Haar Wavelet Method for Solving Differential Equations

Phraewmai Wannateeradet

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

Prince of Songkla University

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I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

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ชื่อวิทยานิพนธ์ วิธีการฮาร์เวฟเล็ตสำหรับการแก้สมการเชิงอนุพันธ์
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บทคัดย่อ

วิธีฮาร์เวฟเล็ต เป็นเครื่องมือที่มีประสิทธิภาพ สำหรับการหาผลเฉลยของปริพันธ์และสมการเชิงอนุพันธ์ ในวิทยานิพนธ์นี้จะประยุกต์ใช้ฟังก์ชันฮาร์ เพื่อหาผลเฉลยของสมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับสองบางสมการ พร้อมด้วยเงื่อนไขเริ่มต้นและเงื่อนไขขอบ อีกทั้งหารูปทั่วไปของการแก้ปัญหาสมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับสอง ด้วยวิธีฮาร์เวฟเล็ต นอกจากนี้เราขยายช่วงของผลเฉลยของสมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับสอง จากช่วงปิด $[0,1]$ ไปยังช่วงปิด $[r, r+1]$ เมื่อ r คือจำนวนเต็ม สุดท้ายในวิทยานิพนธ์ฉบับนี้เรายังหาเงื่อนไข เพื่อหาผลเฉลยของสมการแบล็ค-โชลส์ด้วยวิธีฮาร์เวฟเล็ตอีกด้วย

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ABSTRACT

Haar wavelet method has become an efficient tool for solving various types of integral and differential equations. In this thesis, we apply Haar wavelet method to solve certain second order ordinary differential equations with initial and boundary conditions and we find the general form for solve of second order linear ordinary differential equations by Haar wavelet method. Moreover, we extend the interval of second order ordinary differential equation's solutions from $[0, 1]$ to $[r, r + 1]$ when r is an integer. Finally, we look into conditions for solve of the Black-Scholes equation by Haar wavelet method.

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Phraewmai Wannateeradet

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Introduction

1.1 Literature Review

Wavelet, being a powerful mathematical tool, has been widely used in signal processing and numerical analysis. In the recent years wavelet approach has become more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of initial and boundary value problems. Alfred Haar [4] introduced a group of square waves with magnitude of -1 and 1 in some intervals and zeros elsewhere then we called Haar function [16]. Among all these wavelet types Haar wavelet is the simplest orthonormal wavelet. Chen and Hsiao [3] have gained popularity, due to their useful contribution in wavelet. They first derived a Haar operational matrix for the integrals of Haar function and used it for solving lumped and distributed-parameter systems.

Next time, most people applied Haar wavelet in solving differential equations by using Haar method which is based on the operational matrices defined by them.

In 2007-2008, Lepik [9] and [10] presented methods based on Haar wavelet for solved differential equations. In 2009, Hariharan [6] shown the numerical solution of Fisher's equation using Haar wavelet. Next year, Hariharan and Kannan [7] solved Fitzhugh-Nagumo equation. In 2013, Berwal [2] using Haar wavelet for solved Telegraph equation.

Fischer Black and Myron Scholes [1] constructed the famous theoretical scheme for options which earned them the 1997 Nobel Prize in Economics. It is celled the Black-Scholes equation. Since some people applied some type of wavelet in solving Black-Scholes equation [5] excepted Haar wavelet, so in this thesis we will use Haar wavelet method for solving the Black-Scholes equation.

1.2 Procedure

This thesis consists of Chapter 1 Introduction, Chapter 2 Preliminaries, Chapter 3 Haar wavelet method for solving differential equations and Chapter 4 Conclusion and Discussion.

In Chapter 1 Introduction, we review literature and research about wavelets, Haar wavelet, Haar wavelet method for solving differential equations and method for solving Black-Scholes equation.

In Chapter 2 Preliminaries, we briefly describe Haar wavelets, Integration of Haar wavelets, The operational matrix of integration, Method for solving differential equations via Haar wavelet and The Black-Scholes equation.

Next, in Chapter 3 Haar wavelet method for solving differential equations, we have 2 sections. The first section is Generalized Haar wavelet method for solving linear ordinary differential equations and the second section is Haar wavelet method for solving the Black-Scholes equation. In section of Generalized Haar wavelet method for solving linear ordinary differential equations, we apply Haar wavelets to solve some second order linear ordinary differential equations with initial and boundary conditions and we find the general form for solve of second order linear ordinary differential equations by Haar wavelet method. Moreover, we look into conditions for extend the interval of second order ordinary differential equation's solutions from $[0, 1]$ to $[r, r+1]$ when r is an integer. The second section is Haar wavelet method for solving the Black-Scholes equation. This section we look into conditions for solve of the Black-Scholes equation in the case of a European call option by Haar wavelet method.

Finally, in Chapter 4 Conclusions and Suggestions, we summarize the results and make more suggestions in Chapter 3.

CHAPTER 2

Preliminaries

We will introduce the notion and the basic properties of Haar wavelets included The Black-Scholes equation.

2.1 Haar wavelets

Haar function was initially introduced by the Hungarian mathematician Alfred Haar [4] in 1910. Later on, it is known as the simplest example of an orthogonal wavelet, which is defined by a square wave function on the unit interval $[0, 1]$. The first Haar wavelet is denoted by

$$h_0(t) = \begin{cases} 1 & ; 0 \leq t \leq 1 \\ 0 & ; t > 1 \end{cases}$$

and called the *scaling function*. The second Haar wavelet is

$$h_1(t) = \begin{cases} 1 & ; 0 \leq t < \frac{1}{2} \\ -1 & ; \frac{1}{2} \leq t < 1 \\ 0 & ; t \geq 1 \end{cases}$$

In addition, $h_1(t)$ is the fundamental square wave, or the mother wavelet which also spans the whole interval $(0, 1)$. All the other subsequent curves are generated from $h_n(t)$ with two operations: translation and dilation. In general,

$$h_n(t) = h_1\left(2^j t - \frac{k}{2^j}\right), \tag{2.1}$$

$$n = 2^j + k, \;\; j \geq 0, \;\; 0 \leq k < 2^j.$$

In particular, these of Haar wavelets are orthogonal functions on $[0, 1]$, i.e.

$$\int_0^1 h_i(t)h_l(t)\,dt = \begin{cases} 2^{-j} & ; i = l = 2^j + k \\ 0 & ; i \neq l \end{cases}$$

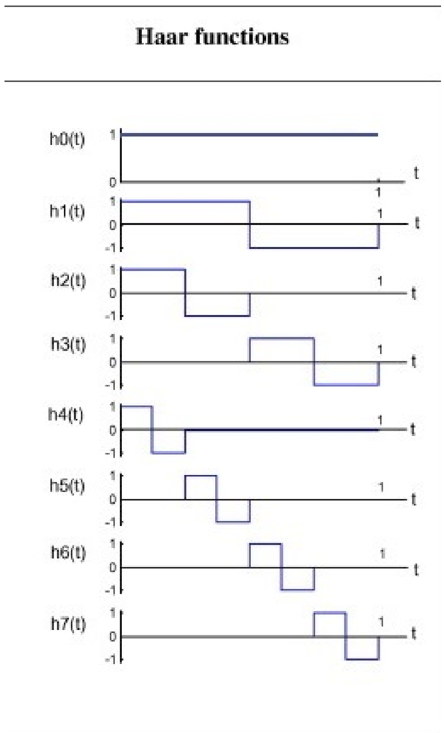

Figure 1. Eight first Haar functions

Moreover, the set of Haar wavelets is on orthogonal basis of $\mathcal{L}^2\,[0, 1]$, [12], that is any square integrable function on the interval $[0, 1]$ $y(t)$ can be represented by the series of Haar wavelets, [16]

$$\begin{aligned} y(t) &= a_0h_0(t) + a_1h_1(t) + a_2h_2(t) + \ldots \\ &= \sum_{n=0}^{\infty} a_nh_n(t) \end{aligned}$$

where

$$a_i = 2^j \int_0^1 y(t)h_i(t)\,dt.$$

Observe that if $y(t)$ is piecewise constant by itself, or can be approximated as piecewise constant during each subinterval, then this equation can be terminated at finite terms,

$$\begin{aligned} y(t) &= a_0h_0(t) + a_1h_1(t) + a_2h_2(t) + \ldots + a_{m-1}h_{m-1}(t) \\ &= \sum_{n=0}^{m-1} a_nh_n(t) \\ &\equiv a^tH_m(t) \end{aligned}$$

where $a^t = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{m-1} \end{bmatrix}$ and $H_m(t)$ is called the *Haar matrix of order* m, in which each row consists of Haar functions $h_0(t), h_1(t), h_2(t), \ldots, h_{m-1}(t)$, i.e.,

$$H_m(t) \equiv \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ \vdots \\ h_{m-1}(t) \end{bmatrix}.$$

In particular, the interval $[0, 1]$ is divided into m subintervals with length $\dfrac{1}{m}$ where $m = 2^k$ for some $k \in \mathbb{N}$. We denote the collocation points by

$$t_l = \frac{2l-1}{2m}$$

where $l = 1, 2, \ldots, m$

and the first row vector (Haar function) can be expressed as

$$h_0(t) \equiv \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \end{bmatrix},$$

and for $n = 1, 2, 3, \ldots, m-1$,

$$h_n(t) \equiv \begin{bmatrix} h_n(\frac{1}{2m}) & h_n(\frac{3}{2m}) & \cdots & h_n(\frac{2l-1}{2m}) & \cdots & h_n(\frac{2m-1}{2m}) \end{bmatrix},$$

where h_n is defined by (2.1).

For example when $m = 4$,

$$H_4(t) \equiv \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix}.$$

We can find, $h_0(t)$, $h_1(t)$, $h_2(t)$ and $h_3(t)$ in a matrix form.

For $n = 0$, we have

$$h_0(t) \equiv \begin{bmatrix} h_0(t_1) & h_0(t_2) & h_0(t_3) & h_0(t_4) \end{bmatrix}.$$

Since $h_0(t) = 1$ when $0 \leq t < 1$, $h_0(t_1) = h_0(t_2) = h_0(t_3) = h_0(t) = 1$,

we obtain

$$h_0(t) \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

For $n = 1$

$$h_1(t) \equiv \begin{bmatrix} h_1(t_1) & h_1(t_2) & h_1(t_2) & h_1(t_2) \end{bmatrix}.$$

we have

$$h_1(t) = \begin{cases} 1 & ; 0 \leq t < \dfrac{1}{2} \\ -1 & ; \dfrac{1}{2} \leq t < 1 \\ 0 & ; t \geq 1 \end{cases}$$

thus $h_1(t_1) = h_1(t_2) = 1$ and $h_1(t_3) = h_1(t_4) = -1$,

so that

$$h_1(t) \equiv \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}.$$

For $n = 2, 3$, the Haar function can be similarly expressed,

$$h_2(t) \equiv \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},$$

$$h_3(t) \equiv \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}.$$

Hence the first four Haar functions can be expressed as follows:

$$H_4 \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2.2 Integration of Haar wavelets

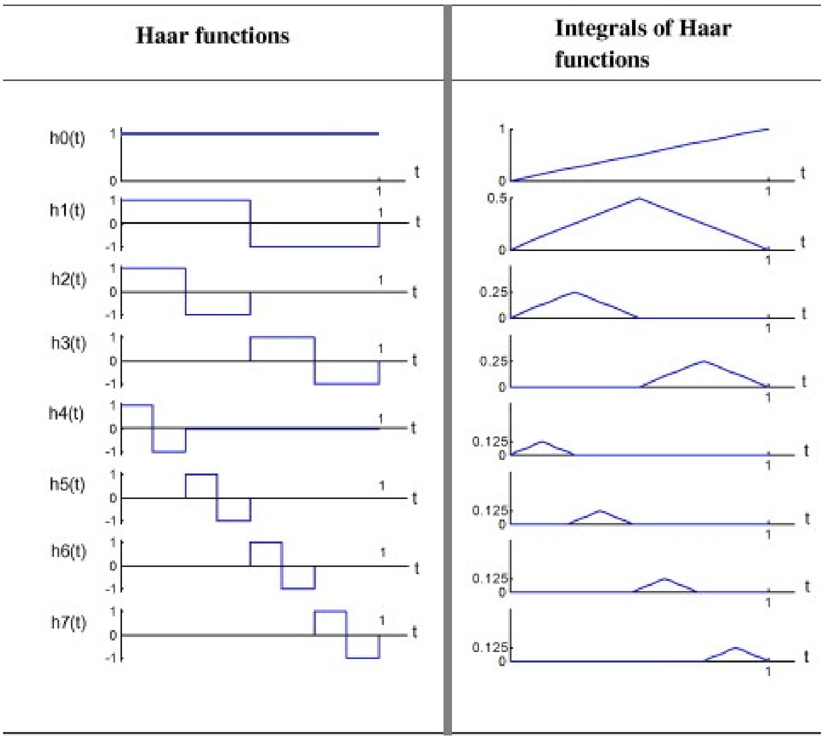

Figure 2. Eight first Haar functions and corresponding integrals [8]

In order to solve differential equations the easy technique is to integrate the equation. Chen and Hsiao [3] introduced the method using the integrals of the first four Haar wavelets :

$$\int_0^t h_0(\tau)\,d\tau = \begin{cases} t & ; 0 \le t < 1 \\ 0 & ; t \ge 1 \end{cases}$$

$$\int_0^t h_1(\tau)\,d\tau = \begin{cases} t & ; 0 \le t < \dfrac{1}{2} \\ 1 - t & ; \dfrac{1}{2} \le t < 1 \\ 0 & ; t \ge 1 \end{cases}$$

$$\int_0^t h_2(\tau)\,d\tau = \begin{cases} t & ; 0 \le t < \dfrac{1}{4} \\ \dfrac{1}{2} - t & ; \dfrac{1}{4} \le t < 1 \\ 0 & ; t \ge 1 \end{cases}$$

$$\int_0^t h_3(\tau)\,d\tau = \begin{cases} t-\frac{1}{2} & ;\frac{1}{2}\leq t<\frac{3}{4} \\ 1-t & ;\frac{3}{4}\leq t<1 \\ 0 & ;t\geq 1 \end{cases}$$

In particular, the integrals of the first four Haar wavelet vectors can be represented by the following row vectors.

$$\int_0^t h_0(\tau)\,d\tau \equiv \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8} \end{bmatrix},$$

$$\int_0^t h_1(\tau)\,d\tau \equiv \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix},$$

$$\int_0^t h_2(\tau)\,d\tau \equiv \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 & 0 \end{bmatrix},$$

$$\int_0^t h_3(\tau)\,d\tau \equiv \begin{bmatrix} 0 & 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}.$$

Therefore, the integral of the Haar matrix H_4 is

$$\int_0^t H_4(\tau)\,d\tau \equiv \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

In general, The integral of the Haar Matrix H_m of order m is defined by

$$\int_0^t H_m(\tau)\,d\tau \equiv \begin{bmatrix} \int_0^t h_0(\tau)\,d\tau \\ \int_0^t h_1(\tau)\,d\tau \\ \vdots \\ \int_0^t h_{m-1}(\tau)\,d\tau \end{bmatrix}. \tag{2.2}$$

where the first row vector is

$$\int_0^t h_0(\tau)\,d\tau \equiv \begin{bmatrix} \frac{1}{2^{m-1}} & \frac{3}{2^{m-1}} & \frac{5}{2^{m-1}} & \cdots & \frac{2m-1}{2^{m-1}} \end{bmatrix},$$

and for $n=1,2,\ldots,m-1$, the other subsequence rows are expressed by

$$\int_0^t h_n(\tau)\,d\tau \equiv \begin{bmatrix} \int_0^{\frac{1}{2^{m-1}}} h_n(\tau)\,d\tau & \int_0^{\frac{3}{2^{m-1}}} h_n(\tau)\,d\tau & \int_0^{\frac{5}{2^{m-1}}} h_n(\tau)\,d\tau & \cdots & \int_0^{\frac{2m-1}{2^{m-1}}} h_n(\tau)\,d\tau \end{bmatrix}.$$

For example, the second row of H_4 can be computed by

$$\begin{aligned}\int_0^t h_1(\tau)\,d\tau &\equiv \begin{bmatrix}\int_0^{\frac{1}{8}} h_1(\tau)\,d\tau & \int_0^{\frac{3}{8}} h_1(\tau)\,d\tau & \int_0^{\frac{5}{8}} h_1(\tau)\,d\tau & \int_0^{\frac{7}{8}} h_1(\tau)\,d\tau\end{bmatrix}\\ &= \begin{bmatrix}\int_0^{\frac{1}{8}} 1\,d\tau & \int_0^{\frac{3}{8}} 1\,d\tau & \int_0^{\frac{1}{2}} 1\,d\tau + \int_{\frac{1}{2}}^{\frac{5}{8}} -1\,d\tau & \int_0^{\frac{1}{2}} 1\,d\tau + \int_{\frac{1}{2}}^{\frac{7}{8}} -1\,d\tau\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{8} & \frac{3}{8} & \frac{1}{2}-\frac{1}{8} & \frac{1}{2}-\frac{3}{8}\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}\end{bmatrix}.\end{aligned}$$

The third row of $\int_0^t H_4(\tau)\,d\tau$ is

$$\begin{aligned}\int_0^t h_2(\tau)\,d\tau &\equiv \begin{bmatrix}\int_0^{\frac{1}{8}} h_2(\tau)\,d\tau & \int_0^{\frac{3}{8}} h_2(\tau)\,d\tau & \int_0^{\frac{5}{8}} h_2(\tau)\,d\tau & \int_0^{\frac{7}{8}} h_2(\tau)\,d\tau\end{bmatrix}\\ &= \begin{bmatrix}\int_0^{\frac{1}{8}} 1\,d\tau & \int_0^{\frac{1}{4}} 1\,d\tau + \int_{\frac{1}{4}}^{\frac{3}{8}} -1\,d\tau & \int_0^{\frac{1}{4}} 1\,d\tau + \int_{\frac{1}{4}}^{\frac{1}{2}} -1\,d\tau & \int_0^{\frac{1}{4}} 1\,d\tau + \int_{\frac{1}{4}}^{\frac{1}{2}} -1\,d\tau\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{8} & \frac{1}{4}-\frac{1}{8} & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{8} & \frac{1}{8} & 0 & 0\end{bmatrix}\end{aligned}$$

and the fourth row can be similarly expressed,

$$\int_0^t h_3(\tau)\,d\tau \equiv \begin{bmatrix}0 & 0 & \dfrac{1}{8} & \dfrac{1}{8}\end{bmatrix}.$$

2.3 The operational matrix of integration

The fundamental idea starts from the approximation of the integral of a vector $\phi(t)$ [17],

$$\int_0^t \phi(\tau)\,d\tau \cong Q\phi(t)$$

where

$$\phi(t) = \begin{bmatrix}\phi_0(t)\\ \phi_1(t)\\ \phi_2(t)\\ \vdots\\ \phi_{m-1}(t)\end{bmatrix}$$

$\phi_i(t)$ are the orthogonal functions on a some interval $[a, b]$ and Q is called the *operational matrix of integration* uniquely determined by the orthogonal functions $\phi_i(t)$.

Chen and Hsiao [3] were frist authors who introduced an operational matrix of integration, denoted by P_m. The notion of this operational P_m matrix is based on the integrals of Haar matrix (2.2), i.e.,

$$\int_0^t H_m(\tau)\,d\tau = P_m H_m(t).$$

Note that P_m is a $2m$ square matrix which can be computed by

$$P_m = \left[\int_0^t H_m(\tau)\,d\tau\right] H_m^{-1}(t).$$

For example,

$$\begin{aligned}
P_4 &= \left[\int_0^t H_4(\tau)\,d\tau\right] H_4^{-1} \\
&\equiv \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^{-1} \\
&= \frac{1}{8}\begin{bmatrix} 4 & -2 & -1 & -1 \\ 2 & 0 & -1 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

By the same process, we obtain

$$\begin{aligned}
P_8 &= \left[\int_0^t H_8(\tau)\,d\tau\right] H_8^{-1} \\
&\equiv \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 & -1 & -1 & -1 & -1 \\ 4 & 0 & -2 & 2 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1/4 & 1/4 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & -1/4 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & -1/4 & 0 & -1/2 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Chen and Hsiao [3] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_m = \begin{bmatrix} P_{ma} & P_{mb} \\ P_{mc} & P_{md} \end{bmatrix}$$

where

$$\begin{aligned} P_{ma} &= P_{m/2}, \\ P_{mb} &= -\frac{1}{2m} H_{m/2}, \\ P_{mc} &= \frac{1}{2m} H_{m/2}^{-1}, \\ P_{md} &= null\, matrix. \end{aligned}$$

It should be noted that calculations for P_m and $H_m(t)$ must be carried out only once; after that they will be applicable for solving whatever differential equations. Since $H_m(t)$ and $H_m^{-1}(t)$ comprise many zeros, this case makes the Haar wavelet transform must faster than the Fourier transform. This is one of the reasons for fast convergence of the Haar wavelet transform.

2.4 Method for solving differential equations via Haar wavelet

In principle, we apply the Haar wavelet in the time domain, and solve the space domain problem with the conventional method. Let us indicate this idea with the following example [3].

Example 2.1. Consider the diffusion equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = RC \frac{\partial v(x,t)}{\partial t} \tag{2.3}$$

where R resistance, C capacitance per unit length, and the voltage v.

Haar wavelets are introduced to solve these partial differential equations.

Idea for solving: In the Haar domain, let us assume $\frac{\partial v(x,t)}{\partial t}$ can be expanded in a Haar series as

$$\frac{\partial v(x,t)}{\partial t} \equiv A^t(x)H(t) \tag{2.4}$$

Integrating and applying the integration matrix P of previous section (The operational matrix of integration), we have

$$v(x,t) \equiv A^t(x)PH(t) + v(x,0) \tag{2.5}$$

Entering equation (2.4) and equation (2.5) into equation (2.3) yields,

$$\frac{d^2A^t(x)}{dx^2} \equiv RCA^t(x)P^{-1} \tag{2.6}$$

Next, solving equation (2.6), we get $A^t(x)$.
Finally, substitute $A^t(x)$ into equation (2.4), we obtain $v(x,t)$.

2.5 The Black-Scholes equation

In mathematical finance, the BlackScholes equation [14], [18] is a partial differential equation (PDE) governing the price evolution of a European call or European put under the BlackScholes model. Widely speaking, the term may refer to a same PDE that can be derived for a variety of options, or more generally, derivatives. For a European call or put [15] on an underlying stock paying no dividends, the equation is

$$\frac{\partial}{\partial t}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2}u(s,t) - ru(s,t) = 0$$

where $u(s,t)$ is the option price at time t with $0 \leq t \leq T$, s is the price of stock at time t, r is the interest rate and σ is the volatility of stock.

CHAPTER 3

Haar wavelet method for solving differential equations

The idea in this chapter starts from using an operational matrix for integration via Haar wavelets for apply this operational matrix to some differential equations, to demonstrate the other method of the approach.

3.1 Generalized Haar wavelet method for solving linear ordinary differential equations

In this section, we apply Haar wavelet method to solve some second order linear ordinary differential equations with initial conditions and boundary conditions. Moreover, we extend this approach for solving the general from of second order linear boundary value problem. Finally, we look into conditions for extend the interval of second order ordinary differential equation's solutions from $[0, 1]$ to $[r, r + 1]$ when r is an integer.

3.1.1 Methods of solution of linear ordinary differential equations with initial conditions

The Initial Value Problem is in the form

$$y^{''} = \phi(t, y, y^{'}) \tag{3.1}$$

with initial conditions

$$y^{'}(0) = \alpha,\ y(0) = \beta.$$

Assume that $y^{''}(t)$ is square integrable in the interval $0 \leq t < 1$.
Its Haar wavelets expansion can be expressed as

$$y^{''}(t) = \sum_{n=0}^{m-1} a_n h_n(t) \equiv A^t H_m(t) \tag{3.2}$$

where $A^t = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{m-1} \end{bmatrix}$ is the unknown vector of real numbers and $H_m(t)$ is Haar matrix. The m in $H_m(t)$ will be dropped to simplify the notation.

$$y^{''}(t) \equiv A^t H(t).$$

Integrating over $[0, t]$ yields

$$y^{'}(t) = \int_0^t y^{''}(\tau)\, d\tau + y^{'}(0) \equiv \int_0^t A^t H(\tau)\, d\tau + y^{'}(0).$$

Since $\int_0^t H(\tau)\, d\tau = PH(t)$, we get

$$y^{'}(t) \equiv A^t PH(t) + \alpha. \tag{3.3}$$

Integrating again, we obtain the solution

$$\begin{aligned} y(t) &= \int_0^t y^{'}(\tau)\, d\tau + y(0) \\ &\equiv \int_0^t \left(A^t H(\tau) + \alpha \right)\, d\tau + \beta. \end{aligned}$$

Hence

$$y(t) \equiv A^t P^2 H(t) + \alpha t + \beta. \tag{3.4}$$

Next, substituting (3.2), (3.3) and (3.4) into the equation (3.1), we obtain the linear system with unknown vector A^t. Finally, solving this system gives the solution of the form (3.4)

Example 3.1. Consider the differential equation

$$y''(t) + y'(t) + y(t) = -\sin t - \sin t(\cos t)^2 \tag{3.5}$$

with initial conditions

$$y'(0) = 0,\ y(0) = 1.$$

The exact solution is given by

$$\begin{aligned} y(t) = & \frac{-31\sqrt{3}}{219} \exp\left(\frac{-t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) \\ & + \frac{1}{73}\left(-19\exp\left(\frac{-t}{2}\right)\cos\left(\frac{\sqrt{3}t}{2}\right)\right) \\ & + \frac{1}{73}\left(8\sin(t)\cos^2(t) - 2\sin(t) + 3\cos^3(t) + 89\cos(t)\right). \end{aligned}$$

Assume that

$$y''(t) \equiv A^t H(t)$$

where $A^t = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{m-1} \end{bmatrix}$ and $H(t)$ is Haar matrix.
So we have

$$\begin{aligned} y'(t) &\equiv A^t P H(t) \\ y(t) &\equiv A^t P^2 H(t) + 1. \end{aligned}$$

Substituting $y''(t) \equiv A^t H(t)$, $y'(t) \equiv A^t P H(t)$ and $y(t) \equiv A^t P^2 H(t) + 1$ into (3.5) we obtain that

$$AH(t) + APH(t) + AP^2 H(t) \equiv -\sin t - \sin t(\cos t)^2 - 1.$$

Since we know matrices P and $H(t)$, we can compute the vector A^t and get the solution $y(t)$ as in Table 1, Table 2 and Table 3.

Time (t)	**Exact Solution**	**Haar Solution**	**Error**
0	1.0	0.9963369963	0.003663003663
0.125	0.9918840512	0.9821680151	0.009716036028
0.375	0.9229326726	0.9552691267	0.03233645414
0.625	0.7811469077	0.8707645007	0.08961759292
0.875	0.5749830640	0.7570535347	0.1820704707

Table 1. The comparison of exact solution and Haar solution when $m = 8$

Time (t)	**Exact Solution**	**Haar Solution**	**Error**
0	1.0	0.9990539262	0.0009460737938
0.125	0.9918840512	0.9880658880	0.003818163116
0.375	0.9229326726	0.9283967839	0.02449980899
0.625	0.7811469077	0.8308578420	0.07686851128
0.875	0.5749830640	0.7091856004	0.1661928251

Table 2. The comparison of exact solution and Haar solution when $m = 16$

Time (t)	**Exact Solution**	**Haar Solution**	**Error**
0	1.0	0.9999394417	0.00006055834797
0.125	0.9918840512	0.9915467363	0.0003373148315
0.375	0.9229326726	0.9360012442	0.01306857163
0.625	0.7811469077	0.8412507834	0.06010387567
0.875	0.5749830640	0.7211795486	0.1461964846

Table 3. The comparison of exact solution and Haar solution when $m = 64$

We see that Haar solutions approximate to the exact solution when m is increasing. In other word, error values are inversely proportional to m.

3.1.2 Methods of solution of linear ordinary differential equations with boundary conditions

The Boundary Value Problem is in the form

$$y^{''} = \phi(t, y, y^{'}) \tag{3.6}$$

with boundary conditions $a \leq b$.

Case 1: $y(a) = \alpha, y(b) = \beta$

Case 2: $y'(a) = \alpha, y(b) = \beta$

Case 3: $y(a) = \alpha, y'(b) = \beta$

Case 4: $y'(a) = \alpha, y'(b) = \beta$.

Similar to previous subsection, we assume that

$$y^{''}(t) \equiv A^t H(t). \tag{3.7}$$

By using the integration over $[a, t]$, we obtain

$$
\begin{aligned}
y^{'}(t) &= \int_a^t y^{''}(\tau)\, d\tau + y^{'}(a) && (3.8)\\
&= \int_0^t y^{''}(\tau)\, d\tau - \int_0^a y^{''}(\tau)\, d\tau + y^{'}(a) && (3.9)\\
&\equiv \int_0^t A^t H(\tau)\, d\tau - \left(y^{'}(a) - y^{'}(0)\right) + y^{'}(a) && (3.10)\\
y^{'}(t) &= A^t P H(t) + y^{'}(0). && (3.11)
\end{aligned}
$$

Integrating again, we get

$$y(t) \equiv A^t P^2 H(t) + y^{'}(0)t + y(0), \tag{3.12}$$

and

$$
\begin{aligned}
y^{'}(0) &\equiv y^{'}(a) + A^t P H(a), && (3.13)\\
y(0) &\equiv y(a) - A^t P^2 H(t) - y^{'}(0)a. && (3.14)
\end{aligned}
$$

Before substituting (3.7), (3.11) and (3.12) into (3.6), we need to compute $y(0)$ and $y^{'}(0)$, which depend on the choice of boundary conditions.

Case 1: $y(a) = \alpha, y(b) = \beta$.

Entering $t = a$ and $t = b$ into (3.12) yields,

$$
\begin{aligned}
y(a) &\equiv A^t P^2 H(a) + y^{'}(0)a + y(0),\\
y(b) &\equiv A^t P^2 H(b) + y^{'}(0)b + y(0).
\end{aligned}
$$

By solving this equation system, we get

$$
\begin{aligned}
y^{'}(0) &\equiv \frac{1}{(a-b)} \left[(y(a) - y(b)) - \left(A^t P^2 H(a) - A P^2 H(b)\right)\right]\\
&= \frac{1}{(a-b)} \left[(\alpha - \beta) - \left(A^t P^2 H(a) - A P^2 H(b)\right)\right].
\end{aligned}
$$

Entering $y^{'}(0)$ into (3.14) yields,

$$
\begin{aligned}
y(0) &\equiv y(a) - A^t P^2 H(a) - \frac{a}{(a-b)} \left[(y(a) - y(b)) - \left(A^t P^2 H(a) - A^t P^2 H(b) \right) \right] \\
&= \alpha - A^t P^2 H(a) - \frac{a}{(a-b)} \left[(\alpha - \beta) - \left(A^t P^2 H(a) - A^t P^2 H(b) \right) \right].
\end{aligned}
$$

Case 2: $y'(a) = \alpha, y(b) = \beta$.

Put $t = a$ and $t = b$ into (3.11) and (3.12) respectively,

$$
\begin{aligned}
y^{'}(a) &\equiv A^t P H(a) + y^{'}(0) \\
y(b) &\equiv A^t P^2 H(b) + y^{'}(0) b + y(0).
\end{aligned}
$$

We get

$$
\begin{aligned}
y^{'}(0) &\equiv y^{'}(a) - A^t P H(a) \\
&= \alpha - A^t P H(a),
\end{aligned}
$$

and

$$
\begin{aligned}
y(0) &\equiv y(b) - A^t P^2 H(b) - y^{'}(0) b \\
&\equiv y(b) - A^t P^2 H(b) - \left(y^{'}(a) - A^t P H(a) \right) b \\
&= \beta - A^t P^2 H(b) - \left(\alpha - A^t P H(a) \right) b.
\end{aligned}
$$

Case 3: $y(a) = \alpha, y'(b) = \beta$.

Put $t = b$ and $t = a$ into (3.11) and (3.12) respectively,

$$
\begin{aligned}
y^{'}(b) &\equiv A^t P H(b) + y^{'}(0) \\
y(a) &\equiv A^t P^2 H(a) + y^{'}(0) a + y(0).
\end{aligned}
$$

We get

$$
\begin{aligned}
y^{'}(0) &\equiv y^{'}(b) - A^t P H(b) \\
&= \beta - A^t P H(b),
\end{aligned}
$$

and

$$
\begin{aligned}
y(0) &\equiv y(a) - A^t P^2 H(a) - y^{'}(0) a \\
&\equiv y(a) - A^t P^2 H(a) - \left(y^{'}(b) - A^t P H(b) \right) a \\
&= \alpha - A^t P^2 H(a) - \left(\beta - A^t P H(b) \right) a.
\end{aligned}
$$

Case 4: $y'(a) = \alpha, y'(b) = \beta$.

In this case a or b must be equal to 0, then we can find $y(0)$ by substituting $y^{'}(0)$ in (3.12).

All of cases, we have P, $H(t)$, $y(0)$, $y'(0)$ and we can compute the vector A^t. Continuously, combine everything into (3.12) we obtained $y(t)$.

Example 3.2. Consider the differential equation

$$y^{''}(t) + y^{'}(t) + y(t) = \sin t \tag{3.15}$$

with boundary conditions

$$y^{'}(0) = 0,\ y(0.2) = 1.$$

The exact solution is given by

$$\begin{aligned} y(t) = & \frac{\exp(-0.5t)\sin\left(0.5\sqrt{3}t\right)(\cos(0.2)+1)}{\exp(-0.5t)\left(\sqrt{3}\cos\left(0.2\sqrt{3}\right)+\sin\left(0.2\sqrt{3}\right)\right)} \\ & + \frac{\exp(-0.5t)\cos\left(0.5\sqrt{3}t\right)(\cos(0.2)+1)}{\exp(-0.5t)\left(\sqrt{3}\cos\left(0.2\sqrt{3}\right)+\sin\left(0.2\sqrt{3}\right)\right)} - \cos(t). \end{aligned}$$

Assume that

$$y^{''}(t) \equiv A^t H(t).$$

where $A^t = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{m-1} \end{bmatrix}$ and $H(t)$ is Haar matrix.

So

$$\begin{aligned} y^{'}(t) &\equiv A^t P H(t) \\ y(t) &\equiv A^t P^2 H(t) + y(0). \end{aligned}$$

We know that

$$y(0) \equiv y(0.2) - A^t P^2 H(0.2) - \left(y^{'}(0) - A^t P H(0)\right)(0.2)$$

then

$$y(t) \equiv A^t P^2 H(t) + 1 - A^t P^2 H(0.2) + 0.2 A^t P H(0).$$

Substituting $y^{''}(t)$, $y^{'}(t)$ and $y(t)$ into (3.15).

We get

$$A^t H(t) + A^t P H(t) + A^t P^2 H(t) = \sin t - \left(1 - A^t P^2 H(0.2) + 0.2 A^t P H(0)\right).$$

We can compute the vector A^t because we have P and $H(t)$, then we get $y(t)$ as in Figure 3.

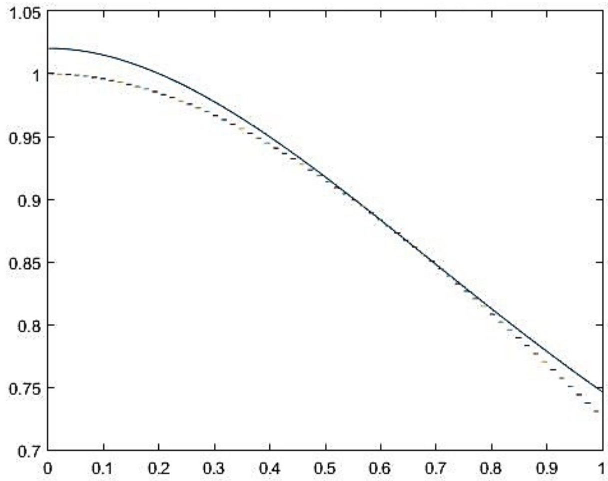

Figure 3. Haar solution when $m = 64$ (Dashed line) and exact solution (Straight line)

We see that Haar solutions (Dashed line) approximate to the exact solution (Straight line).

3.1.3 Methods of solution of linear ordinary differential equations

In this subsection, we look into conditions for extend the interval of second order ordinary differential equation's solutions from $[0,1]$ to $[r, r+1]$ when r is an integer [11].

We have

$$\int_r^{r+t} H_m(\tau)\, d\tau = P_m H_m(t).$$

Consider the initial value problem with conditions

$$y^{'}(a) = \alpha,\; y(a) = \beta$$

where $a \in [r, r+1]$.

Let us apply the Haar transform in the time domain and expand $y^{''}(t)$ into Haar wavelets

$$y^{''}(t) \equiv A^t H(t).$$

Integrating $y^{''}(t)$ from $a+r$ to $r+t$,

$$\int_{a+r}^{r+t} y^{''}(\tau)\, d\tau = y^{'}(t+r) - y^{'}(a+r)$$

so that

$$
\begin{aligned}
y^{'}(t+r) &= \int_{a+r}^{r+t} y^{''}(\tau)\,d\tau + y^{'}(a+r) \\
&= \left[\int_{0}^{r+t} y^{''}(\tau)\,d\tau - \int_{0}^{a+r} y^{''}(\tau)\,d\tau\right] + y^{'}(a+r) \\
&= \left[\left(\int_{0}^{r} y^{''}(\tau)\,d\tau + \int_{r}^{r+t} y^{''}(\tau)\,d\tau\right) - \int_{0}^{a+r} y^{''}(\tau)\,d\tau\right] + y^{'}(a+r) \\
&= \left(y^{'}(r) - y^{'}(0)\right) + \int_{r}^{r+t} y^{''}(\tau)\,d\tau - \left(y^{'}(a+r) - y^{'}(0)\right) + y^{'}(a+r) \\
&= \int_{r}^{r+t} y^{''}(\tau)\,d\tau + y^{'}(r) \\
&\equiv \int_{r}^{r+t} A^{t}H(\tau)\,d\tau + y^{'}(r) \\
&= A^{t}\int_{r}^{r+t} H(\tau)\,d\tau + y^{'}(r) \\
&= A^{t}PH(t) + y^{'}(r).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
y^{'}(t+r) &\equiv A^{t}PH(t) + y^{'}(r) \\
y(t+r) &\equiv A^{t}P^{2}H(t) + y^{'}(r)t + y(r).
\end{aligned}
$$

Note that, we can find $y^{'}(r)$ and $y(r)$ by the same method of calculating $y^{'}(0)$ and $y(0)$.

3.2 Haar wavelet method for solving the Black-Scholes equation

This section we look into conditions for solve of the Black-Scholes equation by Haar wavelet method.

The Black-Scholes equation is given by

$$
\frac{\partial}{\partial t}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) + \frac{\sigma^{2}s^{2}}{2}\frac{\partial^{2}}{\partial s^{2}}u(s,t) - ru(s,t) = 0
$$

where $u(s,t)$ is the option price at time t with $0 \leq t \leq T$, s is the price of stock at time t, r is the interest rate and σ is the volatility of stock which is $r > \dfrac{\sigma^{2}}{2}$.

Consider the case of a European call option.

Boundary conditions for a call option [15] are

$$u(0,t) = 0$$
$$u(s,t) = s \text{ when } s \to \infty$$
$$u(s,T) = \max(s - s_p, 0) \text{ when } s_p \text{ is a strike price.}$$

For simplicity, we set,

$$\begin{aligned}
u(s,t) &= u, \\
\frac{\partial}{\partial t}u(s,t) &= u_t, \\
\frac{\partial}{\partial s}u(s,t) &= u_s, \\
\frac{\partial^2}{\partial s^2}u(s,t) &= u_{ss}.
\end{aligned}$$

Consider

$$u_t + rsu_s + \frac{\sigma^2 s^2}{2}u_{ss} - ru = 0. \tag{3.16}$$

Let

$$u_{ss} \equiv a^t(t)H(s)$$

where $a^t(t)$ is an m-vector function of t.

So

$$\begin{aligned}
u_s &\equiv a^t(t)PH(s) + u_s(0,t) \\
u &\equiv a^t(t)P^2H(s) + su_s(0,t) + u(0,t).
\end{aligned}$$

Thus

$$u_t \equiv \dot{a}^t(t)P^2H(s) + s\dot{u}_s(0,t).$$

Since $\lim_{s\to 0} u(s,t) = 0$, so in this case is not interested.

We know that

$$\lim_{s\to\infty} u(s,t) = s.$$

So

$$u_s(s,t) = 1 = u_s(0,t).$$

Thus

$$\dot{u}_s(0,t) = 0.$$

Hence

$$u_t \equiv \dot{a}^t(t)P^2H(s).$$

Substituting u_t, u, u_{ss} and u_s into (3.16).

We get

$$\dot{a}^t(t)P^2H(s) + rs\left(a^t(t)PH(s) + 1\right) + \frac{\sigma^2 s^2}{2}\left(a^t(t)H(s)\right) - rs \equiv 0.$$

Finally, we obtain that

$$\dot{a}^t(t)P^2 + \left(rsP + \frac{\sigma^2 s^2}{2} + rP^2 1\right)a^t \equiv 0. \tag{3.17}$$

Solving equation (3.17), we get

$$a^t(t) \equiv a^t(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}t\right].$$

Hence

$$u(s,t) \equiv a^t(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}t\right]P^2H(s) + s.$$

Now, we can find $a^t(0)$ by the condition $u(s,T) = s^\alpha$ when $\alpha \in \mathbb{R}$ [13].

$$s^\alpha = u(s,T) \equiv a^T(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}T\right]P^2H(s) + s.$$

Finally, we obtain that

$$u(s,t) \equiv a^t(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}t\right]P^2H(s) + s.$$

CHAPTER 4

Conclusions and Suggestions

In chapter 3, we have 2 sections. The first section is Generalized Haar wavelet method for solving linear ordinary differential equations and the second section is Haar wavelet method for solving the Black-Scholes equation.

In section of Generalized Haar wavelet method for solving linear ordinary differential equations, we have 3 subsections.

1. Methods of solution of linear ordinary differential equations with initial conditions

The Initial Value Problem is in the form

$$y^{''} = \phi(t, y, y^{'})$$

with initial conditions

$$y^{'}(a) = \alpha,\, y(a) = \beta.$$

When $a \in [0, 1]$.

In this subsection, we study Haar wavelet method by apply Haar wavelets to solve some second order linear ordinary differential equation with initial conditions. Now, we obtained

$$\begin{aligned} y^{'}(t) &\equiv A^t PH(t) + y^{'}(0) \\ y(t) &\equiv A^t P^2 H(t) + y^{'}(0)t + y(0). \end{aligned}$$

Substituting P, $H(t)$, initial conditions and then compute A^t, we obtained $y(t)$.

Observe that we must have initial conditions $y^{'}(0)$ and $y(0)$. We can find initial conditions at 0 by this equation

$$y^{'}(0) \equiv y^{'}(a) - A^t PH(a)$$

and

$$y(0) \equiv y(a) - A^t P^2 H(a) - y^{'}(0)a.$$

$H(a)$ is The Haar vector at the point a for an mth-order system is defined with m Haar functions in row vector. It means that,

$$H(a) \equiv H_m(a) \equiv \begin{bmatrix} h_0(a) \\ h_1(a) \\ h_2(a) \\ \vdots \\ h_{m-1}(a) \end{bmatrix}.$$

2. Methods of solution of linear ordinary differential equations with boundary conditions

The Boundary Value Problem is in the form

$$y^{''} = \phi(t, y, y^{'})$$

with boundary conditions

Case 1: $y(a) = \alpha, y(b) = \beta$

Case 2: $y'(a) = \alpha, y(b) = \beta$

Case 3: $y(a) = \alpha, y'(b) = \beta$

Case 4: $y'(a) = \alpha, y'(b) = \beta$.

By the first subsection, we get

$$\begin{aligned} y^{'}(t) &\equiv A^t PH(t) + y^{'}(0) \\ y(t) &\equiv A^t P^2 H(t) + y^{'}(0)t + y(0), \end{aligned}$$

and

$$\begin{aligned} y^{'}(0) &\equiv y^{'}(a) - A^t PH(a) \\ y(0) &\equiv y(a) - A^t P^2 H(a) - y^{'}(0)a. \end{aligned}$$

In this subsection we find $y^{'}(0)$ and $y(0)$.

Finally, we obtained.

Case 1: $y(a) = \alpha, y(b) = \beta$.

$$y^{'}(0) \equiv \frac{1}{(a-b)} \left[(\alpha - \beta) - \left(A^t P^2 H(a) - A^t P^2 H(b) \right) \right]$$

and

$$y(0) \equiv \alpha - A^t P^2 H(a) - \frac{a}{(a-b)} \left[(\alpha - \beta) - \left(A^t P^2 H(a) - A^t P^2 H(b) \right) \right].$$

Case 2: $y'(a) = \alpha, y(b) = \beta$.

$$y^{'}(0) \equiv \alpha - A^t P H(a)$$

and

$$y(0) \equiv \beta - A^t P^2 H(b) - \left(\alpha - A^t P H(a) \right) b.$$

Case 3: $y(a) = \alpha, y'(b) = \beta$.

$$y^{'}(0) \equiv \beta - A^t P H(b)$$

and

$$y(0) \equiv \alpha - A^t P^2 H(a) - \left(\beta - A^t P H(b) \right) a.$$

Case 4: $y'(a) = \alpha, y'(b) = \beta$.

In this case a or b must be equal to 0, then we can find $y(0)$ by substituting $y^{'}(0)$ in $y(t) \equiv A^t P^2 H(t) + y^{'}(0)t + y(0)$.

All of cases, we have P, $H(t)$, $y(0)$, $y'(0)$ and we can compute the vector A^t. Continuously, combine everything into $y(t) \equiv A^t P^2 H(t) + y^{'}(0)t + y(0)$ we obtained $y(t)$.

3. Methods of solution of linear ordinary differential equations

In this subsection, we consider the solutions of second order ordinary differential equations in $[r, r+1]$ when r is an integer.

We have

$$\int_r^{r+t} H_m(\tau)\, d\tau = P_m H_m(t)$$

Consider the initial value problem with conditions

$$y^{'}(a) = \alpha,\, y(a) = \beta$$

where $a \in [r, r+1]$.

Let us apply the Haar transform in the time domain and expand $y^{''}(t)$ into Haar wavelets and integrating $y^{''}(t)$ from $a+r$ to $r+t$.
Finally, we get

$$\begin{aligned} y^{'}(t+r) &\equiv A^t P H(t) + y^{'}(r) \\ y(t+r) &\equiv A^t P^2 H(t) + y^{'}(r)t + y(r). \end{aligned}$$

We can find $y^{'}(r)$ and $y(r)$ by the same method of calculating $y^{'}(0)$ and $y(0)$.

The second section is Haar wavelet method for solving the Black-Scholes equation.

The Black-Scholes equation is given by

$$\frac{\partial}{\partial t}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2}u(s,t) - ru(s,t) = 0$$

where $u(s,t)$ is the option price at time t with $0 \leq t \leq T$, s is the price of stock at time t, r is the interest rate and σ is the volatility of stock which is $r > \dfrac{\sigma^2}{2}$.

This section we solve the Black-Scholes equation in the case of a European call option by Haar wavelet method then we obtained that

$$u(s,t) \equiv a^t(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}t\right]P^2 H(s) + s.$$

We can find $a^t(0)$ by this equation

$$s^\alpha = u(s,T) \equiv a^T(0)\exp\left[-\left(rsP + \frac{\sigma^2 s^2}{2} - rP^2\right)P^{-2}T\right]P^2 H(s) + s.$$

The main goal of this thesis is to present that the Haar wavelet method is a effective tool for solving differential equations. Approximate solution of second order differential equations, are compared with exact solution. Among the well-known wavelets, the Haar wavelet is the simplest one. And you can use the Haar function for construct the orthogonal basis by the process that not too hard.

We will make more suggestions in Example 3.1 and Example 3.2. Since we have a problem in program for find solutions of the vector A^t, so we can choose $m \leq 64$. If some people can choose the big m then Haar solutions approximate to the exact solution because error values are inversely proportional to m. Moreover, in the section of Haar wavelet method for solving the Black-Scholes equation, the suggestion is solving the Black-Scholes equation in the case of a European put option by Haar wavelet method.

Bibliography

[1] Black F. and Scholes M. S. (1973). The pricing of options and corporate liabilities, *J. Polit. Econ*, 81, 637-654.

[2] Berwal N., Panchal D. and Parihar C. L. (2013). Haar wavelet method for numerical solution of Telegraph equation, *Italian Journal of Pure and Applied Mathematics*, 30, 317-328.

[3] Chen C. F. and Hsiao C. H. (1997). Haar wavelet method for solving lumped and distributed-parameter systems, *IEE Proceedings-Control Theory and Applications*, 144.1, 87-94.

[4] Haar A. (1910). Zur theorie der orthogonalen Funktionsysteme, *Math.Annal*, 69, 331-371.

[5] Hariharan G. (2013). An efficient wavelet based approximation method to time fractional Black-Scholes European option pricing problem arising in financial market, *Appl. Math. Sci*, 69, 3445-3456.

[6] Hariharan G. (2013). An overview of Haar wavelet method for solving differential and integral equations, *World Applied Sciences Journal*, 23, 01-14.

[7] Hariharan G. and Kannan K. (2010). Haar wavelet method for solving Fitzhugh-Nagumo equation, *World Academy of Sciences, Engineering and Technology*, 43, 560-563.

[8] Hariharan G., Kannan K. and Sharma K.R. (2009). Haar wavelet method for solving Fishers equation, *Appl. Math. Comput.*, 284292.

[9] Lepik U. (2007). Application of Haar wavelet transform to solving integral and differential equations, *Proc. Estonian Acad. Sci. Phys. Math*, 56, No. 1, 28-46.

[10] Lepik U. (2008). Solving differential and integral equations by Haar wavelet method, revisted, *International Journal of Mathematics and Computation*, 1, No. 8, 43-52.

[11] Lepik U. and Hein H. (2014). *Haar wavelets*, Springer International Publishing.

[12] Mallet S. G. (1989). Approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, *Transactions of the American mathematical society*, 315.1, 69-87.

[13] Bohner M. and Zheng Y. (2009). On analytical solution of the Black-Scholes equation, *Appl. Math. Lett.*, 22, pp. 309-313.

[14] Michael J. (2001). *Stochastic Calculus and Financial Applications*, Springer-Verlag, QA 274.2 S 74.

[15] Ugur. O. (2008). *Introduction to computational finance*, Imperial College Press and World Scientific.

[16] Walnut. D. F. (2004). *An introduction to Wavelet Analysis*, Springer Science+Business Media, LLC.

[17] Wu. J. L. (2009). A wavelet operational method for solving fractional partial differential equations numerically, *Applied Mathematics and Computation*, 214.1, 31-40.

[18] Wilmott. P., Howison. S. and Dewynne. J. *The Mathematics of Financial Derivatives*, Cambridge University Press.

APPENDIX

In this Chapter, we show codes of program for solving a vector A^t in examples of Chapter 3.

```
function [H]=Haar(m)

H=zeros(m);
for i =1:m
            H(1,i)=1;
end
n=1;
a=m/2;
while a>=1
    c=1;
    b=a*2;
    d=a;
    while b<=m
        n=n+1;

        for i =c:d
            H(n,i)=1;
        end
        for i =d+1:b
            H(n,i)=-1;
        end
        H;
        c=c+(2*a);
        d=d+(2*a);
        b=b+(2*a);
    end
    a=a/2;
end

end
```

```
function [P]=Pe(m)
P=1/2;
i=1;
if m>1
    while i<m

    H=Haar(i);
    InvH=inv(H);

    P=[P;InvH/(2*i*2)];
    Hz=[-H/(2*i*2);zeros(i)];
    P=[P,Hz];
    i=i*2;
    end
end
end
```

```
function SolveAnotzeroBVP(m)
syms a
A=sym('a',[1 m]);
H=Haar(m);
P=Pe(m);
```

```
UPP=(A*H);
UP=(A*P*H);
U=(A*P*P*H);
for i =1:m
            Ha=H(i,13)
end
for i =1:m
            Hb=H(i,1)
end
V=UPP+UP+U
%
S=solve(V==A*P*P*Ha-1+sin(0.2)-(0.2*(A*P*Hb)))
% A=[S.a1,S.a2,S.a3,S.a4,S.a5,S.a6,S.a7,S.a8];
A=[S.a1,S.a2,S.a3,S.a4,S.a5,S.a6,S.a7,S.a8,S.a9,S.a10,S.a11,S.a1
2,S.a13,S.a14,S.a15,S.a16,S.a17,S.a18,S.a19,S.a20,S.a21,S.a22,S.
a23,S.a24,S.a25,S.a26,S.a27,S.a28,S.a29,S.a30,S.a31,S.a32,S.a33,
S.a34,S.a35,S.a36,S.a37,S.a38,S.a39,S.a40,S.a41,S.a42,S.a43,S.a4
4,S.a45,S.a46,S.a47,S.a48,S.a49,S.a50,S.a51,S.a52,S.a53,S.a54,S.
a55,S.a56,S.a57,S.a58,S.a59,S.a60,S.a61,S.a62,S.a63,S.a64];
```

```
Haarsol=A*P*P*H+ones(1,m)-(A*P*P*Ha)+0.2*(A*P*Hb)

 syms u(t)
 Du = diff(u);
%
 ode = diff(u,t,2)+diff(u,t)+u == sin(t);
cond1 = u(0.2) == 1;
cond2 = Du(0) == 0;
%
conds = [cond1 cond2]
uSol(t) = dsolve(ode,conds)
uSol = simplify(uSol)
```

```
t = linspace(0,1);
t1  =   0   :0.01:  0.015625    ;
t2  =    0.015625   :0.01:  0.031250    ;
t3  =    0.031250   :0.01:  0.046875    ;
t4  =    0.046875   :0.01:  0.062500    ;
t5  =    0.062500   :0.01:  0.078125    ;
t6  =    0.078125   :0.01:  0.093750    ;
t7  =    0.093750   :0.01:  0.109375    ;
t8  =    0.109375   :0.01:  0.125000    ;
t9  =    0.125000   :0.01:  0.140625    ;
t10 =    0.140625   :0.01:  0.156250    ;
t11 =    0.156250   :0.01:  0.171875    ;
t12 =    0.171875   :0.01:  0.187500    ;
t13 =    0.187500   :0.01:  0.203125    ;
t14 =    0.203125   :0.01:  0.218750    ;
t15 =    0.218750   :0.01:  0.234375    ;
t16 =    0.234375   :0.01:  0.250000    ;
t17 =    0.250000   :0.01:  0.265625    ;
t18 =    0.265625   :0.01:  0.281250    ;
t19 =    0.281250   :0.01:  0.296875    ;
t20 =    0.296875   :0.01:  0.312500    ;
t21 =    0.312500   :0.01:  0.328125    ;
t22 =    0.328125   :0.01:  0.343750    ;
t23 =    0.343750   :0.01:  0.359375    ;
t24 =    0.359375   :0.01:  0.375000    ;
t25 =    0.375000   :0.01:  0.390625    ;
t26 =    0.390625   :0.01:  0.406250    ;
t27 =    0.406250   :0.01:  0.421875    ;
t28 =    0.421875   :0.01:  0.437500    ;
t29 =    0.437500   :0.01:  0.453125    ;
t30 =    0.453125   :0.01:  0.468750    ;
t31 =    0.468750   :0.01:  0.484375    ;
t32 =    0.484375   :0.01:  0.500000    ;
t33 =    0.500000   :0.01:  0.515625    ;
t34 =    0.515625   :0.01:  0.531250    ;
t35 =    0.531250   :0.01:  0.546875    ;
t36 =    0.546875   :0.01:  0.562500    ;
t37 =    0.562500   :0.01:  0.578125    ;
t38 =    0.578125   :0.01:  0.593750    ;
t39 =    0.593750   :0.01:  0.609375    ;
t40 =    0.609375   :0.01:  0.625000    ;
t41 =    0.625000   :0.01:  0.640625    ;
t42 =    0.640625   :0.01:  0.656250    ;
```

```
t43 =    0.656250   :0.01:  0.671875    ;
t44 =    0.671875   :0.01:  0.687500    ;
t45 =    0.687500   :0.01:  0.703125    ;
t46 =    0.703125   :0.01:  0.718750    ;
t47 =    0.718750   :0.01:  0.734375    ;
t48 =    0.734375   :0.01:  0.750000    ;
t49 =    0.750000   :0.01:  0.765625    ;
t50 =    0.765625   :0.01:  0.781250    ;
t51 =    0.781250   :0.01:  0.796875    ;
t52 =    0.796875   :0.01:  0.812500    ;
t53 =    0.812500   :0.01:  0.828125    ;
t54 =    0.828125   :0.01:  0.843750    ;
t55 =    0.843750   :0.01:  0.859375    ;
t56 =    0.859375   :0.01:  0.875000    ;
t57 =    0.875000   :0.01:  0.890625    ;
t58 =    0.890625   :0.01:  0.906250    ;
t59 =    0.906250   :0.01:  0.921875    ;
t60 =    0.921875   :0.01:  0.937500    ;
t61 =    0.937500   :0.01:  0.953125    ;
t62 =    0.953125   :0.01:  0.968750    ;
t63 =    0.968750   :0.01:  0.984375    ;
t64 =    0.984375   :0.01:  1.000000    ;
```

```
u = uSol(t);
u1  =   Haarsol(1,  1   );
u2  =   Haarsol(1,  2   );
u3  =   Haarsol(1,  3   );
u4  =   Haarsol(1,  4   );
u5  =   Haarsol(1,  5   );
u6  =   Haarsol(1,  6   );
u7  =   Haarsol(1,  7   );
u8  =   Haarsol(1,  8   );
u9  =   Haarsol(1,  9   );
u10 =   Haarsol(1,  10  );
u11 =   Haarsol(1,  11  );
u12 =   Haarsol(1,  12  );
u13 =   Haarsol(1,  13  );
u14 =   Haarsol(1,  14  );
u15 =   Haarsol(1,  15  );
u16 =   Haarsol(1,  16  );
u17 =   Haarsol(1,  17  );
u18 =   Haarsol(1,  18  );
u19 =   Haarsol(1,  19  );
u20 =   Haarsol(1,  20  );
u21 =   Haarsol(1,  21  );
u22 =   Haarsol(1,  22  );
u23 =   Haarsol(1,  23  );
u24 =   Haarsol(1,  24  );
u25 =   Haarsol(1,  25  );
u26 =   Haarsol(1,  26  );
u27 =   Haarsol(1,  27  );
u28 =   Haarsol(1,  28  );
u29 =   Haarsol(1,  29  );
u30 =   Haarsol(1,  30  );
u31 =   Haarsol(1,  31  );
u32 =   Haarsol(1,  32  );
u33 =   Haarsol(1,  33  );
u34 =   Haarsol(1,  34  );
u35 =   Haarsol(1,  35  );
u36 =   Haarsol(1,  36  );
u37 =   Haarsol(1,  37  );
u38 =   Haarsol(1,  38  );
u39 =   Haarsol(1,  39  );
u40 =   Haarsol(1,  40  );
u41 =   Haarsol(1,  41  );
u42 =   Haarsol(1,  42  );
u43 =   Haarsol(1,  43  );
u44 =   Haarsol(1,  44  );
u45 =   Haarsol(1,  45  );
u46 =   Haarsol(1,  46  );
u47 =   Haarsol(1,  47  );
u48 =   Haarsol(1,  48  );
u49 =   Haarsol(1,  49  );
u50 =   Haarsol(1,  50  );
```

```
u51 =   Haarsol(1,   51  );
u52 =   Haarsol(1,   52  );
u53 =   Haarsol(1,   53  );
u54 =   Haarsol(1,   54  );
u55 =   Haarsol(1,   55  );
u56 =   Haarsol(1,   56  );
u57 =   Haarsol(1,   57  );
u58 =   Haarsol(1,   58  );
u59 =   Haarsol(1,   59  );
u60 =   Haarsol(1,   60  );
u61 =   Haarsol(1,   61  );
u62 =   Haarsol(1,   62  );
u63 =   Haarsol(1,   63  );
u64 =   Haarsol(1,   64  );


figure % opens new figure window
plot(t,u,t1,ones(size(t1))*u1,t2,ones(size(t2))*u2,t3,ones(size(
t3))*u3,t4,ones(size(t4))*u4,t5,ones(size(t5))*u5,t6,ones(size(t
6))*u6,t7,ones(size(t7))*u7,t8,ones(size(t8))*u8,t9,ones(size(t9
))*u9,t10,ones(size(t10))*u10,t11,ones(size(t11))*u11,t12,ones(s
ize(t12))*u12,t13,ones(size(t13))*u13,t14,ones(size(t14))*u14,t1
5,ones(size(t15))*u15,t16,ones(size(t16))*u16,t17,ones(size(t17)
)*u17,t18,ones(size(t18))*u18,t19,ones(size(t19))*u19,t20,ones(s
ize(t20))*u20,t21,ones(size(t21))*u21,t22,ones(size(t22))*u22,t2
3,ones(size(t23))*u23,t24,ones(size(t24))*u24,t25,ones(size(t25)
)*u25,t26,ones(size(t26))*u26,t27,ones(size(t27))*u27,t28,ones(s
ize(t28))*u28,t29,ones(size(t29))*u29,t30,ones(size(t30))*u30,t3
1,ones(size(t31))*u31,t32,ones(size(t32))*u32,t33,ones(size(t33)
)*u33,t34,ones(size(t34))*u34,t35,ones(size(t35))*u35,t36,ones(s
ize(t36))*u36,t37,ones(size(t37))*u37,t38,ones(size(t38))*u38,t3
9,ones(size(t39))*u39,t40,ones(size(t40))*u40,t41,ones(size(t41)
)*u41,t42,ones(size(t42))*u42,t43,ones(size(t43))*u43,t44,ones(s
ize(t44))*u44,t45,ones(size(t45))*u45,t46,ones(size(t46))*u46,t4
7,ones(size(t47))*u47,t48,ones(size(t48))*u48,t49,ones(size(t49)
)*u49,t50,ones(size(t50))*u50,t51,ones(size(t51))*u51,t52,ones(s
ize(t52))*u52,t53,ones(size(t53))*u53,t54,ones(size(t54))*u54,t5
5,ones(size(t55))*u55,t56,ones(size(t56))*u56,t57,ones(size(t57)
)*u57,t58,ones(size(t58))*u58,t59,ones(size(t59))*u59,t60,ones(s
ize(t60))*u60,t61,ones(size(t61))*u61,t62,ones(size(t62))*u62,t6
3,ones(size(t63))*u63,t64,ones(size(t64))*u64)
title('Exact solution and Haar solution (64)')
```

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P. Wannateeradet and K. Nualtong, Generalized Haar wavelet method for solving linear ordinary differential equations, *The* 43^{rd} *Congress on Science and Technology of thailand.,* Chulalongkorn University, Bangkok, Thailand, October 17-19, 2017, 658-666