New Properties for Ramanujan's Continued Fractions

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ABSTRACT

Ramanujan was able to find the factorisation theorem for those two important identities about the Rogers-Ramanujan continued fraction, resulting into new identities. In his notebook, he also wrote the 2- and 5-dissections of the Rogers-Ramanujan continued fraction and its reciprocal.

In this thesis, we factorise some identities about Ramanujan's cubic continued fraction analogously to those of the famous Rogers-Ramanujan continued fraction, following the lead of Ramanujan. As applications, we established as corollaries the early works on Ramanujan's cubic continued fractions. Moreover, we also recorded several new results for Ramanujan's cubic continued fraction and provided a 3-dissection property for $(q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$. Lastly, we proved the 2-dissection of the Göllnitz-Gordon functions.

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Saburi Tolulope Rasheed

CONTENTS

CHAPTER 1

Introduction

Srinivasa Ramanujan recorded many interesting results for theta functions and continued fractions. Among his discoveries are Rogers-Ramanujan, Ramanujan-Göllnitz-Gordon and Ramanujan's cubic continued fractions.

As customary and throughout this thesis, we assume that $|q| < 1$ and use the standard q-product notation

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad \text{for} \quad n \geq 1, \qquad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots (a_k;q)_\infty.$$

The symbol $(a;q)_\infty$ is the q-analogue of the rising or shifted factorial which is sometimes called the Pochhammer symbol.

Ramanujan's symmetric two variable general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.1}$$

which can also be expressed in terms of Jacobi's triple product identity [6, p.35] by

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{1.2}$$

The three significant special cases of $f(a,b)$ [6, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.5}$$

where the product representations in (1.3)-(1.5) follow from (1.2).

After Ramanujan, we define

$$\chi(q) = (-q; q^2)_\infty. \tag{1.6}$$

Note that $\chi(q)$ cannot be written as a special case of $f(a, b)$ but it's very useful to study Ramanujan's theta function. The famous identity credited to Euler [21, p. 164] is

$$\chi(-q) = \frac{1}{(q; -q)_\infty}. \tag{1.7}$$

The Rogers-Ramanujan continued fraction is defined by

$$R(q) := \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \dots}}}}$$

In compact form,

$$R(q) := \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1 + \dots}$$

Two of the most important formulas about $R(q)$ [3, p. 11], [24, pp. 135,238] are

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}, \tag{1.8}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}. \tag{1.9}$$

G.N. Watson discovered (1.8) and (1.9) in Ramanujan's notebook and provided proofs of them in his work [26]. On page 206 of Ramanujan's lost notebook, Ramanujan claimed that (1.8) and (1.9) can be factorised and the resulting factorisations are new identities. As established by Ramanujan [3, pp. 21-22], [24, p. 206], the factorisations of (1.8) and (1.9) are

$$\frac{1}{\sqrt{R(q)}} - \gamma\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1 + \gamma q^{n/5} + q^{2n/5}},$$

$$\frac{1}{\sqrt{R(q)}} - \delta\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1 + \delta q^{n/5} + q^{2n/5}},$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\gamma\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\gamma q^{n/5}+q^{2n/5})^5},$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\delta\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\delta q^{n/5}+q^{2n/5})^5},$$

where $\gamma = \dfrac{1-\sqrt{5}}{2}$ and $\delta = \dfrac{1+\sqrt{5}}{2}$.

On page 229 of his second notebook [23], (see [6, p. 221]), the Ramanujan-Göllnitz-Gordon continued fraction is defined as

$$G(q) := \frac{q^{1/2}}{1+q+}\,\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^5}{1+q^7+\ldots}$$

The two Göllnitz-Gordon functions $S(q)$ and $T(q)$ are defined [29] as

$$S(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2} \qquad \text{and} \qquad T(q) := \sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2+2n}. \tag{1.10}$$

So, the Göllnitz-Gordon identities are given by

$$S(q) = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}, \tag{1.11}$$

$$T(q) = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}. \tag{1.12}$$

It is easy to see that

$$G(q) = q^{1/2}\frac{T(q)}{S(q)}.$$

$S(q)$ and $T(q)$ were discovered independently by Göllnitz [15] and Gordon [14] when they worked on the convergence of a continued fraction. In L.J. Slater's famous list of 130 identities for Rogers-Ramanujan type [25], $S(q)$ and $T(q)$ are equation (36) and (37). The proofs of (1.11) and (1.12) was provided by G.E. Andrews [2]. K. Alladi [1] considered the odd-even split of the Euler Pentagonal Series and the Triangular Series of Gauss to establish elementary proofs of (1.11) and (1.12). In the spirit of Ramanujan, B. Yuttanan in 2012 extended the factorisation theorem to $G(q)$. He factorised some Ramanujan, T. Horie and N. Kanou's identities and established several new identities for $G(q)$ [29].

The factorisations for $G(q)$ provided by B. Yuttanan are

$$\frac{1}{\sqrt{G(q)}} + i\sqrt{G(q)} = \frac{f^2(q)}{q^{1/4}f(-iq^{1/2})f(-q^8)\sqrt{\chi(-q)}},$$
$$\frac{1}{\sqrt{G(q)}} - i\sqrt{G(q)} = \frac{f^2(q)}{q^{1/4}f(iq^{1/2})f(-q^8)\sqrt{\chi(-q)}},$$
$$\frac{1}{\sqrt{G(q)}} + \sqrt{G(q)} = \frac{\chi(q^2)\chi(-q^4)}{q^{1/4}\sqrt{\chi(-q)}}\prod_{n=1}^{\infty}\left(1+\left(\frac{n}{2}\right)q^{n/2}\right),$$
$$\frac{1}{\sqrt{G(q)}} - \sqrt{G(q)} = \frac{\chi(q^2)\chi(-q^4)}{q^{1/4}\sqrt{\chi(-q)}}\prod_{n=1}^{\infty}\left(1-\left(\frac{n}{2}\right)q^{n/2}\right),$$

where $\left(\frac{n}{2}\right)$ is the Kronecker symbol.
If $\alpha := \sqrt{2}-1, \beta := \sqrt{2}+1$ then

$$\frac{1}{\sqrt{G(q)}} - \alpha\sqrt{G(q)} = \frac{\prod_{n=1}^{\infty}(1+\alpha(-1)^n q^{n/2} - \alpha q^n - (-1)^n q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},$$
$$\frac{1}{\sqrt{G(q)}} + \alpha\sqrt{G(q)} = \frac{\prod_{n=1}^{\infty}(1+\alpha q^{n/2} - \alpha q^n - q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},$$
$$\frac{1}{\sqrt{G(q)}} - \beta\sqrt{G(q)} = \frac{\prod_{n=1}^{\infty}(1-\beta q^{n/2} + \beta q^n - q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},$$
$$\frac{1}{\sqrt{G(q)}} + \beta\sqrt{G(q)} = \frac{\prod_{n=1}^{\infty}(1-\beta(-1)^n q^{n/2} + \beta q^n - (-1)^n q^{3n/2})}{q^{1/4}f(-q^8)\sqrt{\chi(-q)}},$$

Recorded on page 366 of Ramanujan's lost notebook is the cubic continued fraction defined by

$$v(q) := \frac{q^{1/3}}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}$$

Ramanujan claimed that both $v(q)$ and $G(q)$ have many properties similar to those established for $R(q)$. In his notebooks [6, p. 345], [22] and in his lost notebooks [3, p. 94], [24], Ramanujan found that

$$v(q) = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{1.13}$$

Moreover, Ramanujan recorded several identities involving $v(q)$ [6, p.

345], namely,

$$
\begin{aligned}
1+\frac{1}{v(q)} &= \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)},\\
1-2v(q) &= \frac{\varphi(-q^{1/3})}{\varphi(-q^3)},\\
1+\frac{1}{v^3(q)} &= \frac{\psi^4(q)}{q\psi^4(q^3)},\\
1-8v^3(q) &= \frac{\varphi^4(-q)}{\varphi^4(-q^3)},\\
\frac{1}{v(q)}+4v^2(q) &= \left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3},\\
\frac{1}{v(q)}+4v^2(q) &= 3+\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}.
\end{aligned} \tag{1.14}
$$

This continued fraction recently has been studied by several authors. In 1995, H. H. Chan [12] derived the formula

$$
v^3(q) = v(q^3)\frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)}. \tag{1.15}
$$

In 2008, M. S. Mahadeva Naika [20] proved some identities

$$
\begin{aligned}
\frac{1}{v^2(q)}-2v(q) &= \left(27+\frac{f^{12}(-q^2)}{q^2f^{12}(-q^6)}\right)^{1/3},\\
\frac{1}{v^2(q)}-2v(q) &= 3+\frac{f^3(-q^{2/3})}{q^{2/3}f^3(-q^6)}.
\end{aligned} \tag{1.16}
$$

The cubic analogues of (1.8) and (1.9) were proved by H-C. Chan [9],[10, p. 113]. They are

$$
\begin{aligned}
\frac{1}{v(q)}-1-2v(q) &= \frac{f(-q^{1/3})f(-q^{2/3})}{q^{1/3}f(-q^3)f(-q^6)},\\
\frac{1}{v^3(q)}-7-8v^3(q) &= \frac{f^4(-q)f^4(-q^2)}{qf^4(-q^3)f^4(-q^6)}.
\end{aligned} \tag{1.17}
$$

Before we end this chapter, we give three important definitions.

Definition 1.1. *For all $n \geq 0$ and n is an integer, the partition of n denoted by $p(n)$ is the number of ways n can be represented as a sum of positive integers regardless of the order of the summands.*

For example, $p(4) = 5$, i.e. , $4 = 3+1 = 2+1+1 = 2+2 = 1+1+1+1$.

The generating function for $p(n)$ is

$$
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.
$$

Definition 1.2. *For all $n \geq 0$, $c(n)$ is the number of two colour partitions of n with colours r and b such that colour b only appears in even parts. This is called the cubic partition function.*

For example, $c(2) = 3, c(3) = 4, c(4) = 9$ and $c(n) = \sum_{n=x+2y} p(x).p(y)$. The generating function for $c(n)$ is

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{1.18}$$

Definition 1.3. [8]. *If $P(q)$ is a power series in q denoted by $\sum_{k=0}^{\infty} a_k q^k$, then the $n-$dissection of $P(q)$ is defined by*

$$P(q) := \sum_{k=0}^{n-1} q^k P_k(q^n),$$

where the parts of dissections are P_k.

In 2011, H. Zhao and Z. Zhong [30] established the 3-dissection property for the cubic partition function.

The object of this thesis is to provide a factorisation theorem for $v(q)$ which is analogous to those established for $R(q)$ and $G(q)$. After that, we use the results of our factorisations to give another proof of early works by Ramanujan in (1.14), Chan in (1.15), Mahadeva Naika in (1.16) and Chan in (1.17). Furthermore, we use (1.13) to establish several new identities for $v(q)$. Moreover, we proved a 3-dissection property of (1.18). Finally, we establish the 2-dissection of the Göllnitz-Gordon identities motivated by the work of Hirschhorn [17] on another Ramanujan's identities.

CHAPTER 2

Some Preliminary Results

We recall a simple fact about the primitive cube root of unity. Let $\zeta = e^{2\pi i/3}$. Then, we have

$$\zeta^{2n} + \zeta^n + 1 = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ 0 & \text{if } n \not\equiv 0 \pmod 3 \end{cases}$$

and, for any $q \in \mathbb{C}$,

$$(1-q)(1-\zeta q)(1-\zeta^2 q) = 1 - q^3.$$

Next, we introduce some preliminary results concerning Ramanujan's theta functions and Ramanujan's cubic continued fraction. Ramanujan recorded several identities for $f(a,b), \varphi(q), \psi(q), f(-q)$ and $\chi(q)$ in his lost notebooks. The following lemmas provide such identities used in this literature.

Lemma 2.1. ([6, p. 48]). *Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n. Then*

$$f(U_1, V_1) = \sum_{r=0}^{k-1} U_r f\left(\frac{U_{k+r}}{U_r}, \frac{V_{k-r}}{U_r}\right), \tag{2.1}$$

for every positive integer k.

Lemma 2.2. ([6, p. 34]). *We have*

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f(a,a^3) \tag{2.3}$$

and if n is an integer, then

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.4}$$

Lemma 2.3. ([6, pp. 39-40]). *We have*

$$f(-q^2)=\psi(-q)\chi(q), \tag{2.5}$$

$$f(q)=\chi(q)f(-q^2), \tag{2.6}$$

$$\chi(q)f(q)=\varphi(q), \tag{2.7}$$

$$\varphi(q)\psi(-q)=f(q)f(-q^2), \tag{2.8}$$

$$f(q)\psi(-q)=f^2(-q^2), \tag{2.9}$$

$$\varphi(-q)\psi^2(q)=f^3(-q^2), \tag{2.10}$$

$$\varphi^2(-q)\psi(q)=f^3(-q). \tag{2.11}$$

Theorem 2.4. ([20]). *We have*

$$\frac{f^6(-q)}{f^6(-q^3)}=\frac{\psi^2(q)}{\psi^2(q^3)}\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)},$$
$$\frac{f^6(-q^2)}{qf^6(-q^6)}=\frac{\varphi^2(-q)}{\varphi^2(-q^3)}\frac{\varphi^4(-q)-9\varphi^4(-q^3)}{\varphi^4(-q)-\varphi^4(-q^3)},$$
$$\frac{f^{12}(-q)}{qf^{12}(-q^3)}=\frac{\varphi^8(-q)}{\varphi^8(-q^3)}\frac{\varphi^4(-q)-9\varphi^4(-q^3)}{\varphi^4(-q)-\varphi^4(-q^3)},$$
$$\frac{f^{12}(-q^2)}{f^{12}(-q^6)}=\frac{\psi^8(q)}{\psi^8(q^3)}\frac{\psi^4(q)-9q\psi^4(q^3)}{\psi^4(q)-q\psi^4(q^3)}.$$

Theorem 2.5. ([20]). *We have*

$$\frac{f^3(-q)}{f^3(-q^9)}=\frac{\psi(q)}{\psi(q^9)}\left(\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}\right)^2,$$
$$\frac{f^3(-q^2)}{f^3(-q^{18})}=\frac{\psi^2(q)}{\psi^2(q^9)}\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)},$$
$$\frac{f^3(-q)}{qf^3(-q^9)}=\frac{\varphi^2(-q)}{\varphi^2(-q^9)}\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)},$$
$$\frac{f^3(-q^2)}{q^2f^3(-q^{18})}=\frac{\varphi(-q)}{\varphi(-q^9)}\left(\frac{\varphi(-q)-3\varphi(-q^9)}{\varphi(-q)-\varphi(-q^9)}\right)^2.$$

Theorem 2.6. ([20]). *If* $\alpha=\dfrac{1-i\sqrt{3}}{2}$ *and* $\beta=\dfrac{1+i\sqrt{3}}{2}$*, then*

$$\varphi(-q)+i\sqrt{3}\varphi(-q^3)=\frac{(1+i\sqrt{3})\chi(-q)f(-q^4)}{\prod_{n\equiv 0,2,3 \pmod 4}(1-\alpha q^n)\prod_{n\equiv 0,1,2 \pmod 4}(1-\beta q^n)},$$
$$\varphi(-q)-i\sqrt{3}\varphi(-q^3)=\frac{(1-i\sqrt{3})\chi(-q)f(-q^4)}{\prod_{n\equiv 0,1,2 \pmod 4}(1-\alpha q^n)\prod_{n\equiv 0,2,3 \pmod 4}(1-\beta q^n)},$$
$$\varphi^2(-q)+3\varphi^2(-q^3)=4\chi^2(-q)f(-q^4)\prod_{n\equiv 0 \pmod 3}(1+q^n)(1+q^{2n}).$$

Theorem 2.7. ([20]). *We have*

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} = 1 + \sqrt[3]{\frac{4\mu(q)}{(1-\mu(q))^2}},$$
$$\frac{\psi(q^{2/3})}{q^{2/3}\psi(q^6)} = 1 + \sqrt[3]{\frac{4(1-\mu(q))}{\mu(q)}},$$

where $\mu(q) = 2v(q)v(q^2)$.

In the next section in this chapter, we write briefly on the Chan's cubic analogue of an identity that was considered the Ramanujan's best.

2.1 The cubic analogue of Ramanujan's "most beautiful identity"

Out of nearly 3,900 results that Ramanujan independently compiled, one of them stood out. And both G.H. Hardy (Ramanujan's chief collaborator) and Major Macmahon agreed that this identity;

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$$

is the Ramanujan's "Most Beautiful Identity". The consequence of this stunning identity of Ramanujan is that

$$p(5n+4) \equiv 0 \pmod 5.$$

Motivated by the work of Ramanujan, H.C. Chan [9] in 2010 discovered the cubic analogue of Ramanujan's "Most Beautiful Identity having worked extensively on Ramanujan's cubic continued fraction.

Theorem 2.8. [9](The cubic analogue of Ramanujan's "most beautiful identity"). *For* $n \geq 0$ *and let* $c(n)$ *be a cubic partition function with a generating function* $\sum_{n\geq 0} c(n)q^n = (q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$. *Then*

$$\sum_{n=0}^{\infty} c(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}. \tag{2.12}$$

Corollary 2.9. *We have*

$$c(3n+2) \equiv 0 \pmod 3. \tag{2.13}$$

Several authors have recently proved (2.12) following the lead of H.C. Chan [9]. In 2010, Z. Cao [8] proved it by the 3-dissection of $(q;q)_\infty(q^2;q^2)_\infty$ and a strategy similar to the one M.D. Hirschhorn [16] used to establish the Ramanujan's "most beautiful identity". H.H. Chan and P.C. Toh [11] in 2010 proved the existence of (2.12) by the theory of modular functions. In 2011, H. Zhao and Z. Zhong [30] established the result by using some Borweins' cubic theta functions. In 2011, X. Xiong [27], [28] gave an elementary proofs by using the 3-dissection of $\frac{1}{\varphi(-q)}$ and $\frac{1}{\psi(q)}$ and meromorphic modular functions respectively. In 2015, N.D. Baruah and K.K. Ojah [5] gave a new proof by the 3-dissection of $(q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$. W. Chu and R.R. Zhou [13] in 2015 gave a computational proof of (2.12). K. Banerjee and P.D. Ardhikary [4] in 2017 also provided an elementary alternative proof.

Theorem 2.10. [19](The Jacobi's triple product identity). *For each pair of complex numbers z and q, with $z \neq 0$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n = (q, z, qz^{-1}; q)_\infty. \tag{2.14}$$

Theorem 2.11. [13](The quintuple product identity). *For the indeterminate q and z,*

$$\sum_{n=-\infty}^{\infty} q^{3\binom{n}{2}+n}(1-q^n z)z^{3n} = (q, z, qz^{-1}; q)_\infty (qz^2, qz^{-2}; q^2)_\infty. \tag{2.15}$$

Theorem 2.12. [13]. *For $z \neq 0$,*

$$\begin{aligned}(q, z, qz^{-1}; q)_\infty (q^2, z^2, q^2 z^{-2}; q^2)_\infty &= (q^6, q^3, q^3; q^6)_\infty \sum_{n=-\infty}^{\infty} q^{3\binom{n}{2}} z^{3n} \\ &\quad - (q^6, q, q^5; q^6)_\infty \sum_{n=-\infty}^{\infty} q^{3\binom{n}{2}+n}(1+q^n z) z^{3n+1}.\end{aligned} \tag{2.16}$$

2.1.1 The 3-dissection of $(q;q)_\infty(q^2;q^2)_\infty$

Theorem 2.13.

$$(q;q)_\infty(q^2;q^2)_\infty = (q^9;q^9)_\infty(q^{18};q^{18})_\infty \left\{ \frac{(q^9;q^{18})_\infty^3}{(q^3;q^6)_\infty} - q - 2q^2 \frac{(q^3;q^6)_\infty}{(q^9;q^{18})_\infty^3} \right\}. \tag{2.17}$$

Proof. Here, we show the proof of W. Chu and R.R. Zhou [13]. See also [8], [9]. By changing q to q^3 and z to q in (2.16), we have

$$(q;q)_\infty(q^2;q^2)_\infty = (q^{18},q^9,q^9;q^{18})_\infty \sum_{n=-\infty}^{\infty} q^{9\binom{n}{2}+3n} - q(q^{18},q^3,q^{15};q^{18})_\infty \sum_{n=-\infty}^{\infty} (1+q^{3n+1})q^{9\binom{n}{2}+6n}.$$

Using (2.14), then

$$(q;q)_\infty(q^2;q^2)_\infty = (q^{18},q^9,q^9;q^{18})_\infty(q^9,-q^3,-q^6;q^9)_\infty - q(q^{18},q^3,q^{15};q^{18})_\infty (q^9,-q^3,-q^6;q^9)_\infty - 2q^2(q^{18},q^3,q^{15};q^{18})_\infty(q^9,-q^9,-q^9;q^9)_\infty. \tag{2.18}$$

After we divide (2.18) through by $(q^9;q^9)_\infty(q^{18},q^{18})_\infty$, we notice that

$$\frac{(q^{18},q^3,q^{15};q^{18})_\infty(q^9,-q^3,-q^6;q^9)_\infty}{(q^{18};q^{18})_\infty(q^9;q^9)_\infty} = (q^3,q^{15};q^{18})_\infty(-q^3,-q^6;q^9)_\infty = \frac{(q^3;q^6)_\infty(-q^3;q^3)_\infty}{(q^9;q^{18})_\infty(-q^9;q^9)_\infty} = 1.$$

Hence,

$$\begin{aligned}\frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty} &= \frac{(q^{18},q^9,q^9;q^{18})_\infty}{(q^{18},q^3,q^{15};q^{18})_\infty} - q - 2q^2\frac{(q^9,-q^9,-q^9;q^9)_\infty}{(q^9,-q^3,-q^6;q^9)_\infty}\\ &= \frac{(q^9;q^{18})_\infty^3}{(q^3;q^6)_\infty} - q - 2q^2\frac{(-q^9;q^9)_\infty^3}{(-q^3;q^3)_\infty}\\ &= \frac{(-q^3;q^3)_\infty}{(-q^9;q^9)_\infty^3} - q - 2q^2\frac{(-q^9;q^9)_\infty^3}{(-q^3;q^3)_\infty}.\end{aligned}$$

Note that we get,

$$\frac{1}{F(q^3)} - q - 2q^2F(q^3) = \frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty},$$

where $F(q) = q^{-1/3}v(q)$. Also, by changing q^3 to q and subsequently dividing through by $q^{1/3}$ yields the first equality in (1.17). □

2.1.2 The 3-dissection of $(q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$

We show H.-C. Chan's proof here [9]. See also [5]. Replacing q by q^3 in the second identity of (1.17) and multiplying through by q^3, we obtain

$$\frac{1}{F^3(q^3)} - 7q^3 - 8q^6F^3(q^3) = \left(\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}\right)^4.$$

So,

$$\begin{aligned}\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} &= \frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q;q)_\infty(q^2;q^2)_\infty} \times \frac{1}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}\\ &= \frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4} \times \frac{F^{-3}(q^3) - 7q^3 - 8q^6F^3(q^3)}{F^{-1}(q^3) - q - 2q^2F(q^3)}\\ &= \frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}\left\{\left(\frac{1}{F^2(q^3)} - 2q^3F(q^3)\right)\right.\\ &\quad \left. + q\left(4q^3F^2(q^3) + \frac{1}{F(q^3)}\right) + 3q^2\right\}. \qquad (2.19)\end{aligned}$$

It should be noted that we have used the long division in the last equality. Therefore, the third part of this dissection is equivalent to

$$\sum_{n=0}^{\infty} c(3n+2)q^{3n+2} = 3q^2\frac{(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}. \qquad (2.20)$$

To prove (2.12) we divide both sides of (2.20) by q^2 and change q to $q^{1/3}$.

2.2 The 2-, 3-, 4- and 6-dissections of Ramanujan's cubic continued fraction $v(q)$ and its reciprocal

All the results presented here are those of M.D. Hirschhorn and Roselin [18].

Theorem 2.14. *The 2-dissections $v(q)$ and its reciprocal are given by*

$$v(q) = \frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4} - q\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^6},$$

$$1/v(q) = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty} + q\frac{(q^2;q^2)_\infty(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty(q^6;q^6)_\infty^3}.$$

Theorem 2.15. *The 3-dissections of $v(q)$ and its reciprocal are given by*

$$\begin{aligned}v(q) &= \frac{(q^6;q^6)_\infty(q^{21},q^{33},q^{54};q^{54})_\infty}{(q^3;q^3)_\infty^2}\\ &\quad - q\frac{(q^6;q^6)_\infty(q^{15},q^{39},q^{54};q^{54})_\infty}{(q^3;q^3)_\infty^2} - q^5\frac{(q^6;q^6)_\infty(q^3;q^{51},q^{54};q^{54})_\infty}{(q^3;q^3)_\infty^2},\end{aligned}$$

$$\begin{aligned}1/v(q) &= \frac{(q^3;q^3)_\infty(-q^{12},-q^{15},q^{27};q^{27})_\infty}{(q^6;q^6)_\infty^2}\\ &\quad + q\frac{(q^3;q^3)_\infty(-q^{6},-q^{21},q^{27};q^{27})_\infty}{(q^6;q^6)_\infty^2} + q^2\frac{(q^3;q^3)_\infty(-q^{3},-q^{24},q^{27};q^{27})_\infty}{(q^6;q^6)_\infty^2}.\end{aligned}$$

Theorem 2.16. *The 4-dissections of $v(q)$ and its reciprocal are given by*

$$\begin{aligned}v(q) &= \frac{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty^{14}}{(q^{12};q^{12})_\infty^{12}(q^{48};q^{48})_\infty^4}\\ &\quad - q\left(\frac{(q^8;q^8)_\infty^4(q^{24};q^{24})_\infty^4}{(q^{12};q^{12})_\infty^8} + q^4\frac{(q^4;q^4)_\infty^4(q^{24};q^{24})_\infty^{12}}{(q^8;q^8)_\infty^4(q^{12};q^{12})_\infty^{12}}\right)\\ &\quad + 4q^6\frac{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty^2(q^{48};q^{48})_\infty^4}{(q^{12};q^{12})_\infty^8} + 2q^3\frac{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty^8}{(q^{12};q^{12})_\infty^{10}},\end{aligned}$$

$$\begin{aligned}1/v(q) &= \frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty^5}{(q^8;q^8)_\infty(q^{12};q^{12})_\infty^5(q^{48};q^{48})_\infty^2} + q\frac{(q^8;q^8)_\infty^2(q^{24};q^{24})_\infty^2}{(q^{12};q^{12})_\infty^4}\\ &\quad + q^2\frac{(q^8;q^8)_\infty^5(q^{48};q^{48})_\infty^2}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty^3(q^{16};q^{16})_\infty^2(q^{24};q^{24})_\infty}\\ &\quad - q^3\frac{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty^6}{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^6}.\end{aligned}$$

Theorem 2.17. *The 6-dissections of $v(q)$ and its reciprocal are given by*

$$
\begin{aligned}
v(q) = \Bigg(&\frac{(q^{12};q^{12})_\infty^3(q^{18};q^{18})_\infty^2(q^{24},q^{30},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^5(q^{36};q^{36})_\infty} \\
&- q^6\frac{(q^{12};q^{12})_\infty^2(q^{36};q^{36})_\infty^2(q^6,q^{48},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^4(q^{18};q^{18})_\infty}\Bigg) \\
- q\Bigg(&\frac{(q^{12};q^{12})_\infty^2(q^{42},q^{66},q^{108};q^{108})_\infty^2}{(q^6;q^6)_\infty^4} \\
&+ 2q^{12}\frac{(q^{12};q^{12})_\infty^2(q^6,q^{30},q^{78},q^{102},q^{108},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^4}\Bigg) \\
- q^2\Bigg(&\frac{(q^{12};q^{12})_\infty^3(q^{18};q^{18})_\infty^2(q^{12},q^{42},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^5(q^{36};q^{36})_\infty} \\
&- \frac{(q^{12};q^{12})_\infty^2(q^{36};q^{36})_\infty^2(q^{24},q^{30},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^4(q^{18};q^{18})_\infty}\Bigg) \\
+ q^3\Bigg(&2\frac{(q^{12};q^{12})_\infty^2(q^{30},q^{42},q^{66},q^{78},q^{108},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^4} \\
&- q^{18}\frac{(q^{12};q^{12})_\infty^2(q^6,q^{102},q^{108};q^{108})_\infty^2}{(q^6;q^6)_\infty^4}\Bigg) \\
- q^4\Bigg(&\frac{(q^{12};q^{12})_\infty^3(q^{18};q^{18})_\infty^2(q^6,q^{48},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^5(q^{36};q^{36})_\infty} \\
&+ \frac{(q^{12};q^{12})_\infty^2(q^{36};q^{36})_\infty^2(q^{12},q^{42},q^{54};q^{54})_\infty}{(q^6;q^6)_\infty^4(q^{18};q^{18})_\infty}\Bigg) \\
- q^5\Bigg(&\frac{(q^{12};q^{12})_\infty^2(q^{30},q^{78},q^{108};q^{108})_\infty^2}{(q^6;q^6)_\infty^4} \\
&- 2q^6\frac{(q^{12};q^{12})_\infty^2(q^6,q^{42},q^{66},q^{102},q^{108},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^4}\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
1/v(q) = &\Bigg(\frac{(q^{18};q^{18})_\infty(q^{48},q^{60},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2(q^{36};q^{36})_\infty} - q^6\frac{(q^{36};q^{36})_\infty^2(q^{24},q^{84},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\Bigg) \\
&+ q\frac{(q^{12};q^{12})_\infty(q^{42},q^{66},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2} \\
&+ q^2\Bigg(\frac{(q^{36};q^{36})_\infty^2(q^{48},q^{60},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty} - q^6\frac{(q^{18};q^{18})_\infty(q^{12},q^{96},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2(q^{36};q^{36})_\infty}\Bigg) \\
&- q^3\frac{(q^{12};q^{12})_\infty(q^{30},q^{78},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2} \\
&- q^4\Bigg(\frac{(q^{18};q^{18})_\infty(q^{24},q^{84},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2(q^{36};q^{36})_\infty} + q^6\frac{(q^{36};q^{36})_\infty^2(q^{12},q^{96},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\Bigg) \\
&- q^{11}\frac{(q^{12};q^{12})_\infty(q^6,q^{102},q^{108};q^{108})_\infty}{(q^6;q^6)_\infty^2}.
\end{aligned}
$$

Corollary 2.18. *If we write*

$$v(q) =: \sum_{n\geq 0} a_n q^n, \quad 1/v(q) =: \sum_{n\geq 0} b_n q^n,$$

then the sign of the a_n is periodic with period 3, and the sign of the b_n is periodic with period 6. Indeed,

$$b_{6n} > 0, a_{3n} = b_{6n+1} > 0, b_{6n+4} > 0, a_{3n+1} = b_{6n+3} < 0, b_{6n+4} < 0, a_{3n+2} = b_{6n+5} < 0$$

except $a_2 = b_5 = b_8 = 0.$

CHAPTER 3

New properties for Ramanujan's continued fractions

In this chapter, we derive new identities involving Ramanujan's cubic continued fraction that are analogous to those of the famous Rogers-Ramanujan continued fraction. Using these new identities, we are able to give the new proofs of several early results of this continued fraction and also establish a 3-dissection property for the generating function of the cubic partition function. Lastly, we establish the 2-dissection of the Ramanujan-Göllnitz-Gordon functions.

3.1 The factorisations of Ramanujan's cubic continued fraction

Using (1.13), the continued fraction $v(q)$ can be re-expressed as

$$\begin{aligned} v(q) &= q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} \\ &= q^{1/3}\frac{(q;q^2)_\infty(q^6;q^6)_\infty}{(q^3;q^6)_\infty^2(q^6;q^6)_\infty(q^3;q^6)_\infty} \\ &= q^{1/3}\frac{(q;q^2)_\infty\psi(q^3)}{\varphi(-q^3)} \\ &= q^{1/3}\frac{\psi(q^3)}{\varphi(-q^3)(-q;q)} \\ &= q^{1/3}\frac{\psi(q^3)(q^3;q^3)_\infty}{\varphi(-q^3)(-q^3;q^3)_\infty f(q,q^2)} \\ &= q^{1/3}\frac{\psi(q^3)}{f(q,q^2)} \\ &= q^{1/3}\frac{f(q^3,q^9)}{f(q,q^2)}. \end{aligned} \tag{3.1}$$

Throughout this section, we let $\zeta := e^{\pi i/3}$.

Theorem 3.1. *We have*

$$\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}=\frac{\psi(q^{1/3})}{q^{1/6}}\sqrt{\frac{\chi(-q)}{f(-q^3)f(-q^6)}}, \tag{3.2}$$

$$\frac{1}{\sqrt{v(q)}}-\zeta\sqrt{v(q)}=\frac{\psi(-\zeta q^{1/3})}{q^{1/6}}\sqrt{\frac{\chi(-q)}{f(-q^3)f(-q^6)}}, \tag{3.3}$$

$$\frac{1}{\sqrt{v(q)}}+\zeta^2\sqrt{v(q)}=\frac{\psi(\zeta^2 q^{1/3})}{q^{1/6}}\sqrt{\frac{\chi(-q)}{f(-q^3)f(-q^6)}}, \tag{3.4}$$

$$\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}=\frac{\varphi(-q^{1/3})}{q^{1/6}\sqrt{\chi(-q)f(-q^3)f(-q^6)}}, \tag{3.5}$$

$$\frac{1}{\sqrt{v(q)}}+2\zeta\sqrt{v(q)}=\frac{\varphi(\zeta q^{1/3})}{q^{1/6}\sqrt{\chi(-q)f(-q^3)f(-q^6)}}, \tag{3.6}$$

$$\frac{1}{\sqrt{v(q)}}-2\zeta^2\sqrt{v(q)}=\frac{\varphi(-\zeta^2 q^{1/3})}{q^{1/6}\sqrt{\chi(-q)f(-q^3)f(-q^6)}}, \tag{3.7}$$

Proof of (3.2). By (3.1), the left hand side of (3.2) becomes

$$\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}=\frac{f(q,q^2)+q^{1/3}f(q^3,q^9)}{q^{1/6}\sqrt{f(q,q^2)f(q^3,q^9)}}. \tag{3.8}$$

Using Jacobi's triple product identity (1.2), we have

$$\begin{aligned}
f(q,q^2)f(q^3,q^9) &= (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty\\
&= (q^3;q^3)_\infty(q^6;q^6)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(-q^6;q^6)_\infty\\
&= f(-q^3)f(-q^6)(-q;q)_\infty\\
&= \frac{f(-q^3)f(-q^6)}{\chi(-q)},
\end{aligned} \tag{3.9}$$

where the last equality is obtained by (1.7). Putting $k=3$, $a=1$, $b=q^{1/3}$ in (2.1) together with (2.2) and (2.3), it follows that

$$\begin{aligned}
f(1,q^{1/3}) &= f(q,q^2)+f(q^2,q)+q^{1/3}f(q^3,1)\\
2f(q^{1/3},q) &= 2f(q,q^2)+2q^{1/3}f(q^3,q^9)\\
\psi(q^{1/3}) &= f(q,q^2)+q^{1/3}f(q^3,q^9).
\end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain the result. □

Proof of (3.3). By (3.1), the left hand side of (3.3) becomes

$$\frac{1}{\sqrt{v(q)}}-\zeta\sqrt{v(q)}=\frac{f(q,q^2)-\zeta q^{1/3}f(q^3,q^9)}{q^{1/6}\sqrt{f(q,q^2)f(q^3,q^9)}}. \tag{3.11}$$

Putting $k = 3, a = 1, b = -\zeta q^{1/3}$ in (2.1) together with (2.2) and (2.3), it follows that

$$\begin{aligned} f(1, -\zeta q^{1/3}) &= f(q, q^2) + f(q^2, q) - \zeta q^{1/3} f(q^3, 1) \\ 2f(-\zeta q^{1/3}, q) &= 2f(q, q^2) - 2\zeta q^{1/3} f(q^3, q^9) \\ \psi(-\zeta q^{1/3}) &= f(q, q^2) - \zeta q^{1/3} f(q^3, q^9). \end{aligned} \tag{3.12}$$

Substituting (3.12) and (3.9) into (3.11), we complete the proof of (3.3). □

Proof of (3.4). By (3.1), the left hand side of (3.4) becomes

$$\frac{1}{\sqrt{v(q)}} + \zeta^2 \sqrt{v(q)} = \frac{f(q, q^2) + \zeta^2 q^{1/3} f(q^3, q^9)}{q^{1/6} \sqrt{f(q, q^2) f(q^3, q^9)}}. \tag{3.13}$$

Putting $k = 3, a = 1, b = \zeta^2 q^{1/3}$ in (2.1) together with (2.2) and (2.3), it follows that

$$\begin{aligned} f(1, \zeta^2 q^{1/3}) &= f(q, q^2) + f(q^2, q) + \zeta^2 q^{1/3} f(q^3, 1) \\ 2f(\zeta^2 q^{1/3}, q) &= 2f(q, q^2) + 2\zeta^2 q^{1/3} f(q^3, q^9) \\ \psi(\zeta^2 q^{1/3}) &= f(q, q^2) + \zeta^2 q^{1/3} f(q^3, q^9). \end{aligned} \tag{3.14}$$

Substituting (3.14) and (3.9) into (3.13), we complete the proof of (3.4). □

Proof of (3.5). By (3.1) and (3.9), we have

$$\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} = \frac{(f(q, q^2) - 2q^{1/3} f(q^3, q^9))\sqrt{\chi(-q)}}{q^{1/6} \sqrt{f(-q^3) f(-q^6)}}. \tag{3.15}$$

Take $k = 3, a = \zeta^2, b = -\zeta q^{1/3}$ in (2.1) and obtain

$$f(\zeta^2, -\zeta q^{1/3}) = f(q, q^2) + \zeta^2 f(q, q^2) - \zeta q^{1/3} f(q^3, 1)$$

Since $\dfrac{\zeta}{1 + \zeta^2} = 1$, we arrive at

$$\frac{f(\zeta^2, -\zeta q^{1/3})}{1 + \zeta^2} = f(q, q^2) - 2q^{1/3} f(q^3, q^9). \tag{3.16}$$

Using (1.2), we deduce that

$$
\begin{aligned}
\frac{f(\zeta^2, -\zeta q^{1/3})}{1+\zeta^2} &= \frac{(-\zeta^2; q^{1/3})_\infty (\zeta q^{1/3}; q^{1/3})_\infty (q^{1/3}; q^{1/3})_\infty}{1+\zeta^2} \\
&= (-\zeta^2 q^{1/3}; q^{1/3})_\infty (\zeta q^{1/3}; q^{1/3})_\infty (q^{1/3}; q^{1/3})_\infty \\
&= f(-q^{1/3}) \prod_{k=1}^{\infty} (1+\zeta^2 q^{k/3})(1-\zeta q^{k/3}) \\
&= f(-q^{1/3}) \prod_{k=1}^{\infty} (1 - q^{k/3} + q^{2k/3}) \\
&= f(-q^{1/3}) \prod_{k=1}^{\infty} \frac{1+q^k}{1+q^{k/3}} \\
&= \frac{f(-q^{1/3})(-q;q)_\infty}{(-q^{1/3}; q^{1/3})_\infty} \\
&= \frac{f(-q^{1/3})\chi(-q^{1/3})}{\chi(-q)}, \qquad (3.17)
\end{aligned}
$$

where the last equality is obtained by (1.7). By (3.15), (3.16) and (3.17), we complete the proof of (3.5). □

Proof of (3.6). By (3.1) and (3.9), the left hand side of (3.6) becomes

$$
\frac{1}{\sqrt{v(q)}} + 2\zeta\sqrt{v(q)} = \frac{(f(q,q^2) + 2\zeta q^{1/3} f(q^3, q^9))\sqrt{\chi(-q)}}{q^{1/6}\sqrt{f(-q^3)f(-q^6)}}. \qquad (3.18)
$$

Putting $k = 3, a = -\zeta, b = q^{1/3}$ in (2.1) together with (2.2) and (2.3), it follows that

$$
\begin{aligned}
f(-\zeta, q^{1/3}) &= f(q,q^2) - \zeta f(q^2, q) + q^{1/3} f(q^3, 1) \\
\frac{f(-\zeta, q^{1/3})}{1-\zeta} &= f(q,q^2) + \frac{2q^{1/3}}{1-\zeta} f(q^3, q^9) \\
&= f(q,q^2) + 2\zeta q^{1/3} f(q^3, q^9). \qquad (3.19)
\end{aligned}
$$

Using (1.2), we deduce that

$$
\begin{aligned}
\frac{f(-\zeta,q^{1/3})}{1-\zeta} &= \frac{(\zeta;-\zeta q^{1/3})_\infty(-q^{1/3};-\zeta q^{1/3})_\infty(-\zeta q^{1/3};-\zeta q^{1/3})_\infty}{1-\zeta}\\
&= f(-\zeta q^{1/3})\prod_{k=0}^{\infty}\left(1+\zeta^2 q^{1/3}(-\zeta q^{1/3})^k\right)\left(1+q^{1/3}(-\zeta q^{1/3})^k\right)\\
&= f(-\zeta q^{1/3})\prod_{k=0}^{\infty}\left(1+(1+\zeta^2)q^{1/3}(-\zeta q^{1/3})^k+\zeta^2 q^{2/3}(-\zeta q^{1/3})^{2k}\right)\\
&= f(-\zeta q^{1/3})\prod_{k=0}^{\infty}\left(1+\zeta q^{1/3}(-\zeta q^{1/3})^k+(-\zeta q^{1/3})^{2(k+1)}\right)\\
&= f(-\zeta q^{1/3})\prod_{k=1}^{\infty}\left(1-(-\zeta q^{1/3})^k+(-\zeta q^{1/3})^{2k}\right)\\
&= f(-\zeta q^{1/3})\prod_{k=1}^{\infty}\frac{1+q^k}{1+(-\zeta q^{1/3})^k}\\
&= \frac{f(-\zeta q^{1/3})(-q;q)_\infty}{(\zeta q^{1/3};-\zeta q^{1/3})_\infty}\\
&= \frac{f(-\zeta q^{1/3})\chi(\zeta q^{1/3})}{\chi(-q)}. \qquad (3.20)
\end{aligned}
$$

Using (3.18), (3.19) and (3.20), we finish the proof of (3.6). □

Proof of (3.7). By (3.1) and (3.9), the left hand side of (3.7) becomes

$$\frac{1}{\sqrt{v(q)}}-2\zeta^2\sqrt{v(q)}=\frac{(f(q,q^2)-2\zeta^2 q^{1/3}f(q^3,q^9))\sqrt{\chi(-q)}}{q^{1/6}\sqrt{f(-q^3)f(-q^6)}}. \qquad (3.21)$$

Putting $k=3, a=\zeta^2, b=q^{1/3}$ in (2.1) together with (2.2) and (2.3), it follows that

$$
\begin{aligned}
f(\zeta^2,q^{1/3}) &= f(q,q^2)+\zeta^2 f(q^2,q)+q^{1/3}f(q^3,1)\\
\frac{f(\zeta^2,q^{1/3})}{1+\zeta^2} &= f(q,q^2)+\frac{2q^{1/3}}{1+\zeta^2}f(q^3,q^9)\\
&= f(q,q^2)-2\zeta^2 q^{1/3}f(q^3,q^9). \qquad (3.22)
\end{aligned}
$$

Using (1.2), we deduce that

$$\begin{aligned}
\frac{f(\zeta^2,q^{1/3})}{1+\zeta^2} &= \frac{(-\zeta^2;\zeta^2q^{1/3})_\infty(-q^{1/3};\zeta^2q^{1/3})_\infty(\zeta^2q^{1/3};\zeta^2q^{1/3})_\infty}{1+\zeta^2}\\
&= f(\zeta^2q^{1/3})\prod_{k=0}^{\infty}\Big(1+\zeta^4q^{1/3}(\zeta^2q^{1/3})^k\Big)\Big(1+q^{1/3}(\zeta^2q^{1/3})^k\Big)\\
&= f(\zeta^2q^{1/3})\prod_{k=0}^{\infty}\Big(1+(1+\zeta^4)q^{1/3}(\zeta^2q^{1/3})^k+\zeta^4q^{2/3}(\zeta^2q^{1/3})^{2k}\Big)\\
&= f(\zeta^2q^{1/3})\prod_{k=0}^{\infty}\Big(1-\zeta^2q^{1/3}(\zeta^2q^{1/3})^k+(\zeta^2q^{1/3})^{2(k+1)}\Big)\\
&= f(\zeta^2q^{1/3})\prod_{k=1}^{\infty}\Big(1-(\zeta^2q^{1/3})^k+(\zeta^2q^{1/3})^{2k}\Big)\\
&= f(\zeta^2q^{1/3})\prod_{k=1}^{\infty}\frac{1+q^k}{1+(\zeta^2q^{1/3})^k}\\
&= \frac{f(\zeta^2q^{1/3})(-q;q)_\infty}{(-\zeta^2q^{1/3};\zeta^2q^{1/3})_\infty}\\
&= \frac{f(\zeta^2q^{1/3})\chi(-\zeta^2q^{1/3})}{\chi(-q)}. \qquad (3.23)
\end{aligned}$$

Using (3.21), (3.22) and (3.23), we finish the proof of (3.7). □

Corollary 3.2. *We have*

$$\frac{1}{v(q)}-1-2v(q)=\frac{f(-q^{1/3})f(-q^{2/3})}{q^{1/3}f(-q^3)f(-q^6)}, \qquad (3.24)$$

$$\frac{1}{v(q)}-1+v(q)=\frac{\psi^4(q)\chi(-q)}{q^{1/3}\psi(q^{1/3})\psi(q^3)f(-q^3)f(-q^6)}, \qquad (3.25)$$

$$\frac{1}{v(q)}+2+4v(q)=\frac{f^4(-q)\chi^3(-q)}{q^{1/3}\varphi(-q^{1/3})f^2(-q^3)\chi(-q^3)f(-q^6)}. \qquad (3.26)$$

Proof of (3.24). We see that

$$\frac{1}{v(q)}-1-2v(q)=\left(\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}\right)\left(\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}\right).$$

By (3.2) and (3.5), it follows that

$$\frac{1}{v(q)}-1-2v(q)=\frac{\psi(q^{1/3})\varphi(-q^{1/3})}{q^{1/3}f(-q^3)f(-q^6)}.$$

Using (2.8) with q replaced by $-q^{1/3}$, we complete the proof of (3.24). □

Proof of (3.25). We observe that

$$\frac{1}{v(q)}-1+v(q)=\left(\frac{1}{\sqrt{v(q)}}-\zeta\sqrt{v(q)}\right)\left(\frac{1}{\sqrt{v(q)}}+\zeta^2\sqrt{v(q)}\right).$$

Employing (3.3) and (3.4), we arrive at

$$\frac{1}{v(q)} - 1 + v(q) = \frac{\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3})\chi(-q)}{q^{1/3} f(-q^3) f(-q^6)}. \tag{3.27}$$

We find that

$$\begin{aligned}\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3}) &= \frac{(\zeta^2 q^{2/3};\zeta^2 q^{2/3})_\infty(\zeta^4 q^{2/3};\zeta^4 q^{2/3})_\infty}{(-\zeta q^{1/3};\zeta^2 q^{2/3})_\infty(\zeta^2 q^{1/3};\zeta^4 q^{2/3})_\infty}\\ &= \prod_{k=0}^{\infty}\frac{(1-\zeta^{2(k+1)}q^{2(k+1)/3})(1-\zeta^{4(k+1)}q^{2(k+1)/3})}{(1+\zeta^{2k+1}q^{(2k+1)/3})(1-\zeta^{4k+2}q^{(2k+1)/3})}\\ &= \prod_{k=0}^{\infty}\frac{1-(\zeta^{4(k+1)}+\zeta^{2(k+1)})q^{2(k+1)/3}+q^{4(k+1)/3}}{1-(\zeta^{4k+2}-\zeta^{2k+1})q^{(2k+1)/3}+q^{(4k+2)/3}}. \end{aligned} \tag{3.28}$$

Since

$$\zeta^{4(k+1)}+\zeta^{2(k+1)} = \begin{cases} 2 & \text{if } 3 \mid k+1 \\ -1 & \text{if } 3 \nmid k+1 \end{cases}$$

and

$$\zeta^{4k+2}-\zeta^{2k+1} = \begin{cases} 2 & \text{if } 3 \mid k+2 \\ -1 & \text{if } 3 \nmid k+2 \end{cases},$$

the equation (3.28) becomes

$$\begin{aligned}&\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3})\\ &\quad= \prod_{k=0}^{\infty}\frac{(1-2q^{2k+2}+q^{4k+4})(1+q^{(6k+2)/3}+q^{(12k+4)/3})(1+q^{(6k+4)/3}+q^{(12k+8)/3})}{(1-2q^{2k+1}+q^{4k+2})(1+q^{(6k+1)/3}+q^{(12k+2)/3})(1+q^{(6k+5)/3}+q^{(12k+10)/3})}\\ &\quad= \prod_{k=0}^{\infty}\frac{(1-q^{2k+2})^2}{(1-q^{2k+1})^2}\left(\frac{1-q^{6k+2}}{1-q^{(6k+2)/3}}\right)\left(\frac{1-q^{6k+4}}{1-q^{(6k+4)/3}}\right)\left(\frac{1-q^{(6k+1)/3}}{1-q^{6k+1}}\right)\left(\frac{1-q^{(6k+5)/3}}{1-q^{6k+5}}\right)\\ &\quad= \frac{(q^2;q^2)_\infty^2(q^2;q^6)_\infty(q^4;q^6)_\infty(q^{1/3};q^2)_\infty(q^{5/3};q^2)_\infty}{(q;q^2)_\infty^2(q;q^6)_\infty(q^5;q^6)_\infty(q^{2/3};q^2)_\infty(q^{4/3};q^2)_\infty}\\ &\quad= \frac{(q^2;q^2)_\infty^4(q^3;q^6)_\infty(q^{1/3};q^{2/3})_\infty}{(q;q^2)_\infty^4(q^6;q^6)_\infty(q^{2/3};q^{2/3})_\infty}.\end{aligned}$$

By (1.4), it follows that

$$\psi(-\zeta q^{1/3})\psi(\zeta^2 q^{1/3}) = \frac{\psi^4(q)}{\psi(q^{1/3})\psi(q^3)}. \tag{3.29}$$

Substituting (3.29) into (3.27), the proof is complete.

Proof of (3.26). We find that

$$\frac{1}{v(q)}+2+4v(q)=\left(\frac{1}{\sqrt{v(q)}}+2\zeta\sqrt{v(q)}\right)\left(\frac{1}{\sqrt{v(q)}}-2\zeta^2\sqrt{v(q)}\right).$$

Utilising (3.6) and (3.7), we deduce that

$$\frac{1}{v(q)}+2+4v(q)=\frac{\varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3})}{q^{1/3}\chi(-q)f(-q^3)f(-q^6)}. \tag{3.30}$$

For the numerator of (3.30), we see that

$$\begin{aligned}\varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3})&=\frac{(-\zeta q^{1/3};-\zeta q^{1/3})_\infty(\zeta^2 q^{1/3};\zeta^2 q^{1/3})_\infty}{(\zeta q^{1/3};-\zeta q^{1/3})_\infty(-\zeta^2 q^{1/3};\zeta^2 q^{1/3})_\infty}\\&=\prod_{k=1}^{\infty}\frac{\big(1-(-\zeta q^{1/3})^k\big)\big(1-(\zeta^2 q^{1/3})^k\big)}{\big(1+(-\zeta q^{1/3})^k\big)\big(1+(\zeta^2 q^{1/3})^k\big)}\\&=\prod_{k=1}^{\infty}\frac{1-(\zeta^{2k}+(-\zeta)^k)q^{k/3}+q^{2k/3}}{1+(\zeta^{2k}+(-\zeta)^k)q^{k/3}+q^{2k/3}}.\end{aligned}$$

Since $\zeta^{2k}+(-\zeta)^k=2$ if $3\mid k$ and $\zeta^{2k}+(-\zeta)^k=-1$ if $3\nmid k$, it follows that

$$\begin{aligned}&\varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3})\\&=\prod_{k=0}^{\infty}\frac{(1-2q^{k+1}+q^{2(k+1)})(1+q^{(3k+1)/3}+q^{2(3k+1)/3})(1+q^{(3k+2)/3}+q^{2(3k+2)/3})}{(1+2q^{k+1}+q^{2(k+1)})(1-q^{(3k+1)/3}+q^{2(3k+1)/3})(1-q^{(3k+2)/3}+q^{2(3k+2)/3})}\\&=\prod_{k=0}^{\infty}\frac{(1-q^{k+1})^2}{(1+q^{k+1})^2}\left(\frac{1-q^{3k+1}}{1-q^{(3k+1)/3}}\right)\left(\frac{1-q^{3k+2}}{1-q^{(3k+2)/3}}\right)\left(\frac{1+q^{(3k+1)/3}}{1+q^{3k+1}}\right)\left(\frac{1+q^{(3k+2)/3}}{1+q^{3k+2}}\right)\\&=\left(\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\right)\left(\frac{(q;q^3)_\infty(q^2;q^3)_\infty}{(-q;q^3)_\infty(-q^2;q^3)_\infty}\right)\left(\frac{(-q^{1/3};q)_\infty(-q^{2/3};q)_\infty}{(q^{1/3};q)_\infty(q^{2/3};q)_\infty}\right)\\&\left(\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}\right)\left(\frac{(q;q)_\infty(-q^3;q^3)_\infty}{(q^3;q^3)_\infty(-q;q)_\infty}\right)\left(\frac{(q;q)_\infty(-q^{1/3};q^{1/3})_\infty}{(q^{1/3};q^{1/3})_\infty(-q;q)_\infty}\right)\\&=\frac{f^4(-q)\chi^4(-q)}{f(-q^3)\chi(-q^3)f(-q^{1/3})\chi(-q^{1/3})}.\end{aligned}$$

Using (2.7) with q replaced by $-q^{1/3}$, we deduce that

$$\varphi(\zeta q^{1/3})\varphi(-\zeta^2 q^{1/3})=\frac{f^4(-q)\chi^4(-q)}{f(-q^3)\chi(-q^3)\varphi(-q^{1/3})}. \tag{3.31}$$

Substituting (3.31) into (3.30), we complete the proof.

Corollary 3.3. *We have*

$$\frac{1}{\sqrt{v^3(q)}}+\sqrt{v^3(q)}=\frac{\psi^4(q)}{q^{1/2}\psi(q^3)}\sqrt{\frac{\chi^3(-q)}{f^3(-q^3)f^3(-q^6)}}, \tag{3.32}$$

$$\frac{1}{\sqrt{v^3(q)}}-8\sqrt{v^3(q)}=\frac{f^4(-q)}{q^{1/2}\chi(-q^3)}\sqrt{\frac{\chi^5(-q)}{f^5(-q^3)f^3(-q^6)}}. \tag{3.33}$$

Proof of (3.32). We see that

$$\frac{1}{\sqrt{v^3(q)}}+\sqrt{v^3(q)}=\left(\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}\right)\left(\frac{1}{v(q)}-1+v(q)\right).$$

Multiplying (3.2) and (3.25) yields the desired result. □

Proof of (3.33). Observe that

$$\frac{1}{\sqrt{v^3(q)}}-8\sqrt{v^3(q)}=\left(\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}\right)\left(\frac{1}{v(q)}+2+4v(q)\right).$$

By (3.5) and (3.26), we readily arrive at (3.33). □

Corollary 3.4. *We have*

$$\frac{1}{v^3(q)}-7-8v^3(q)=\frac{f^4(-q)f^4(-q^2)}{qf^4(-q^3)f^4(-q^6)}. \tag{3.34}$$

Proof. Note that

$$\frac{1}{v^3(q)}-7-8v^3(q)=\left(\frac{1}{\sqrt{v^3(q)}}+\sqrt{v^3(q)}\right)\left(\frac{1}{\sqrt{v^3(q)}}-8\sqrt{v^3(q)}\right).$$

Multiply (3.32) and (3.33) and obtain

$$\frac{1}{v^3(q)}-7-8v^3(q)=\frac{f^4(-q)\psi^4(q)\chi^4(-q)}{q\psi(q^3)\chi(-q^3)f^4(-q^3)f^3(-q^6)}.$$

Utilising (2.5) for $\psi^4(q)\chi^4(-q)$ and $\psi(q^3)\chi(-q^3)$, the result follows immediately. □

Before proceeding further, we will write $v(q)$ in another form. By (2.5) and (2.6), we have

$$\chi(-q^3)=\frac{f(-q^6)}{\psi(q^3)} \qquad \text{and} \qquad \chi(-q^3)=\frac{f(-q^3)}{f(-q^6)}, \tag{3.35}$$

respectively. Utilising (3.35), $v(q)$ in (3.1) can be written as

$$\begin{aligned} v(q) &= q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} \\ &= q^{1/3}\chi(-q)\left(\frac{\psi^2(q^3)}{f^2(-q^6)}\right)\left(\frac{f(-q^6)}{f(-q^3)}\right) \\ &= q^{1/3}\frac{\psi^2(q^3)\chi(-q)}{f(-q^3)f(-q^6)}. \end{aligned} \tag{3.36}$$

Corollary 3.5. *We have*

$$1+v(q)=\frac{\psi(q^{1/3})\chi(-q)}{\varphi(-q^3)}, \tag{3.37}$$

$$\frac{1}{v(q)}+1=\frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \tag{3.38}$$

$$1-2v(q)=\frac{\varphi(-q^{1/3})}{\varphi(-q^3)}, \tag{3.39}$$

$$\frac{1}{v(q)}-2=\frac{\varphi(-q^{1/3})}{q^{1/3}\chi(-q)\psi(q^3)}, \tag{3.40}$$

$$1+v^3(q)=\frac{\varphi(-q)\psi^3(q)}{\varphi^3(-q^3)\psi(q^3)}, \tag{3.41}$$

$$\frac{1}{v^3(q)}+1=\frac{\psi^4(q)}{q\psi^4(q^3)}, \tag{3.42}$$

$$1-8v^3(q)=\frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \tag{3.43}$$

$$\frac{1}{v^3(q)}-8=\frac{\varphi^3(-q)\psi(q)}{q\varphi(-q^3)\psi^3(q^3)}. \tag{3.44}$$

Proof of (3.37). Note that

$$1+v(q)=\sqrt{v(q)}\left(\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}\right).$$

By (3.2) and (3.36), we conclude that

$$1+v(q)=\frac{\psi(q^{1/3})\psi(q^3)\chi(-q)}{f(-q^3)f(-q^6)}.$$

Using (2.5) and (2.7), the result follows immediately. □

Proof of (3.38). We see that

$$\frac{1}{v(q)}+1=\frac{1}{\sqrt{v(q)}}\left(\frac{1}{\sqrt{v(q)}}+\sqrt{v(q)}\right).$$

Employing (3.2) and (3.36), we finish the proof. □

Proof of (3.39). Since

$$1-2v(q)=\sqrt{v(q)}\left(\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}\right).$$

by (3.5) and (3.36), it follows that

$$1-2v(q)=\frac{\varphi(-q^{1/3})\psi(q^3)}{f(-q^3)f(-q^6)},$$

Utilising (2.5) and (2.7), we complete the proof. □

Proof of (3.40). Since

$$\frac{1}{v(q)} - 2 = \frac{1}{\sqrt{v(q)}}\left(\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)}\right).$$

Employing (3.5) and (3.36), we achieve the proposed identity.

Proof of (3.41). Notice that

$$1 + v^3(q) = \sqrt{v^3(q)}\left(\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)}\right).$$

Using (3.32) and (3.36), it follows that

$$1 + v^3(q) = \frac{\psi^2(q^3)\psi^4(q)\chi^3(-q)}{f^3(-q^3)f^3(-q^6)}.$$

Utilising (2.5)-(2.8), we finish the proof. □

Proof of (3.42). We observe that

$$\frac{1}{v^3(q)} + 1 = \frac{1}{\sqrt{v^3(q)}}\left(\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)}\right).$$

Using (3.32) and (3.36), we complete the proof. □

Proof of (3.43). We find that

$$1 - 8v^3(q) = \sqrt{v^3(q)}\left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)}\right).$$

By (3.33) and (3.36), we get

$$1 - 8v^3(q) = \frac{\psi^3(q^3)\chi^4(-q)f^4(-q)}{f^4(-q^3)\chi(-q^3)f^3(-q^6)}.$$

By (2.5) and (2.7), we deduce the desired result. □

Proof of (3.44). Since

$$\frac{1}{v^3(q)} - 8 = \frac{1}{\sqrt{v^3(q)}}\left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)}\right),$$

by (3.33) and (3.36), we obtain

$$\frac{1}{v^3(q)} - 8 = \frac{\chi(-q)f^4(-q)}{q\psi^3(q^3)\chi(-q^3)f^3(-q^3)}.$$

Using (2.5), (2.7) and (2.9)-(2.11), we deduce the desired result. □

Corollary 3.6. *We have*

$$\frac{1}{v(q)}+4v^2(q)=\left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}, \tag{3.45}$$

$$\frac{1}{v^2(q)}-2v(q)=\left(27+\frac{f^{12}(-q^2)}{q^2f^{12}(-q^6)}\right)^{1/3}, \tag{3.46}$$

$$\frac{1}{v(q)}+4v^2(q)=3+\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}, \tag{3.47}$$

$$\frac{1}{v^2(q)}-2v(q)=3+\frac{f^3(-q^{2/3})}{q^{2/3}f^3(-q^6)}. \tag{3.48}$$

Proof of (3.45). Observe that

$$\left(\frac{1}{v(q)}+4v^2(q)\right)^3-27=(1+v^3(q))\left(\frac{1}{\sqrt{v^3(q)}}-8\sqrt{v^3(q)}\right)^2.$$

By (3.41) and (3.33), we arrive at

$$\left(\frac{1}{v(q)}+4v^2(q)\right)^3-27=\frac{\psi^4(q)\chi^8(-q)f^8(-q)}{q\psi(q^3)\varphi^3(-q^3)\chi^2(-q^3)f^5(-q^3)f^3(-q^6)}. \tag{3.49}$$

Using (2.5), (2.6) and (2.7), we have

$$\psi^4(q)\chi^8(-q)=f^4(-q), \tag{3.50}$$

and

$$\psi(q^3)\varphi^3(-q^3)\chi^2(-q^3)f^3(-q^6)=f^7(-q^3). \tag{3.51}$$

Substitute (3.50) and (3.51) into (3.49) and obtain the desired result. □

Proof of (3.46). Employing (3.32), (3.33) and (3.36), it follows that

$$\begin{aligned}\left(\frac{1}{v^2(q)}-2v(q)\right)^3-27&=\frac{1}{\sqrt{v^3(q)}}\left(\frac{1}{\sqrt{v^3(q)}}+\sqrt{v^3(q)}\right)^2\left(\frac{1}{\sqrt{v^3(q)}}-8\sqrt{v^3(q)}\right)\\&=\frac{\psi^8(q)\chi^4(-q)f^4(-q)}{q^2\psi^5(q^3)\chi(-q^3)f^4(-q^3)f^3(-q^6)}.\end{aligned} \tag{3.52}$$

Using (2.5) and (2.9), we get

$$\psi^8(q)\chi^4(-q)f^4(-q)=f^{12}(-q^2) \tag{3.53}$$

and

$$\psi^5(q^3)\chi(-q^3)f^4(-q^3)=f^9(-q^6). \tag{3.54}$$

Putting (3.53), (3.54) into (3.52), we finish the proof. □

Proof of (3.47). Using (3.37) and (3.5), we deduce that

$$\begin{aligned}\frac{1}{v(q)}+4v^2(q)-3&=(1+v(q))\left(\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}\right)^2\\&=\frac{\psi(q^{1/3})\varphi^2(-q^{1/3})}{q^{1/3}\varphi(-q^3)f(-q^3)f(-q^6)}.\end{aligned}\tag{3.55}$$

By (2.11), we obtain

$$\psi(q^{1/3})\varphi^2(-q^{1/3})=f^3(-q^{1/3})\tag{3.56}$$

and by (2.6) and (2.7), we get

$$\varphi(-q^3)f(-q^3)f(-q^6)=f^3(-q^3).\tag{3.57}$$

We achieve the proposed formula after substituting (3.56) and (3.57) into (3.55). □

Proof of (3.48). By (3.5), (3.36), and (3.37), we see that

$$\begin{aligned}\frac{1}{v^2(q)}-2v(q)-3&=\frac{1}{\sqrt{v^3(q)}}(1+v(q))^2\left(\frac{1}{\sqrt{v(q)}}-2\sqrt{v(q)}\right)\\&=\frac{\psi^2(q^{1/3})\varphi(-q^{1/3})f(-q^3)f(-q^6)}{q^{2/3}\psi^3(q^3)\varphi^2(-q^3)}.\end{aligned}\tag{3.58}$$

Utilising (2.10), we find that

$$\psi^2(q^{1/3})\varphi(-q^{1/3})=f^3(-q^{2/3}),\tag{3.59}$$

and employing (2.8), (2.9), we obtain

$$\frac{f(-q^3)f(-q^6)}{\psi^3(q^3)\varphi^2(-q^3)}=\frac{1}{f^3(-q^6)}.\tag{3.60}$$

Substituting (3.59) and (3.60) into (3.58), we complete the proof. □

Corollary 3.7. *We have*

$$v^3(q)=v(q^3)\frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)}.$$

Proof. Using (3.25), (3.26) and (3.36), we arrive at

$$v(q^3)\frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)}=q\left(\frac{\varphi(-q)}{\psi(q)}\right)\left(\frac{\psi^4(q^3)}{f^4(-q^3)\chi(-q^3)}\right)\left(\frac{\psi(q^9)\chi(-q^9)}{f(-q^{18})}\right).\tag{3.61}$$

By (2.5), we have

$$\psi(q^9)\chi(-q^9) = f(-q^{18}). \tag{3.62}$$

Utilising (2.5), (2.7), and (2.6), respectively, we get

$$\frac{\varphi(-q)}{\psi(q)} = \frac{\chi^2(-q)f(-q)}{f(-q^2)} = \chi^3(-q), \tag{3.63}$$

and finally by (2.6), we have

$$\frac{\psi^4(q^3)}{f^4(-q^3)\chi(-q^3)} = \frac{\psi^4(q^3)}{\chi^4(-q^3)f^4(-q^6)\chi(-q^3)} = \frac{1}{\chi^9(-q^3)}. \tag{3.64}$$

Hence substituting (3.62), (3.63), (3.64) into (3.61) together with (1.8), we conclude that

$$v(q^3)\frac{1-v(q^3)+v^2(q^3)}{1+2v(q^3)+4v^2(q^3)} = q\frac{\chi^3(-q)}{\chi^9(-q^3)} = v^3(q).$$

□

Corollary 3.8. (The 3-dissection of $(q;q)^{-1}(q^2;q^2)^{-1}$).

$$\begin{aligned}\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{f^3(-q^9)f^3(-q^{18})}{f^4(-q^3)f^4(-q^6)}\Bigg(&\frac{f^6(-q^9)f^2(-q^6)}{f^6(-q^{18})f^2(-q^3)} + \frac{qf(-q^6)f^3(-q^9)}{f(-q^3)f^3(-q^{18})} \\ &+ 3q^2 - \frac{2q^3f(-q^3)f^3(-q^{18})}{f(-q^6)f^3(-q^9)} + \frac{4q^4f^2(-q^3)f^6(-q^{18})}{f^2(-q^6)f^6(-q^9)}\Bigg).\end{aligned}$$

Proof. Multiplying (3.25) and (3.26) together and changing q to q^3, we obtain

$$\begin{aligned}\frac{1}{\varphi(-q)\psi(q)} = \frac{q^2\psi(q^9)f^3(-q^9)f^2(-q^{18})\chi(-q^9)}{\psi^4(q^3)\chi(-q^3)f^4(-q^3)\chi^3(-q^3)} \times \Bigg(&\frac{1}{v^2(q^3)} + \frac{1}{v(q^3)} \\ &+ 3 - 2v(q^3) + 4v^2(q^3)\Bigg).\end{aligned} \tag{3.65}$$

Utilising (1.4) and (2.6) in (3.65), we achieve

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{q^2f^3(-q^9)f^3(-q^{18})}{f^4(-q^3)f^4(-q^6)} \times \left(\frac{1}{v^2(q^3)} + \frac{1}{v(q^3)} + 3 - 2v(q^3) + 4v^2(q^3)\right). \tag{3.66}$$

Using (1.13) and (2.6), we end the proof. □

3.2 The 2-dissection of the Göllnitz-Gordon functions

Theorem 3.9.

$$
\begin{aligned}
S(q) = {} & \Bigg(\frac{(q^8, q^{24}, q^{36}, q^{40}, q^{56}, q^{60}, q^{64}, q^{64}, q^{68}, q^{72}, q^{88}, q^{92}, q^{104}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^2, q^{14}, q^{16}, q^{18}, q^{18}, q^{30}, q^{30}, q^{32}, q^{34}, q^{34}, q^{46}, q^{46}, q^{48}, q^{50}, q^{62}, q^{64}; q^{64})_\infty} \\
& + q^8 \frac{(q^4, q^8, q^{24}, q^{28}, q^{40}, q^{56}, q^{64}, q^{64}, q^{72}, q^{88}, q^{100}, q^{104}, q^{120}, q^{124}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^2, q^2, q^{14}, q^{14}, q^{16}, q^{18}, q^{30}, q^{32}, q^{34}, q^{46}, q^{48}, q^{50}, q^{50}, q^{62}, q^{62}, q^{64}; q^{64})_\infty} \Bigg) \\
& + q \Bigg(\frac{(q^8, q^{24}, q^{28}, q^{40}, q^{56}, q^{60}, q^{64}, q^{64}, q^{68}, q^{72}, q^{88}, q^{100}, q^{104}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^2, q^{14}, q^{14}, q^{16}, q^{18}, q^{30}, q^{30}, q^{32}, q^{34}, q^{34}, q^{46}, q^{46}, q^{48}, q^{50}, q^{62}, q^{64}; q^{64})_\infty} \\
& + q^6 \frac{(q^4, q^8, q^{24}, q^{36}, q^{40}, q^{56}, q^{64}, q^{64}, q^{72}, q^{88}, q^{92}, q^{104}, q^{120}, q^{124}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^2, q^2, q^{14}, q^{16}, q^{18}, q^{18}, q^{30}, q^{32}, q^{34}, q^{46}, q^{46}, q^{48}, q^{50}, q^{62}, q^{62}, q^{64}; q^{64})_\infty} \Bigg).
\end{aligned}
\tag{3.67}
$$

$$
\begin{aligned}
T(q) = {} & \Bigg(\frac{(q^8, q^{24}, q^{40}, q^{44}, q^{52}, q^{56}, q^{64}, q^{64}, q^{72}, q^{76}, q^{84}, q^{88}, q^{104}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^6, q^{10}, q^{16}, q^{22}, q^{22}, q^{26}, q^{26}, q^{32}, q^{38}, q^{38}, q^{42}, q^{42}, q^{48}, q^{54}, q^{58}, q^{64}; q^{64})_\infty} \\
& + q^8 \frac{(q^8, q^{12}, q^{20}, q^{24}, q^{40}, q^{56}, q^{64}, q^{64}, q^{72}, q^{88}, q^{104}, q^{108}, q^{116}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^6, q^6, q^{10}, q^{10}, q^{16}, q^{22}, q^{26}, q^{32}, q^{38}, q^{42}, q^{48}, q^{54}, q^{54}, q^{58}, q^{58}, q^{64}; q^{64})_\infty} \Bigg) \\
& + q \Bigg(q^2 \frac{(q^8, q^{20}, q^{24}, q^{40}, q^{52}, q^{56}, q^{64}, q^{64}, q^{72}, q^{76}, q^{88}, q^{104}, q^{108}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^6, q^{10}, q^{10}, q^{16}, q^{22}, q^{26}, q^{26}, q^{32}, q^{38}, q^{38}, q^{42}, q^{48}, q^{54}, q^{54}, q^{58}, q^{64}; q^{64})_\infty} \\
& + q^4 \frac{(q^8, q^{12}, q^{24}, q^{40}, q^{44}, q^{56}, q^{64}, q^{64}, q^{72}, q^{84}, q^{88}, q^{104}, q^{116}, q^{120}, q^{128}, q^{128}; q^{128})_\infty}{(q^4;q^4)_\infty (q^6, q^6, q^{10}, q^{16}, q^{22}, q^{22}, q^{26}, q^{32}, q^{38}, q^{42}, q^{42}, q^{48}, q^{54}, q^{58}, q^{58}, q^{64}; q^{64})_\infty} \Bigg).
\end{aligned}
\tag{3.68}
$$

Proof of (3.67).

$$
\begin{aligned}
S(q) =& \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \\
&= \frac{1}{(q,q^4,q^7,q^9,q^{12},q^{15};q^{16})_\infty} \\
&= \frac{1}{(q^4,q^{12};q^{16})_\infty} \prod_{j=1}^{\infty} \frac{1}{(1-q^{16j-9})(1-q^{16j-7})} \prod_{j=1}^{\infty} \frac{1}{(1-q^{16j-1})(1-q^{16j-15})} \\
&= \frac{1}{(q^4,q^{12},q^{16},q^{16};q^{16})_\infty} \prod_{j=1}^{\infty} \frac{(1+q^{16j-9})(1+q^{16j-7})(1-q^{16j})}{(1-q^{32j-18})(1-q^{32j-14})} \\
&\quad \prod_{j=1}^{\infty} \frac{(1+q^{16j-1})(1+q^{16j-15})(1-q^{16j})}{(1-q^{32j-2})(1-q^{32j-30})} \\
&= \frac{1}{(q^2,q^4,q^{12},q^{14},q^{16},q^{16},q^{18},q^{20},q^{28},q^{30},q^{32},q^{32};q^{32})_\infty} \times \sum_{j=-\infty}^{\infty} q^{8j^2-j} \times \sum_{j=-\infty}^{\infty} q^{8j^2-7j} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^2,q^{14},q^{16},q^{18},q^{30},q^{32};q^{32})_\infty} \times \sum_{j=-\infty}^{\infty} q^{8j^2-j} \times \sum_{j=-\infty}^{\infty} q^{8j^2-7j} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^2,q^{14},q^{16},q^{18},q^{30},q^{32};q^{32})_\infty} \times \{ \sum_{j=-\infty}^{\infty} q^{32j^2-2j} + q^7 \sum_{j=-\infty}^{\infty} q^{32j^2+30j} \} \\
&\quad \times \{ \sum_{j=-\infty}^{\infty} q^{32j^2-14j} + q \sum_{j=-\infty}^{\infty} q^{32j^2+18j} \} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^2,q^{14},q^{16},q^{18},q^{30},q^{32};q^{32})_\infty} \\
&\times \{ (-q^{30},-q^{34},q^{64};q^{64})_\infty + q^7(-q^2,-q^{62},q^{64};q^{64})_\infty \} \\
&\times \{ (-q^{18},-q^{46},q^{64};q^{64})_\infty + q(-q^{14},-q^{50},q^{64};q^{64})_\infty \} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^2,q^{14},q^{16},q^{18},q^{30},q^{32};q^{32})_\infty} \\
&\times \{ \frac{(q^{60},q^{64},q^{68},q^{128};q^{128})_\infty}{(q^{30},q^{34};q^{64})_\infty} + q^7 \frac{(q^4,q^{64},q^{124},q^{128};q^{128})_\infty}{(q^2,q^{62};q^{64})_\infty} \} \\
&\times \{ \frac{(q^{36},q^{64},q^{92},q^{128};q^{128})_\infty}{(q^{18},q^{46};q^{64})_\infty} + q \frac{(q^{28},q^{64},q^{100},q^{128};q^{128})_\infty}{(q^{14},q^{50};q^{64})_\infty} \} \\
&= \frac{(q^8,q^{24},q^{36},q^{40},q^{56},q^{60},q^{64},q^{64},q^{68},q^{72},q^{88},q^{92},q^{104},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty (q^2,q^{14},q^{16},q^{18},q^{18},q^{30},q^{30},q^{32},q^{34},q^{34},q^{46},q^{46},q^{48},q^{50},q^{62},q^{64};q^{64})_\infty} \\
&+ q \frac{(q^8,q^{24},q^{28},q^{40},q^{56},q^{60},q^{64},q^{64},q^{68},q^{72},q^{88},q^{100},q^{104},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty (q^2,q^{14},q^{14},q^{16},q^{18},q^{30},q^{30},q^{32},q^{34},q^{34},q^{46},q^{46},q^{48},q^{50},q^{62},q^{64};q^{64})_\infty} \\
&+ q^7 \frac{(q^4,q^8,q^{24},q^{36},q^{40},q^{56},q^{64},q^{64},q^{72},q^{88},q^{92},q^{104},q^{120},q^{124},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty (q^2,q^2,q^{14},q^{16},q^{18},q^{18},q^{30},q^{32},q^{34},q^{46},q^{46},q^{48},q^{50},q^{62},q^{62},q^{64};q^{64})_\infty} \\
&+ q^8 \frac{(q^4,q^8,q^{24},q^{28},q^{40},q^{56},q^{64},q^{64},q^{72},q^{88},q^{100},q^{104},q^{120},q^{124},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty (q^2,q^2,q^{14},q^{14},q^{16},q^{18},q^{30},q^{32},q^{34},q^{46},q^{48},q^{50},q^{50},q^{62},q^{62},q^{64};q^{64})_\infty}.
\end{aligned}
$$

□

Proof of (3.68).

$$
\begin{aligned}
T(q) =& \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty} \\
&= \frac{1}{(q^3,q^4,q^5,q^{11},q^{12},q^{13};q^{16})_\infty} \\
&= \frac{1}{(q^4,q^{12};q^{16})_\infty} \prod_{k=1}^{\infty} \frac{1}{(1-q^{16k-13})(1-q^{16k-3})} \prod_{k=1}^{\infty} \frac{1}{(1-q^{16k-5})(1-q^{16k-11})} \\
&= \frac{1}{(q^4,q^{12},q^{16},q^{16};q^{16})_\infty} \prod_{k=1}^{\infty} \frac{(1+q^{16k-13})(1+q^{16j-3})(1-q^{16k})}{(1-q^{32k-26})(1-q^{32k-6})} \\
&\quad \prod_{k=1}^{\infty} \frac{(1+q^{16k-5})(1+q^{16k-11})(1-q^{16k})}{(1-q^{32j-10})(1-q^{32j-22})} \\
&= \frac{1}{(q^4,q^6,q^{10},q^{12},q^{16},q^{16},q^{20},q^{22},q^{26},q^{28},q^{32},q^{32};q^{32})_\infty} \times \sum_{k=-\infty}^{\infty} q^{8k^2-5k} \times \sum_{k=-\infty}^{\infty} q^{8k^2-3j} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^6,q^{10},q^{16},q^{22},q^{26},q^{32};q^{32})_\infty} \times \sum_{k=-\infty}^{\infty} q^{8k^2-5k} \times \sum_{k=-\infty}^{\infty} q^{8k^2-3j} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^6,q^{10},q^{16},q^{22},q^{26},q^{32};q^{32})_\infty} \times \{\sum_{k=-\infty}^{\infty} q^{32k^2-10k} + q^3 \sum_{k=-\infty}^{\infty} q^{32k^2+22k}\} \\
&\quad \times \{\sum_{k=-\infty}^{\infty} q^{32k^2-6k} + q^5 \sum_{k=-\infty}^{\infty} q^{32k^2+26k}\} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^6,q^{10},q^{16},q^{22},q^{26},q^{32};q^{32})_\infty} \\
&\times \{(-q^{22},-q^{42},q^{64};q^{64})_\infty + q^3(-q^{10},-q^{54},q^{64};q^{64})_\infty\} \\
&\times \{(-q^{26},-q^{38},q^{64};q^{64})_\infty + q^5(-q^{6},-q^{58},q^{64};q^{64})_\infty\} \\
&= \frac{(q^8,q^{24};q^{32})_\infty}{(q^4;q^4)_\infty} \times \frac{1}{(q^6,q^{10},q^{16},q^{22},q^{26},q^{32};q^{32})_\infty} \\
&\times \{\frac{(q^{44},q^{64},q^{84},q^{128};q^{128})_\infty}{(q^{22},q^{42};q^{64})_\infty} + q^3 \frac{(q^{20},q^{64},q^{108},q^{128};q^{128})_\infty}{(q^{10},q^{54};q^{64})_\infty}\} \\
&\times \{\frac{(q^{52},q^{64},q^{76},q^{128};q^{128})_\infty}{(q^{26},q^{38};q^{64})_\infty} + q^5 \frac{(q^{12},q^{64},q^{116},q^{128};q^{128})_\infty}{(q^{6},q^{58};q^{64})_\infty}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(q^6,q^{24},q^{40},q^{44},q^{52},q^{56},q^{64},q^{72},q^{76},q^{84},q^{88},q^{104},q^{120},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty(q^6,q^{10},q^{16},q^{22},q^{22},q^{26},q^{26},q^{32},q^{38},q^{38},q^{42},q^{42},q^{48},q^{54},q^{58},q^{64};q^{64})_\infty}\\
&+ q^5\frac{(q^8,q^{12},q^{24},q^{40},q^{44},q^{56},q^{64},q^{64},q^{72},q^{84},q^{88},q^{104},q^{116},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty(q^6,q^6,q^{10},q^{16},q^{22},q^{22},q^{26},q^{32},q^{38},q^{42},q^{42},q^{48},q^{54},q^{58},q^{58},q^{64};q^{64})_\infty}\\
&+ q^3\frac{(q^8,q^{20},q^{24},q^{40},q^{52},q^{56},q^{64},q^{64},q^{72},q^{76},q^{88},q^{104},q^{108},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty(q^6,q^{10},q^{10},q^{16},q^{22},q^{26},q^{26},q^{32},q^{38},q^{38},q^{42},q^{48},q^{54},q^{54},q^{58},q^{64};q^{64})_\infty}\\
&+ q^8\frac{(q^8,q^{12},q^{20},q^{24},q^{40},q^{56},q^{64},q^{64},q^{72},q^{88},q^{104},q^{108},q^{116},q^{120},q^{128},q^{128};q^{128})_\infty}{(q^4;q^4)_\infty(q^6,q^6,q^{10},q^{10},q^{16},q^{22},q^{26},q^{32},q^{38},q^{42},q^{48},q^{54},q^{54},q^{58},q^{58},q^{64};q^{64})_\infty}.
\end{aligned}
$$

□

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