$ on S by*

$$\tilde{\phi} = \{(a, b) \in S \times S \mid \phi(a) \leq_T \phi(b)\}.$$

Then $\tilde{\phi}$ is a pseudoorder on S.

Proof. Let $(a, b) \in \leq_S$. Since $a \leq_S b$ and ϕ is isotone, $\phi(a) \leq_T \phi(b)$. Then $(a, b) \in \tilde{\phi}$. Next, let $a, b, c \in S$ such that $(a, b), (b, c) \in \tilde{\phi}$. So $\phi(a) \leq_T \phi(b)$ and $\phi(b) \leq_T \phi(c)$. Then $\phi(a) \leq_T \phi(c)$. This implies $(a, c) \in \tilde{\phi}$.

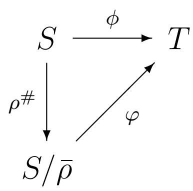
Finally, let $a, b, c \in S$ and $\gamma \in \Gamma$. Assume that $(a, b) \in \tilde{\phi}$. Since $\phi(a) \leq_T \phi(b)$, ϕ is a homomorphism and T is an ordered Γ-semigroup,

$$\phi(a\gamma c) = \phi(a)\gamma\phi(c) \leq_T \phi(b)\gamma\phi(c) = \phi(b\gamma c).$$

Then $(a\gamma c, b\gamma c) \in \tilde{\phi}$. Similarly, $(c\gamma a, c\gamma b) \in \tilde{\phi}$.

Hence $\tilde{\phi}$ is a pseudoorder on S. □

Theorem 2.11. *Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. If ρ is a pseudoorder on S such that $\rho \subseteq \tilde{\phi}$, then the mapping $\varphi : S/\bar{\rho} \to T$ defined by $\varphi(a\bar{\rho}) = \phi(a)$ is a unique homomorphism of $S/\bar{\rho}$ into T such that $ran\varphi = ran\phi$ and the diagram*



commuts (i.e., $\varphi \circ \rho^{\#} = \phi$) where the mapping $\rho^{\#} : S \to S/\bar{\rho}$ defined by $\rho^{\#}(a) = a\bar{\rho}$ for all $a \in S$.

Proof. Define $\varphi : S/\bar{\rho} \to T$ by

$$\varphi(a\bar{\rho}) = \phi(a) \text{ for all } a \in S.$$

We have φ is well-defined since for all $a, b \in S$,

$$\begin{aligned} a\bar{\rho} = b\bar{\rho} &\Rightarrow (a, b) \in \bar{\rho} \\ &\Rightarrow (a, b), (b, a) \in \rho \\ &\Rightarrow (a, b), (b, a) \in \tilde{\phi} \\ &\Rightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a) \\ &\Rightarrow \phi(a) = \phi(b). \end{aligned}$$

Let $a, b \in S$ and $\gamma \in \Gamma$. We have

$$\varphi(a\bar{\rho}\gamma b\bar{\rho}) = \varphi((a\gamma b)\bar{\rho}) = \phi(a\gamma b) = \phi(a)\gamma\phi(b) = \varphi(a\bar{\rho})\gamma\varphi(\bar{\rho})$$

and

$$a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho} \;\Rightarrow\; (a, b) \in \rho \subseteq \tilde{\phi} \;\Rightarrow\; \phi(a) \leq \phi(b).$$

Hence φ is a homomorphism. For each $a \in S$, we have

$$(\varphi \circ \rho^{\#})(a) = \varphi(\rho^{\#}(a)) = \varphi(a\bar{\rho}) = \phi(a).$$

Then $\varphi \circ \rho^{\#} = \phi$.

Next, let $\psi : S/\bar{\rho} \to T$ be any homomorphism such that $\psi \circ \phi^{\#} = \phi$. For $a \in S$, we have

$$\psi(a\bar{\rho}) = \psi(\rho^{\#}(a)) = (\psi \circ \rho^{\#})(a) = \phi(a) = \varphi(a\bar{\rho}).$$

So $\psi = \varphi$. Finally, we have $ran\varphi = \{\varphi(a\bar{\rho}) \mid a \in S\} = \{\phi(a) \mid a \in S\} = ran\phi$.

Hence the theorem is proved. □

Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Define $ker\phi = \{(a, b) \in S \times S \mid \phi(a) = \phi(b)\}$. It is easy to see that $ker\phi$ is a congruence on S. Then we have

$$\begin{aligned} (a, b) \in ker\phi &\Leftrightarrow \phi(a) = \phi(b) \\ &\Leftrightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a) \\ &\Leftrightarrow (a, b) \in \tilde{\phi} \text{ and } (b, a) \in \tilde{\phi} \\ &\Leftrightarrow (a, b) \in \bar{\tilde{\phi}}. \end{aligned}$$

So $ker\phi = \bar{\tilde{\phi}}$.

The following corollary holds.

Theorem 2.12. *(First Isomorphism Theorem) Let $(S, \Gamma, \leq_S)$ and $(T, \Gamma, \leq_T)$ be ordered Γ-semigroups under same Γ and $\phi : S \to T$ a homomorphism. Then $S/ker\phi \cong ran\phi$.*

Proof. We apply the first part of Theorem 2.11 for $\rho = \tilde{\phi}$ and $ker\phi = \bar{\tilde{\phi}}$. Then the mapping $\varphi : S/ker\phi \to T$ defined by $\varphi(aker\phi) = \phi(a)$ is a homomorphism.

Next, we will to show that φ is reverse isotone. Let $a, b \in S$ such that $\phi(a) \leq_T \phi(b)$. Then $(a, b) \in \tilde{\phi}$. Since $\tilde{\phi}$ is a pseudoorder on S, by Theorem 2.7 (1), $aker\phi \preceq_{ker\phi} bker\phi$. Then φ is reverse isotone. Therefore φ is an isomorphism. □

Theorem 2.13. *(Third Isomorphism Theorem) Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup, ρ and σ pseudooders on S such that $\rho \subseteq \sigma$. Then*

(1) σ/ρ is a pseudoorder on $S/\bar{\rho}$;

(2) $(S/\bar{\rho})(\overline{\sigma/\rho}) \cong S/\bar{\sigma}$.

Proof. (1) Let $(a\bar{\rho}, b\bar{\rho}) \in \preceq_{\bar{\rho}}$. Then $(a,b) \in \rho$. Since $\rho \subseteq \sigma$, we have $(a,b) \in \sigma$. So $(a\bar{\rho}, b\bar{\rho}) \in \sigma/\rho$. Therefore $\preceq_{\bar{\rho}} \subseteq \sigma/\rho$.

Next, let $a,b,c \in S$ such that $(a\bar{\rho}, b\bar{\rho}) \in \sigma/\rho$ and $(b\bar{\rho}, c\bar{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$ and $(b,c) \in \sigma$, so $(a,c) \in \sigma$. Therefore $(a\bar{\rho}, c\bar{\rho}) \in \sigma/\rho$.

Finally, let $a,b,c \in S$ and $\gamma \in \Gamma$. Assume $(a\bar{\rho}, b\bar{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$. Since σ is a pseudoorder on S, $(a\gamma c, b\gamma c) \in \sigma$. So $((a\gamma c)\bar{\rho}, (b\gamma c)\bar{\rho}) \in \sigma/\rho$. Similarly, $((c\gamma a)\bar{\rho}, (c\gamma b)\bar{\rho}) \in \sigma/\rho$.

(2) Define $\phi : S/\bar{\rho} \to S/\bar{\sigma}$ by

$$\phi(a\bar{\rho}) = a\bar{\sigma} \text{ for all } a \in S.$$

We have ϕ is well-defined, since for all $a,b \in S$,

$$a\bar{\rho} = b\bar{\rho} \ \Rightarrow \ (a,b) \in \bar{\rho} \ \Rightarrow \ (a,b),(b,a) \in \rho \subseteq \sigma \ \Rightarrow \ (a,b) \in \bar{\sigma} \ \Rightarrow \ a\bar{\sigma} = b\bar{\sigma}.$$

Next, let $a,b \in S$ and $\gamma \in \Gamma$. Then we have

$$\phi(a\bar{\rho}\gamma b\bar{\rho}) = \phi((a\gamma b)\bar{\rho}) = (a\gamma b)\bar{\sigma} = (a\bar{\sigma})\gamma(b\sigma) = \phi(a\bar{\rho})\gamma\phi(b\bar{\rho})$$

and

$$a\bar{\rho} \preceq_{\bar{\rho}} b\bar{\rho} \ \Rightarrow \ (a,b) \in \rho \ \Rightarrow \ (a,b) \in \sigma \ \Rightarrow \ a\bar{\sigma} \preceq_{\bar{\sigma}} b\bar{\sigma}.$$

Hence ϕ is a homomorphism. By the definition of $\tilde{\phi}$, we have

$$\tilde{\phi} = \{(a\bar{\rho}, b\bar{\rho}) \in S/\bar{\rho} \times S/\bar{\rho} \mid \phi(a\bar{\rho}) \preceq_{\bar{\sigma}} \phi(b\bar{\rho})\}.$$

Thus

$$(a\bar{\rho}, b\bar{\rho}) \in \tilde{\phi} \ \Leftrightarrow \ \phi(a\bar{\rho}) \preceq_{\bar{\sigma}} \phi(b\bar{\rho}) \ \Leftrightarrow \ a\bar{\sigma} \preceq_{\bar{\sigma}} b\bar{\sigma} \ \Leftrightarrow \ (a,b) \in \sigma \ \Leftrightarrow \ (a\bar{\rho}, b\bar{\rho}).$$

Then $\tilde{\phi} = \sigma/\rho$, so $ker\phi = \bar{\tilde{\phi}} = \overline{\sigma/\rho}$. It is easy to see that $ran\phi = S/\bar{\sigma}$. By Theorem 2.12, $(S/\bar{\rho})/(\overline{\sigma/\rho}) \cong S/\bar{\sigma}$. □

CHAPTER 3

Bi-ideals in ordered Γ-semigroups

In this chapter, we study bi-ideals in ordered Γ-semigroups. We demonstrate this chapter in three sections. In the first section, we study the notion of bi-ideals in ordered Γ-semigroups. In the second section, we give some characterizations of minimal and 0-minimal bi-ideals in ordered Γ-semigroups, respectively. In the last section, maximal bi-ideals in ordered Γ-semigroups are studied.

3.1 Bi-ideals in ordered Γ-semigroups

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and T a nonempty subset of S. If H is a nonempty subset of T, we denote the set

$$(H]_T = \{t \in T \mid t \leq h \text{ for some } h \in H\}.$$

If $T = S$, then always write $(H]$ by $(H]_S$.

Definition 3.1. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A nonempty subset A of S is called a *sub*Γ*-semigroup* if

(1) $x\gamma y \in A$ for all $x, y \in S$ and $\gamma \in \Gamma$;

(2) $(A] \subseteq A$.

Definition 3.2. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A subΓ-semigroup A of S is called a *left (resp. right) ideal* of S if

(1) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$) ;

(2) $(A] \subseteq A$.

Definition 3.3. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A subΓ-semigroup A of S is called a *bi-ideal* of S if

(1) $A\Gamma S\Gamma A \subseteq A$;

(2) $(A] \subseteq A$.

Definition 3.4. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A subΓ-semigroup A of S is called *left (resp. right) t-simple* if an ordered Γ-semigroup $(A, \Gamma, \leq)$ does not contain proper left (resp. right) ideals.

Definition 3.5. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. S is said to be *left (resp. right) simple* if S does not contain proper left (resp. right) ideals.

Remark 3.1. Equivalent definition is as follow : for every left (resp. right) ideal A of S, we have $A = S$.

Definition 3.6. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. S is called *regular* if, for every $a \in S$, there exist $x \in S$ and $\gamma, \beta \in \Gamma$ such that $a \leq a\gamma x\beta a$.

Remark 3.2. Equivalent definitions are as follow :

(1) $A \subseteq (A\Gamma S\Gamma A]$ for every $A \subseteq S$ or

(2) $a \in (a\Gamma S\Gamma a]$ for every $a \in S$.

Theorem 3.1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. The following statements are true.*

(1) $(S\Gamma a]$ is a left ideal of S for all $a \in S$.

(2) $(a\Gamma S]$ is a right ideal of S for all $a \in S$.

(3) $(S\Gamma a\Gamma S]$ is an ideal of S for all $a \in S$.

Proof. (1) First, we will show that $S\Gamma(S\Gamma a] \subseteq (S\Gamma a]$. Let $x \in S\Gamma(S\Gamma a]$. Then $x = y\alpha b$ for some $y \in S$, $b \in (S\Gamma a]$ and $\alpha \in \Gamma$. Since $b \in (S\Gamma a]$, $b \leq t$ for some $t \in S\Gamma a$. Thus $t = w\beta a$ for some $w \in S$ and $\beta \in \Gamma$. Hence $b \leq w\beta a$. Since S is an ordered Γ-semigroup, $y\alpha b \leq y\alpha w\beta a = (y\alpha w)\beta a$. Since $(y\alpha w)\beta a \in S\Gamma a$ and $x \leq (y\alpha w)\beta a$, $x \in (S\Gamma a]$.

Next, we show that $((S\Gamma a]] \subseteq (S\Gamma a]$. Let $x \in ((S\Gamma a]]$. Then $x \leq t$ for some $t \in (S\Gamma a]$. Since $t \in (S\Gamma a]$, we have $t \leq s$ for some $s \in S\Gamma a$. Therefore $x \leq s$ and $s \in S\Gamma a$. Hence $x \in (S\Gamma a]$.

(2) It is similar to (1).

(3) Claim that $S\Gamma(S\Gamma a\Gamma S] \subseteq (S\Gamma a\Gamma S]$. Let $x \in S\Gamma(S\Gamma a\Gamma S]$. Then $x = y\alpha b$ for some $y \in S$, $b \in (S\Gamma a\Gamma S]$ and $\alpha \in \Gamma$. Since $b \in (S\Gamma a\Gamma S]$, $b \leq t$ for

some $t \in S\Gamma a\Gamma S$. Thus $t = w_1\beta a\theta w_2$ for some $w_1, w_2 \in S$ and $\beta, \theta \in \Gamma$. Hence $b \leq w_1\beta a\theta w_2$. Since S is an ordered Γ-semigroup,

$$y\alpha b \leq y\alpha w_1\beta a\theta w_2 = (y\alpha w_1)\beta a\theta w_2.$$

Then $x = y\alpha b \leq (y\alpha w_1)\beta a\theta w_2$. Since $(y\alpha w_1)\beta a\theta w_2 \in S\Gamma a\Gamma S$, $x \in (S\Gamma a\Gamma S]$.

Next, we show that $(S\Gamma a\Gamma S]\Gamma S \subseteq (S\Gamma a\Gamma S]$. Let $x \in (S\Gamma a\Gamma S]\Gamma S$. Then $x = y\alpha b$ for some $y \in (S\Gamma a\Gamma S], b \in S$, and $\alpha \in \Gamma$. Since $y \in (S\Gamma a\Gamma S]$, we have $y \leq t$ for some $t \in S\Gamma a\Gamma S$. Thus $t = w_1\beta a\theta w_2$ for some $w_1, w_2 \in S$ and $\beta, \theta \in \Gamma$. Hence $y \leq w_1\beta a\theta w_2$. Therefore

$$y\alpha b \leq (w_1\beta a\theta w_2)\alpha b = w_1\beta a\theta(w_2\alpha b) \text{ and } w_1\beta a\theta(w_2\alpha b) \in S\Gamma a\Gamma S$$

because S is an ordered Γ-semigroup. Hence $x \in (S\Gamma a\Gamma S]$.

Finally, we show that $((S\Gamma a\Gamma S]] \subseteq (S\Gamma a\Gamma S]$. Let $x \in ((S\Gamma a\Gamma S]]$. Then $x \leq t$ for some $t \in (S\Gamma a\Gamma S]$. We have $t \leq s$ for some $s \in S\Gamma a\Gamma S$ because $t \in (S\Gamma a\Gamma S]$. Hence $x \leq s$ and $s \in S\Gamma a\Gamma S$, so $x \in (S\Gamma a\Gamma S]$. □

The following theorems give necessary and sufficient condition for an ordered Γ-semigroup to be left (right) simple.

Theorem 3.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. S is left (resp. right) simple if and only if $(S\Gamma a] = S$ (resp. $(a\Gamma S] = S$) for all $a \in S$.*

Proof. Assume that S is left simple and let $a \in S$. By Theorem 3.1(1), we have $(S\Gamma a]$ is a left ideal of S. Then $(S\Gamma a] = S$ because S is left simple.

Conversely, suppose that $(S\Gamma a] = S$ for all $a \in S$. Let L be a left ideal of S and $a \in S$. Clearly, $L \subseteq S$. Next, we will show that $S \subseteq L$. Consider

$$S = (S\Gamma a] \subseteq (S\Gamma L] \subseteq (L] \subseteq L.$$

Thus $S = L$, so S is left simple. □

The following corollary follows by Theorem 3.2.

Corollary 3.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A subΓ-semigroup T of S is left (resp. right) simple if and only if $(T\Gamma a]_T = T$ (resp. $(a\Gamma T]_T = T$) for all $a \in T$.*

Theorem 3.4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If S is left and right simple, then S is regular.*

Proof. Assume that S is left and right simple. By theorem 3.2, we have $(S\Gamma a] = S$ and $(a\Gamma S] = S$. Let $a \in S$. Consider

$$a \in (a\Gamma S] = (a\Gamma(a\Gamma S]] = (a\Gamma(S\Gamma a]] = (a\Gamma S\Gamma a].$$

Hence S is regular. □

Let S be an ordered Γ-semigroup and A a nonempty subset of S. We denote by $L(A), R(A)$ and $B(A)$ the left ideal, right ideal and bi-ideal of S generated by A, respectively.

Theorem 3.5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and A a nonempty subset of S. The following statements hold.*

(1) $L(A) = (A \cup S\Gamma A]$.

(2) $R(A) = (A \cup A\Gamma S]$.

(3) $B(A) = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$.

Proof. (1) Let A be a nonempty subset of an ordered Γ-semigroup S. Let $L = (A \cup S\Gamma A]$. Clearly, $A \subseteq L$. We have that

$$L\Gamma L = (A \cup S\Gamma A]\Gamma(A \cup S\Gamma A] \subseteq (S\Gamma A] \subseteq L.$$

It easy to see that $((L]] \subseteq (L]$. Hence L is a subΓ-semigroup of S.

Claim that $S\Gamma L \subseteq L$. Let $x \in S\Gamma L$. Then $x = y\alpha b$ for some $y \in S$, $b \in L$ and $\alpha \in \Gamma$. From $b \in L$, we have $b \in (A]$ or $b \in (S\Gamma A]$.

Case 1.1. If $b \in (A]$, then $b \leq z$ for some $z \in A$. Thus $x = y\alpha b \leq y\alpha z$. Hence $x \in (S\Gamma A] \subseteq (A \cup S\Gamma A] = L$.

Case 1.2. If $b \in (S\Gamma A]$, then $b \leq m\gamma n$ for some $m \in S$, $n \in A$ and $\gamma \in \Gamma$. Thus $x = y\alpha b \leq (y\alpha m)\gamma n$. Hence $x \in (S\Gamma A] \subseteq (A \cup S\Gamma A] = L$. Therefore B is a left ideal of S.

Let M be any left ideal of S containing A. Since M is a subΓ-semigroup of S and $A \subseteq M$, $(A] \subseteq (M] \subseteq M$. Since M is a left ideal of S and $A \subseteq M$, $S\Gamma A \subseteq S\Gamma M \subseteq M$ and $(M] \subseteq M$. Therefore $L = (A \cup S\Gamma A] \subseteq (M] \subseteq M$.

Hence L is the smallest left ideal of S containing A. Therefore $L(A) = L = (A \cup S\Gamma A]$.

(2) It is similar to (1).

(3) Let A be a nonempty subset of an ordered Γ-semigroup S. Let $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. Clearly, $A \subseteq B$. We have that

$$B\Gamma B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (A\Gamma A \cup A\Gamma S\Gamma A] \subseteq B.$$

It easy to see that $((B]] \subseteq (B]$. Hence B is a subΓ-semigroup of S.

Claim that $B\Gamma S\Gamma B \subseteq B$. Then $x \in B\Gamma S\Gamma B$, so $x = b\alpha m\beta y$ for some $b, y \in B$, $m \in S$ and $\alpha, \beta \in \Gamma$.

Case 3.1. If $b \in (A]$ and $y \in (A]$, then $b \leq k$ and $y \leq t$ for some $k, t \in A$. Thus $x = b\alpha m\beta y \leq k\alpha m\beta t$. Hence $x \in (A\Gamma S\Gamma A] \subseteq B$.

Case 3.2. If $b \in (A]$ and $y \in (A\Gamma A]$, then $b \leq k$ and $y \leq s\mu t$ for some $k, s, t \in A$ and $\mu \in \Gamma$. Thus $x = b\alpha m\beta y \leq k\alpha m\beta(s\mu t) = k\alpha(m\beta s)\mu t$. Hence $x \in (A\Gamma S\Gamma A] \subseteq B$.

Case 3.3. If $b \in (A]$ and $y \in (A\Gamma S\Gamma A]$, then $b \leq k$ and $y \leq r\theta s\gamma t$ for some $k, r, t \in A$, $s \in S$ and $\theta, \gamma \in \Gamma$. Thus $x = b\alpha m\beta y \leq k\alpha m\beta(r\theta s\gamma t) = k\alpha(m\beta r\theta s)\gamma t$. Hence $x \in (A\Gamma S\Gamma A] \subseteq B$.

Case 3.4. It is similar to case 3.2.

Case 3.5. If $b \in (A\Gamma A]$ and $y \in (A\Gamma A]$, then $b \leq c\gamma d$ and $y \leq e\mu f$ for some $c, d, e, f \in A$ and $\gamma, \mu \in \Gamma$. Hence $x = b\alpha m\beta y \leq (c\gamma d)\alpha m\beta(e\mu f) = c\gamma(d\alpha m\beta e)\mu f$. Thus $x \in (A\Gamma S\Gamma A] \subseteq B$.

Case 3.6. If $b \in (A\Gamma A]$ and $y \in (A\Gamma S\Gamma A]$, then $b \leq c\gamma d$ and $y \leq r\theta s\gamma t$ for some $c, d, r, t \in A$, $s \in S$ and $\theta, \gamma \in \Gamma$. Hence $x = b\alpha m\beta y \leq (c\gamma d)\alpha m\beta(r\theta s\gamma t) = c\gamma(d\alpha m\beta r\theta s)\gamma t$. Thus $x \in (A\Gamma S\Gamma A] \subseteq B$.

Case 3.7. It is similar to case 3.3.

Case 3.8. It is similar to case 3.6.

Case 3.9. If $b \in (A\Gamma S\Gamma A]$ and $y \in (A\Gamma S\Gamma A]$, then $b \leq c\gamma d\mu e$ and $y \leq r\theta s\eta t$ for some $c, d, r, t \in A$, $d, s \in S$ and $\gamma, \mu, \theta, \eta \in \Gamma$. Hence $x = b\alpha m\beta y \leq (c\gamma d\mu e)\alpha m\beta(r\theta s\eta t) = c\gamma(d\mu e\alpha m\beta r\theta s)\eta t$. Thus $x \in (A\Gamma S\Gamma A] \subseteq B$.

Then $B\Gamma S\Gamma B \subseteq B$. Therefore B is a bi-ideal of S.

Let M be any bi-ideal of S containing A. Since M is a bi-ideal of S and $A \subseteq M$, $A\Gamma A \subseteq M$, $A\Gamma S\Gamma A \subseteq M\Gamma S\Gamma M \subseteq M$ and $(M] \subseteq M$. Therefore $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (M] \subseteq M$.

Hence B is the smallest bi-ideal of S containing A. Therefore $B(A) = B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. □

Let S be an ordered Γ-semigroup and $a \in S$. For $A = \{a\}$, we write $L(a), R(a)$ and $B(a)$ instead of $L(\{a\}), R(\{a\})$ and $B(\{a\})$, respectively, and we call them the principal left ideal, principal right ideal and principle bi-ideal of S, respectively, generated by a. We have

$$L(a) = \{t \in S \mid t \leq a \text{ or } t \leq y\gamma a \text{ for some } y \in S, \gamma \in \Gamma\},$$

$$R(a) = \{t \in S \mid t \leq a \text{ or } t \leq a\beta x \text{ for some } x \in S, \beta \in \Gamma\},$$

$$B(a) = \{t \in S \mid t \leq a \text{ or } t \leq a\gamma a \text{ or } t \leq a\beta x\mu a \text{ for some } x \in S, \gamma, \beta, \mu \in \Gamma\}.$$

Theorem 3.6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then S is left and right simple if and only if S does not contain proper bi-ideals.*

Proof. Assume that S is left and right simple. Let A be a bi-ideal of S. Clearly, $A \subseteq S$. To show $S \subseteq A$, let $a \in S$ and $b \in A$. Since S is left simple, $S = L(b)$. Then $a \leq b$ or $a \leq x\alpha b$ for some $x \in S$ and $\alpha \in \Gamma$ because $a \in S = L(b)$.

Case 1. $a \leq b$. Since $a \in S$ and $a \leq b \in A$, $a \in A$.

Case 2. $a \leq x\alpha b$ for some $x \in S$ and $\alpha \in \Gamma$. Since S is right simple, we have $S = R(b)$. Then $x \leq b$ or $x \leq b\beta y$ for some $y \in S$ and $\beta \in \Gamma$ because $x \in R(b)$.

Case 2.1. $x \leq b$. Then $a \leq x\alpha b \leq b\alpha b$. Since $a \leq b\alpha b$ and $b\alpha b \in A$, $a \in A$.

Case 2.2. $x \leq b\beta y$ for some $y \in S$ and $\beta \in \Gamma$. Then $a \leq x\alpha b \leq (b\beta y)\alpha b$. Since $(b\beta y)\alpha b \in A\Gamma S\Gamma A \subseteq A$, this implies $a \in A$.

Thus $S = A$. Hence S does not contain proper bi-ideals.

Conversely, let L be a left ideal of S. So $(L] \subseteq L$. We have $L\Gamma S\Gamma L =$

$L\Gamma(S\Gamma L) \subseteq L\Gamma L \subseteq L$. Then L is a bi-ideal of S. By assumption, $L = S$. Therefore S is left simple. Similarly, S is right simple. □

Definition 3.7. An ordered Γ-semigroup $(S, \Gamma, \leq)$ is called an *ordered* Γ*-group* if (S, γ) is a group for some $\gamma \in \Gamma$.

The following theorem is true.

Theorem 3.7. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-group. Then* S *does not contain proper bi-ideals.*

Proof. Let S be an ordered Γ-group and A a bi-ideal of S. Then (S, α) is a group for some $\alpha \in \Gamma$. Clearly, $A \subseteq S$. Let $a \in S$ and $b \in A$. Since (S, α) is a group, there exists $b^{-1} \in S$ such that $b\alpha b^{-1} = e = b^{-1}\alpha b$ for some $\alpha \in \Gamma$. Then

$$a = e\beta a\theta e = (b\alpha b^{-1})\beta a\theta(b^{-1}\alpha b) = b\alpha(b^{-1}\beta a\theta b^{-1})\alpha b \in A\Gamma S\Gamma A \subseteq A.$$

Thus $a \in A$. Therefore $S = A$. Hence S does not contain proper bi-ideals. □

3.2 Minimal and 0-minimal bi-ideals in ordered Γ-semigroups

In this section, we study minimal and 0-minimal bi-ideals in ordered Γ-semigroups.

Definition 3.8. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A bi-ideal B of S is called a *minimal bi-ideal* of S if B does not contain proper bi-ideals of S.

Remark 3.3. Equivalent definition is as follow : for any bi-ideal A of S such that $A \subseteq B$, we have $A = B$.

Definition 3.9. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. A subΓ-semigroup T of S is called *t-simple* if an ordered Γ-semigroup $(T, \Gamma, \leq)$ does not contain proper bi-ideals.

The following theorem holds.

Theorem 3.8. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* B *a sub*Γ*-semigroup of* S*. If* B *is a bi-ideal of* S*, then* $(u\Gamma B\Gamma v]$ *is a bi-ideal of* S *for every* $u, v \in S$*. In particular,* $(u\Gamma S\Gamma v]$ *is a bi-ideal of* S *for every* $u, v \in S$*.*

Proof. Let $x, y \in (u\Gamma B\Gamma v], \gamma, \delta \in \Gamma$ and $s \in S$. Then $x \leq c$ and $y \leq d$ for some $c, d \in u\Gamma B\Gamma v$. Then there exist $a, b \in B$ and $\theta, \nu, \alpha, \beta \in \Gamma$ such that

$$c = u\theta a\nu v \text{ and } d = u\alpha b\beta v.$$

Then $a\nu v\gamma u\alpha b = a\nu(v\gamma u)\alpha b \in B\Gamma S\Gamma B \subseteq B$ and

$$x\gamma y \leq c\gamma d = (u\theta a\nu v)\gamma(u\alpha b\beta v) = u\theta(a\nu v\gamma u\alpha b)\beta v.$$

Thus $x\gamma y \in (u\Gamma B\Gamma v]$. This shows that $(u\Gamma B\Gamma v]$ is a subΓ-semigroup of S.
We have

$$x\gamma s\delta y \leq c\gamma s\delta d = (u\theta a\nu v)\gamma s\delta(u\alpha b\beta v) = u\theta(a\nu v\gamma s\delta u\alpha b)\beta v$$

and $a\nu v\gamma s\delta u\alpha b = a\nu(v\gamma s\delta u\alpha b)\beta v \in B\Gamma S\Gamma B \subseteq B$, this implies $x\gamma s\delta y \in (u\Gamma B\Gamma v]$. Thus $(u\Gamma B\Gamma v]\Gamma S\Gamma(u\Gamma B\Gamma v] \subseteq (u\Gamma B\Gamma v]$.

Next, we show that $((u\Gamma B\Gamma v]] \subseteq (u\Gamma B\Gamma v]$. Let $x \in ((u\Gamma B\Gamma v]]$. Then we have $x \leq t$ for some $t \in (u\Gamma B\Gamma v]$. Since $t \in (u\Gamma B\Gamma v]$, $t \leq s$ for some $s \in u\Gamma B\Gamma v$. Hence $x \in (u\Gamma B\Gamma v]$.

Therefore $(u\Gamma B\Gamma v]$ is a bi-ideal of S, and consequently $(u\Gamma S\Gamma v]$ is a bi-ideal of S because S is a bi-ideal of itself. □

Corollary 3.9. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. For any $a \in S$, $(a\Gamma S\Gamma a]$ is a bi-ideal of S.*

Proof. It follows by Theorem 3.8. □

We now characterize the t-simple subΓ-semigroup in ordered Γ-semigroup.

Theorem 3.10. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and T a subΓ-semigroup of S. The following statements are equivalent :*

(1) T is t-simple.

(2) $(t\Gamma T\Gamma t]_T = T$ for all $t \in T$.

(3) T is a left and right simple subΓ-semigroup of S.

Proof. (1) $\Rightarrow$ (2) : Let $t \in T$. By Theorem 3.8, we have $(t\Gamma T\Gamma t]_T$ is a bi-ideal of S. Since T is t-simple, $(t\Gamma T\Gamma t]_T = T$.

(2) $\Rightarrow$ (3) : For every $t \in T$, we have

$$(t\Gamma T\Gamma t]_T \subseteq (T\Gamma t]_T \subseteq T \text{ and } (t\Gamma T\Gamma t]_T \subseteq (t\Gamma T]_T \subseteq T.$$

By our hypothesis, we have $(T\Gamma t] = T$ and $(t\Gamma T] = T$ for all $t \in T$. By Corollary 3.3, T is both left and right simple.

(3) $\Rightarrow$ (1) : Let B be a bi-ideal of T. By our hypothesis, for each $a \in B$, we have $(a\Gamma T]_T = T = (T\Gamma a]_T$. Thus we have

$$T = (a\Gamma T]_T = (a\Gamma(T\Gamma a]_T]_T \subseteq (a\Gamma T\Gamma a]_T \subseteq (B\Gamma T\Gamma B)_T \subseteq (B]_T \subseteq B.$$

Therefore $B = T$ and hence T is t-simple. □

Theorem 3.11. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. The following statements are equivalent :*

(1) S is t-simple.

(2) $(a\Gamma S\Gamma a] = S$ for all $a \in S$.

(3) $B(a) = S$ for all $a \in S$.

Proof. (1) $\Rightarrow$ (2) : Assume that S is t-simple and let $a \in S$. Since $(a\Gamma S\Gamma a]$ is a bi-ideal of S, we have $(a\Gamma S\Gamma a] = S$.

(2) $\Rightarrow$ (3) : Assume that $(a\Gamma S\Gamma a] = S$ for all $a \in S$. Consider the set

$$B(a) = (a\Gamma S\Gamma a \cup a\Gamma a \cup a] = (a\Gamma S\Gamma a] \cup (a\Gamma a] \cup (a] = S \cup (a\Gamma a] \cup (a] = S.$$

Therefore $B(a) = S$ for all $a \in S$.

(3) $\Rightarrow$ (1) : Assume that $B(a) = S$ for all $a \in S$. Let B be an bi-ideal of S and let $a \in B$. Clearly, $B \subseteq S$. Since $B(a) = S$, we have

$$S = B(a) = (a\Gamma S\Gamma a \cup a\Gamma a \cup a] \subseteq (B\Gamma S\Gamma B \cup B\Gamma B \cup B] \subseteq B.$$

Then $B = S$, that is, S is t-simple. □

Theorem 3.12. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and B a bi-ideal of S. Then B is minimal if and only if B is t-simple.*

Proof. Let B be a minimal bi-ideal of S and $a \in B$. We let $J = (a\Gamma B\Gamma a]_B$. Let $c_1, c_2 \in J$. Then $c_1 \leq a\alpha_1 b_1 \beta_1 a$ and $c_2 \leq a\alpha_2 b_2 \beta_2 a$ for some $b_1, b_2 \in B$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$. Let $x \in S$ and $\gamma, \theta \in \Gamma$. Since $b_1\beta_1 a\gamma x\theta a_2\alpha_2 b_2 \in B\Gamma S\Gamma B \subseteq B$ and

$$c_1\gamma x\theta c_2 \leq (a\alpha_1 b_1\beta_1 a)\gamma x\theta(a\alpha_2 b_2\beta_2 a) = a\alpha_1(b_1\beta_1 a\gamma x\theta a\alpha_2 b_2)\beta_2 a,$$

$c_1\gamma x\theta c_2 \in J$. This shows that $J\Gamma S\Gamma J \subseteq J$. Let $y \in (J]$. Then $y \leq z$ for some $z \in J$. Since $z \in J$, $z \in B$. So $y \in B$. Since $z \in J$, there exist $b \in B$ and $\alpha, \beta \in \Gamma$ such that $z \leq a\alpha b\beta a$. Thus $y \leq a\alpha b\beta a$, hence $y \in J$. Therefore J is a bi-ideal of S.

Since B is minimal and J is a bi-ideal of S contained in B, $B = J = (a\Gamma B\Gamma a]_B$ for all $a \in B$. Hence by Theorem 3.10, B is t-simple.

Conversely, let B be a t-simple bi-ideal of S and N a bi-ideal of S satisfying $N \subseteq B$ and $c \in N$. By Theorem 3.10, $B = (c\Gamma B\Gamma c]_B$. Then

$$B \subseteq (N\Gamma B\Gamma N]_B \subseteq (N\Gamma S\Gamma N] \subseteq (N] \subseteq N.$$

Thus $N = B$. This shows that B is a minimal bi-ideal of S. □

Theorem 3.13. *Let M be a minimal bi-ideal of an ordered Γ-semigroup S and B a bi-ideal of S. Then $M = (u\Gamma B\Gamma v]$ for every $u, v \in M$.*

Proof. By Theorem 3.8, $(u\Gamma B\Gamma v]$ is a bi-ideal of S for every $u, v \in S$. Since M is minimal and

$$(u\Gamma B\Gamma v] \subseteq (M\Gamma B\Gamma M] \subseteq (M\Gamma S\Gamma M] \subseteq (M] \subseteq M.$$

Hence $M = (u\Gamma B\Gamma v]$. □

By Theorem 3.8 and Theorem 3.13, we obtain the following theorem.

Theorem 3.14. *Let M be a minimal bi-ideal of an ordered Γ-semigroup S. Then $(s\Gamma M\Gamma t]$ is a minimal bi-ideal of S for every $s, t \in S$.*

Proof. By Theorem 3.8, $(s\Gamma M\Gamma t]$ is a bi-ideal of S. Let N be a bi-ideal of S such that $N \subseteq (s\Gamma M\Gamma t]$. Consider the set $H = \{h \in M \mid (s\alpha h\beta t] \subseteq N$ for some $\alpha, \beta \in \Gamma\}$. It is obvious that $H \subseteq M$.

Let $x \in S$, $h_1, h_2 \in H$ and $\alpha, \gamma, \eta, \mu, \theta, \beta \in \Gamma$. Since N and M are bi-ideals of S,

$$(s\alpha h_1\gamma t\eta x\mu s\theta h_2\beta t] \subseteq ((s\alpha h_1\gamma t]\eta x\mu(s\theta h_2\beta t]] \subseteq (N\Gamma S\Gamma N] \subseteq (N] \subseteq N$$

and

$$h_1\gamma t\eta x\mu s\theta h_2 = h_1\gamma(t\eta x\mu s)\theta h_2 \in M\Gamma S\Gamma M \subseteq M.$$

Then $h_1\gamma t\eta x\mu s\theta h_2 \in H$. Hence $h_1\Gamma t\Gamma S\Gamma s\Gamma h_2 \subseteq H$. Since M is minimal and $(t\Gamma S\Gamma s]$ is a bi-ideal of S by Theorem 3.8, it follows from Theorem 3.13 that

$$M = (h_1\Gamma(t\Gamma S\Gamma s]\Gamma h_2] \subseteq (h_1\Gamma t\Gamma S\Gamma s\Gamma h_2] \subseteq (H].$$

Now, let $y \in (H]$. Then $y \leq h$ for some $h \in M$ such that $(s\alpha h\gamma t] \subseteq N$ for some $\alpha, \gamma \in \Gamma$. From $(H] \subseteq (M] \subseteq M$ and $s\alpha y\gamma t \leq s\alpha h\gamma t$, we obtain $y \in M$ and $(s\alpha y\gamma t] \subseteq (s\alpha h\gamma t] \subseteq N$, that is, $y \in H$. Then $(H] \subseteq H$. This show that $M \subseteq H$ and $H = M$. Therefore

$$(s\Gamma M\Gamma t] = \bigcup_{\substack{h\in M,\\ \alpha,\gamma\in\Gamma}} (s\alpha h\gamma t] = \bigcup_{\substack{h\in H,\\ \alpha,\gamma\in\Gamma}} (s\alpha h\gamma t] \subseteq N.$$

Thus $N = (s\Gamma M\Gamma t]$, that is, $(s\Gamma M\Gamma t]$ is a minimal bi-ideal of S. □

By Theorem 3.13 and 3.14, we observe the following result.

Theorem 3.15. *Let M be a minimal bi-ideal of an ordered Γ-semigroup S. Then every minimal bi-ideal of S is of the form $(s\Gamma M\Gamma t]$, where $s, t \in S$.*

Definition 3.10. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Let $K(S)$ be the intersection of all ideals of S. If $K(S) \neq \emptyset$, then $K(S)$ is called the *kernel* of S.

It easy to see that $K(S)$ is the smallest ideals of S. We now study the kernel of an ordered Γ-semigroup.

Theorem 3.16. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. The union of all minimal bi-ideals of S is the kernel of S.*

Proof. Let M be a minimal bi-ideal of S and $K = (S\Gamma M\Gamma S]$. Then, it is clear that K is an ideal of S. Since

$$K = (S\Gamma M\Gamma S] = \bigcup_{s,t\in S} (s\Gamma M\Gamma t].$$

Then K is the union of all the minimal bi-ideal of S by Theorem 3.15. Let $a \in K$. Then $a \in B$ for some minimal bi-ideal B of S. Since K is an ideal of S, by Theorem 3.10 and Theorem 3.12, there exist $x \in B$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$ and so $a \leq (a\alpha x\beta a)\alpha x\beta a \in K\Gamma a\Gamma K$.
Hence, we have

$$K \subseteq (K\Gamma a\Gamma K] \subseteq (K\Gamma S\Gamma K] \subseteq (K\Gamma S] \subseteq (K] \subseteq K.$$

Then $K = (K\Gamma a\Gamma K]$ for all $a \in K$. Let I be an ideal of S. Thus $I\Gamma K \subseteq I \cap K$ and $I \cap K \neq \emptyset$. Let $c \in I \cap K$. Then $K = (K\Gamma c\Gamma K]$ because $c \in K$. Since I is an ideal of S and $c \in I$, we have

$$K = (K\Gamma c\Gamma K] \subseteq (K\Gamma I\Gamma K] \subseteq (S\Gamma I\Gamma S] \subseteq (I] \subseteq I.$$

Therefore $K = K(S)$ which is the kernel of S. □

By Theorem 3.12 and Theorem 3.16, we deduce the following result.

Theorem 3.17. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If S has a minimal bi-ideal M, then $K(S) = (S\Gamma M\Gamma S]$ which is a union of t-simple subΓ-semigroups of S.*

Theorem 3.18. *If a bi-ideal B of an ordered Γ-semigroup S is a t-simple subΓ-semigroup of S which $e \in B$ satisfying $e \leq e\alpha e$ for some $\alpha \in \Gamma$, then $B = (e\Gamma S\Gamma e]$ and $(S\Gamma e\Gamma S]$ is the kernel of S which is a union of t-simple subΓ-semigroup of S.*

Proof. Since B is t-simple, it follows from Theorem 3.12 that B is a minimal bi-ideal of S. By Theorem 3.8, $(e\Gamma S\Gamma e]$ is a bi-ideal of S. Since $(e\Gamma S\Gamma e] \subseteq (B\Gamma S\Gamma B] \subseteq B$, we have $B = (e\Gamma S\Gamma e]$.

Also, $e \leq e\alpha e \leq e\alpha e\alpha e$ for some $\alpha \in \Gamma$ implies $e \in (e\Gamma S\Gamma e]$. Thus by Theorem 3.17, we have that $K(S)$ is a union of t-simple subΓ-semigroups of S. Consequently, we have

$$K(S) = (S\Gamma(e\Gamma S\Gamma e]\Gamma S] \subseteq (S\Gamma e\Gamma S\Gamma e\Gamma S] \subseteq (S\Gamma e\Gamma S] \subseteq (S\Gamma(e\Gamma S\Gamma e]\Gamma S],$$

since $e \in (e\Gamma S\Gamma e]$. This show that $K(S) = (S\Gamma e\Gamma S]$. □

In the remainder in this section, we study minimal and 0-minimal bi-ideals of ordered Γ-semigroups analogous to minimal and 0-minimal bi-ideals of semigroups considered by A. Iampan (Iampan, 2008)

Definition 3.11. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. An element a of S with at least two element is called *zero element* of S if $x\alpha a = a = a\alpha x$ for all $x \in S$ and $\alpha \in \Gamma$ and is denoted by 0.

Definition 3.12. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with zero. S is called 0-*t*-*simple* if it does not contain nonzero proper bi-ideals of S and $S\Gamma S \neq \{0\}$.

Lemma 3.19. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and $a \in S$. Then*

$$B(a) = (a\Gamma S^1\Gamma a \cup \{a\}] = (a\Gamma S\Gamma a \cup a\Gamma a \cup \{a\}].$$

Proof. It follows by Theorem 3.5(3). □

Lemma 3.20. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with zero. The following statements hold.*

(1) If S is 0-t-simple, then $B(a) = S$ for all $a \in S \setminus \{0\}$.

(2) If $B(a) = S$ for all $a \in S \setminus \{0\}$, then either $S\Gamma S = \{0\}$ or S is 0-t-simple.

Proof. (1) Assume that S is 0-*t*-simple. Then for any $a \in S \setminus \{0\}$, $B(a)$ is a nonozero bi-ideal of S. Hence $B(a) = S$.

(2) Assume that $B(a) = S$ for all $a \in S \setminus \{0\}$ and $S\Gamma S \neq \{0\}$. Let B be a nonzero bi-ideal of S and $a \in B \setminus \{0\}$. Clearly, $B \subseteq S$. By assumption, we have

$$S = B(a) = (a\Gamma S\Gamma a \cup a\Gamma a \cup \{a\}] \subseteq (B\Gamma S\Gamma B \cup B\Gamma B \cup B] \subseteq B.$$

Therefore $B = S$, that is, S is 0-*t*-simple. □

Lemma 3.21. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and $\{B_\lambda \mid \lambda \in \Lambda\}$ a family of bi-ideals of S. Then $\bigcap_{\lambda \in \Lambda} B_\lambda$ is a bi-ideal of S if $\bigcap_{\lambda \in \Lambda} B_\lambda \neq \emptyset$.*

Proof. Assume that $\bigcap_{\lambda\in\Lambda} B_\lambda \neq \emptyset$. Let $a, b \in \bigcap_{\lambda\in\Lambda} B_\lambda$, $x \in S$, $\gamma, \mu \in \Gamma$. Then $a, b \in B_\lambda$ for all $\lambda \in \Lambda$. Since B_λ is a bi-ideal of S for all $\lambda \in \Lambda$, $a\gamma b \in B_\lambda$ and $a\gamma x\mu b \in B_\lambda$ for all $\lambda \in \Lambda$. Then $a\gamma b \in \bigcap_{\lambda\in\Lambda} B_\lambda$ and $a\gamma x\mu b \in \bigcap_{\lambda\in\Lambda} B_\lambda$. Clearly, $((\bigcap_{\lambda\in\Lambda} B_\lambda]] \subseteq (\bigcap_{\lambda\in\Lambda} B_\lambda]$. Therefore $\bigcap_{\lambda\in\Lambda} B_\lambda$ is a bi-ideal of S. □

Lemma 3.22. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup, B a bi-ideal of S and K a subΓ-semigroup of S. The following statements hold.*

(1) If K is t-simple such that $K \cap B \neq \emptyset$, then $K \subseteq B$.

(2) If K is 0-t-simple such that $(K \setminus \{0\}) \cap B \neq \emptyset$, then $K \subseteq B$.

Proof. (1) Assume that K is t-simple such that $K \cap B \neq \emptyset$. Then there exists $a \in K \cap B$. By Theorem 3.11(2), $K = (a\Gamma K\Gamma a]$. We have

$$K = (a\Gamma K\Gamma a] \subseteq (a\Gamma K\Gamma a] \subseteq (B\Gamma S\Gamma B] \subseteq (B] \subseteq B.$$

Hence $K \subseteq B$.

(2) Assume that K is 0-t-simple such that $(K \setminus \{0\}) \cap B \neq \emptyset$. Then $B(a) \neq \{0\}$. It is easy to show that $B(a) \cap K$ is a nonzero bi-ideal of K. Then $K = B(a) \cap K$. We have

$$\begin{aligned} K = B(a) \cap K = (a\Gamma K\Gamma a \cup a\Gamma a \cup \{a\}] \cap K &\subseteq (a\Gamma K\Gamma a \cup a\Gamma a \cup \{a\}] \\ &\subseteq (a\Gamma S\Gamma a \cup a\Gamma a \cup \{a\}] \\ &= B(a) \subseteq B. \end{aligned}$$

Hence $K \subseteq B$. □

Definition 3.13. Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with zero. A nonzero bi-ideal B of S is called a 0-*minimal bi-ideal* of S if there is no nonzero bi-ideal A of S such that $A \subset B$.

Remark 3.4. Equivalent definition are as follow :

(1) For any nonzero bi-ideal A of S such that $A \subseteq B$, we have $A = B$

or

(2) For any bi-ideal A of S such that $A \subset B$, we have $A = \{0\}$.

Theorem 3.23. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with zero and B a bi-ideal of S. If B is a 0-minimal bi-ideal of S, then either $B\Gamma B = \{0\}$ or B is 0-t-simple.*

Proof. It is similar to the proof of Theorem 3.12. □

Using the same proof of Theorem 3.23 and Lemma 3.22(2), we have theorem.

Theorem 3.24. *If ordered Γ-semigroup S has a zero element and B is a nonzero bi-ideal of S, then the following statements hold.*

(1) If B is a 0-minimal bi-ideal of S, then either $A\Gamma B\Gamma A = \{0\}$ for some nonzero bi-ideal A of B or B is 0-t-simple.

(2) If B is 0-t-simple, then B is a 0-minimal bi-ideal of S.

Theorem 3.25. *If ordered Γ-semigroup S has no zero element but it has proper bi-ideals, then every proper bi-ideals of S is minimal if and only if the intersection of any two distinct proper bi-ideals is empty.*

Proof. Assume that every proper bi-ideals of S is minimal. Let B_1 and B_2 be two distinct proper bi-ideals of S. Then B_1 and B_2 are minimal. If $B_1 \cap B_2 \neq \emptyset$, then $B_1 \cap B_2$ is a bi-ideal of S by Lemma 3.21. Since B_1 and B_2 are minimal, $B_1 = B_2$. It is a contradiction. Therefore $B_1 \cap B_2 = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 3.25, we have Theorem 3.26.

Theorem 3.26. *If ordered Γ-semigroup S has a zero element but it has nonzero proper bi-ideals, then every nonzero proper bi-ideals of S is 0-minimal if and only if the intersection of any two distinct nonzero proper bi-ideals is $\{0\}$.*

Proof. It is similar to the proof of Theorem 3.25. □

3.3 Maximal bi-ideals in ordered Γ-semigroups

In this section, we study maximal bi-ideals of ordered Γ-semigroups analogous to maximal bi-ideals of semigroups considered by A. Iampan (Iampan, 2008)

Definition 3.14. A proper bi-ideal B of an ordered Γ-semigroup S is called a *maximal bi-ideal* of S if for any bi-ideal A of S such that $B \subset A$, we have $A = S$.

Remark 3.5. Equivalent definition is as follow : for any proper bi-ideal A of S such that $B \subseteq A$, we have $A = B$.

Theorem 3.27. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup and* B *a bi-ideal of* S*. If*

(1) $S \setminus B = \{a\}$ *for some* $a \in S$ *or*

(2) $S \setminus B \subseteq (b\Gamma S\Gamma b]$ *for all* $b \in S \setminus B$*,*

then B *is a maximal bi-ideal of* S*.*

Proof. Let A be a bi-ideal of S such that $B \subset A$. Then $A \setminus B \neq \emptyset$.

Case 1. $S \setminus B = \{a\}$ for some $a \in S$.

Then

$$B \cup \{a\} = B \cup (S \setminus B) = B \cup (S \cap B^C) = (B \cup S) \cap (B \cup B^C) = S.$$

Since $B \subset A$ and $A \setminus B \neq \emptyset$, $A \setminus B \subseteq S \setminus B = \{a\}$. Then $A \setminus B = \{a\}$. Consider

$$B \cup \{a\} = B \cup (A \setminus B) = (B \cup A) \cap (B \cup B^C) = A \cap S = A.$$

Then $A = S$ because $B \cup \{a\} = S$. Hence B is a maximal bi-ideal of S.

Case 2. $S \setminus B \subseteq (b\Gamma S\Gamma b]$ for all $b \in S \setminus B$. Let $b \in A \setminus B$.

We have

$$b \in A \setminus B \subseteq S \setminus B \subseteq (b\Gamma S\Gamma b] \subseteq (A\Gamma S\Gamma A] \subseteq A.$$

Hence $S = B \cup S \setminus B \subseteq B \cup A \subseteq A \subseteq S$, so $A = S$. Then B is a maximal bi-ideal of S. □

Theorem 3.28. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup. If* B *is a maximal bi-ideal of* S *and* $B \cup B(a)$ *is a bi-ideal of* S *for all* $a \in S \setminus B$*, then either*

(1) $S \setminus B \subseteq (a\Gamma a \cup a]$ *and* $(a\Gamma a\Gamma a] \subseteq B$ *for some* $a \in S \setminus B$*, and* $(b\Gamma S\Gamma b] \subseteq B$ *for all* $b \in S \setminus B$*, or*

(2) $S \setminus B \subseteq B(a)$ *for all* $a \in S \setminus B$*.*

Proof. Assume that B is a maximal bi-ideal of S and $B \cup B(a)$ is a bi-ideal of S for all $a \in S \setminus B$. Then we have the following two case :

Case (1). $(a\Gamma S\Gamma a] \subseteq B$ for some $a \in S \setminus B$. Then $(a\Gamma a\Gamma a] \subseteq (a\Gamma S\Gamma a] \subseteq B$, so $a\Gamma a\Gamma a \subseteq B$. Consider

$$B \cup (a\Gamma a \cup a] = (B \cup (a\Gamma S\Gamma a]) \cup (a\Gamma a \cup a] = B \cup (a\Gamma S\Gamma a \cup a\Gamma a \cup a] = B \cup B(a).$$

Then $B \cup (a\Gamma a \cup a]$ is a bi-ideal of S because $B \cup B(a)$ is a bi-ideal of S. Since B is a maximal bi-ideal of S and $B \subset B \cup (a\Gamma a \cup a]$, $B \cup (a\Gamma a \cup a] = S$. Hence $S \setminus B \subseteq (a\Gamma a \cup a]$.

Let $b \in S \setminus B$. Then $b \in (a\Gamma a \cup a]$. Hence $b \in (a]$ or $b \in (a\Gamma a]$.

Case 1. $b \in (a]$. Then $b \leq a$. Let $x \in (b\Gamma S\Gamma b]$. Then $x \leq y$ for some $y \in b\Gamma S\Gamma b$. Hence $y = b\theta c\gamma b$ for some $c \in S$ and $\theta, \gamma \in \Gamma$. We have $x \leq y = b\theta c\gamma b \leq a\theta c\gamma a$. Then $x \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a] \subseteq B$

Case 2. $b \in (a\Gamma a]$. Then $b \leq a\eta a$ for some $\eta \in \Gamma$. Let $x \in (b\Gamma S\Gamma b]$. Then $x \leq y$ for some $y \in b\Gamma S\Gamma b$. Hence $y = b\mu r\theta b$ for some $r \in S$ and $\mu, \theta \in \Gamma$, we have

$$x \leq y = b\mu r\theta b \leq (a\eta a)\mu r\theta(a\eta a) = a\eta(a\mu r\theta a)\eta a.$$

Then $x \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a] \subseteq B$. Therefore $(b\Gamma S\Gamma b] \subseteq B$ for all $b \in S \setminus B$.

Case (2). $(a\Gamma S\Gamma a] \nsubseteq B$ for all $a \in S \setminus B$. Let $a \in S \setminus B$. Then $B \subset B \cup (a\Gamma S\Gamma a] \subseteq B \cup B(a)$. Since $B \cup B(a)$ is a bi-ideal of S and B is a maximal bi-ideal of S, we get $B \cup B(a) = S$. Then $S \setminus B \subseteq B(a)$ for all $a \in S \setminus B$. □

Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Define $\mathcal{U}$ is the union of all nonzero proper bi-ideals of S if S has a zero element and $\mathcal{U}$ is the union of all proper bi-ideals if S has no a zero element. In the other words, if

$$R = \{B \mid B \text{ is a proper bi-ideal of } S\},$$

then $\mathcal{U} = \bigcup\{B \mid B \in R\}$.

Lemma 3.29. *$S = \mathcal{U}$ if and only if $B(a) \neq S$ for all $a \in S$.*

Proof. Suppose that there exist $a \in S$ such that $S = B(a)$. Since $a \in S = \mathcal{U}$, it follows that $a \in B$ for some proper bi-ideal B of S and so $S = B(a) \subseteq B$. Since B is a proper bi-ideal of S, it is a contradiction.

Conversly, let $a \in S$. By hypothesis, $B(a) \neq S$. Then $B(a)$ is a proper bi-ideal of S such that $a \in B(a)$. Since $B(a) \in R$, $a \in R$, we have $a \in \mathcal{U}$. Therefore $S = \mathcal{U}$. □

Theorem 3.30. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup without zero. Then one of the following four conditions is satisfied.*

(1) $\mathcal{U}$ is not a bi-ideal of S.

(2) $B(a) \neq S$ for all $a \in S$.

(3) There exists $a \in S$ such that $B(a) = S$, $(a\Gamma a \cup a] \nsubseteq (a\Gamma S\Gamma a]$ and $(a\Gamma a\Gamma a] \subseteq \mathcal{U}$, S is not t-simple, $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$, and $\mathcal{U}$ is the unique maximal bi-ideal of S.

(4) $S \setminus \mathcal{U} \subseteq B(a)$ for all $a \in S \setminus \mathcal{U}$, S is not t-simple, $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$, and $\mathcal{U}$ is the unique maximal bi-ideal of S.

Proof. Assume that $\mathcal{U}$ is a bi-ideal of S. Then $\mathcal{U} \neq \emptyset$. Now, we have consider the following two cases :

Case 1. $\mathcal{U} = S$. By Lemma 3.29, $B(a) \neq S$ for all $a \in S$. In this case, the condition (2) is satisfied.

Case 2. $\mathcal{U} \neq S$. Then S is not t-simple. We want to show that $\mathcal{U}$ is the unique maximal bi-ideal of S, let A be a bi-ideal of S such that $\mathcal{U} \subseteq A$. If $A \neq S$, then A is a proper bi-ideal of S. Thus $A \subset \mathcal{U}$, so it is a contradiction, that is, $A = S$. Hence $\mathcal{U}$ is a maximal bi-ideal of S.

Next, assume that B is a maximal bi-ideal of S. Then $B \subseteq \mathcal{U} \subset S$ because B is a proper bi-ideal of S. Since B is a maximal bi-ideal of S, we have $B = \mathcal{U}$. Then $\mathcal{U}$ is the unique maximal bi-ideal of S.

Since $\mathcal{U} \neq S$, it follows from Lemma 3.29 that $B(a) = S$ for all $a \in S$. Clearly, $B(a) = S$ for all $a \in S \setminus \mathcal{U}$. Thus $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$. So $\mathcal{U} \cup B(a) = S$ is a bi-ideal of S for all $a \in S \setminus \mathcal{U}$. By Theorem 3.28, we have the following two case :

(2.1) $S \setminus \mathcal{U} \subseteq (a\Gamma a \cup a]$ and $(a\Gamma a\Gamma a] \subseteq \mathcal{U}$ for some $a \in S \setminus \mathcal{U}$ and $(b\Gamma S\Gamma b] \subseteq \mathcal{U}$ for all $b \in S \setminus \mathcal{U}$

(2.2) $S \setminus \mathcal{U} \subseteq B(a)$ for all $a \in S \setminus \mathcal{U}$.

Assume that $S \setminus \mathcal{U} \subseteq (a\Gamma a \cup a]$ and $(a\Gamma a\Gamma a] \subseteq \mathcal{U}$ for some $a \in S \setminus \mathcal{U}$ and $(b\Gamma S\Gamma b] \subseteq \mathcal{U}$ for all $b \in S \setminus \mathcal{U}$. If $(a\Gamma a \cup a] \subseteq (a\Gamma S\Gamma a]$, then $S = B(a) = (a\Gamma S\Gamma a \cup a\Gamma a \cup a] = (a\Gamma S\Gamma a]$. By hypothesis, $S = (a\Gamma S\Gamma a] \subseteq \mathcal{U}$ and so $\mathcal{U} = S$. It is a contradiction.

Therefore $(a\Gamma a \cup a] \nsubseteq (a\Gamma S\Gamma a]$. In this case, condition (3) is satisfied. Now, assume $S \setminus \mathcal{U} \subseteq B(a)$ for all $a \in S \setminus \mathcal{U}$. In this case, condition (4) is satisfied. □

Using the same proof of Theorem 3.30, we have Theorem 3.31.

Theorem 3.31. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup with zero and $S\Gamma S \neq \{0\}$. Then one of the following five conditions is satisfied.*

(1) $\mathcal{U}$ is not a bi-ideal of S.

(2) $B(a) \neq S$ for all $a \in S$.

(3) $\mathcal{U} = \{0\}$, $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$, and $\mathcal{U}$ is the unique maximal bi-ideal of S.

(4) There exists $a \in S$ such that $B(a) = S$, $(a\Gamma a \cup a] \nsubseteq (a\Gamma S\Gamma a]$ and $(a\Gamma a\Gamma a] \subseteq \mathcal{U}$, S is not 0-t-simple, $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$, and $\mathcal{U}$ is the unique maximal bi-ideal of S.

(5) $S \setminus \mathcal{U} \subseteq B(a)$ for all $a \in S \setminus \mathcal{U}$, S is not 0-t-simple, $S \setminus \mathcal{U} = \{x \in S \mid B(x) = S\}$, and $\mathcal{U}$ is the unique maximal bi-ideal of S.

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