Group Structure on Cantor $p-$ary Sets

Riduan Waema

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science in Applied Mathematics

Prince of Songkla University

2016

Group Structure on Cantor $p-$ary Sets

Riduan Waema

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics

Prince of Songkla University

2016

Thesis Title Group Structure on Cantor $p-$ary Sets

Author Mr. Riduan Waema

Major Program Applied Mathematics

Major Advisor:

. . . .

(Asst. Prof. Dr. Aniruth Phon-On)

Co-advisor:

. . . .

(Dr. Saofee Busaman)

Examining Committee:

. . . . Chairperson

(Asst. Prof. Dr. Areeyuth Sama-ae)

. . . .

(Asst. Prof. Dr. Aniruth Phon-On)

. . . .

(Dr. Saofee Busaman)

. . . .

(Dr. Pakwan Riyapan)

. . . .

(Dr. Annop Kaewkhao)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science in Applied Mathematics.

. . . .

(Assoc. Prof. Dr. Teerapol Srichana)

Dean of Graduate School

This is to certify that the work here submitted is the result of the candidate's own investigations. Due acknowledgment has been made of any assistance received.

. . . . Signature

(Asst. Prof. Dr. Aniruth Phon-On)

Major Advisor

. . . . Signature

(Mr. Riduan Waema)

Candidate

I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

.... Signature

(Mr. Riduan Waema)

Candidate

ชื่อวิทยานิพนธ์ โครงสร้างกรุปบนเซตพี-อารี คันทอร์

ผู้เขียน นายริฎวน แวมะ

สาขาวิชา คณิตศาสตร์ประยุกต์

ปีการศึกษา 2559

บทคัดย่อ

เซตคันทอร์ (Cantor set) $\mathfrak{C}$ เป็นเซตที่สร้างโดยนักคณิตศาสตร์ชื่อ Georg Cantor (Nelson, n.d.) งานวิจัยนี้เราได้สร้างเซตใหม่ที่ชื่อว่า เซตพี-อารี คันทอร์ (Cantor p-ary set) $\mathfrak{C}_p$ โดย p เป็นจำนวนเฉพาะคี่ ซึ่งเป็นเซตที่มีความนัยทั่วไปกว่าเซตคันทอร์ และเราได้นิยามเซต spawning p-ary A_n^p และเซต child p-ary $A_{p^k n}^p$ เมื่อ $k = 1, 2, 3, \ldots$.

งานวิจัยนี้ประกอบด้วยเนื้อหา 2 ส่วน ส่วนที่ 1 เราได้แสดงการพิสูจน์ความสัมพันธ์ของจำนวนสมาชิกที่อยู่ในเซต spawning p-ary A_n^p และสมาชิกในเซต child p-ary $A_{p^k n}^p$ ดังนี้ $\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot \left|A_n^p\right|$ เมื่อ $K_p^e = \{0, 2, 4, \ldots, p-1\}$ งานวิจัยส่วนที่ 2 เรานิยามฟังก์ชัน R หมายถึงการสลับเลขโดด a_i กับคอมพลีเมนต์ของมันคือ $\breve{a}_i$ และนิยามฟังก์ชัน T คือการเลื่อนเลขโดด a_i แต่ละตัวใน $0.\overline{a_1 a_2 \ldots a_i \ldots a_l}$ ไปยังซ้ายมือหนึ่งครั้ง ต่อมาเราสร้างกรุป G ที่มีสมาชิกเป็นฟังก์ชันที่ถูกสร้างขึ้นจากฟังก์ชัน R และฟังก์ชัน T และพิสูจน์ได้ว่า

(1) $T^l = I$ และ $R^2 = I$ เมื่อ I เป็นฟังก์ชันเอกลักษณ์และ l เป็นความยาวช่วงของสมาชิกที่อยู่ในเซต spawning p-ary A_n^p

(2) T และ R มีสมบัติการสลับที่ภายใต้การดำเนินการ $\circ$

(3) กรุป G ไอโซโมฟิกกับกรุป $\mathbb{Z}_l \times \mathbb{Z}_2$

(4) G_T เป็นกรุปย่อยที่มีสมาชิกประกอบด้วย T เท่านั้น มีสมบัติเป็นกรุปวัฏจักร faithful และมีอันดับ l

นอกจากนี้ เราพิสูจน์ว่า l หารจำนวนสมาชิกทั้งหมดที่อยู่ในเซต spawning p-ary A_n^p ได้ลงตัว พร้อมทั้ง l สามารถหารจำนวนสมาชิกทั้งหมดที่อยู่ในเซต child p-ary $A_{p^k n}^p$ ได้ลงตัวเช่นกัน

Thesis Title Group Structure on Cantor $p-$ary Sets

Author Mr. Riduan Waema

Major Program Applied Mathematics

Academic Year 2016

ABSTRACT

The Cantor set $\mathfrak{C}$ or the Cantor middle thirds set was constructed by Georg Cantor (Nelson, n.d.). In this thesis, we define the generalization of the Cantor set namely Cantor p-ary set $\mathfrak{C}_p$, where p is an odd prime. Then we give the definitions of spawning p-ary set A_n^p and child p-ary set $A_{p^k n}^p$ where $k = 1, 2, 3, \ldots$.

This thesis consists of two parts. The first part, we prove the relation of cardinality of spawning p-ary set A_n^p and child p-ary set $A_{p^k n}^p$, that is $\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot \left|A_n^p\right|$, where $K_p^e = \{0, 2, 4, \ldots, p-1\}$. The second part, we define a transformation R by swapping a digit a_i with its complement $\breve{a}_i$ and denote a transformation T by cycle the digit a_i in $0.\overline{a_1 a_2 \ldots a_i \ldots a_l}$ to the left. Then we construct a group G which its elements are generated by the transformation R and the transformation T and we prove that

(1) $T^l = I, R^2 = I$, where I is an identity function and l be the period length of elements in spawning p-ary sets A_n^p

(2) T and R commute

(3) G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$

(4) G_T, the subgroup generated by T alone, is a faithful cyclic subgroup of order l.

Moreover, we prove that $l \mid \left|A_n^p\right|$ and $l \mid \left|A_{p^k n}^p\right|$.

ACKNOWLEDGEMENTS

In the name of God(Allah), the Most Gracious, the Most Merciful.

This thesis would not have been successful without the help of mentioned people here, I would like to grab this opportunity to give a big thanks for them.

First of all, I would like to address my gratitude to my advisors Asst. Prof. Dr. Aniruth Phon-On and Dr. Saofee Busaman for their invaluable assistance encouragement and helpful guidance during my study.

Secondly, I would also like to acknowledge Faculty of Science and Technology, Prince of Songkla University, Graduate School and the Centre of Excellence in Mathematics (CEM), for providing the scholarship for my master degree studying.

Thirdly, I would like to acknowledge all lecturers of Applied Mathematics at Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus.

Many thanks to my friends for best memories, their best laughs, their jokes, and their joint activities together that help me to be relax and happy during my study.

Finally, I would like to say a huge thanks my family (Usman, Hamidah, Ilham, Asma, Nasree and Imron) for their encouragement in all aspect, their praying, and their understanding throughout this study.

Riduan Waema

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background and rationale

The Cantor set or the Cantor middle thirds set was constructed by Georg Cantor (Nelson, n.d.). It has interesting properties and special construction. In the first step, if set $A_0 = \{[0,1]\}$, then divide the closed interval into three equal subintervals and remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$. It follows that the new set $A_1 = \left\{\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right]\right\}$ is obtained. In the second step, we again subdivide each element in A_1 into three equal subintervals and remove the middle thirds $\left\{\left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right)\right\}$. Hence, the set A_2 will be $\left\{\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right]\right\}$. The next step, we divide again each element in the set A_2 into three equal subintervals and delete the middle thirds, this follows that set A_3 will be as

$$A_3 = \left\{\left[0, \frac{1}{27}\right], \left[\frac{2}{27}, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{7}{27}\right], \left[\frac{8}{27}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{19}{27}\right], \left[\frac{20}{27}, \frac{7}{9}\right], \left[\frac{8}{9}, \frac{25}{27}\right], \left[\frac{26}{27}, 1\right]\right\}.$$

We divide all elements in the set A_3 and remove the middle thirds, this leads to get all elements in a set A_4. If we repeat this process, subdivide each element in A_{n-1}, where $n = 1, 2, 3, \ldots$, and remove the middle thirds respectively, these will generate all elements in A_n. Therefore, the Cantor set $\mathfrak{C}$ defined as

$$\mathfrak{C} = \cap_{n=0}^{\infty} \left(\cup A_n\right),$$

is the intersection of all $\cup A_n$, where $\cup A_n$ is the union of all elements in A_n. To clarify the construction of the set, it will be shown as Figure 1.1.

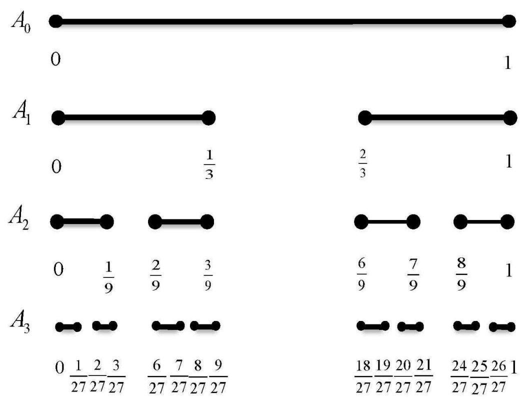

Figure 1.1: The construction of the Cantor set

The Cantor set has several properties that are non-empty set, closed, perfect, compact, nowhere dense and totally disconnected. It was stated in (Nelson, n.d.), (kunczynski, 1968), (Rosen, 1993) and (Woolley, 2008). Moreover, kunczynski (1968), Rosen (1993) and Woolley (2008) described that the complement of $\mathfrak{C}$ is $[0,1]\backslash\mathfrak{C}$ has length 1, this concludes that $\mathfrak{C}$ has measure zero.

There is a property obtaining from the construction of the Cantor set, namely uncountable. We will use the following theorem and corollary for proving the property.

Theorem 1.1.1. *(Sella, n.d.) Let $h : X \longrightarrow Y$ be a surjection. If the set X is countable, then the set Y is countable.*

Corollary 1.1.2. *Let $h : X \longrightarrow Y$ be a surjection. If the set Y is uncountable, then X is uncountable.*

Theorem 1.1.3. *The Cantor set $\mathfrak{C}$ is uncountable.*

Proof. Define $f : \mathfrak{C} \longrightarrow [0,1]$ such that

$$f\left(\sum_{i=1}^{\infty}\frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty}\frac{b_i}{2^i},$$

where

$$b_i = \begin{cases} a_i & \text{if } a_i = 0; \\ a_i - 1 & \text{if } a_i = 2. \end{cases}$$

It is clear that f is well-define.

We will show that f is onto. For $y = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$, consider

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},$$

where

$$a_i = \begin{cases} b_i & \text{if } b_i = 0; \\ b_i + 1 & \text{if } b_i = 1. \end{cases}$$

Since $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \in \mathfrak{C}$, we have

$$f(x) = \sum_{i=1}^{\infty} \frac{b_i}{2^i} = y.$$

Therefore, f is onto. Since $[0, 1]$ is uncountable, by corollary 1.1.2 we imply that $\mathfrak{C}$ is uncountable. □

According to the Cantor set is uncountable, so it contained both rational and irrational numbers. Nevertheless, we will focus our attention on rational numbers in the Cantor set. Now, there is an observation of characteristic of elements in the Cantor set. Before describing the observation, we introduce some definitions that are helpful for understanding the elements in the Cantor set.

Definition 1.1.4. Suppose that x is real number satisfying $0 \leq x < 1$. Then x will be written in the ternary expansion as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},$$

where $a_i = 0, 1$ or 2, for all i.

Now we can see the characteristic of elements in the Cantor set as follows:

Let S be an interval, and let s_0, s_1, s_2 be subintervals of S which are labelled as $0, 1, 2$, respectively. These were ordered from the left hand side to the right of the interval S. Hence, all $x \in \mathfrak{C}$ will be determined with the following process.

(1) Divide the set $C_0 = [0, 1]$ into three subintervals and then remove the middle subinterval or subintervals s_1. Another way to do, delete the interval which ternary expansion of its elements has $a_1 = 1$. Therefore, $x \in C_1$ will be as

$$0.0a_2a_3\ldots$$

$$0.2a_2a_3\ldots$$

(2) We divide each element in C_1 into three subintervals again, and remove all subintervals s_1. This means that we delete the intervals which ternary expansion of its elements has $a_2 = 1$. Hence, we conclude that $x \in C_2$ will be as

$$0.00a_3a_4\ldots$$

$$0.02a_3a_4\ldots$$

$$0.20a_3a_4\ldots$$

$$0.22a_3a_4\ldots$$

(3) Now to construct C_3, we remove subintervals s_1 from each element in C_2. In conclusion, delete the intervals whose elements contain $a_3 = 1$ in the ternary expansion form. So that $x \in C_3$ can be expressed in the form of

$$0.000a_4a_5\ldots,\ 0.002a_4a_5\ldots$$

$$0.020a_4a_5\ldots,\ 0.022a_4a_5\ldots$$

$$0.200a_4a_5\ldots,\ 0.202a_4a_5\ldots$$

$$0.220a_4a_5\ldots,\ 0.222a_4a_5\ldots$$

(4) In the general case n, we continue with removing subintervals s_1 from each element in C_{n-1}, to construct C_n. In the similar way, we delete the intervals which its element contains $a_i = 1$ in the ternary expansion. Therefore, the form of $x \in C_n$ will be shown as

$$0.a_1a_2\ldots a_{i-1}0a_{i+1}\ldots$$

$$0.a_1a_2\ldots a_{i-1}2a_{i+1}\ldots$$

where $i = 1, 2, \ldots, l-1$ and $a_i \in \{0, 2\}$.

The previous process tells us, if $x \in \mathfrak{C}$ then $x = 0.a_1a_2\ldots a_i\ldots$, where $a_i \in \{0, 2\}$.

There were some researches related to with the Cantor set. Nagy (2001) proved that if a prime number $p > 3$ such that 3 is a primitive root modulo p^2, then there is no fractions $\frac{a}{b} \in \mathfrak{C}$ (where a and b are relatively prime numbers) such that b is a power of p. Nevertheless, for each prime $p > 3$, there are finitely many fractions $\frac{a}{b} \in \mathfrak{C}$ such that b is a power of p.

In unpublished paper, Jordan and Schayer (n.d.) described a characteristic of Cantor rationals in the Cantor set by showing that the period length of the ternary expansion of all elements divides the number of all elements with the same denominator.

Also an unpublished paper (Schayer and Jordan, n.d.) showed the sums of all Cantor rational in the spawning set S_i and all its child sets with denominators $1 \leq i \leq N$, denoted by C_N, that is

$$C_N \geq \frac{1}{2} N^{\frac{\log 2}{\log 3}} \sum_{i=1}^{N} |S_i| \, i^{-\frac{\log 2}{\log 3}},$$

where N is a positive integer.

The generalization of the Cantor set will be called The Cantor p-ary set $\mathfrak{C}_p$ and rational numbers in $\mathfrak{C}_p$ will be called Cantor p-ary rationals. Phon-On (2013) showed that the Cantor p-ary set $\mathfrak{C}_p$ is homeomorphic to $\mathfrak{C}$, and the total number of Cantor p-ary rational in $\mathfrak{C}_p$ with denominator $1 \leq i \leq N$, written by T_N,

$$T_N \geq \frac{1}{p-1} L_N,$$

where

$$L_N = \sum_{i \in S(N)} |A_i^p| \left(p - 3 + \frac{4}{p+1} \left(\frac{N}{i} \right)^{\frac{\log\left(\frac{p+1}{2}\right)}{\log p}} \right)$$

and $S(N) = \{ i \in \mathbb{N} \mid 1 \leq i \leq N, \gcd(i,p) = 1, \text{and } A_i^p \text{ is a spawning } p\text{-ary set} \}$.

The following theorem leads us to the first objective of this thesis.

Theorem 1.1.5. *(Phon-On, 2013) Let $A_n^p = \{a_1, a_2, \ldots, a_k\}$ be a spawning p-ary set, where p does not divide n and $a_i \in \mathfrak{C}_p$. Let $A_{pn}^p = \{b_1, b_2, \ldots, b_r\}$ be a child p-ary set of A_n^p, where $b_i \in \mathfrak{C}_p$. Then, for each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, k\}$ and $l \in K_p^e$ such that $pb_i - l = a_j$. Consequently, $\left|A_{pn}^p\right| \geq |A_n^p|$ and $\left|A_{p^k n}^p\right| = \left|K_p^e\right|^{k-1} \left|A_{pn}^p\right|$ for all $k \geq 2$, and if $p = 3$, then $|A_{3n}^3| \geq |A_n^3|$ and hence $\left|A_{3^k n}^3\right| = 2^{k-1} |A_n^3|$ for all $k \geq 1$.*

We know that $\left|A_{pn}^p\right| \geq |A_n^p|$ and $\left|A_{p^k n}^p\right| = \left|K_p^e\right|^{k-1} \left|A_{pn}^p\right|$, where $K_p^e = \{0, 2, 4, \ldots, p-1\}$. Therefore, the first objective of this thesis is to determine a positive integer k which satisfying $\left|A_{pn}^p\right| = k \cdot |A_n^p|$. Consequently, a relationship of $\left|A_{p^k n}^p\right|$ and $|A_n^p|$ will be presented.

One of the important research mentioned to the Cantor set and group action on the set, Jordan and Schayer (n.d.) give the period length l of elements in spawning set A_q can divided the number of all Cantor rationals in A_q. Moreover, they constructed a transformation R and a transformation T which generated a group action G on each spawning set A_q. The results are given as follows,

(1) $T^l = I, R^2 = I$

(2) T and R commute

(3) G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$

(4) G_T, the subgroup generated by T alone, is a faithful cyclic subgroup of order l.

Follows the work of Jordan and Schayer above, we obtain a motivation that we can consider a group structure of the generalization of the spawning set A_q. Therefore, the second objective of this thesis is to find the group structure on spawning p-ary sets A_n^p.

1.2 Basic definitions and notations

We provide here a list of basic definitions and notations that will be used throughout this thesis.

1.2.1 Functions

Definition 1.2.1. Let A and B be a nonempty sets. A relation f from A into B is called a **function** from A into B if

(i) dom $(f) = A$

(ii) for all $(x, y), (x', y') \in f, x = x'$ implies $y = y'$.

When (ii) is satisfied by a relation f, we say that f is **well defined**.

Definition 1.2.2. Let f be a function from a set A into a set B. Then

- f is called **one-one** $(injective)$ if for all $x, x' \in A, f(x) = f(x')$ implies $x = x'$
- f is called **onto** $(surjective)$ if Im $(f) = B$.

Definition 1.2.3. A function $f : A \to B$ is **bijective** (a bijection) if it is both one-one and onto.

Definition 1.2.4. Let A, B and C be nonempty sets and $f : A \to B$ and $g : B \to C$. The **composition** $\circ$ of f and g, written $g \circ f$, is the relation from A into C defined as follows:

$$g \circ f = \{(x, z) \mid x \in A, z \in C, \text{there exists } y \in B \text{ such that } f(x) = y \text{ and } g(y) = z\}.$$

Definition 1.2.5. Let x be a real number. $\lfloor x \rfloor$ is a **floor function** of x, it is the greatest integer number less than or equal to x.

Definition 1.2.6. Let n be a positive integer. Define the **Euler** $\phi-$**function** $\phi(n)$ to be the number of integer j with $1 \le j \le n$ such that gcd $(j, n) = 1$.

Theorem 1.2.7. *(Rosen,1993) If p is a prime, then $\phi(p) = p - 1$. Conversely, if p is a positive integer with $\phi(p) = p - 1$, then p is prime.*

Theorem 1.2.8. *(Rosen, 1993) Let p be a prime and n be a positive integer. Then $\phi(p^n) = p^n - p^{n-1} = p^{n-1}(p - 1)$.*

Theorem 1.2.9. *(Rosen,1993) Let $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ be the prime-power factorization of the positive integer n. Then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$.*

1.2.2 Countable and Uncountable sets

Definition 1.2.10. Let A and B be sets. We say that A and B have the **same cardinality** if there is a function f from A to B which is both one-one and onto. We write card$(A) =$ card(B) , or $|A| = |B|$.

Theorem 1.2.11. *(Kraft and Washington, 2015) Let A, B be sets. If there is a one-one function $f : A \to B$ and a one-one function $g : B \to A$, then A and B have the same cardinality.*

Definition 1.2.12. If a set A has the same cardinality as $\mathbb{N}$ then we say that A is **countable**.

Definition 1.2.13. A set A will be said **uncountable** if A is not countable.

1.2.3 Divisibility

Definition 1.2.14. If a and b are integers, we say that a **divides** b if there is an integer c such that $b = ac$. If a divides b, we say that a is a divisor or factor of b.

Definition 1.2.15. (The Division Algorithm) Let a and b be integers with $b > 0$. Then there exist unique integers q (the quotient) and r (the remainder) so that

$$a = bq + r$$

with $0 \leq r < b$.

1.2.4 Relation

Definition 1.2.16. A **binary relation** or a **relation** $\sim$ from a set A into a set B is a subset of $A \times B$.

Definition 1.2.17. Let $\sim$ be a binary relation on a set A. Then $\sim$ is called

- **reflexive** for all $x \in A, x \sim x$.
- **symmetric** for all $x, y \in A$, if $x \sim y$, then $y \sim x$.
- **transitive** for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.2.18. A binary relation $\sim$ on a set A is called an **equivalence relation** on A if $\sim$ is reflexive, symmetric, and transitive.

Definition 1.2.19. Let $\sim$ be an equivalence relation on a set A. For all $x \in A$, let $[x]$ denote the set

$$[x] = \{y \in A \mid y \sim x\}.$$

The set $[x]$ is called the **equivalence class** (with respect to $\sim$) determined by x.

Theorem 1.2.20. *(Malik et al., 1997) Let $\sim$ be an equivalence relation on the set A. Then*

(i) for all $x \in A, [x] \neq \phi$,

(ii) if $y \in [x]$, then $[x] = [y]$, where $x, y \in A$,

(iii) for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \phi$,

(iv) $A = \cup_{x \in A}[x]$, i.e., A is the union of all equivalence classes with respect to $\sim$.

Definition 1.2.21. Let A be a set and $\mathcal{P}$ be a collection of nonempty subsets of A. Then $\mathcal{P}$ is called a **partition** of A if the following properties are satisfied:

(i) for all $B, C \in \mathcal{P}$, either $B = C$ or $B \cap C = \phi$.

(ii) $A = \cup_{B \in \mathcal{P}} B$.

Theorem 1.2.22. *(Malik et al., 1997) Let $\sim$ be an equivalence relation on the set A. Then*

$$\mathcal{P} = \{[x] \mid x \in A\}$$

is a partition of A.

1.2.5 Congruences

Definition 1.2.23. Two numbers a and b are **congruent** $\pmod{m}$, written $a \equiv b \pmod{m}$, if $a - b$ is a multiple of m. The integer m is called the **modulus** of the congruence and is assumed to be positive.

Proposition 1.2.24. *(Kraft and Washington, 2015) $a \equiv b \pmod{m}$ if and only if $a = b + km$ for some integers k.*

Proposition 1.2.25. *(Kraft and Washington, 2015) If a is an integer and m is a positive integer, then there is a unique integer r with $0 \leq r \leq m - 1$ so that $a \equiv r \pmod{m}$. This integer r is called the* ***least nonnegative residue*** *of $a \pmod{m}$.*

Proposition 1.2.26. *(Kraft and Washington, 2015) If a,b,c and m are integers with $m > 0$, then*

(1) $a \equiv a \pmod{m}$

(2) If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$

(3) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

Proposition 1.2.27. *(Kraft and Washington, 2015) Assume that a,b,c,d and m are integers with m positive. If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then*

(1) $a + c \equiv b + d \pmod{m}$.

(2) $a - c \equiv b - d \pmod{m}$.

(3) $ac \equiv bd \pmod{m}$.

Corollary 1.2.28. *(Kraft and Washington, 2015) If* $a \equiv b \pmod{m}$, *then* $a^n \equiv b^n \pmod{m}$ *for any positive integers* n.

Theorem 1.2.29. *(Euler's Theorem) Let* n *be a positive integer and let* b *be an integer with gcd* $(b, n) = 1$. *Then*

$$b^{\phi(n)} \equiv 1 \pmod{n}.$$

1.2.6 Group

Definition 1.2.30. A **group** is an ordered pair $(G, *)$, where G is a nonempty set and $*$ is a binary operation on G such that the following properties hold:

(i) For all $a, b, c \in G$ satisfy $a * (b * c) = (a * b) * c$.

(ii) For all $a \in G$, there exists $e \in G$ such that $a * e = a = e * a$.

(iii) For all $a \in G$, there exists $b \in G$ such that $a * b = e = b * a$.

Definition 1.2.31. Let $(G, *)$ and $(G_1, *_1)$ be groups and f is a function from G into G_1. Then f is called a **homomorphism** of G into G_1 if for all $a, b \in G$, $f(a * b) = f(a) *_1 f(b)$.

Definition 1.2.32. A homomorphism f of a group G into a group G_1 is called an **isomorphism** of G onto G_1, if f is one-one and onto G_1. We write $G \cong G_1$ and say that G and G_1 are isomorphic.

Definition 1.2.33. Let $(G, *)$ be a group and H be a nonempty subset of G. Then $(H, *)$ is called a **subgroup** of $(G, *)$ if $(H, *)$ is a group.

Definition 1.2.34. A group $(G, *)$ is called a **finite group** if G has only a finite number of elements. The **order**, written $|G|$, of a group $(G, *)$ is the number of elements of G.

Definition 1.2.35. Let $(G, *)$ be a group and $a \in G$. If there exists a positive integer n such that $a^n = e$, then the smallest such that positive integer is called the **order** of a. If no such positive integer n exists, then we say that a is of **infinite order.** We denote the order of an element a of a group $(G, *)$ by $\circ(a)$.

Definition 1.2.36. A group G is called a **cyclic group** if there exists $a \in G$ such that

$$G = \langle a \rangle,$$

where $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$.

Theorem 1.2.37. *(Malik et al., 1997) Let $\langle a \rangle$ be a finite cyclic group of order n. Then $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$.*

Corollary 1.2.38. *(Malik et al., 1997) Let $\langle a \rangle$ be a finite cyclic group. Then $\circ(a) = |\langle a \rangle|$.*

Corollary 1.2.39. *(Malik et al., 1997) A finite group G is a cyclic group if and only if there exists an element $a \in G$ such that $\circ(a) = |G|$.*

Definition 1.2.40. Let G be a group and S a nonempty set. **A (left) action** of G on S is a function $\cdot : G \times S \to S$ such that

(i) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

(ii) $e \cdot x = x$, where e is the identity of G

for all $x \in S$ and $g_1, g_2 \in G$.

Definition 1.2.41. Let X be a nonempty set. A **permutation** π **of** X is an one-one function from X to X.

Definition 1.2.42. Let S be a nonempty set, an action of G on S is **faithful**, if any two distinct elements $g, h \in G$ give distinct permutations of S.

Chapter 2

The Cantor $p-$ary Sets

We begin this chapter with the construction of the Cantor p-ary set $\mathfrak{C}_p$. Its construction is similar to the construction of the Cantor set $\mathfrak{C}$.

2.1 Construction of the Cantor $p-$ary sets

The construction of the Cantor p-ary set $\mathfrak{C}_p$ will be described as follows: given p an odd prime.

(1) Denote $C_0 = \{[0,1]\}$.

(2) Divide the closed interval in C_0 into p equal subintervals, and denote open subintervals $s_1, s_3, \ldots, s_{(p-4)}, s_{(p-2)}$ of C_0 be the odd parts.

(3) Remove the odd parts

$$\left(\frac{1}{p}, \frac{2}{p}\right), \left(\frac{3}{p}, \frac{4}{p}\right), \ldots, \left(\frac{p-4}{p}, \frac{p-3}{p}\right), \left(\frac{p-2}{p}, \frac{p-1}{p}\right).$$

Hence, set C_1 as

$$C_1 = \left\{\left[0, \frac{1}{p}\right], \left[\frac{2}{p}, \frac{3}{p}\right], \left[\frac{4}{p}, \frac{5}{p}\right], \ldots, \left[\frac{p-3}{p}, \frac{p-2}{p}\right], \left[\frac{p-1}{p}, 1\right]\right\}.$$

It follows that C_1 consists of $\frac{p+1}{2}$ closed subintervals.

(4) Subdivide again all the remain intervals in C_1 into p equal subintervals, and remove the odd parts of C_1. Thus C_2 will be a set that contains the following closed subintervals,

$$C_2 = \{ \left[0, \frac{1}{p^2}\right], \left[\frac{2}{p^2}, \frac{3}{p^2}\right], \ldots, \left[\frac{p-3}{p^2}, \frac{p-2}{p^2}\right], \left[\frac{p-1}{p^2}, \frac{p}{p^2}\right],$$

$$\left[\frac{2p}{p^2}, \frac{2p+1}{p^2}\right], \left[\frac{2p+2}{p^2}, \frac{2p+3}{p^2}\right], \ldots, \left[\frac{3p-3}{p^2}, \frac{3p-2}{p^2}\right], \left[\frac{3p-1}{p^2}, \frac{3p}{p^2}\right], \ldots,$$

$$\left[\frac{(p-3)p}{p^2}, \frac{(p-3)p+1}{p^2}\right], \left[\frac{(p-3)p+2}{p^2}, \frac{(p-3)p+3}{p^2}\right], \ldots, \left[\frac{(p-2)p-1}{p^2}, \frac{(p-2)p}{p^2}\right],$$

$$\left[\frac{(p-1)p}{p^2}, \frac{(p-1)p+1}{p^2}\right], \left[\frac{(p-1)p+2}{p^2}, \frac{(p-1)p+3}{p^2}\right], \ldots, \left[\frac{p^2-1}{p^2}, 1\right] \}.$$

Consequently, we have that C_2 consists of $\left(\frac{p+1}{2}\right)^2$ elements.

(5) To find the set C_3, we can subdivide each element in C_2 into p equal subintervals and then remove the odd parts.

(6) Repeating this process, for each n, subdivide each elements in C_{n-1} into p equal subintervals and remove the odd parts, the remain intervals will be the elements in C_n. Hence, C_n contains $\left(\frac{p+1}{2}\right)^n$ elements.

(7) Finally, for each set C_n, the Cantor p-ary set $\mathfrak{C}_p$ will be defined as

$$\mathfrak{C}_p = \cap_{n=0}^{\infty} (\cup C_n),$$

is the intersection of all $\cup C_n$, where $\cup C_n$ is the union of all elements in C_n.

Phon-On (2013) showed that the Cantor $p-$ary set is homeomorphic to Cantor set, hence the Cantor p-ary set also satisfies the following properties:

(1) nonempty set

(2) closed

(3) perfect

(4) compact

(5) nowhere dense

(6) totally disconnected

(7) uncountable.

Considering the constructions of the Cantor p-ary set $\mathfrak{C}_p$. The initial interval of these construction is $[0, 1]$. So then a real number γ which is in $\mathfrak{C}_p$ satisfies $0 \leq \gamma \leq 1$, and can be written as base p-expansion according to the following theorem.

Theorem 2.1.1. *(Rosen, 1993) Let γ be a real number with $0 \leq \gamma \leq 1$, and let p be an odd prime. Then γ can be uniquely written as*

$$\gamma = \sum_{i=1}^{\infty} \frac{a_i}{p^i},$$

where the coefficients a_i are integers with $0 \leq a_i \leq p-1$ for $i = 1, 2, 3, \ldots$ with the restriction that for every positive integer N there is n with $n \geq N$ and $a_n \neq p-1$.

In the chapter 1, we have shown that all elements in the Cantor set can be written as the ternary expansion $\sum_{i=1}^{\infty} \frac{a_i}{3^i}$ where $a_i \in \{0, 2\}$ for all $i \in \mathbb{N}$. While, in this chapter we will show that each element in Cantor p-ary set $\mathfrak{C}_p$ can be written as the base p expansion $\sum_{i=1}^{\infty} \frac{a_i}{p^i}$ where $a_i \in \{0, 2, \ldots, p-1\}$ for all $i \in \mathbb{N}$. The observation will be described in the following process.

Begin our process by denote S be an interval, and divide S to be $s_0, s_1, \ldots, s_{p-1}$ subintervals of S which are labelled as $0, 1, 2, \ldots, p-1$, respectively. These were ordered from the left hand side to the right of the interval S. We can see the elements in Cantor p-ary sets $\mathfrak{C}_p$ as follows:

(1) The initial set $C_0 = [0, 1]$ will be divided into p equal subintervals. Then subintervals $s_1, s_3, \dots, s_{p-2}$ will be removed. This highlights delete the intervals which its element have $a_1 = 1, 3, 5, \dots,$ or $p - 2$ in the base p expansion. Therefore, $x \in C_1$ will be as

$$0.0a_2a_3\dots$$

$$0.2a_2a_3\dots$$

$$0.4a_2a_3\dots$$

$$\vdots$$

$$0.(p-1)\,a_2a_3\dots$$

(2) Now, we divide each element in C_1 into p equal subintervals and remove the subintervals $s_1, s_3, s_5, \dots, s_{p-2}$ to create C_2. This means that we delete the intervals, which its elements contain $a_2 = 1, 3, 5, \dots$ or $p - 2$ in the base p expansion. Hence, this implies that $x \in C_2$ will be as

$$0.00a_3a_4\dots,\ 0.02a_3a_4\dots,\ 0.04a_3a_4\dots,\ \dots, 0.0\,(p-1)\,a_3a_4\dots$$

$$0.20a_3a_4\dots,\ 0.22a_3a_4\dots,\ 0.24a_3a_4\dots,\ \dots, 0.2\,(p-1)\,a_3a_4\dots$$

$$\vdots$$

$$0.(p-1)\,0a_3a_4\dots,\ 0.(p-1)\,2a_3a_4,\ \dots, 0.(p-1)\,(p-1)\,a_3a_4\dots$$

(3) Next, to construct C_3, we remove the subintervals $s_1, s_3, s_5, \dots, s_{p-2}$ from each element in C_2. These considerations imply that delete the intervals whose elements contain $a_3 = 1, 3, 5, \dots,$ or $p - 2$ in the base p expansion form. So that

$x \in C_3$ can be shown in the form as

$$0.a_1a_2a_3a_4\ldots$$

where $a_1, a_2, a_3 \in \{0, 2, \ldots, p-1\}$.

(4) In the general case n, we continue with removing subintervals $s_1, s_3, s_5, \ldots, s_{p-2}$ from each element in C_{n-1}, to construct C_n. Otherwise, we delete the intervals which its element contain $a_i = 1, 3, 5, \ldots,$ or $p-2$ in the base p expansion. Therefore, the form of $x \in C_n$ will be shown as

$$0.a_1a_2a_3\ldots a_i\ldots$$

where $\forall i, a_i \in \{0, 2, 4, \ldots, p-1\}$.

From above description, it is easy to see that $x \in \mathfrak{C}_p$ will be as

$$0.a_1a_2a_3\ldots a_i\ldots$$

where $a_i \in \{0, 2, \ldots, p-1\}$. We state the theorem as follows:

Theorem 2.1.2. *For each $x \in \mathfrak{C}_p$, then x can be written uniquely in the base p expansions of the form*

$$x = \sum_{i=1}^{\infty} \frac{a_i}{p^i} \text{ or } x = 0.a_1a_2a_3\ldots a_i\ldots$$

where $a_i \in \{0, 2, 4, \ldots, p-1\}$.

Example 2.1.3. $0.024024\ldots$ is a rational number in $\mathfrak{C}_5$.

Example 2.1.4. $0.044226024\ldots$ is an irrational number in $\mathfrak{C}_7$.

Chapter 3

Rational Points in Cantor $p-$ary Sets

One of the properties of the Cantor p-ary set is uncountable. It shows that the set consists of rational and irrational numbers. We begin this chapter by recalling the definitions and examples related to the rational and irrational numbers.

Definition 3.0.1. A real number $\gamma = \frac{m}{n}$ will be called **rational number**, if m, n be integers with $n \neq 0$ and the greatest common divisor (gcd) of m and n equal to 1.

Example 3.0.2. $2, 6, \frac{1}{3}, \frac{12}{25}$ and $\frac{31}{175}$ all are rational numbers, since the greatest common divisor of numerators and denominators equal to 1. Also, a decimal expansion $3.2020\ldots$ can be written in the rational form as $\frac{317}{99}$.

In real system, a number which is not rational number will be called **irrational number.** The following example shows us some irrational numbers.

Example 3.0.3. We cannot write $\pi = 3.1415926\ldots$, $e = 2.7182818\ldots$ and $0.3214752\ldots$ in form $\frac{m}{n}$ with $n \neq 0$. Then these are irrational numbers.

At this point, we will focus our attention on rational numbers in Cantor p-ary set, since it has obvious pattern and simple understanding.

Before we find out the rational numbers in the Cantor p-ary set, we introduce the important formula that will convert a rational number $\frac{m}{n}$ to the base p expansions. It was introduced by Rosen in (Rosen, 1993).The formula is given as follows:

$$a_i = \lfloor p \cdot \gamma_{i-1} \rfloor, \qquad \gamma_i = p \cdot \gamma_{i-1} - \lfloor p \cdot \gamma_{i-1} \rfloor \tag{3.0.1}$$

where $\gamma_0 = \frac{m}{n}$, and $i = 1, 2, 3, \ldots$

Therefore, the base p expansion can be expressed in the form

$$\gamma = \frac{m}{n} = \sum_{i=1}^{\infty} \frac{a_i}{p^i} = \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \ldots$$

Next, Examples 3.0.4 and 3.0.5 will illustrate the transformation $\frac{11}{171}$ to the base 7 expansion and $\frac{23}{122}$ to the base 11 expansion.

Example 3.0.4. Applying the formula, convert $\frac{11}{171}$ to the base 7 expansion.

$$\begin{aligned}
a_1 &= \left\lfloor 7 \cdot \frac{11}{171} \right\rfloor = 0, & \gamma_1 &= 7 \cdot \frac{11}{171} - 0 = \frac{77}{171}, \\
a_2 &= \left\lfloor 7 \cdot \frac{77}{171} \right\rfloor = 3, & \gamma_2 &= 7 \cdot \frac{77}{171} - 3 = \frac{26}{171}, \\
a_3 &= \left\lfloor 7 \cdot \frac{26}{171} \right\rfloor = 1, & \gamma_3 &= 7 \cdot \frac{26}{171} - 1 = \frac{11}{171}, \\
a_4 &= \left\lfloor 7 \cdot \frac{11}{171} \right\rfloor = 0, & \gamma_4 &= 7 \cdot \frac{11}{171} - 0 = \frac{77}{171}, \\
a_5 &= \left\lfloor 7 \cdot \frac{77}{171} \right\rfloor = 3, & \gamma_5 &= 7 \cdot \frac{77}{171} - 3 = \frac{26}{171}, \\
a_6 &= \left\lfloor 7 \cdot \frac{26}{171} \right\rfloor = 1, & \gamma_6 &= 7 \cdot \frac{26}{171} - 1 = \frac{11}{171},
\end{aligned}$$

and so on. Therefore,

$$\frac{11}{171} = \frac{0}{7} + \frac{3}{7^2} + \frac{1}{7^3} + \frac{0}{7^4} + \frac{3}{7^5} + \frac{1}{7^6} + \ldots = \left(0.\overline{031}\right)_7.$$

Example 3.0.5. Using the formula to convert $\frac{23}{122}$ to the base 11 expansion.

$$\begin{aligned}
a_1 &= \left\lfloor 11 \cdot \frac{23}{123} \right\rfloor = 2, & \gamma_1 &= 11 \cdot \frac{23}{122} - 2 = \frac{9}{122}, \\
a_2 &= \left\lfloor 11 \cdot \frac{9}{122} \right\rfloor = 0, & \gamma_2 &= 11 \cdot \frac{9}{122} - 0 = \frac{99}{122}, \\
a_3 &= \left\lfloor 11 \cdot \frac{99}{122} \right\rfloor = 8, & \gamma_3 &= 11 \cdot \frac{99}{122} - 8 = \frac{113}{122},
\end{aligned}$$

$$a_4 = \left\lfloor 11 \cdot \frac{113}{122} \right\rfloor = 10, \qquad \gamma_4 = 11 \cdot \frac{113}{122} - 10 = \frac{23}{122},$$
$$a_5 = \left\lfloor 11 \cdot \frac{23}{122} \right\rfloor = 2, \qquad \gamma_5 = 11 \cdot \frac{23}{122} - 2 = \frac{9}{122},$$
$$a_6 = \left\lfloor 11 \cdot \frac{9}{122} \right\rfloor = 0, \qquad \gamma_6 = 11 \cdot \frac{9}{122} - 0 = \frac{99}{122},$$

In the base 11, denote $10 = A$ we have

$$\frac{23}{122} = \frac{2}{11} + \frac{0}{11^2} + \frac{8}{11^3} + \frac{A}{11^4} + \frac{2}{11^5} + \frac{0}{11^6} + \ldots = \left(0.\overline{208A}\right)_{11}.$$

We now give some definitions and examples that useful to understanding this thesis.

Definition 3.0.6. (Phon-On, 2013) A rational number $\frac{m}{n} \in \mathbb{Q}$ is called a **Cantor p-ary rational** if it satisfies the following conditions:

(1) $\frac{m}{n}$ is in the Cantor p-ary set.

(2) m and n are relatively prime, i.e. gcd $(m, n) = 1$.

Example 3.0.7. $\frac{25}{62}$ is a Cantor 5-ary rational, since $\frac{25}{62} \in \mathfrak{C}_5$ and $(25, 62) = 1$.

For convenience, we denote K_p^e the set of zero and even numbers less than p. For example, where $p = 13$, then $K_{13}^e = \{0, 2, 4, 6, 8, 10, 12\}$.

In paper (Phon-On, 2013), the author categorized Cantor p-ary rationals to three types. If $\frac{m}{n} \in \mathfrak{C}_p$, denote

$$\frac{m}{n} = (0.b_1 b_2 \ldots b_k \overline{a_1 a_2 \ldots a_l})_p\,,$$

where $b_k, a_l \in K_p^e, k \in \{0\} \cup \mathbb{N}$ and $n \in \mathbb{N}$. We call $(b_1 b_2 \ldots b_k)_p$ a pre-period part and $(a_1 a_2 \ldots a_l)_p$ a period part of $\frac{m}{n}$. The values k, l are defined as a pre-period length and l a period length, respectively. Moreover, $\frac{m}{n}$ will be called

(1) **Terminating** if no period part. Then $\frac{m}{n} = (0.b_1b_2\ldots b_k)_p$.

(2) **Purely periodic** if no pre-period part. Then $\frac{m}{n} = (0.\overline{a_1a_2\ldots a_l})_p$.

(3) **Mix periodic** if there are both pre-period and period parts.

Then $\frac{m}{n} = (0.b_1b_2\ldots b_k\overline{a_1a_2\ldots a_l})_p$.

Example 3.0.8. Consider on the base 7 expansion, we obtain $\frac{116}{343} = 0.224$, $\frac{29}{50} = 0.\overline{4026}$ and $\frac{19}{392} = 0.02\overline{24}$. Consequently, we say that $\frac{14}{25}$ is terminate, $\frac{29}{50}$ is purely periodic and $\frac{19}{392}$ is mix periodic.

Example 3.0.9. The base 5 expansion of $\frac{23}{130} = 0.042\overline{0242}$. Thus $(042)_5$ and $(0242)_5$ represent the pre-period and period part of $\frac{23}{130}$, respectively. Furthermore, the pre-period length is 3 and the period length is 4.

Note that all Cantor $p-$ary rationals can be written in the reduce form $\frac{m}{n}$ which $\gcd(m,n) = 1$. We collect the elements whose denominator is n in a set namely spawning $p-$ary set and denoted by A_n^p. Next, we define and illustrate the spawning $p-$ary set with the following definition and example.

Definition 3.0.10. (Phon-On, 2013) **The spawning p-ary set** (A_n^p) is a set of Cantor p-ary rational which satisfies two conditions:

(1) $A_n^p \neq \phi$.

(2) p does not divide n, where $n \in \mathbb{N}$.

Example 3.0.11. $A_{60}^{11} = \left\{\frac{1}{60}, \frac{11}{60}, \frac{13}{60}, \frac{23}{60}, \frac{37}{60}, \frac{47}{60}, \frac{49}{60}, \frac{59}{60}\right\}$ is a spawning 11-ary set.

Phon-On (2013) gave a condition on n so that A_n^p is a spawning p-ary set stated as follows:

Theorem 3.0.12. *Let p be an odd prime and $n \in \mathbb{N}$. Then, A_n^p will be a spawning $p-$ary set if and only if there exist $k, l \in \mathbb{N}$ and $a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_k \in K_p^e$ such that*

$$n = \frac{p^k\left(p^{k+l}-1\right)}{d}$$

where $d = gcd\left(\left(p^{k+l}-1\right)\sum_{i=1}^{k} b_i p^{k-i} + \sum_{i=1}^{l} a_i p^{k+l-i}, p^k\left(p^{k+l}-1\right)\right)$.

Theorem 3.0.13. $A^p_{p^n+1}$ *and* $A^p_{\frac{p^n-1}{2}}$ *are spawning p-ary sets.*

Definition 3.0.14. Let A_n^p be the spawning p-ary set, the sets $A^p_{p^k n}$ are **child p-ary sets** of A_n^p, where $k = 1, 2, 3, \ldots$.

The following tables show us some spawning p-ary sets and child p-ary sets.

Table 3.1: Some spawning p-ary sets

p	5		7		11		13	
n	$A^5_{5^n+1}$	$A^5_{\frac{5^n-1}{2}}$	$A^7_{7^n+1}$	$A^7_{\frac{7^n-1}{2}}$	$A^{11}_{11^n+1}$	$A^{11}_{\frac{11^n-1}{2}}$	$A^{13}_{13^n+1}$	$A^{13}_{\frac{13^n-1}{2}}$
1	A^5_6	A^5_2	A^7_8	A^7_3	A^{11}_{12}	A^{11}_5	A^{13}_{14}	A^{13}_6
2	A^5_{26}	A^5_{12}	A^7_{50}	A^7_{24}	A^{11}_{122}	A^{11}_{60}	A^{13}_{170}	A^{13}_{84}
3	A^5_{126}	A^5_{62}	A^7_{344}	A^7_{171}	A^{11}_{1332}	A^{11}_{665}	A^{13}_{2198}	A^{13}_{1098}
4	A^5_{626}	A^5_{312}	A^7_{2402}	A^7_{1200}	A^{11}_{14642}	A^{11}_{7320}	A^{13}_{28562}	A^{13}_{14280}

Table 3.2: Some child p-ary sets, where $p = 5, 7, 11$ and 13

spawning 5-ary sets	child 5-ary sets	spawning 7-ary sets	child 7-ary sets
A_n^5	$A_{5^k n}^5$	A_n^7	$A_{7^k n}^7$
A_2^5	$A_{10}^5, A_{50}^5, A_{250}^5, A_{1250}^5, \ldots$	A_3^7	$A_{21}^7, A_{147}^7, A_{1029}^7, \ldots$
A_6^5	$A_{30}^5, A_{150}^5, A_{750}^5, A_{3750}^5, \ldots$	A_8^7	$A_{56}^7, A_{392}^7, A_{2744}^7, \ldots$
A_{12}^5	$A_{60}^5, A_{300}^5, A_{1500}^5, \ldots$	A_{24}^7	$A_{168}^7, A_{1176}^7, A_{8232}^7, \ldots$
A_{26}^5	$A_{130}^5, A_{650}^5, A_{3250}^5, \ldots$	A_{50}^7	$A_{350}^7, A_{2450}^7, A_{17150}^7, \ldots$
A_{62}^5	$A_{310}^5, A_{1550}^5, A_{7750}^5, \ldots$	A_{171}^7	$A_{1197}^7, A_{8379}^7, A_{58653}^7, \ldots$
spawning 11-ary sets	child 11-ary sets	spawning 13-ary sets	child 13-ary sets
A_n^{11}	$A_{11^k n}^{11}$	A_n^{13}	$A_{13^k n}^{13}$
A_5^{11}	$A_{55}^{11}, A_{605}^{11}, A_{6655}^{11}, \ldots$	A_6^3	$A_{78}^{13}, A_{1014}^{13}, A_{13182}^{13}, \ldots$
A_{12}^{11}	$A_{132}^{11}, A_{1452}^{11}, A_{15972}^{11}, \ldots$	A_{14}^{13}	$A_{182}^{13}, A_{2366}^{13}, A_{30758}^{13}, \ldots$
A_{60}^{11}	$A_{660}^{11}, A_{7260}^{11}, A_{79860}^{11}, \ldots$	A_{84}^{13}	$A_{1092}^{13}, A_{14196}^{13}, A_{184548}^{13}, \ldots$
A_{122}^{11}	$A_{1342}^{11}, A_{14762}^{11}, A_{162382}^{11}, \ldots$	A_{170}^{13}	$A_{2210}^{13}, A_{28730}^{13}, A_{373490}^{13}, \ldots$

There are some useful theorems that concern with types of rational numbers in spawning p-ary sets and child p-ary sets.

Theorem 3.0.15. *(Rosen, 1993) The real number $\gamma, 0 \leq \gamma < 1$, has a terminating base b expansion if and only if γ is rational and $\gamma = \frac{m}{n}$, where $0 \leq m < n$ and every prime factor of n also divides b.*

Theorem 3.0.16. *(Kraft and Washington, 2015) A decimal expansion of a real number γ is eventually $($purely or mix$)$ periodic if and only if γ is rational.*

Remark 3.0.17. Given n be a positive integer and gcd $(a, n) = 1$. Define $ord_n a$ the order of $a \pmod{n}$, it means the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$.

To determine the $\text{ord}_n a$, we need the following corollary.

Corollary 3.0.18. *(Kraft and Washington, 2015)*

(1) Let p be prime and let a be an integer with $a \not\equiv 0 \pmod{p}$. Then $ord_p a \mid (p-1)$.

(2) Let n be a positive integer and let a be an integer with gcd $(a, n) = 1$. Then $ord_n a \mid \phi(n)$.

Example 3.0.19. Calculate the $\text{ord}_{29} 5$, we know that $\text{ord}_p a \mid (p-1)$, then $\text{ord}_{29} 5 \mid 28$. The possible answers are $1, 2, 7, 14$ or 28. By computation, the smallest number is 14 such that $5^{14} \equiv 1 \pmod{29}$. Therefore, $\text{ord}_{29} 5 = 14$.

Theorem 3.0.20. *(Rosen, 1993) Denote $0 < \gamma < 1, \gamma = \frac{m}{n}$, where m and n are relatively prime positive integers, and $n = TU$ where every prime factor of T divides b and $(U, b) = 1$, then the period length of the base b expansion of γ is $ord_U b$, and the pre-period length is N, where N is the smallest positive such that $T \mid b^N$.*

By Theorem 3.0.20, we have the following corollaries.

Corollary 3.0.21. *Let A_n^p be a spawning p-ary set. If $x \in A_n^p$, then x is a purely periodic.*

Corollary 3.0.22. *Let $A_{p^k n}^p$, where $k \geq 1$, be a child p-ary set. If $x \in A_{p^k n}^p$, then x is a mix periodic.*

Example 3.0.23. Determine the pre-period and the period length of rational number $\frac{1}{2450}$ in the base 7 expansion. Since $2450 = (2 \cdot 5^2) \cdot 7^2$, we have $T = 7^2 = 49$ and

$U = 2 \cdot 5^2 = 50$ and gcd $(50, 7) = 1$. Then $49 \mid 7^2$ and $\text{ord}_{50} 7 = 4$, the pre-period length is 2 and period length is 4. These are corresponding with $\frac{1}{2450} = \left(0.00\overline{0066}\right)_7$.

The following example illustrates the way of checking rational numbers in spawning p-ary sets.

Example 3.0.24. Let A_{62}^5 be a spawning 5-ary set. Then we convert $\frac{7}{62}$ and $\frac{3}{62}$ to the base 5 expansion. We have

$$\frac{7}{62} = \frac{0}{5} + \frac{2}{5^2} + \frac{4}{5^3} + \frac{0}{5^4} + \frac{2}{5^5} + \frac{4}{5^6} + \ldots = \left(0.\overline{024}\right)_5$$

and

$$\frac{3}{62} = \frac{0}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{0}{5^4} + \frac{1}{5^5} + \frac{1}{5^6} + \ldots = \left(0.\overline{011}\right)_5 .$$

We see that for all a_i in base 5 expansion of $\frac{7}{62}$ are elements in $\{0, 2, 4\}$, then we have $\frac{7}{62} \in A_{62}^5$. In contrast, there exist a_i in the base 5 expansion of $\frac{3}{62}$ are not elements in $\{0, 2, 4\}$, then we conclude that $\frac{3}{62} \notin A_{62}^5$.

Example 3.0.25. Let $n = 62, p = 5$. Consider $\frac{m}{62}$ where $1 \leq m < 62$ and $\gcd(m, 62) = 1$. Using the formula to convert $\frac{m}{62}$ to the base 5 expansion, we have the following table.

Table 3.3: Rational numbers with denominator 62 on base 5

$\frac{1}{62} = 0.\overline{002}$	$\frac{61}{62} = 0.\overline{442}$	$\frac{3}{62} = 0.\overline{011}$	$\frac{21}{62} = 0.\overline{132}$	$\frac{43}{62} = 0.\overline{321}$
$\frac{5}{62} = 0.\overline{020}$	$\frac{57}{62} = 0.\overline{424}$	$\frac{9}{62} = 0.\overline{033}$	$\frac{23}{62} = 0.\overline{141}$	$\frac{45}{62} = 0.\overline{330}$
$\frac{25}{62} = 0.\overline{200}$	$\frac{37}{62} = 0.\overline{244}$	$\frac{13}{62} = 0.\overline{101}$	$\frac{29}{62} = 0.\overline{213}$	$\frac{47}{62} = 0.\overline{334}$
$\frac{7}{62} = 0.\overline{024}$	$\frac{55}{62} = 0.\overline{420}$	$\frac{15}{62} = 0.\overline{110}$	$\frac{33}{62} = 0.\overline{231}$	$\frac{49}{62} = 0.\overline{343}$
$\frac{35}{62} = 0.\overline{240}$	$\frac{27}{62} = 0.\overline{204}$	$\frac{17}{62} = 0.\overline{114}$	$\frac{39}{62} = 0.\overline{303}$	$\frac{53}{62} = 0.\overline{411}$
$\frac{51}{62} = 0.\overline{402}$	$\frac{11}{62} = 0.\overline{042}$	$\frac{19}{62} = 0.\overline{123}$	$\frac{41}{62} = 0.\overline{312}$	$\frac{59}{62} = 0.\overline{433}$

According to Table 3.3, rational numbers which are in the first two column, each a_i is $0, 2$ or 4 in base 5 expansion. This means that those are Cantor p-ary rationals in A^5_{62}. Whereas, all remain rational numbers in the next three columns are not in A^5_{62} since they are not in $\mathfrak{C}_5$.

To obtain the Cantor p-ary rationals in child p-ary sets, we used the same method that we check elements in spawning p-ary sets. The following table illustrates some of Cantor p-ary rationals in child p-ary sets.

Table 3.4: Rational numbers with denominator 56 on base 7

$\frac{1}{56} = 0.0\overline{60}$	$\frac{55}{56} = 0.6\overline{60}$	$\frac{9}{56} = 0.1\overline{06}$	$\frac{29}{56} = 0.3\overline{42}$
$\frac{3}{56} = 0.0\overline{24}$	$\frac{53}{56} = 0.6\overline{42}$	$\frac{11}{56} = 0.1\overline{24}$	$\frac{31}{56} = 0.3\overline{60}$
$\frac{5}{56} = 0.0\overline{42}$	$\frac{51}{56} = 0.6\overline{24}$	$\frac{13}{56} = 0.1\overline{42}$	$\frac{41}{56} = 0.5\overline{06}$
$\frac{17}{56} = 0.2\overline{06}$	$\frac{39}{56} = 0.4\overline{60}$	$\frac{15}{56} = 0.1\overline{60}$	$\frac{43}{56} = 0.5\overline{24}$
$\frac{19}{56} = 0.2\overline{24}$	$\frac{37}{56} = 0.4\overline{42}$	$\frac{25}{56} = 0.3\overline{06}$	$\frac{45}{56} = 0.5\overline{42}$
$\frac{23}{56} = 0.2\overline{60}$	$\frac{33}{56} = 0.4\overline{06}$	$\frac{27}{56} = 0.3\overline{24}$	$\frac{47}{56} = 0.5\overline{60}$

Tables 3.3 and 3.4 give important observation, Cantor p-ary rationals in spawning p-ary sets will be purely periodic and mix periodic in child p-ary sets.

Chapter 4

Cardinality of Child $p-$ary Sets

This chapter presents an interesting theorem concerning with cardinality of child p-ary sets. The theorem gives a relation of Cantor p-ary rationals in spawning p-ary sets and child p-ary sets. It is useful for determining the number of elements in child p-ary sets.

In the first section, we show the cardinality of some spawning p-ary sets and child p-ary sets. Then, we prove the main theorem. The second section, we apply the main theorem by giving the several numbers of Cantor p-ary rationals in the sets.

Our aim of the first section is to determine a relationship of cardinality of Cantor p-ary set and their child p-ary sets.

4.1 Spawning $p-$ary sets and child $p-$ary sets

In order to find out a cardinality of a spawning p-ary sets and child p-ary sets, we firstly consider some example of those sets.

Table 4.1: Cardinality of some spawning p-ary sets

spawning 5-ary set		spawning 7-ary set		spawning 11-ary set		spawning 13-ary set	
A_n^5	$\lvert A_n^5\rvert$	A_n^7	$\lvert A_n^7\rvert$	A_n^{11}	$\lvert A_n^{11}\rvert$	A_n^{13}	$\lvert A_n^{13}\rvert$
A_2^5	1	A_3^7	2	A_5^{11}	4	A_6^{13}	2
A_6^5	2	A_8^7	4	A_{12}^{11}	4	A_{14}^{13}	6
A_{12}^5	4	A_{24}^7	4	A_{60}^{11}	8	A_{84}^{13}	16
A_{26}^5	8	A_{50}^7	12	A_{122}^{11}	36	A_{170}^{13}	40
A_{62}^5	12	A_{171}^7	60	A_{665}^{11}	146		

We now particularly focus on elements in child p-ary sets $A^p_{p^k n}$ where $k = 1$. Table 4.2 presents the number of all Cantor p-ary rationals in child p-ary sets A^p_{pn} where $p = 5, 7, 11$ and 13.

Table 4.2: Cardinality of some child p-ary sets

spawning 5-ary sets	child 5-ary sets		spawning 7-ary sets	child 7-ary sets	
A^5_n	A^5_{5n}	$\lvert A^5_{5n}\rvert$	A^7_n	A^7_{7n}	$\lvert A^7_{7n}\rvert$
A^5_2	$A^5_{5\cdot 2} = A^5_{10}$	2	A^7_3	$A^7_{7\cdot 3} = A^7_{21}$	6
A^5_6	$A^5_{5\cdot 6} = A^5_{30}$	4	A^7_8	$A^7_{7\cdot 8} = A^7_{56}$	12
A^5_{12}	$A^5_{5\cdot 12} = A^5_{60}$	8	A^7_{24}	$A^7_{7\cdot 24} = A^7_{168}$	12
A^5_{26}	$A^5_{5\cdot 26} = A^5_{130}$	16	A^7_{50}	$A^7_{7\cdot 50} = A^7_{350}$	36
spawning 11-ary sets	child 11-ary sets		spawning 13-ary sets	child 13-ary sets	
A^{11}_n	A^{11}_{11n}	$\lvert A^{11}_{11n}\rvert$	A^{13}_n	A^{13}_{13n}	$\lvert A^{13}_{13n}\rvert$
A^{11}_5	$A^{11}_{11\cdot 5} = A^{11}_{55}$	20	A^{13}_6	$A^{13}_{13\cdot 6} = A^{13}_{78}$	12
A^{11}_{12}	$A^{11}_{11\cdot 12} = A^{11}_{132}$	20	A^{13}_{14}	$A^{13}_{13\cdot 14} = A^{13}_{182}$	36
A^{11}_{60}	$A^{11}_{11\cdot 60} = A^{11}_{660}$	40			

We now compare the cardinalities of spawning p-ary sets and their child p-ary sets, where $k = 1$. Table 4.3 will show the cardinalities of some spawning p-ary sets and their child p-ary sets.

Table 4.3: Cardinality of some spawning p-ary sets and child p-ary sets

$p = 5$		$p = 7$	
$\|A_n^5\|$	$\|A_{5n}^5\|$	$\|A_n^7\|$	$\|A_{7n}^7\|$
$\|A_2^5\| = 1$	$\|A_{10}^5\| = 2$	$\|A_3^7\| = 2$	$\|A_{21}^7\| = 6$
$\|A_6^5\| = 2$	$\|A_{30}^5\| = 4$	$\|A_8^7\| = 4$	$\|A_{56}^7\| = 12$
$\|A_{12}^5\| = 4$	$\|A_{60}^5\| = 8$	$\|A_{24}^7\| = 4$	$\|A_{168}^7\| = 12$
$\|A_{26}^5\| = 8$	$\|A_{130}^5\| = 16$	$\|A_{50}^7\| = 12$	$\|A_{350}^7\| = 36$
$p = 11$		$p = 13$	
$\|A_n^{11}\|$	$\|A_{11n}^{11}\|$	$\|A_n^{13}\|$	$\|A_{13n}^{13}\|$
$\|A_5^{11}\| = 4$	$\|A_{55}^{11}\| = 20$	$\|A_6^{13}\| = 2$	$\|A_{78}^{13}\| = 12$
$\|A_{12}^{11}\| = 4$	$\|A_{132}^{11}\| = 20$	$\|A_{14}^{13}\| = 6$	$\|A_{182}^{13}\| = 36$
$\|A_{60}^{11}\| = 8$	$\|A_{660}^{11}\| = 40$	$\|A_{84}^{13}\| = 16$	$\|A_{1092}^{13}\| = 96$

As we can see from Table 4.3, we conjecture that $\left|A_{pn}^p\right| = \left(\left|K_p^e\right| - 1\right) \cdot |A_n^p|$, where $K_p^e = \{0, 2, 4, \ldots, p-1\}$. The conjecture will be proved in Theorem 4.1.2 and the relationship of cardinality of spawning p-ary sets A_n^p and child p-ary sets $A_{p^k n}^p$ will be stated as the corollary.

Before proving the main theorem, we need the following theorem.

Theorem 4.1.1. *(Phon-On, 2013) Let $A_n^p = \{a_1, a_2, \ldots, a_k\}$ be a spawning p-ary set, where p does not divide n and $a_i \in \mathfrak{C}_p$. Let $A_{pn}^p = \{b_1, b_2, \ldots, b_r\}$ be a child p-ary set of A_n^p, where $b_i \in \mathfrak{C}_p$. Then, for each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, k\}$ and $l \in K_p^e$ such that $pb_i - l = a_j$. Consequently, $\left|A_{pn}^p\right| \geq |A_n^p|$ and $\left|A_{p^k n}^p\right| = \left|K_p^e\right|^{k-1} \left|A_{pn}^p\right|$ for all $k \geq 2$, and if $p = 3$, then $|A_{3n}^3| \geq |A_n^3|$ and hence $\left|A_{3^k n}^3\right| = 2^{k-1} |A_n^3|$ for all $k \geq 1$.*

Theorem 4.1.2. *Let p be an odd prime and $K_p^e = \{0, 2, 4, \ldots, p-1\}$. If A_{pn}^p is a child p-ary set of spawning $p-$ary set A_n^p, then $\left|A_{pn}^p\right| = \left(\left|K_p^e\right| - 1\right) \cdot \left|A_n^p\right|$.*

Proof. By Theorem 4.1.1, we have

$$pb_i - l = a_{_j},$$

where $l \in K_p^e$. Thus

$$b_i = \frac{a_j + l}{p}.$$

Since $a_j = \frac{m}{n} \in A_n^p$, then

$$\begin{aligned} b_i &= \frac{\frac{m}{n} + l}{p} \\ &= \frac{m + nl}{pn}. \end{aligned}$$

Let

$$B = \left\{ \frac{m + nl}{pn} \mid l \in K_p^e, \frac{m}{n} \in A_n^p \right\},$$

We will assert that, if $\frac{m_1}{n}, \frac{m_2}{n} \in A_n^p, l_1, l_2 \in K_p^e$ and

$$\frac{m_1 + nl_1}{pn} = \frac{m_2 + nl_2}{pn},$$

then $m_1 = m_2$ and $l_1 = l_2$.

Consider

$$\frac{m_1 + nl_1}{pn} = \frac{m_2 + nl_2}{pn},$$

then, we have

$$m_1 - m_2 = n\left(l_2 - l_1\right).$$

Consequently,

$$\left|m_1 - m_2\right| = \left|n\left(l_2 - l_1\right)\right|$$

$$|m_1 - m_2| = n\,|l_2 - l_1|\,.$$

Since both $\frac{m_1}{n}, \frac{m_2}{n} \in A_n^p$, it follows that $0 < m_1 < n$ and $0 < m_2 < n$.

Hence

$$|m_1 - m_2| < n$$
$$n\,|l_2 - l_1| < n$$
$$0 \le |l_2 - l_1| < 1.$$

Since $l_1, l_2 \in K_p^e$, then

$$l_2 - l_1 = 0$$
$$l_1 = l_2.$$

Therefore,

$$|m_1 - m_2| = 0$$
$$m_1 - m_2 = 0$$
$$m_1 = m_2.$$

It is clear that

$$|B| = \left|K_p^e\right| \cdot |A_n^p| \tag{4.1.1}$$

The elements in the set B can be categorized by considering the greatest common divisor of $m + nl$ and p, that is

$$\gcd(m + nl, p) = \begin{cases} p, & \text{if } m + nl \equiv 0 \pmod{p}; \\ 1, & \text{if } m + nl \not\equiv 0 \pmod{p}. \end{cases}$$

Let

$$A^p_{p^*n} = \left\{ \frac{m+nl}{pn} \mid \text{gcd}\,(m+nl,p) = p \right\},$$

and

$$A^p_{pn} = \left\{ \frac{m+nl}{pn} \mid \text{gcd}\,(m+nl,p) = 1 \right\}.$$

Note that $A^p_{p^*n}$ and A^p_{pn} are disjoint sets. We then have $B = A^p_{p^*n} \cup A^p_{pn}$, where $A^p_{p^*n} \cap A^p_{pn} = \phi$.

For $A^p_{p^*n}$, we will claim that for each $\frac{m}{n} \in A^p_n$, there exists a unique $l_0 \in K^e_p$ such that

$$p \mid (m + nl_0).$$

Let $\frac{m}{n} \in A^p_n$ and consider

$$\begin{aligned}
\frac{m}{n} &= (0.\overline{c_1 c_2 \ldots c_n})_p \\
&= \left(\frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_n}{p^n}\right) + \left(\frac{c_1}{p^{n+1}} + \frac{c_2}{p^{n+2}} + \ldots + \frac{c_n}{p^{2n}}\right) \\
&\quad + \left(\frac{c_1}{p^{2n+1}} + \frac{c_2}{p^{2n+2}} + \ldots + \frac{c_n}{p^{3n}}\right) + \ldots \\
&= \frac{p^n}{p^n}\left(\frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_n}{p^n}\right) + \frac{p^n}{p^n}\left(\frac{c_1}{p^{n+1}} + \frac{c_2}{p^{n+2}} + \ldots + \frac{c_n}{p^{2n}}\right) \\
&\quad + \frac{p^n}{p^n}\left(\frac{c_1}{p^{2n+1}} + \frac{c_2}{p^{2n+2}} + \ldots + \frac{c_n}{p^{3n}}\right) + \ldots \\
&= \left(\frac{c_1 p^{n-1}}{p^n} + \frac{c_2 p^{n-2}}{p^n} + \ldots + \frac{c_n}{p^n}\right) + \left(\frac{c_1 p^{n-1}}{p^{2n}} + \frac{c_2 p^{n-2}}{p^{2n}} + \ldots + \frac{c_n}{p^{2n}}\right) \\
&\quad + \left(\frac{c_1 p^{n-1}}{p^{3n}} + \frac{c_2 p^{n-2}}{p^{3n}} + \ldots + \frac{c_n}{p^{3n}}\right) + \ldots \\
&= \frac{(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n)}{p^n} + \frac{(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n)}{p^{2n}} \\
&\quad + \frac{(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n)}{p^{3n}} + \ldots \\
&= (c_1 p^{n-1} + c_2 p^{n-2} + c_3 p^{n-3} + \ldots + c_n)\left(\frac{1}{p^n} + \frac{1}{p^{2n}} + \frac{1}{p^{3n}} + \frac{1}{p^{4n}} + \ldots\right) \\
&= (c_1 p^{n-1} + c_2 p^{n-2} + c_3 p^{n-3} + \ldots + c_n)\left(\frac{\frac{1}{p^n}}{1 - \frac{1}{p^n}}\right)
\end{aligned}$$

$$\begin{aligned}\frac{m}{n} &= \left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n\right)\left(\frac{1}{p^n-1}\right)\\ &= \frac{c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n}{p^n-1}\\ \frac{m}{n} &= \frac{c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n}{p^n-1}.\end{aligned}$$

Thus,

$$n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n\right)=m\left(p^n-1\right)$$

$$n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right)+nc_n=mp^n-m$$

$$n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right)-mp^n=-m-nc_n$$

$$mp^n-n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right)=m+nc_n$$

$$mp^n-np\left(c_1p^{n-2}+c_2p^{n-3}+c_3p^{n-4}+\ldots+c_{n-1}\right)=m+nc_n.$$

Since $p \mid mp^n$ and $p \mid np\left(c_1p^{n-2}+c_2p^{n-3}+c_3p^{n-4}+\ldots+c_{n-1}p\right)$,

this implies that

$$p \mid (m+nc_n).$$

Therefore, there exists $l_0=c_n\in K_p^e$ such that

$$p \mid (m+nl_0).$$

Afterwards, we will prove the uniqueness of l_0.

Assume that

$$p \mid (m+nl_1),$$

and

$$p \mid (m+nl_2),$$

where $l_1, l_2 \in K_p^e$.

Then

$$p \mid (m + nl_1) - (m + nl_2)$$

$$p \mid (nl_1 - nl_2)$$

$$p \mid n (l_1 - l_2)$$

Since $(p, n) = 1$, it follows that

$$p \mid (l_1 - l_2).$$

Since $l_1, l_2 \in K_p^e$, we have

$$l_1 - l_2 = 0$$

$$l_1 = l_2.$$

By the claim above, for each $\frac{m}{n} \in A_n^p$, there exists a unique $l = l_0 \in K_p^e$ such that

$$p \mid (m + nl_0),$$

which implies that

$$\text{gcd} (m + nl_0, p) = p.$$

Since the number of $\frac{m}{n} \in A_n^p$ is $|A_n^p|$, this implies that

$$\left|A_{p^*n}^p\right| = |A_n^p| \tag{4.1.2}$$

Since $B = A_{p^*n}^p \cup A_{pn}^p$ and $A_{p^*n}^p \cap A_{pn}^p = \phi$, this concludes that

$$|B| = \left|A_{p^*n}^p\right| + \left|A_{pn}^p\right|. \tag{4.1.3}$$

Substituting the equations (4.1.1) and (4.1.2) into the equation (4.1.3), we have

$$|B| = \left|A_{p^*n}^p\right| + \left|A_{pn}^p\right|$$

$$\left|K_p^e\right| \cdot \left|A_n^p\right| = \left|A_n^p\right| + \left|A_{pn}^p\right|$$

$$\left|A_{pn}^p\right| = \left|K_p^e\right| \cdot \left|A_n^p\right| - \left|A_n^p\right|$$

$$= \left(\left|K_p^e\right| - 1\right) \cdot \left|A_n^p\right|.$$

Therefore, $\left|A_{pn}^p\right| = \left(\left|K_p^e\right| - 1\right) \cdot \left|A_n^p\right|.$ □

From the previous theorem and Theorem 4.1.1, we will show a new relation of cardinality of child $p-$ary set $A_{p^k n}^p$ and A_{pn}^p stated as the following corollary.

Corollary 4.1.3. *If* $A_{p^k n}^p, k = 1, 2, 3, \ldots,$ *are child* p*-ary sets and* $K_p^e = \{0, 2, 4, \ldots, p-1\}$, *then*

$$\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot \left|A_n^p\right|.$$

Proof. Follows from Theorems 4.1.1 and 4.1.2. □

4.2 Examples

As mentioned in the Corollary 4.1.3, the cardinality of child p-ary sets can be found as $\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot \left|A_n^p\right|$. The next four tables show that cardinality of spawning p-ary sets and child p-ary sets, where $k = 1, 2, 3$.

Table 4.4: Cardinality of spawning 5-ary sets A_n^5 and child 5-ary sets $A_{5^k n}^5$, where $k = 1, 2, 3$

spawning 5-ary sets	child 5-ary sets		
$\lvert A_n^5 \rvert$	$\lvert A_{5n}^5 \rvert$	$\lvert A_{5^2 n}^5 \rvert$	$\lvert A_{5^3 n}^5 \rvert$
$\lvert A_2^5 \rvert = 1$	$\lvert A_{10}^5 \rvert = 2$	$\lvert A_{50}^5 \rvert = 6$	$\lvert A_{250}^5 \rvert = 18$
$\lvert A_6^5 \rvert = 2$	$\lvert A_{30}^5 \rvert = 4$	$\lvert A_{150}^5 \rvert = 12$	$\lvert A_{750}^5 \rvert = 36$
$\lvert A_{12}^5 \rvert = 4$	$\lvert A_{60}^5 \rvert = 8$	$\lvert A_{300}^5 \rvert = 24$	$\lvert A_{1500}^5 \rvert = 72$
$\lvert A_{26}^5 \rvert = 8$	$\lvert A_{130}^5 \rvert = 16$	$\lvert A_{650}^5 \rvert = 48$	$\lvert A_{3250}^5 \rvert = 144$
$\lvert A_{62}^5 \rvert = 12$	$\lvert A_{310}^5 \rvert = 24$	$\lvert A_{1550}^5 \rvert = 72$	$\lvert A_{7750}^5 \rvert = 216$
$\lvert A_{126}^5 \rvert = 18$	$\lvert A_{630}^5 \rvert = 36$	$\lvert A_{3150}^5 \rvert = 108$	$\lvert A_{15750}^5 \rvert = 324$
$\lvert A_{312}^5 \rvert = 32$	$\lvert A_{1560}^5 \rvert = 64$	$\lvert A_{7800}^5 \rvert = 192$	$\lvert A_{39000}^5 \rvert = 576$

Table 4.5: Cardinality of spawning 7-ary sets A_n^7 and child 7-ary sets $A_{7^k n}^7$, where $k = 1, 2, 3$

spawning 7-ary sets	child 7-ary sets		
$\lvert A_n^7 \rvert$	$\lvert A_{7n}^7 \rvert$	$\lvert A_{7^2 n}^7 \rvert$	$\lvert A_{7^3 n}^7 \rvert$
$\lvert A_3^7 \rvert = 2$	$\lvert A_{21}^7 \rvert = 6$	$\lvert A_{147}^7 \rvert = 24$	$\lvert A_{1029}^7 \rvert = 96$
$\lvert A_8^7 \rvert = 4$	$\lvert A_{50}^7 \rvert = 12$	$\lvert A_{392}^7 \rvert = 48$	$\lvert A_{2744}^7 \rvert = 192$
$\lvert A_{24}^7 \rvert = 4$	$\lvert A_{168}^7 \rvert = 12$	$\lvert A_{1176}^7 \rvert = 48$	$\lvert A_{8232}^7 \rvert = 192$
$\lvert A_{50}^7 \rvert = 12$	$\lvert A_{350}^7 \rvert = 36$	$\lvert A_{2450}^7 \rvert = 144$	$\lvert A_{17150}^7 \rvert = 576$
$\lvert A_{171}^7 \rvert = 60$	$\lvert A_{1197}^7 \rvert = 180$	$\lvert A_{8379}^7 \rvert = 720$	$\lvert A_{58653}^7 \rvert = 2880$

Table 4.6: Cardinality of spawning 11-ary sets A_n^{11} and child 11-ary sets $A_{11^k n}^{11}$, where $k = 1, 2, 3$

spawning 11-ary sets	child 11-ary sets		
$\lvert A_n^{11}\rvert$	$\lvert A_{11n}^{11}\rvert$	$\lvert A_{11^2n}^{11}\rvert$	$\lvert A_{11^3n}^{11}\rvert$
$\lvert A_5^{11}\rvert = 4$	$\lvert A_{55}^{11}\rvert = 20$	$\lvert A_{605}^{11}\rvert = 120$	$\lvert A_{6655}^{11}\rvert = 720$
$\lvert A_{12}^{11}\rvert = 4$	$\lvert A_{132}^{11}\rvert = 20$	$\lvert A_{1452}^{11}\rvert = 120$	$\lvert A_{15972}^{11}\rvert = 720$
$\lvert A_{60}^{11}\rvert = 8$	$\lvert A_{660}^{11}\rvert = 40$	$\lvert A_{7260}^{11}\rvert = 240$	$\lvert A_{79860}^{11}\rvert = 1440$
$\lvert A_{122}^{11}\rvert = 36$	$\lvert A_{1342}^{11}\rvert = 180$	$\lvert A_{14762}^{11}\rvert = 1080$	$\lvert A_{162382}^{11}\rvert = 6480$
$\lvert A_{665}^{11}\rvert = 146$	$\lvert A_{7315}^{11}\rvert = 730$	$\lvert A_{80465}^{11}\rvert = 4380$	$\lvert A_{885115}^{11}\rvert = 26280$

Table 4.7: Cardinality of spawning 13-ary sets A_n^{13} and child 13-ary sets $A_{13^k n}^{13}$, where $k = 1, 2, 3$

spawning 13-ary sets	child 13-ary sets		
$\lvert A_n^{13}\rvert$	$\lvert A_{13n}^{13}\rvert$	$\lvert A_{13^2n}^{13}\rvert$	$\lvert A_{13^3n}^{13}\rvert$
$\lvert A_6\rvert = 2$	$\lvert A_{78}^{13}\rvert = 12$	$\lvert A_{1014}^{13}\rvert = 84$	$\lvert A_{13182}^{13}\rvert = 588$
$\lvert A_{14}\rvert = 6$	$\lvert A_{182}^{13}\rvert = 36$	$\lvert A_{2366}^{13}\rvert = 252$	$\lvert A_{30758}^{13}\rvert = 1764$
$\lvert A_{84}\rvert = 16$	$\lvert A_{1092}^{13}\rvert = 96$	$\lvert A_{14196}^{13}\rvert = 672$	$\lvert A_{184548}^{13}\rvert = 4704$
$\lvert A_{170}\rvert = 40$	$\lvert A_{2210}^{13}\rvert = 240$	$\lvert A_{28730}^{13}\rvert = 1680$	$\lvert A_{373490}^{13}\rvert = 11760$

Chapter 5

Group Structure on Cantor $p-$ary Sets

This chapter consists of three sections. The first section begins with definitions of some transformation which later will be called transformation R and transformation T. The second section provides a group action G, which is generated by transformation R and transformation T, that acts on spawning p-ary sets. Finally, in the section three, we prove the main result involving group structure on Cantor p-ary sets.

We begin this chapter by presenting the definitions of transformation R and transformation T on a spawning set A_q as in (Jordan and Schayer, n.d.). Then, we can extend the definitions of the transfromations on the spawning set to the spawning p-ary sets, where p is an odd prime.

5.1 Transformation R and transformation T

Before giving the definition of transformation R, we introduce some notations that are helpful to understanding the transformation.

Let p be an odd prime, define a complement digit of a_i by

$$\breve{a}_i = (p-1) - a_i,$$

where $a_i \in K_p^e$.

Table 5.1 will show the complement digit in base $3, 5, 7$ and 11.

Table 5.1: The complement digits in base $3, 5, 7$ and 11

$p = 3$		$p = 5$		$p = 7$		$p = 11$	
a_i	$\breve{a}_i$	a_i	$\breve{a}_i$	a_i	$\breve{a}_i$	a_i	$\breve{a}_i$
0	2	0	4	0	6	0	10
2	0	2	2	2	4	2	8
		4	0	4	2	4	6
				6	0	6	4
						8	2
						10	0

We next consider the definition of the transformation R.

5.1.1 Transformation R

Let A_n^p be a spawning p-ary set and suppose that $\frac{m}{n} = 0.\overline{a_1a_2\ldots a_l} \in A_n^p$. Then define the transformation $R : A_n^p \to A_n^p$ as follows:

$$R\left(\frac{m}{n}\right) = \frac{n-m}{n},$$

or

$$R\left((0.\overline{a_1a_2\cdots a_l})_p\right) = \left(0.\overline{\breve{a}_1\breve{a}_2\cdots\breve{a}_l}\right)_p.$$

It is easy to see that, the transformation R means swapping a_i with its complement $\breve{a}_i$. We illustrate the use of the transformation with the following example.

Example 5.1.1. $R\left(\frac{17}{171}\right) = \frac{171-17}{171} = \frac{154}{171}$ or $R\left(\frac{17}{171}\right) = R\left((0.\overline{046})_7\right) = 0.\overline{620}_7$.

5.1.2 Transformation T

Given $\frac{m}{n} \in A_n^p$, where A_n^p is a spawning p-ary set. The transformation $T : A_n^p \to A_n^p$, where $p = 5, 7, 11$ and 13, respectively, can be defined as follows:

$$T\left(\frac{m}{n}\right) = \begin{cases} 5\frac{m}{n}, & \text{if } 5m \leq n; \\ 5\left(\frac{m}{n} - \frac{2}{5}\right), & \text{if } \frac{5}{3}m \leq n < 5m; \\ R\left(5R\left(\frac{m}{n}\right)\right), & \text{if } n < \frac{5}{3}m. \end{cases}$$

$$T\left(\frac{m}{n}\right) = \begin{cases} 7 \cdot \frac{m}{n}, & \text{if } 7m \leq n; \\ 7\left(\frac{m}{n} - \frac{2}{7}\right), & \text{if } \frac{7}{3}m \leq n < 7m; \\ 7\left(\frac{m}{n} - \frac{4}{7}\right), & \text{if } \frac{7}{5}m \leq n < \frac{7}{3}m; \\ R\left(7R\left(\frac{m}{n}\right)\right), & \text{if } n < \frac{7}{5}m. \end{cases}$$

$$T\left(\frac{m}{n}\right) = \begin{cases} 11 \cdot \frac{m}{n}, & \text{if } 11m \leq n; \\ 11\left(\frac{m}{n} - \frac{2}{11}\right), & \text{if } \frac{11}{3}m \leq n < 11m; \\ 11\left(\frac{m}{n} - \frac{4}{11}\right), & \text{if } \frac{11}{5}m \leq n < \frac{11}{3}m; \\ 11\left(\frac{m}{n} - \frac{6}{11}\right), & \text{if } \frac{11}{7}m \leq n < \frac{11}{5}m; \\ 11\left(\frac{m}{n} - \frac{8}{11}\right), & \text{if } \frac{11}{9}m \leq n < \frac{11}{7}m; \\ R\left(11R\left(\frac{m}{n}\right)\right), & \text{if } n < \frac{11}{9}m. \end{cases}$$

.

$$T\left(\frac{m}{n}\right)=\begin{cases}13\cdot\frac{m}{n}, & \text{if } 13m\le n;\\ 13\left(\frac{m}{n}-\frac{2}{13}\right), & \text{if } \frac{13}{3}m\le n<13m;\\ 13\left(\frac{m}{n}-\frac{4}{13}\right), & \text{if } \frac{13}{5}m\le n<\frac{13}{3}m;\\ 13\left(\frac{m}{n}-\frac{6}{13}\right), & \text{if } \frac{13}{7}m\le n<\frac{13}{5}m;\\ 13\left(\frac{m}{n}-\frac{8}{13}\right), & \text{if } \frac{13}{9}m\le n<\frac{13}{7}m;\\ 13\left(\frac{m}{n}-\frac{10}{13}\right), & \text{if } \frac{13}{11}m\le n<\frac{13}{9}m;\\ R\left(13R\left(\frac{m}{n}\right)\right), & \text{if } n<\frac{13}{11}m.\end{cases}$$

Then, a general form of the transformation T can be written as

$$T\left(\frac{m}{n}\right)=\begin{cases}p\frac{m}{n}, & \text{if } pm\le n;\\ p\left(\frac{m}{n}-\frac{2i}{p}\right), & \text{if } \frac{pm}{2i+1}\le n<\frac{pm}{2i-1}, \text{where } i=1,2,\ldots,\lceil\frac{p}{2}-2\rceil\\ R\left(pR\left(\frac{m}{n}\right)\right), & \text{if } n<\frac{p}{p-2}m.\end{cases}$$

It is great to point out that transformation T is cycle the digit a_i to the left in its period form. This process is shown in Example 5.1.2

Example 5.1.2. Consider $\frac{m}{n}=\frac{23}{122}\in A^{11}_{122}$ on base 11. Since $\frac{11}{3}\cdot 23\le 122<\frac{11}{1}\cdot 23$, it follows that

$$\begin{aligned}T\left(\frac{23}{122}\right)&=11\cdot\left(\frac{23}{122}-\frac{2}{11}\right)\\&=11\cdot\left(\frac{23\cdot(11)-2\cdot(122)}{122\cdot 11}\right)\\&=\frac{253-244}{122}=\frac{9}{122}.\end{aligned}$$

On the other hand,

$$T\left(\frac{23}{122}\right) = T\left(\left(0.\overline{208A}\right)_{11}\right)$$
$$= 0.\overline{08A2}_{11}$$
$$= \frac{9}{122}.$$

Hence, $T\left(\frac{23}{122}\right) = \frac{9}{122}$.

5.2 Group action on spawning $p-$ary sets

Let $(G, \circ)$ be a group with $G = \{T^i R^j, 1 \leq i \leq l, 0 \leq j \leq 1\}$, where l is the period length of a rational number $\frac{m}{n}$ in the spawning p-ary sets A_n^p. It is straightforward to check that $(G, \circ)$ is a group.

By determining a group action G on spawning set A_q in (Jordan and Schayer, n.d.), we can extend the set into the spawning p-ary set A_n^p. This section will show a group action G on spawning p-ary sets A_n^p and will state some observations.

We first show that G acts on spawning p-ary sets. Define an action G on A_n^p,

$$* : G \times A_n^p \to A_n^p$$

via

$$* : (g, x) \to g \cdot x = g(x),$$

for $g \in G$ and $x \in A_n^p$.

We claim that

(1) $(g_1 \circ g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

(2) $e \cdot x = x$ where e is an identity of G.

Consider

$$
\begin{aligned}
*\left(\left(g_1 \circ g_2\right), x\right) &= \left(g_1 \circ g_2\right) \cdot x \\
&= \left(g_1 \circ g_2\right)(x) \\
&= g_1\left(g_2(x)\right) \\
&= g_1 \cdot \left(g_2(x)\right) \\
&= g_1 \cdot \left(g_2 \cdot x\right).
\end{aligned}
$$

Clearly, the first condition for a group action holds.

Note that $e = I$ is the identity function of G, then

$$
\begin{aligned}
*(I, x) &= I \cdot x \\
&= I(x) \\
&= x.
\end{aligned}
$$

Thus, $I \cdot x = x$

We see that the second condition is also satisfied.

The following theorem reveals some interesting observations of group action G on spawning p-ary sets A_n^p.

Theorem 5.2.1. *Let l be the period length of elements in a spawning p-ary set A_n^p. T and R generate a group action G on A_n^p with the following properties:*

(1) $T^l = I, R^2 = I$.

(2) T and R commute.

(3) G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$.

(4) G_T, the subgroup generated by T alone, is a faithful cyclic subgroup of order l.

Proof. (1) $T^l = I, R^2 = I$.

We will show that T^l, R^2 are identity functions. This means that $T^l\left(\frac{m}{n}\right) = \frac{m}{n}$ and $R^2\left(\frac{m}{n}\right) = \frac{m}{n}$.

Given $\frac{m}{n} = (0.\overline{a_1a_2\ldots a_l})_p$ has the period length l.

Consider

$$\begin{aligned}
\frac{m}{n} &= (0.\overline{a_1a_2\ldots a_l})_p \\
T\left(\frac{m}{n}\right) &= (0.\overline{a_2\ldots a_la_1})_p \\
T^2\left(\frac{m}{n}\right) = T\left(T\left(\frac{m}{n}\right)\right) &= (0.\overline{a_3\ldots a_la_1a_2})_p \\
&\vdots \\
T^{l-1}\left(\frac{m}{n}\right) = T\left(\underbrace{\ldots T}_{l-2}\left(\frac{m}{n}\right)\right) &= (0.\overline{a_la_1\ldots a_{l-1}})_p \\
T^{l}\left(\frac{m}{n}\right) = T\left(\underbrace{\ldots T}_{l-1}\left(\frac{m}{n}\right)\right) &= (0.\overline{a_1a_2\ldots a_l})_p \\
T^l\left(\frac{m}{n}\right) &= (0.\overline{a_1a_2\ldots a_l})_p \\
T^l\left(\frac{m}{n}\right) &= \frac{m}{n}.
\end{aligned}$$

Hence, $T^l = I$.

Next, we consider

$$\begin{aligned}
\frac{m}{n} &= (0.\overline{a_1a_2\ldots a_l})_p \\
R\left(\frac{m}{n}\right) &= \left(0.\overline{\breve{a}_1\breve{a}_2\ldots \breve{a}_l}\right)_p \\
R^2\left(\frac{m}{n}\right) = R\left(R\left(\frac{m}{n}\right)\right) &= \left(0.\overline{\breve{\breve{a}}_1\breve{\breve{a}}_2\ldots \breve{\breve{a}}_l}\right)_p \\
R^2\left(\frac{m}{n}\right) &= (0.\overline{a_1a_2\ldots a_l})_p
\end{aligned}$$

$$R^2\left(\frac{m}{n}\right)=\frac{m}{n}.$$

Hence $R^2 = I$, therefore, $T^l = I, R^2 = I$.

(2) T and R commute.

We claim that $T\left(R\left(\frac{m}{n}\right)\right) = R\left(T\left(\frac{m}{n}\right)\right)$.

Let $\frac{m}{n} = (0.\overline{a_1a_2\ldots a_l})_p$.

Notice that

$$\begin{aligned}
T\left(R\left(\frac{m}{n}\right)\right) &= T\left(R\left((0.\overline{a_1a_2\ldots a_l})_p\right)\right)\\
&= T\left(\left(0.\overline{\breve{a}_1\breve{a}_2\ldots\breve{a}_l}\right)_p\right)\\
&= \left(0.\overline{\breve{a}_2\breve{a}_3\ldots\breve{a}_l\breve{a}_1}\right)_p\\
&= R\left((0.\overline{a_2a_3\ldots a_la_1})_p\right)\\
&= R\left(T\left((0.\overline{a_1a_2\ldots a_l})_p\right)\right)\\
&= R\left(T\left(\frac{m}{n}\right)\right).
\end{aligned}$$

Hence $T\left(R\left(\frac{m}{n}\right)\right) = R\left(T\left(\frac{m}{n}\right)\right)$. It implies that T and R commute.

(3) G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$.

Let $g = T^iR^j \in G$ and assume that a function $\phi\colon G \to \mathbb{Z}_l \times \mathbb{Z}_2$ such that $\phi\left(T^iR^j\right) = \left(\bar{i}, \bar{j}\right)$ for $i = 1, 2, \ldots, l$ and $j = 0, 1$. Where $\bar{i} \in \mathbb{Z}_l$ or $\bar{i} = \{p \mid i \equiv p \pmod{l}\}$ and $\bar{j} \in \mathbb{Z}_2$ or $\bar{j} = \{q \mid j \equiv q \pmod{2}\}$. This is well defined. Then we will prove that $\phi\left(T^iR^j\right) = (\bar{i}, \bar{j})$ is isomorphism, by showing that ϕ is 1-1, onto and homomorphism $(\phi(gh) = \phi(g) + \phi(h))$.

- To see that ϕ is $1-1$, we assume that if $\phi\left(T^iR^j\right) = \phi\left(T^pR^q\right)$ then $T^iR^j = T^pR^q$.

  Since

$$\phi\left(T^iR^j\right) = (\bar{i}, \bar{j})$$

and

$$\phi\left(T^{p}R^{q}\right)=\left(\bar{p},\bar{q}\right).$$

By assumption, we have

$$\left(\bar{i},\bar{j}\right)=\left(\bar{p},\bar{q}\right).$$

Thus

$$i\equiv p \pmod{l}.$$

This concludes that there exists $t\in\mathbb{Z}$ such that $i=p+lt$.

Hence

$$\begin{aligned}T^{i}&=T^{p+lt}\\&=T^{p}\left(T^{l}\right)^{t}\\&=T^{p}\left(I\right)^{t}\\&=T^{p}.\end{aligned}$$

Then $T^{i}=T^{p}$.

Also we have

$$j\equiv q \pmod{2}.$$

This implies that there exists $t\in\mathbb{Z}$ such that $j=q+2t$.

Then

$$\begin{aligned}R^{j}&=R^{q+2t}\\&=R^{q}\left(R^{2}\right)^{t}\\&=R^{q}\left(I\right)^{t}\\&=R^{q}.\end{aligned}$$

Thus $R^j = R^q$.

From the two previous results, we have $T^iR^j = T^pR^q$. Consequently ϕ is 1-1.

- Let $(\bar{i}, \bar{j}) \in \mathbb{Z}_l \times \mathbb{Z}_2$. There exists $T^iR^j \in G$ such that $\phi\left(T^iR^j\right) = (\bar{i}, \bar{j})$, therefore ϕ is onto.

- Let $g, h \in G$ such that $g = T^iR^j$ and $h = T^pR^q$, for some integers i, j, p and q. To show that ϕ is homomorphism, we claim that $\phi(gh) = \phi(g) + \phi(h)$.

  Observe that

$$
\begin{aligned}
\phi(gh) &= \phi\left[\left(T^iR^j\right)\left(T^pR^q\right)\right] \\
&= \phi\left[T^iR^jT^pR^q\right] \\
&= \phi\left[\left(T^iT^p\right)\left(R^jR^q\right)\right] \\
&= \phi\left(T^{i+p}R^{j+q}\right) \\
&= \left(\overline{i+p}, \overline{j+q}\right) \\
&= (\bar{i}+\bar{p}, \bar{j}+\bar{q}) \\
&= \phi\left(T^iR^j\right) + \phi\left(T^pR^q\right) \\
&= \phi(g) + \phi(h) \\
\phi(gh) &= \phi(g) + \phi(h).
\end{aligned}
$$

  Therefore, ϕ is homomorphism.

Since ϕ is 1-1, onto and homomorphism, it implies that G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$. Further, the number of elements in the group G must be equal to the number of elements in $\mathbb{Z}_l \times \mathbb{Z}_2$.

(4) G_T, the subgroup generated by T alone, is a faithful cyclic subgroup of order l.

Any two distinct elements $T^i, T^j \in G_T$ where $i \neq j$ and $i, j \in \{1, 2, \ldots, l\}$ give distinct permutation of A_n^p. $< T >$ is a generator of G_T, and the number of all elements in G_T is l, these implies that G_T is a faithful cyclic subgroup of order l. □

Furthermore, we then discuss a partition of spawning p-ary sets A_n^p. This topic gives us numbers of elements in each partition and a characteristic of the elements in the partition. All observation will be explained in section 5.3.

5.3 Group structure on Cantor $p-$ary sets

We begin this section by defining equivalence relation on A_n^p.

5.3.1 Equivalence relation

Definition 5.3.1. Let $x = \frac{a}{n}$ and $y = \frac{b}{n}$ be elements in A_n^p and denote $\sim$ be an equivalence relation with the following condition:

$$x \sim y \leftrightarrow a \sim b \leftrightarrow b \equiv (-1)^m ap^j \pmod{n},$$

for some $j \in \{0, 1, \ldots, l-1\}, m \in \{0, 1\}$.

Then, we now show that $\sim$ is an equivalence relation.

Reflexive: $a \sim a, \forall a$.

Since

$$a \equiv (-1)^0 ap^0 \pmod{n}$$

$$a \equiv a \pmod{n}.$$

Then $a \sim a$.

Symmetric: If $a \sim b$, then $b \sim a$.

We will prove in 2 cases.

Case 1. Assume that $b \equiv ap^j \pmod{n}$, for some $j \in \{0, 1, \ldots, l-1\}$.

To show that $\sim$ is symmetric, we claim that

$$a \equiv bp^i \pmod{n}$$

for some $i \in \{0, 1, \ldots, l-1\}$.

We know that $(p, n) = 1$, by Euler's theorem, we have

$$p^{\phi(n)} \equiv 1 \pmod{n}.$$

By division Algorithm, $\exists r, s \in \mathbb{Z}$ such that

$$j = r(\phi(n)) + s,$$

where $0 \leq s < \phi(n)$.

Notice that

$$\begin{aligned} b &\equiv ap^j \pmod{n} \\ &\equiv ap^{r(\phi(n))+s} \pmod{n} \\ &\equiv ap^{r(\phi(n)} \cdot p^s \pmod{n} \end{aligned}$$

Since $p^{\phi(n)} \equiv 1 \pmod{n}$, hence

$$b \equiv ap^s \pmod{n}$$

$$bp^{\phi(n)-s} \equiv ap^s \cdot p^{\phi(n)-s} \pmod{n}$$

$$\equiv ap^{\phi(n)} \pmod{n}$$

$$\equiv a \pmod{n}$$

$$bp^{\phi(n)-s} \equiv a \pmod{n}$$

$$a \equiv bp^{\phi(n)-s} \pmod{n}$$

It is clear that $a \equiv bp^i \pmod{n}$, where $i = \phi(n) - s$.

Therefore, $b \sim a$.

Case 2. Let $b \equiv -ap^j \pmod{n}$, for some j.

We will show that

$$a \equiv -bp^i \pmod{n},$$

for some i.

We know that $(p, n) = 1$, and by Euler's theorem, we have

$$p^{\phi(n)} \equiv 1 \pmod{n}.$$

By division algorithm, $\exists r, s \in \mathbb{Z}$ such that

$$j = r\left(\phi\left(n\right)\right) + s$$

where $0 \leq s < \phi\left(n\right)$.

Consider

$$b \equiv -ap^j \pmod{n}$$

$$\equiv -ap^{r(\phi(n))+s} \pmod{n}$$

$$\equiv -ap^{r(\phi(n))} \cdot p^s \pmod{n}.$$

Since $p^{\phi(n)} \equiv 1 \pmod{n}$, we have

$$b \equiv -ap^s \pmod{n}$$

$$b \cdot p^{\phi(n)-s} \equiv -ap^s \cdot p^{\phi(n)-s} \pmod{n}$$

$$\equiv -ap^{\phi(n)} \pmod{n}$$

$$\equiv -a \pmod{n}$$

$$-a \equiv b \cdot p^{\phi(n)-s} \pmod{n}$$

$$a \equiv -b \cdot p^{\phi(n)-s} \pmod{n}.$$

Hence, $a \equiv (-1)^1 bp^i \pmod{n}$, where $i = \phi(n) - s$.

Therefore, $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$ then $a \sim c$.

We will prove in 4 cases.

Case 1. $a \sim b \Leftrightarrow b \equiv ap^j \pmod{n}$ and $b \sim c \Leftrightarrow c \equiv bp^i \pmod{n}$.

Suppose that

$$b \equiv ap^j \pmod{n},$$

then

$$bp^i \equiv ap^{j+i} \pmod{n}.$$

Since $c \equiv bp^i \pmod{n}$, we have

$$c \equiv ap^{j+i} \pmod{n}$$

$$c \equiv (-1)^0 ap^{j+i} \pmod{n}.$$

Therefore, $c \equiv (-1)^0 ap^k \pmod{n}$, where $k = j + i$.

Consequently, $a \sim c$.

Case 2. $a \sim b \Leftrightarrow b \equiv ap^j \pmod{n}$ and $b \sim c \Leftrightarrow c \equiv -bp^i \pmod{n}$.

Given

$$b \equiv ap^j \pmod{n},$$

hence

$$bp^i \equiv ap^{j+i} \pmod{n}$$

$$-bp^i \equiv -ap^{j+i} \pmod{n}.$$

Also given $c \equiv -bp^i \pmod{n}$, then

$$c \equiv -ap^{j+i} \pmod{n}$$

$$c \equiv (-1)^1 ap^{j+i} \pmod{n}.$$

Therefore, $c \equiv (-1)^1 ap^k \pmod{n}$, where $k = j + i$.

Consequently, $a \sim c$.

Case 3. $a \sim b \Leftrightarrow b \equiv -ap^j \pmod{n}$ and $b \sim c \Leftrightarrow c \equiv bp^i \pmod{n}$.

Assume that

$$b \equiv -ap^j \pmod{n},$$

then

$$bp^i \equiv -ap^{j+i} \pmod{n}.$$

We give $c \equiv bp^i \pmod{n}$, so then

$$c \equiv -ap^{j+i} \pmod{n}$$

$$c \equiv (-1)^1 ap^{j+i} \pmod{n}.$$

Therefore, $c \equiv (-1)^1 ap^k \pmod{n}$,where $k = j + i$.

Consequently, $a \sim c$.

Case 4. $a \sim b \Leftrightarrow b \equiv -ap^j \pmod{n}$ and $b \sim c \Leftrightarrow c \equiv -bp^i \pmod{n}$.

Consider

$$b \equiv -ap^j \pmod{n}$$

$$b(-p^i) \equiv -ap^j(-p^i) \pmod{n}$$

$$-bp^i \equiv ap^{j+i} \pmod{n}.$$

Since $c \equiv -bp^i \pmod{n}$, then

$$c \equiv ap^{j+i} \pmod{n}$$

$$c \equiv (-1)^0 ap^{j+i} \pmod{n}.$$

Therefore, $c \equiv (-1)^0 ap^k \pmod{n}$, where $k = j + i$.

Consequently, $a \sim c$.

As we have shown the proofs of **Reflexive**, **Symmetric** and **Transitive**, we conclude that $\sim$ is an equivalence relation.

5.3.2 Partition of spawning $p-$ary sets

Recall that $\sim$ is an equivalence relation. Define a partition of A_n^p by

$$A^p_{n/\sim} = \{[m_1], [m_2], \ldots, [m_i]\},$$

where i be a positive integer and $[m_i]$ be an equivalence class (with respect to $\sim$) determined by m_i.

By the definition of the partition of spawning p-ary sets above, we discover some interesting observations that will be shown in Table 5.2. The table shows some spawning p-ary sets together with the period length of elements, the number of elements in the partition, the number of the partition and the number of all elements in the sets.

Table 5.2: Some results of partition of some spawning $p-$ary sets

A_n^p	period length (l)	number of elements in each partition	number of partition	$\|A_n^p\|$
spawning 5-ary set				
A_2^5	1	1	1	1
A_6^5	2	2	1	2
A_{12}^5	2	4	1	4
A_{26}^5	4	4	2	8
A_{62}^5	3	6	2	12
A_{126}^5	6	6	3	18
A_{312}^5	4	8	4	32
spawning 7-ary set				
A_3^7	1	2	1	2
A_8^7	2	2	2	4
A_{24}^7	2	4	1	4
A_{50}^7	4	4	3	12
A_{171}^7	3	6	10	60
spawning 11-ary set				
A_5^{11}	1	2	2	4
A_{12}^{11}	2	2	2	4
A_{60}^{11}	2	4	2	8
A_{122}^{11}	4	4	9	36
A_{665}^{11}	3	6	24	150

A_n^p	period length (l)	number of elements in each partition	number of partition	$\|A_n^p\|$
spawning 13-ary set				
A_6^{13}	1	2	1	2
A_{14}^{13}	2	2	3	6
A_{84}^{13}	2	4	4	16
A_{170}^{13}	4	4	10	40

Table 5.2 gives two guideline results about the number of element in partition. First, we can see that each $A^p_{n/\sim}$ contains either l or $2l$ elements, where l is the period length of the elements in A_n^p. The second guideline result is l divides $|A_n^p|$.

The following theorem and two corollaries will assert the two guideline results.

Theorem 5.3.2. *Each partition of A_n^p by group G consists of l or $2l$ elements.*

Proof. Recall that $(G, \circ)$ be a group with $G = \{T^i R^j\}$, where $i \in \{1, 2, \ldots, l\}$, $j \in \{0, 1\}$ and $G_T = \left\{T, T^2, \ldots, T^{l-1}, I\right\}$ be a subgroup of G. Note that G_T acts faithfully on the partition, that is

$$T^i\left(\frac{m}{n}\right) \neq T^j\left(\frac{m}{n}\right)$$

where $i \neq j, i, j \in \{1, 2, \ldots, l\}$ and $\frac{m}{n} \in A_n^p$.

It concludes that the set

$$\left\{T\left(\frac{m}{n}\right), T^2\left(\frac{m}{n}\right), \ldots, T^{l-1}\left(\frac{m}{n}\right), I\left(\frac{m}{n}\right)\right\}$$

give l distinct elements.

We classify the group action G on the partition of A_n^p into two cases.

(1) Group G acts faithfully on the partition.

(2) Group G does not acts faithfully on the partition.

Case 1: Group G acts faithfully.

Let $x = \frac{m}{n} \in A_n^p$

Then

$$T(x) = x_1$$

$$T^2(x) = x_2$$

$$\vdots$$

$$T^{l-1}(x) = x_{l-1}$$

$$I(x) = T^l(x) = x_l$$

give l distinct elements, and

$$RT(x) = x_{l+1}$$

$$RT^2(x) = x_{l+2}$$

$$\vdots$$

$$RT^{l-1}(x) = x_{l+(l-1)}$$

$$R(x) = RT^l(x) = x_{l+l} = x_{2l}$$

also give l distinct elements.

Hence

$$\left|\{T(x), T^2(x), \ldots, T^l(x), RT(x), \ldots, RT^l(x)\}\right| = 2l.$$

Consequently, the size of the partition is $2l$.

Case 2: Group G does not acts faithfully.

Let k be a size of the partition.

By assumption, G does not acts faithfully, we have $k < 2l$ and there exist $x \in A_n^p$ and $m < l$ such that $R\left(T^m\right)x = x$.

Notice that

$$\begin{aligned} x &= R\left(T^m\right)x \\ &= R\left(T^m\right)\left(R\left(T^m\right)x\right) \\ &= \left(RT^m \cdot RT^m\right)x \\ &= R^2T^{2m}x \\ &= IT^{2m}x \\ &= T^{2m}x. \end{aligned}$$

Hence,

$$x = T^{2m}x. \tag{5.3.1}$$

Since $T^{2m} \in G_T$ and G_T acts faithfully on the partition, we have

$$x = T^l x \tag{5.3.2}$$

and hence by the equations we have

$$T^{2m}x = T^l x.$$

It follows that

$$\begin{aligned} 2m &= l \\ m &= \frac{l}{2}. \end{aligned}$$

Therefore, $RT^{\frac{l}{2}}(x) = x$ or $RT^{\frac{l}{2}}\left(\frac{m}{n}\right) = \frac{m}{n}$.

Assume that $\frac{m}{n} = \left(0.\overline{a_1 a_2 \ldots a_{\frac{l}{2}} \ldots a_{l-1} a_l}\right)_p$.

Thus,

$$RT^{\frac{l}{2}}\left(\frac{m}{n}\right) = \frac{m}{n}$$
$$\left(0.\overline{\breve{a}_{\frac{l}{2}+1}\breve{a}_{\frac{l}{2}+2} \ldots \breve{a}_n \breve{a}_1 \ldots \breve{a}_{\frac{l}{2}}}\right)_p = \left(0.\overline{a_1 a_2 \ldots a_{\frac{l}{2}} a_{\frac{l}{2}+1} \ldots a_l}\right)_p.$$

Comparing each digit and taking its complements, we obtain

$$\breve{a}_{\frac{l}{2}+1} = a_1 \longrightarrow a_{\frac{l}{2}+1} = \breve{a}_1,$$
$$\breve{a}_{\frac{l}{2}+2} = a_2 \longrightarrow a_{\frac{l}{2}+2} = \breve{a}_2,$$
$$\vdots$$
$$\breve{a}_l = a_{\frac{l}{2}} \longrightarrow a_l = \breve{a}_{\frac{l}{2}}.$$

Hence

$$\begin{aligned}\frac{m}{n} &= \left(0.\overline{a_1 a_2 \ldots a_{\frac{l}{2}} a_{\frac{l}{2}+1} a_{\frac{l}{2}+2} \ldots a_l}\right)_p \\ &= \left(0.\overline{a_1 a_2 \ldots a_{\frac{l}{2}} \breve{a}_1 \breve{a}_2 \ldots \breve{a}_{\frac{l}{2}}}\right)_p.\end{aligned}$$

Now we show that

$$RT^j\left(\frac{m}{n}\right) = T^{\frac{l}{2}+j}\left(\frac{m}{n}\right)$$

where $j = 1, 2, \ldots, l$.

Notice that

$$\begin{aligned}RT^1\left(\frac{m}{n}\right) &= R\left(\left(0.\overline{a_2 \ldots a_{\frac{l}{2}} \breve{a}_1 \ldots \breve{a}_{\frac{l}{2}} a_1}\right)_p\right) \\ &= \left(0.\overline{\breve{a}_2 \ldots \breve{a}_{\frac{l}{2}} a_1 \ldots a_{\frac{l}{2}} \breve{a}_1}\right)_p \\ &= T^{\frac{l}{2}+1}\left(\left(0.\overline{a_1 \ldots a_{\frac{l}{2}} \breve{a}_1 \ldots \breve{a}_{\frac{l}{2}}}\right)_p\right)\end{aligned}$$

$$RT^1\left(\frac{m}{n}\right) = T^{\frac{l}{2}+1}\left(\frac{m}{n}\right).$$

We conclude that $RT^1\left(\frac{m}{n}\right) = T^{\frac{l}{2}+1}\left(\frac{m}{n}\right).$

Applying the transformation T on both sides and commute T and R, we have

$$RT^2\left(\frac{m}{n}\right) = T\left(RT^1\left(\frac{m}{n}\right)\right) = T\left(T^{\frac{l}{2}+1}\left(\frac{m}{n}\right)\right) = T^{\frac{l}{2}+2}\left(\frac{m}{n}\right).$$

Consequently, $RT^j\left(\frac{m}{n}\right) = T^{\frac{l}{2}+j}\left(\frac{m}{n}\right)$ for all $j = 1, 2, \ldots, l.$

Therefore, all $RT^j(x)$, where $x = \frac{m}{n}$, are redundant. That are

$$\begin{aligned} RT(x) &= T^{\frac{l}{2}+1}(x) \\ &\vdots \\ RT^{\frac{l}{2}-1}(x) &= T^{l-1}(x) \\ RT^{\frac{l}{2}}(x) &= T^{l}(x) \\ RT^{\frac{l}{2}+1}(x) &= T(x) \\ &\vdots \\ RT^{l-1}(x) &= T^{\frac{l}{2}-1}(x) \\ RT^{l}(x) &= T^{\frac{l}{2}}(x). \end{aligned}$$

It is clear that the elements in the partition reduce into l elements, hence

$$\left\{T(x), T^2(x), \ldots, T^l(x), RT(x), \ldots, RT^l(x)\right\} = \left\{T(x), T^2(x), \ldots, T^l(x)\right\}.$$

Then, we have

$$\left|\left\{T(x), T^2(x), \ldots, T^n(x)\right\}\right| = l,$$

this implies that the size of the partition is l.

Finally, from the proof of the **Case 1** and the **Case 2**, we conclude that each partition of A_n^p by group G contains l or $2l$ elements. $\square$

The following corollaries provide the group structure on Cantor p-ary sets.

Corollary 5.3.3. *Let A_n^p be a spawning p-ary set and l be the period length of elements in the set, then $l \mid |A_n^p|$.*

Proof. Suppose that k_1, k_2 are numbers of the partition that contains l and $2l$ elements, respectively. It is obvious that k_1 and k_2 are non-negative integers.

Then

$$\begin{aligned}|A_n^p| &= k_1(l) + k_2(2l)\\ &= l(k_1 + 2k_2).\end{aligned}$$

It implies that $l \mid |A_n^p|$. □

Corollary 5.3.4. *Let $A_{p^k n}^p$ be the childs p-ary sets of spawning p-ary set A_n^p, then $l \mid \left|A_{p^k n}^p\right|$.*

Proof. Corollary 4.1.3 in Chapter 4 tells us

$$\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot |A_n^p|.$$

It follows that

$$|A_n^p| \mid \left|A_{p^k n}^p\right|. \tag{5.3.3}$$

The above theorem shows that

$$l \mid |A_n^p|. \tag{5.3.4}$$

Following the results in (5.3.3) and (5.3.4), we then have

$$l \mid \left|A_{p^k n}^p\right|.$$

□

By using the equivalence relation $\sim$ from the section 5.3.1 to partition some spawning p-ary sets A_n^p, we can see the relationship between elements and the smallest element in their partition.

Therefore, the characteristic of elements in each partition can be expressed and proved in the Theorem 5.3.5

Theorem 5.3.5. *Let A_n^p be a spawning p-ary set with a period length l of elements in A_n^p. Let $P = \left\{\frac{m_0}{n}, \frac{m_1}{n}, \dots, \frac{m_k}{n}\right\}$, where $k \leq 2l$ and m_0 is the least element among $m_0, \dots, m_k$, be a partition of A_n^p. Then*

$$m_j \equiv \begin{cases} m_0 p^j \pmod{n}, & \text{if } 0 \leq j < l; \\ -m_0 p^j \pmod{n}, & \text{if } l \leq j < 2l. \end{cases}$$

Proof. As mentioned in Theorem 5.3.2, each partition will contain l or $2l$ elements. Then we have to consider two cases.

(1) $|P| = l$

(2) $|P| = 2l$

<u>**Case 1**</u>: Assume that $|P| = l$, and $P = \left\{\frac{m_0}{n}, \frac{m_1}{n}, \dots, \frac{m_{l-1}}{n}\right\}$.

let $0 \leq j < l$, $\frac{m_0}{n} = (0.\overline{a_1 \dots a_l})_p$ and $\frac{m_0}{n} = \min P$.

Suppose that $\frac{m_j}{n} = T^j\left(\frac{m_0}{n}\right)$, and we know that by the division algorithm, there exist $r, s \in \mathbb{Z}^+$ such that

$$m_0 p^j = rn + s$$

where $0 \leq s < n$. Then

$$s = m_0 p^j - rn.$$

Consider

$$\begin{aligned}
\frac{m_0p^j - rn}{n} &= \frac{m_0p^j}{n} - \frac{rn}{n} \\
&= \frac{m_0p^j}{n} - r \\
&= p^j\left(\frac{m_0}{n}\right) - r \\
&= p^j\left(0.\overline{a_1\ldots a_l}\right)_p - r \\
\frac{m_0p^j - rn}{n} &= p^j\left(\frac{a_1}{p} + \frac{a_2}{p^2} + \ldots + \frac{a_l}{p^l} + \frac{a_1}{p^{l+1}} + \ldots\right) - r \\
&= p^j\left(\frac{a_1}{p} + \frac{a_2}{p^2} + \ldots + \frac{a_j}{p^j}\right) \\
&\quad + p^j\left(\frac{a_{j+1}}{p^{j+1}} + \frac{a_{j+2}}{p^{j+2}} + \ldots + \frac{a_l}{p^l} + \frac{a_1}{p^{l+1}} + \ldots\right) - r \\
&= \left(a_1p^{j-1} + a_2p^{j-2} + \ldots + a_j\right) \\
&\quad + \left(\frac{a_{j+1}}{p} + \frac{a_{j+2}}{p^2} + \ldots + \frac{a_l}{p^{l-j}} + \frac{a_1}{p^{l+1-j}} + \ldots\right) - r \\
&= \left(a_1p^{j-1} + a_2p^{j-2} + \ldots + a_j\right) - r \\
&\quad + \left(\frac{a_{j+1}}{p} + \frac{a_{j+2}}{p^2} + \ldots + \frac{a_l}{p^{l-j}} + \frac{a_1}{p^{l+1-j}} + \ldots\right).
\end{aligned}$$

Since $a_1p^{j-1} + a_2p^{j-2} + \ldots + a_j \in \mathbb{Z}$, $r \in \mathbb{Z}$ and $0 \leq \frac{m_op^j - rn}{n} < 1$, we have

$\left(a_1p^{j-1} + a_2p^{j-2} + \ldots + a_j\right) - r = 0.$

Then

$$\begin{aligned}
\frac{m_0p^j - rn}{n} &= \frac{a_{j+1}}{p} + \frac{a_{j+2}}{p^2} + \ldots + \frac{a_l}{p^{l-j}} + \frac{a_1}{p^{l+1-j}} + \ldots + \frac{a_j}{p^l} + \frac{a_{j+1}}{p^{l+1}} + \ldots \\
&= \left(0.\overline{a_{j+1}a_{j+2}\ldots a_j}\right)_p \\
&= T^j\left(\left(0.\overline{a_1\ldots a_l}\right)_p\right) \\
&= T^j\left(\frac{m_0}{n}\right) \\
&= \frac{m_j}{n}.
\end{aligned}$$

Hence, $\frac{m_0p^j-rn}{n} = \frac{m_j}{n}$. This implies that $m_j = m_0p^j - rn$.

Since $m_0p^j - rn \equiv m_0p^j \pmod{n}$, Therefore, $m_j \equiv m_0p^j \pmod{n}$.

Case 2: Assume that $|P| = 2l$, and $P = \left\{\frac{m_0}{n}, \frac{m_1}{n}, \ldots, \frac{m_{l-1}}{n}, \frac{m_l}{n}, \frac{m_{l+1}}{n}, \ldots, \frac{m_{2l-1}}{n}\right\}$.

We will prove on 2 cases.

Subcase 2.1 If $0 \leq j < l$, then the way to prove for this case is similarly with **case 1**.

Subcase 2.2 If $l \leq j < 2l$, then we will describe as follows.

We observe that

$$\begin{aligned}
T^j &= T^{l+(j-l)} \\
&= T^l \cdot T^{j-l} \\
&= I \cdot T^{j-l} \\
&= T^{j-l}.
\end{aligned}$$

Hence $T^j = T^{j-l}$.

We know that by division algorithm, there exists $r > 0$ and $s \in \mathbb{Z}$ such that

$$-m_0p^j = (-r)\,n + s,$$

where $0 \leq s < n$.

Then

$$\begin{aligned}
s &= -m_0p^j - (-rn) \\
&= -m_0p^j + rn.
\end{aligned}$$

Notice that

$$\frac{-m_0p^j + rn}{n} = \frac{-m_0p^j}{n} + \frac{rn}{n}$$

$$\begin{aligned}
\frac{-m_0p^j+rn}{n} &= \frac{-m_0p^j}{n}+r \\
&= -p^j\left(\frac{m_0}{n}\right)+r \\
&= -p^j\left(\frac{a_1}{p}+\ldots+\frac{a_l}{p^l}+\frac{a_1}{p^{l+1}}+\ldots+\frac{a_{j-l}}{p^j}+\ldots+\frac{a_l}{p^{2l}}+\frac{a_1}{p^{2l+1}}+\ldots\right) \\
&\quad +r \\
&= -p^j\left(\frac{a_1}{p}+\ldots+\frac{a_l}{p^l}+\frac{a_1}{p^{l+1}}+\ldots+\frac{a_{j-l}}{p^j}\right) \\
&\quad -p^j\left(\frac{a_{j-l+1}}{p^{j+1}}+\ldots+\frac{a_l}{p^{2l}}+\frac{a_1}{p^{2l+1}}+\ldots\right)+r \\
&= -\left(a_1p^{j-1}+a_2p^{j-2}+\ldots+a_lp^{j-l}+\ldots+a_{j-l}\right) \\
&\quad -\left(\frac{a_{j-l+1}}{p}+\frac{a_{j-l+2}}{p^2}+\ldots+\frac{a_l}{p^{2l-j}}+\frac{a_1}{p^{2l+1-j}}+\frac{a_{j-l}}{p^{2l}}+\ldots\right)+r \\
&= r-\left(a_1p^{j-1}+a_2p^{j-2}+\ldots+a_lp^{j-n}+\ldots+a_{j-l}\right) \\
&\quad -\left(\frac{a_{j-l+1}}{p}+\frac{a_{j-l+2}}{p^2}+\ldots+\frac{a_l}{p^{2l-j}}+\frac{a_1}{p^{2l+1-j}}+\frac{a_{j-l}}{p^{2l}}+\ldots\right)
\end{aligned}$$

Since $a_1p^{j-1}+a_2p^{j-2}+\ldots+a_np^{j-l}+\ldots+a_{j-l}\in\mathbb{Z}$, $r\in\mathbb{Z}$ and $0\leq\frac{-m_op^j+rn}{n}<1$, we have

$$r-\left(a_1p^{j-1}+a_2p^{j-2}+\ldots+a_lp^{j-l}+\ldots+a_{j-l}\right)=1.$$

It follows that

$$\begin{aligned}
\frac{-m_0p^j+rn}{n} &= 1-\left(\frac{a_{j-l+1}}{p}+\frac{a_{j-l+2}}{p^2}+\ldots+\frac{a_l}{p^{2l-j}}+\frac{a_1}{p^{2l+1-j}}+\ldots+\frac{a_{j-l}}{p^{2l}}+\ldots\right) \\
&= 1-(0.\overline{a_{j-l+1}a_{j-l+2}\ldots a_la_1a_2\ldots a_{j-l}})_p \\
&= 1-T^{j-l}\left((0.\overline{a_1\ldots a_l})_p\right) \\
&= 1-T^{j-l}\left(\frac{m_0}{n}\right)
\end{aligned}$$

Since $l\leq j<2l$, then $0\leq j-l<l$. By assumption in the **Case 1**, $T^j\left(\frac{m_0}{n}\right)=\frac{m_j}{n}$, where $0\leq j<l$, we have

$$\frac{-m_0p^j+rn}{n}=1-\frac{m_{j-l}}{n}$$

$$\frac{-m_0p^j + rn}{n} = \frac{n - m_{j-l}}{n} = \frac{m_j}{n}$$

Hence, $\frac{-m_0p^j+rn}{n} = \frac{m_j}{n}$. This implies that $m_j = -m_0p^j + rn$.

Since $-m_0p^j + rn \equiv -m_0p^j \pmod{n}$, Therefore, $m_j \equiv -m_0p^j \pmod{n}$.

This completes the proof. □

Finally, we give the advantage of the previous theorem. The theorem will be used to identify the elements in the same partition and we then determine all elements in spawning p-ary sets A_n^p. This observation is shown in Tables 5.3 and 5.4.

Table 5.3: Characteristic of elements in each partition of the set A_{122}^{11}

P_1	P_2	P_3
$1 \equiv 1 \cdot 11^0 \pmod{122}$	$3 \equiv 3 \cdot 11^0 \pmod{122}$	$5 \equiv 5 \cdot 11^0 \pmod{122}$
$11 \equiv 1 \cdot 11^1 \pmod{122}$	$33 \equiv 3 \cdot 11^1 \pmod{122}$	$55 \equiv 5 \cdot 11^1 \pmod{122}$
$121 \equiv 1 \cdot 11^2 \pmod{122}$	$119 \equiv 3 \cdot 11^2 \pmod{122}$	$117 \equiv 5 \cdot 11^2 \pmod{122}$
$111 \equiv 1 \cdot 11^3 \pmod{122}$	$89 \equiv 3 \cdot 11^3 \pmod{122}$	$67 \equiv 5 \cdot 11^3 \pmod{122}$
P_4	P_5	P_6
$7 \equiv 7 \cdot 11^0 \pmod{122}$	$9 \equiv 9 \cdot 11^0 \pmod{122}$	$25 \equiv 25 \cdot 11^0 \pmod{122}$
$77 \equiv 7 \cdot 11^1 \pmod{122}$	$99 \equiv 9 \cdot 11^1 \pmod{122}$	$31 \equiv 25 \cdot 11^1 \pmod{122}$
$115 \equiv 7 \cdot 11^2 \pmod{122}$	$113 \equiv 9 \cdot 11^2 \pmod{122}$	$97 \equiv 25 \cdot 11^2 \pmod{122}$
$45 \equiv 7 \cdot 11^3 \pmod{122}$	$23 \equiv 9 \cdot 11^3 \pmod{122}$	$91 \equiv 25 \cdot 11^3 \pmod{122}$
P_7	P_8	P_9
$27 \equiv 27 \cdot 11^0 \pmod{122}$	$29 \equiv 29 \cdot 11^0 \pmod{122}$	$49 \equiv 49 \cdot 11^0 \pmod{122}$
$53 \equiv 27 \cdot 11^1 \pmod{122}$	$75 \equiv 29 \cdot 11^1 \pmod{122}$	$51 \equiv 49 \cdot 11^1 \pmod{122}$
$95 \equiv 27 \cdot 11^2 \pmod{122}$	$93 \equiv 29 \cdot 11^2 \pmod{122}$	$73 \equiv 49 \cdot 11^2 \pmod{122}$
$69 \equiv 27 \cdot 11^3 \pmod{122}$	$47 \equiv 29 \cdot 11^3 \pmod{122}$	$71 \equiv 49 \cdot 11^3 \pmod{122}$

To clarify the observation, we found that the partition P_1 consist of four elements, these are $\frac{1}{122}$, $\frac{11}{122}$, $\frac{121}{122}$ and $\frac{111}{122}$. Also we observe that $\frac{5}{122}$, $\frac{55}{122}$, $\frac{117}{122}$ and $\frac{67}{122}$ are all elements in the partition P_3.

Table 5.4: Characteristic of elements in each partition of the set A^5_{312}

P_1	P_2
$1 \equiv 1 \cdot 5^0 \pmod{312}$	$7 \equiv 7 \cdot 5^0 \pmod{312}$
$5 \equiv 1 \cdot 5^1 \pmod{312}$	$35 \equiv 7 \cdot 5^1 \pmod{312}$
$25 \equiv 1 \cdot 5^2 \pmod{312}$	$175 \equiv 7 \cdot 5^2 \pmod{312}$
$125 \equiv 1 \cdot 5^3 \pmod{312}$	$251 \equiv 7 \cdot 5^3 \pmod{312}$
$311 \equiv -1 \cdot 5^4 \pmod{312}$	$305 \equiv -7 \cdot 5^4 \pmod{312}$
$307 \equiv -1 \cdot 5^5 \pmod{312}$	$277 \equiv -7 \cdot 5^5 \pmod{312}$
$287 \equiv -1 \cdot 5^6 \pmod{312}$	$137 \equiv -7 \cdot 5^6 \pmod{312}$
$187 \equiv -1 \cdot 5^7 \pmod{312}$	$61 \equiv -7 \cdot 5^7 \pmod{312}$
P_3	P_4
$11 \equiv 11 \cdot 5^0 \pmod{312}$	$31 \equiv 31 \cdot 5^0 \pmod{312}$
$55 \equiv 11 \cdot 5^1 \pmod{312}$	$155 \equiv 31 \cdot 5^1 \pmod{312}$
$275 \equiv 11 \cdot 5^2 \pmod{312}$	$151 \equiv 31 \cdot 5^2 \pmod{312}$
$127 \equiv 11 \cdot 5^3 \pmod{312}$	$131 \equiv 31 \cdot 5^3 \pmod{312}$
$301 \equiv -11 \cdot 5^4 \pmod{312}$	$281 \equiv -31 \cdot 5^4 \pmod{312}$
$257 \equiv -11 \cdot 5^5 \pmod{312}$	$157 \equiv -31 \cdot 5^5 \pmod{312}$
$37 \equiv -11 \cdot 5^6 \pmod{312}$	$161 \equiv -31 \cdot 5^6 \pmod{312}$
$185 \equiv -11 \cdot 5^7 \pmod{312}$	$181 \equiv -31 \cdot 5^7 \pmod{312}$

The above table tell us $\frac{1}{312}$, $\frac{5}{312}$, $\frac{25}{312}$, $\frac{125}{312}$, $\frac{311}{312}$, $\frac{307}{312}$, $\frac{287}{312}$ and $\frac{187}{312}$ will be elements in the partition P_1. Further, the partition P_2 contains the elements $\frac{7}{312}$, $\frac{35}{312}$, $\frac{175}{312}$, $\frac{251}{312}$, $\frac{305}{312}$, $\frac{277}{312}$, $\frac{137}{312}$ and $\frac{61}{312}$.

Conclusion

This thesis consists of five chapters.

Chapter 1: We give the definition of the Cantor set $\mathfrak{C}$ and show that it is uncountable. Then we present objectives of the thesis. The first objective is to determine the relation of cardinality of spawning p-ary set A_n^p and child p-ary sets $A_{p^k n}^p$. The second objective is to find a group structure on Cantor p-ary sets $\mathfrak{C}_p$. Finally, we introduce some definitions and notation that will be used throughout this thesis.

Chapter 2: We show the construction of the Cantor p-ary sets $\mathfrak{C}_p$ and prove that all elements in $\mathfrak{C}_p$ can be written in the base p expansion of the form $x = \sum_{i=1}^{\infty} \frac{a_i}{p^i}$ or $x = 0.a_1 a_2 a_3 \ldots a_i \ldots$ where $a_i \in \{0, 2, 4, \ldots, p-1\}$.

Chapter 3: We illustrate the using of the formula for transformation rational numbers to the base p expansion. The formula is

$$a_i = \lfloor p \cdot \gamma_{i-1} \rfloor, \qquad \gamma_i = p \cdot \gamma_{i-1} - \lfloor p \cdot \gamma_{i-1} \rfloor$$

where $\gamma_0 = \frac{m}{n}$, and $k = 1, 2, 3, \ldots$. Then, we categorized the elements in Cantor p-ary sets into three types by considering the pre-period and the period length. The elements either be terminating, purely periodic or mix periodic. Moreover, we discover that if $x \in A_n^p$, then x is a purely periodic.

Chapter 4: This chapter presents the first objective of thesis. We determine the relationship of cardinality of spawning p-ary set A_n^p and their child p-ary sets A_{pn}^p . The theorem states that

> Let p be an odd prime and $K_p^e = \{0, 2, 4, \ldots, p-1\}$. If A_{pn}^p is a child p-ary set of spawning $p-$ary set A_n^p, then $|A_{pn}^p| = \left(|K_p^e| - 1\right) \cdot |A_n^p|$.

Furthermore, the relation of cardinality of spawning p-ary set A_n^p and their child p-ary sets $A_{p^k n}^p$ will be as

$$\left|A_{p^k n}^p\right| = \left(\left|K_p^e\right|^k - \left|K_p^e\right|^{k-1}\right) \cdot \left|A_n^p\right|.$$

Chapter 5: This chapter presents the second objective of the thesis. We construct a group G which its elements are generated by the transformation R and the transformation T and we prove that

(1) $T^l = I, R^2 = I$, where I is an identity function and l be the period length of elements in spawning p-ary sets A_n^p

(2) T and R commute

(3) G is isomorphic to $\mathbb{Z}_l \times \mathbb{Z}_2$

(4) G_T, the subgroup generated by T alone, is a faithful cyclic subgroup of order l.

Then we define the equivalence relation $\sim$ as follows:

Let $x = \frac{a}{n}$ and $y = \frac{b}{n}$ be elements in A_n^p and denote

$$x \sim y \leftrightarrow a \sim b \leftrightarrow b \equiv (-1)^m ap^j \pmod{n},$$

for some $j \in \{0, 1, \dots, l-1\}, m \in \{0, 1\}$.

We found that each $A_{n/\sim}^p$ contains l or $2l$ elements, $l \mid |A_n^p|$ and $l \mid \left|A_{pn}^p\right|$. Finally, the characteristic of elements in each partition expressed by letting A_n^p be a spawning p-ary set with a period length l of elements in A_n^p. Let $P = \left\{\frac{m_0}{n}, \frac{m_1}{n}, \dots, \frac{m_k}{n}\right\}$, where $k \leq 2l$ and m_0 is the least element among $m_0, \dots, m_k$, be a partition of A_n^p. Then

$$m_j \equiv \begin{cases} m_0 p^j \pmod{n}, & \text{if } 0 \leq j < l; \\ -m_0 p^j \pmod{n}, & \text{if } l \leq j < 2l. \end{cases}$$

References

Jordan, D. and Schayer, R. n.d. Group Structure on Cantor Rationals. Available online: https://www.math.psu.edu/mass/reu/2003/DaRi/proj.pdf [January 15, 2015]

Kraft, J.S., and Washington, L.C. 2015. Elementary Number Theory, CRC Press, 56-100.

Kunczynski, L.F. 1968. The Cantor Set: Some Properties, Variation, and Homeomorphisms. Master Thesis, Southern Connecticut State University.

Malik, D.S., Mordeson, J.M. and Sen, M.K. 1997. Fundamentals of Abstract Algebra, McGraw-Hill, New York, U.S.A., 36-116.

Nagy, J. 2001. Rational points in cantor sets. Fibonacci Quarterly. 39(3), 238-241.

Nelson, D.R. n.d. The Cantor set-A Brief introduction. Available online: https:// www .cfa.harvard.edu/ dnelson/storage/dnelson.cantor-set.pdf [February 10, 2015]

Phon-On, A. 2013. Rational points in cantor p-ary sets. International Journal of Mathematical Analysis. 7(31), 1497-1511.

Rosen, K.H. 1993. Elementary Number Theory and Its Applications, 3^{rd} ed, Addison-Wesley Publishing Company, Boston, U.S.A., 338-341.

Schayer, R. and Jordan, D. n.d. Rational points in the cantor middle thirds set. Available online: https://www.math.psu.edu/mass/reu/2003/DaRi/proj1.pdf [January 30, 2015]

Sella, Y. n.d. Countability. Available online: http:// www.math.ucla.edu/ ysella/ Math115AHCardinality.pdf [January 30, 2015]

Woolley, D.A. 2008. Generic Continuous Functions and Other Strange Functions in Classical Real Analysis. Master Thesis, College of Arts and Sciences, Georgia State University.

Appendix

(1) Conference Certificate and paper

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ขอมอบเกียรติบัตรเพื่อแสดงว่า

ริฎวน แวมะ

ได้นำเสนอผลงานและเข้าร่วม

การประชุมวิชาการทางคณิตศาสตร์ประจำปี 2559 ครั้งที่ 21 (AMM 2016)
การประชุมวิชาการคณิตศาสตร์บริสุทธิ์และประยุกต์ประจำปี 2559 (APAM 2016)
ณ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ระหว่างวันที่ 23 – 25 พฤษภาคม 2559

ศาสตราจารย์ ดร.กฤษณะ เนียมมณี)
ประธานคณะกรรมการจัดการประชุม

(ศาสตราจารย์ ดร.ชิดชนก เหลือสินทรัพย์)
หัวหน้าภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

Cardinality of Child $p-$ary Sets in Cantor $p-$ary Sets

Riduan Waema*[1], Aniruth Phon-On[2] and Saofee Busaman[3]

[1,2,3]Prince of Songkla University, Pattani Campus

[1]akhuimron@gmail.com, [2]aniruth.p@psu.ac.th, [3]saofee.b@psu.ac.th

Abstract

Let p be an odd prime number. For a positive integer n with $p \nmid n$, denote the set of all Cantor p-ary rationals with denominator n by A_n, and call it a spawning p-ary set. For $k \in \mathbb{N}$, define $A_{p^k n}$ to be the set of all Cantor p-ary rationals with denominator $p^k n$, and call it a child p-ary set of A_n. In this paper, we show that for $k \in \mathbb{N}$, $|A_{p^k n}| = \left(|K_p^e|^k - |K_p^e|^{k-1}\right) \cdot |A_n|$, where $K_p^e = \{0, 2, \ldots, p-1\}$. In addition, examples of spawning p-ary sets with child p-ary sets are provided.

Mathematics Subject Classification: 03E10

Keywords: cantor p-ary sets, spawning p-ary sets, child p-ary sets, cardinality of sets

1 Introduction

The Cantor set or the Cantor middle thirds set was constructed by Georg Cantor [5]. It has interesting properties and special construction. In the first step, we set $A_0 = [0,1]$, then divide the closed interval into 3 equal intervals and remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$. It follows that the new set $A_1 = \left\{\left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right]\right\}$ is obtained. In the second step, we again subdivide each element in A_1 into three subintervals and remove the middle thirds $\left\{\left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right)\right\}$. Hence, the set A_2 will be $\left\{\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right]\right\}$. If we repeat this process, subdivide each element in A_{n-1}, where $n = 1, 2, 3, \ldots$, and remove the middle thirds respectively, these will generate all elements in A_n. Therefore, the Cantor set $\mathfrak{C}$ defined as $\mathfrak{C} = \cap_{n=0}^{\infty} (\cup A_n)$ is the intersection of all $\cup A_n$, where $\cup A_n$ is the union of all elements in A_n. For more details see [2] and [4]. We generalize the Cantor set by dividing the interval $[0,1]$ and each subinterval which are subsets of the interval into p subintervals, where p is an odd prime and call it a Cantor p-ary set. See more details in [1] and [6].

In [2], [3] and [5], the authors proved that the Cantor set is an uncountable set. Then, in [6], Phon-On defined the set of all rational numbers with denominator n and call it a spawning p-ary set A_n, where p does not divide n and $A_n \neq \phi$. Furthermore, the set of all rational numbers with denominator

* Corresponding author

$p^k n$, where $k = 1, 2, 3, \ldots$ will be called a child p-ary set $A_{p^k n}$ of A_n. Also he showed that $|A_{pn}| \geq |A_n|$ and $|A_{p^k n}| = |K_p^e|^{k-1} \cdot |A_{pn}|$, where $K_p^e = \{0, 2, 4, \ldots, p-1\}$. In this paper, we aim to improve the previous formula in order to complete the results.

2 Preliminaries

Before proving the main results, we should understand some definitions and theorems in [6] that will be used in this paper.

Definition 2.1. Let $n \in \mathbb{Z}^+$ and $p \in \mathbb{N}$, where p is an odd prime. Define an interval $\Theta(k_1, \ldots, k_n)$ by

$$\Theta(k_1, \ldots, k_n) = \left[\sum_{i=1}^{n} \frac{k_i}{p^i}, \sum_{i=1}^{n} \frac{k_i}{p^i} + \frac{1}{p^n}\right]$$

for $k_i \in K_p$, where $K_p = \{0, 1, 2, \ldots, p-1\}$ and denote $K_p^e = \{0, 2, 4, \ldots, p-1\}, K_p^o = \{1, 3, 5, \ldots, p-2\}$ and $C_n^p = \{\Theta(k_1, k_2, \ldots, k_n) \, | k_i \in K_p^e\}$. **The Cantor p-ary set $\mathfrak{C}_p$** is defined as:

$$\mathfrak{C}_p = \cap_{n=0}^{\infty} (\cup C_n^p).$$

Definition 2.2. A rational number $\frac{m}{n} \in \mathbb{Q}$ is a **Cantor p-ary rational** if it satisfies the following conditions:

1. $\frac{m}{n}$ is in the Cantor p-ary set $\mathfrak{C}_p$.
2. m and n are relatively prime, i.e. gcd $(m, n) = 1$.

Denote A_n the set of all Cantor p-ary rationals with denominator n and $|A_n|$ the number of all Cantor p-ary rationals with denominator n.

Definition 2.3. Let $n \in \mathbb{N}$ be such that p does not divide n. Then A_n is said to be a **spawning p-ary set** if $A_n \neq \phi$ and the sets $A_{pn}, A_{p^2 n}, A_{p^3 n}, \ldots$ are called the **child p-ary sets** of A_n.

Example 2.4. In the base 5 expansion, we found that $\frac{1}{12}, \frac{5}{12}, \frac{7}{12}$ and $\frac{11}{12}$ all are Cantor 5-ary rationals in the set A_{12}, whereas $\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \ldots, \frac{10}{11}$ are not Cantor 5-ary rationals in the set A_{11}. Since $A_{12} \neq \phi$ and gcd $(5, 12) = 1, A_{12}$ will be called a spawning 5-ary set. In contrast, $A_{11} = \phi$ and gcd $(5, 11) = 1, A_{11}$ will not be said a spawning 5-ary set.

3 Cardinality of Child $p-$ary sets

In order to determine a cardinality of a spawning p-ary set and child p-ary sets, we firstly consider some example of those sets. According to [6], for given n, A_{p^n+1} and $A_{\frac{p^n-1}{2}}$ are general forms of spawning p-ary sets. We substitute $p = 5, 7, 11, 13$ and $n = 1, 2, 3, 4$ on the forms. Then some spawning p-ary sets will be shown in the following table.

p	5		7		11		13	
n	A_{5^n+1}	$A_{\frac{5^n-1}{2}}$	A_{7^n+1}	$A_{\frac{7^n-1}{2}}$	A_{11^n+1}	$A_{\frac{11^n-1}{2}}$	A_{13^n+1}	$A_{\frac{13^n-1}{2}}$
1	A_6	A_2	A_8	A_3	A_{12}	A_5	A_{14}	A_6
2	A_{26}	A_{12}	A_{50}	A_{24}	A_{122}	A_{60}	A_{170}	A_{84}
3	A_{126}	A_{62}	A_{344}	A_{171}	A_{1332}	A_{665}	A_{2198}	A_{1098}
4	A_{626}	A_{312}	A_{2402}	A_{1200}	A_{14642}	A_{7320}	A_{28562}	A_{14280}

Table 1: Some spawning p-ary sets

We illustrate the child p-ary sets, where $p = 5, 7, 11$, and 13, of some A_n in Table 2.

A_n	$A_{5^k n}$	A_n	$A_{7^k n}$
A_2	$A_{10}, A_{50}, A_{250}, A_{1250}, \ldots$	A_3	$A_{21}, A_{147}, A_{1029}, \ldots$
A_6	$A_{30}, A_{150}, A_{750}, A_{3750}, \ldots$	A_8	$A_{56}, A_{392}, A_{2744}, \ldots$
A_{12}	$A_{60}, A_{300}, A_{1500}, \ldots$	A_{24}	$A_{168}, A_{1176}, A_{8232}, \ldots$
A_{26}	$A_{130}, A_{650}, A_{3250}, \ldots$	A_{50}	$A_{350}, A_{2450}, A_{17150}, \ldots$
A_{62}	$A_{310}, A_{1550}, A_{7750}, \ldots$	A_{171}	$A_{1197}, A_{8379}, A_{58653}, \ldots$
A_n	$A_{11^k n}$	A_n	$A_{13^k n}$
A_5	$A_{55}, A_{605}, A_{6655}, \ldots$	A_6	$A_{78}, A_{1014}, A_{13182}, \ldots$
A_{12}	$A_{132}, A_{1452}, A_{15972}, \ldots$	A_{14}	$A_{182}, A_{2366}, A_{30758}, \ldots$
A_{60}	$A_{660}, A_{7260}, A_{79860}, \ldots$	A_{84}	$A_{1092}, A_{14196}, A_{184548}, \ldots$
A_{122}	$A_{1342}, A_{14762}, A_{162382}, \ldots$	A_{170}	$A_{2210}, A_{28730}, A_{373490}, \ldots$

Table 2: Some child p-ary sets, where $p = 5, 7, 11$ and 13

To investigate the elements in a spawning p-ary set A_n and its child p-ary sets, we use the following method.

First we check all rational numbers with denominator n which are Cantor p-ary rationals. Given a rational number $\frac{m}{n}$, if gcd $(m, n) \neq 1$, it suffices to say that $\frac{m}{n} \notin A_n$. Since A_n is a spawning p-ary set and it is a subset of Cantor p-ary set $\mathfrak{C}_p$. If $\frac{m}{n} \in A_n$, then $0 < \frac{m}{n} < 1$ and we have that $0 < m < n$.

Consequently, we conclude that gcd $(m,n)=1$. We then use the following formula in [2] to convert $\frac{m}{n}$ to the base p expansions. The main formula can be expressed in the following form,

$$c_k = \lfloor p\cdot\gamma_{k-1}\rfloor\,, \qquad \gamma_k = p\cdot\gamma_{k-1} - \lfloor p\cdot\gamma_{k-1}\rfloor \tag{3.1}$$

where $\gamma_0=\frac{m}{n}$, and $k=1,2,3,\ldots$

From the above formula a rational number $\frac{m}{n}$ can either be written in the base p expansions as $\frac{m}{n} = \frac{c_1}{p}+\frac{c_2}{p^2}+\frac{c_3}{p^3}+\ldots+\frac{c_n}{p^n}+\frac{c_1}{p^{n+1}}+\frac{c_2}{p^{n+2}}+\ldots$ or the period form as $\frac{m}{n} = (0.\overline{c_1c_2\ldots c_n})_p$. Then, if $c_k\notin\{0,2,4,...,p-1\}$ for some k, it implies that $\frac{m}{n}\notin A_n$. While, if $c_k\in\{0,2,4,...,p-1\}$ for all k, we conclude that $\frac{m}{n}\in A_n$. To clarify the transformation to the base p expansions, we will illustrate three following examples.

Example 3.1. On base 7, the numbers $\frac{2}{50},\frac{10}{50},\frac{25}{50}\notin A_{50}$. Since gcd $(2,50)$, gcd $(10,50)$ and gcd$(25,50)$ are not equal to 1.

Example 3.2. We convert a rational number $\frac{3}{50}$ to the base 7 expansion by applying the above formula.

$$c_1=\left\lfloor 7\cdot\frac{3}{50}\right\rfloor=0, \qquad \gamma_1=7\cdot\frac{3}{50}-0=\frac{21}{50},$$

$$c_2=\left\lfloor 7\cdot\frac{21}{50}\right\rfloor=2, \qquad \gamma_2=7\cdot\frac{21}{50}-2=\frac{47}{50},$$

$$c_3=\left\lfloor 7\cdot\frac{47}{50}\right\rfloor=6, \qquad \gamma_3=7\cdot\frac{47}{50}-6=\frac{29}{50},$$

$$c_4=\left\lfloor 7\cdot\frac{29}{50}\right\rfloor=4, \qquad \gamma_4=7\cdot\frac{29}{50}-4=\frac{3}{50},$$

$$c_5=\left\lfloor 7\cdot\frac{3}{50}\right\rfloor=0, \qquad \gamma_5=7\cdot\frac{3}{50}-0=\frac{21}{50},$$

$$c_6=\left\lfloor 7\cdot\frac{21}{50}\right\rfloor=2, \qquad \gamma_6=7\cdot\frac{21}{50}-2=\frac{47}{50},$$

and so on. Hence, the rational number $\frac{3}{50}$ can be written either in base 7 expansion as $\frac{3}{50}=\frac{0}{7}+\frac{2}{7^2}+\frac{6}{7^3}+\frac{4}{7^4}+\frac{0}{7^5}+\frac{2}{7^6}+\ldots$ or the period form as $\frac{3}{50}=\left(0.\overline{0264}\right)_7$.

Example 3.3. We will convert $\frac{97}{171}$ to the base 7 expansion.

$$c_1=\left\lfloor 7\cdot\frac{97}{171}\right\rfloor=3, \qquad \gamma_1=7\cdot\frac{97}{171}-3=\frac{166}{171},$$

$$c_2 = \left\lfloor 7 \cdot \frac{166}{171} \right\rfloor = 6, \qquad \gamma_2 = 7 \cdot \frac{166}{171} - 6 = \frac{136}{171},$$

$$c_3 = \left\lfloor 7 \cdot \frac{136}{171} \right\rfloor = 5, \qquad \gamma_3 = 7 \cdot \frac{136}{171} - 5 = \frac{97}{171},$$

$$c_4 = \left\lfloor 7 \cdot \frac{97}{171} \right\rfloor = 3, \qquad \gamma_4 = 7 \cdot \frac{97}{171} - 3 = \frac{166}{171},$$

$$c_5 = \left\lfloor 7 \cdot \frac{166}{171} \right\rfloor = 6, \qquad \gamma_5 = 7 \cdot \frac{166}{171} - 6 = \frac{136}{171},$$

$$c_6 = \left\lfloor 7 \cdot \frac{136}{171} \right\rfloor = 5, \qquad \gamma_6 = 7 \cdot \frac{136}{171} - 5 = \frac{97}{171},$$

Thus, we can write $\frac{97}{171} = \frac{3}{7} + \frac{6}{7^2} + \frac{5}{7^3} + \frac{3}{7^4} + \frac{6}{7^5} + \frac{5}{7^6} + \ldots = \left(0.\overline{365}\right)_7$. Since there exists $c_k \notin \{0, 2, 4, 6\}$ for some k, this implies that $\frac{97}{171} \notin A_{171}$.

Secondly, we will determine all elements in spawning p-ary sets. The initial step has shown a transformation $\frac{m}{n}$ to the period form as $(0.\overline{c_1 c_2 \ldots c_n})_p$. In this step, all elements $\frac{m}{n} = (0.\overline{c_1 c_2 \ldots c_n})_p$, which all c_k are in $\{0, 2, 4, \ldots, p-1\}$, will be collected in spawning p-ary set. Then the following tables show us all Cantor p-ary rationals in some spawning p-ary sets.

$\frac{1}{50} = 0.\overline{0066}$	$\frac{49}{50} = 0.\overline{6600}$
$\frac{3}{50} = 0.\overline{0264}$	$\frac{47}{50} = 0.\overline{6402}$
$\frac{7}{50} = 0.\overline{0660}$	$\frac{43}{50} = 0.\overline{6006}$
$\frac{17}{50} = 0.\overline{2244}$	$\frac{33}{50} = 0.\overline{4422}$
$\frac{19}{50} = 0.\overline{2442}$	$\frac{31}{50} = 0.\overline{4224}$
$\frac{21}{50} = 0.\overline{2640}$	$\frac{29}{50} = 0.\overline{4026}$

Table 3: All elements in a spawning 7-ary set with denominator 50

Assume that A and C represent the number 10 and the number 12, respectively, in the base 13 expansion. Then Table 4 shows us all Cantor 13-ary rationals in spawning 13-ary set with denominator 84.

$\frac{1}{84} = 0.\overline{02}$	$\frac{83}{84} = 0.\overline{CA}$	$\frac{19}{84} = 0.\overline{2C}$	$\frac{65}{84} = 0.\overline{A0}$
$\frac{5}{84} = 0.\overline{0A}$	$\frac{79}{84} = 0.\overline{C2}$	$\frac{29}{84} = 0.\overline{46}$	$\frac{55}{84} = 0.\overline{86}$
$\frac{13}{84} = 0.\overline{20}$	$\frac{71}{84} = 0.\overline{AC}$	$\frac{31}{84} = 0.\overline{4A}$	$\frac{53}{84} = 0.\overline{82}$
$\frac{17}{84} = 0.\overline{28}$	$\frac{67}{84} = 0.\overline{A4}$	$\frac{41}{84} = 0.\overline{64}$	$\frac{43}{84} = 0.\overline{68}$

Table 4: All elements in a spawning 13-ary set with denominator 84

Consequently, we show the number of Cantor p-ary rationals in some spawning p-ary sets in the following table.

spawning 5-ary set		spawning 7-ary set		spawning 11-ary set		spawning 13-ary set	
A_n	$\|A_n\|$	A_n	$\|A_n\|$	A_n	$\|A_n\|$	A_n	$\|A_n\|$
A_2	1	A_3	2	A_5	4	A_6	2
A_6	2	A_8	4	A_{12}	4	A_{14}	6
A_{12}	4	A_{24}	4	A_{60}	8	A_{84}	16
A_{26}	8	A_{50}	12	A_{122}	36	A_{170}	40
A_{62}	12	A_{171}	60	A_{665}	150		

Table 5: Cardinality of some spawning p-ary sets

Thirdly, we will find out elements in the child p-ary sets $A_{p^k n}$. We use the same formula to convert all elements $\frac{m}{p^k n}$, where $k = 1, 2, 3, \ldots$ and gcd $(m, p^k n) = 1$ to the base p expansions. Also, all elements $\frac{m}{p^k n}$ whose all c_k are in $\{0, 2, 4, \ldots, p-1\}$ will be collected into $A_{p^k n}$. Table 6 represents the number of all elements in some child p-ary sets, where $k = 1$.

spawning 5-ary set	child 5-ary set		spawning 7-ary set	child 7-ary set	
A_n	A_{5n}	$\|A_{5n}\|$	A_n	A_{7n}	$\|A_{7n}\|$
A_2	$A_{5\cdot 2}$	2	A_3	$A_{7\cdot 3}$	6
A_6	$A_{5\cdot 6}$	4	A_8	$A_{7\cdot 8}$	12
A_{12}	$A_{5\cdot 12}$	8	A_{24}	$A_{7\cdot 28}$	12
A_{26}	$A_{5\cdot 26}$	16	A_{50}	$A_{7\cdot 50}$	36
spawning 11-ary set	child 11-ary set		spawning 13-ary set	child 13-ary set	
A_n	A_{11n}	$\|A_{11n}\|$	A_n	A_{13n}	$\|A_{13n}\|$
A_5	$A_{11\cdot 5}$	20	A_6	$A_{13\cdot 6}$	12
A_{12}	$A_{11\cdot 12}$	20	A_{14}	$A_{13\cdot 14}$	36
A_{60}	$A_{11\cdot 60}$	40			

Table 6: Cardinality of some child p-ary sets

We have illustrated the method for investigating the elements in spawning p-ary sets and child p-ary sets. In the last process we will compare the number of elements in a spawning set with its child p-ary sets.

Finally, we now compare the cardinalities of spawning p-ary sets and their child p-ary sets, where $k = 1$. Table 7 will show the cardinalities of some spawning p-ary sets and their child p-ary sets.

$p=5$		$p=7$	
$\lvert A_n \rvert$	$\lvert A_{5n} \rvert$	$\lvert A_n \rvert$	$\lvert A_{7n} \rvert$
$\lvert A_2 \rvert = 1$	$\lvert A_{10} \rvert = 2$	$\lvert A_3 \rvert = 2$	$\lvert A_{21} \rvert = 6$
$\lvert A_6 \rvert = 2$	$\lvert A_{30} \rvert = 4$	$\lvert A_8 \rvert = 4$	$\lvert A_{56} \rvert = 12$
$\lvert A_{12} \rvert = 4$	$\lvert A_{60} \rvert = 8$	$\lvert A_{24} \rvert = 4$	$\lvert A_{168} \rvert = 12$
$\lvert A_{26} \rvert = 8$	$\lvert A_{130} \rvert = 16$	$\lvert A_{50} \rvert = 12$	$\lvert A_{350} \rvert = 36$
$p=11$		$p=13$	
$\lvert A_n \rvert$	$\lvert A_{11n} \rvert$	$\lvert A_n \rvert$	$\lvert A_{13n} \rvert$
$\lvert A_5 \rvert = 4$	$\lvert A_{55} \rvert = 20$	$\lvert A_6 \rvert = 2$	$\lvert A_{78} \rvert = 12$
$\lvert A_{12} \rvert = 4$	$\lvert A_{132} \rvert = 20$	$\lvert A_{14} \rvert = 6$	$\lvert A_{182} \rvert = 36$
$\lvert A_{60} \rvert = 8$	$\lvert A_{660} \rvert = 40$		

Table 7: Cardinality of spawning p-ary sets and child p-ary sets

As we can see from the table, we conjecture that $|A_{pn}| = \left(|K_p^e| - 1\right) \cdot |A_n|$, where $K_p^e = \{0, 2, 4, \ldots, p-1\}$. The conjecture will be proved in the following main theorem and the relationship of cardinality of spawning p-ary sets and child p-ary sets will be stated as the corollary.

4 Main Theorem

Before proving the main result, we need the following theorem.

Theorem 4.1. *[6] Let $A_n = \{a_1, a_2, \ldots, a_k\}$ be a spawning p-ary set, where p does not divide n and $a_i \in \mathfrak{C}_p$. Let $A_{pn} = \{b_1, b_2, \ldots, b_r\}$ be a child p-ary set of A_n, where $b_i \in \mathfrak{C}_p$. Then, for each $i \in \{1, \ldots, r\}$, there exist $j \in \{1, \ldots, k\}$ and $l \in K_p^e$ such that $pb_i - l = a_j$. Consequently, $|A_{pn}| \geq |A_n|$ and $|A_{p^k n}| = |K_p^e|^{k-1} |A_{pn}|$ for all $k \geq 2$, and if $p = 3$, then $|A_{3n}| \geq |A_n|$ and hence $|A_{3^k n}| = 2^{k-1} |A_n|$ for all $k \geq 1$.*

We then prove Theorem 4.2 as the main theorem of this article.

Theorem 4.2. *Let p be an odd prime and $K_p^e = \{0, 2, 4, \ldots, p-1\}$. If A_{pn} is a child p-ary set of spawning p–ary set A_n, then $|A_{pn}| = \left(|K_p^e| - 1\right) \cdot |A_n|$.*

Proof. By Theorem 4.1, we have

$$pb_i - l = a_j,$$

where $l \in K_p^e$. Thus

$$pb_i - l = a_j$$
$$pb_i = a_j + l$$
$$b_i = \frac{a_j + l}{p}.$$

Since $a_j = \frac{m}{n} \in A_n$, then

$$b_i = \frac{\frac{m}{n} + l}{p} = \frac{m + nl}{pn}.$$

Let

$$B = \left\{ \frac{m+nl}{pn} \mid l \in K_p^e, \frac{m}{n} \in A_n \right\},$$

We will assert that, if $\frac{m_1}{n}, \frac{m_2}{n} \in A_n, l_1, l_2 \in K_p^e$ and

$$\frac{m_1 + nl_1}{pn} = \frac{m_2 + nl_2}{pn},$$

then $m_1 = m_2$ and $l_1 = l_2$.

Consider

$$\frac{m_1 + nl_1}{pn} = \frac{m_2 + nl_2}{pn},$$

then

$$m_1 + nl_1 = m_2 + nl_2$$
$$m_1 - m_2 = nl_2 - nl_1$$
$$m_1 - m_2 = n\left(l_2 - l_1\right).$$

Consequently,

$$|m_1 - m_2| = |n\left(l_2 - l_1\right)| = n\left|l_2 - l_1\right|.$$

Since both $\frac{m_1}{n}, \frac{m_2}{n} \in A_n$, it follows that $0 < m_1 < n$ and $0 < m_2 < n$.

Hence

$$|m_1 - m_2| < n$$
$$n\left|l_2 - l_1\right| < n$$
$$0 \leq \left|l_2 - l_1\right| < 1.$$

Since $l_1, l_2 \in \mathbb{Z}^+$, then

$$l_2 - l_1 = 0$$

$$l_1 = l_2.$$

Therefore,

$$\begin{aligned} |m_1 - m_2| &= 0 \\ m_1 - m_2 &= 0 \\ m_1 &= m_2. \end{aligned}$$

It is clear that

$$|B| = \left|K_p^e\right| \cdot |A_n| \tag{4.1}$$

The elements in the set B can be categorized by considering the greatest common divisor of $m + nl$ and p, that is

$$\gcd(m + nl, p) = \begin{cases} p, & \text{if } m + nl \equiv 0 \pmod{p}; \\ 1, & \text{if } m + nl \not\equiv 0 \pmod{p}. \end{cases}$$

Let

$$A_{p^*n} = \left\{ \frac{m + nl}{pn} \mid \gcd(m + nl, p) = p \right\}$$

and

$$A_{pn} = \left\{ \frac{m + nl}{pn} \mid \gcd(m + nl, p) = 1 \right\}.$$

Note that A_{p^*n} and A_{pn} are disjoint sets. We then have $B = A_{p^*n} \cup A_{pn}$ where $A_{p^*n} \cap A_{pn} = \phi$.

For A_{p^*n}, we will claim that for each $\frac{m}{n} \in A_n$, there exists a unique $l_0 \in K_p^e$ such that

$$p \mid (m + nl_0).$$

Let $\frac{m}{n} \in A_n$ and consider

$$\begin{aligned} \frac{m}{n} &= (0.\overline{c_1 c_2 \ldots c_n})_p \\ &= \left(\frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_n}{p^n} \right) + \left(\frac{c_1}{p^{n+1}} + \frac{c_2}{p^{n+2}} + \ldots + \frac{c_n}{p^{2n}} \right) + \left(\frac{c_1}{p^{2n+1}} + \frac{c_2}{p^{2n+2}} + \ldots + \frac{c_n}{p^{3n}} \right) + \ldots \\ &= \frac{p^n}{p^n} \left(\frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_n}{p^n} \right) + \frac{p^n}{p^n} \left(\frac{c_1}{p^{n+1}} + \frac{c_2}{p^{n+2}} + \ldots + \frac{c_n}{p^{2n}} \right) \\ &+ \frac{p^n}{p^n} \left(\frac{c_1}{p^{2n+1}} + \frac{c_2}{p^{2n+2}} + \ldots + \frac{c_n}{p^{3n}} \right) + \ldots \\ &= \left(\frac{c_1 p^{n-1}}{p^n} + \frac{c_2 p^{n-2}}{p^n} + \ldots + \frac{c_n}{p^n} \right) + \left(\frac{c_1 p^{n-1}}{p^{2n}} + \frac{c_2 p^{n-2}}{p^{2n}} + \ldots + \frac{c_n}{p^{2n}} \right) \\ &+ \left(\frac{c_1 p^{n-1}}{p^{3n}} + \frac{c_2 p^{n-2}}{p^{3n}} + \ldots + \frac{c_n}{p^{3n}} \right) + \ldots \\ &= \frac{\left(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n\right)}{p^n} + \frac{\left(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n\right)}{p^{2n}} \\ &+ \frac{\left(c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n\right)}{p^{3n}} + \ldots \\ &= \left(c_1 p^{n-1} + c_2 p^{n-2} + c_3 p^{n-3} + \ldots + c_n\right) \left(\frac{1}{p^n} + \frac{1}{p^{2n}} + \frac{1}{p^{3n}} + \frac{1}{p^{4n}} + \ldots \right) \end{aligned}$$

$$\begin{aligned}
\frac{m}{n} &= \left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n\right)\left(\frac{\frac{1}{p^n}}{1-\frac{1}{p^n}}\right)\\
&= \left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n\right)\left(\frac{1}{p^n-1}\right)\\
&= \frac{c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n}{p^n-1}\\
\frac{m}{n} &= \frac{c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n}{p^n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_n\right) &= m\left(p^n-1\right)\\
n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right)+nc_n &= mp^n-m\\
n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right)-mp^n &= -m-nc_n\\
mp^n-n\left(c_1p^{n-1}+c_2p^{n-2}+c_3p^{n-3}+\ldots+c_{n-1}p\right) &= m+nc_n\\
mp^n-np\left(c_1p^{n-2}+c_2p^{n-3}+c_3p^{n-4}+\ldots+c_{n-1}\right) &= m+nc_n.
\end{aligned}$$

Since $p \mid mp^n$ and $p \mid np\left(c_1p^{n-2}+c_2p^{n-3}+c_3p^{n-4}+\ldots+c_{n-1}p\right)$,
this implies that

$$p \mid (m+nc_n).$$

Therefore, there exists $l_0 = c_n \in K_p^e$ such that

$$p \mid (m+nl_0).$$

Afterwards, we will prove the uniqueness of l_0.
Assume that

$$p \mid (m+nl_1),$$

and

$$p \mid (m+nl_2),$$

where $l_1, l_2 \in K_p^e$.
Then

$$\begin{aligned}
&p \mid (m+nl_1)-(m+nl_2)\\
&p \mid (nl_1-nl_2)\\
&p \mid n\left(l_1-l_2\right)
\end{aligned}$$

Since $(p,n)=1$, it follows that

$$p \mid (l_1-l_2).$$

Since $l_1, l_2 \in K_p^e$, we have

$$l_1-l_2=0$$

$$l_1 = l_2.$$

By the claim above, for each $\frac{m}{n} \in A_n$, there exists a unique $l = l_0 \in K_p^e$ such that

$$p \mid (m + nl_0),$$

which implies that

$$\gcd(m + nl_0, p) = p.$$

Since the number of $\frac{m}{n} \in A_n$ is $|A_n|$, this implies that

$$|A_{p^*n}| = |A_n| \tag{4.2}$$

Since $B = A_{p^*n} \cup A_{pn}$ and $A_{p^*n} \cap A_{pn} = \phi$, this conclude that

$$|B| = |A_{p^*n}| + |A_{pn}|. \tag{4.3}$$

Substituting the equations 4.1 and 4.2 into the equation 4.3, we have

$$\begin{aligned} |B| &= |A_{p^*n}| + |A_{pn}| \\ |K_p^e| \cdot |A_n| &= |A_n| + |A_{pn}| \\ |A_{pn}| &= |K_p^e| \cdot |A_n| - |A_n| \\ &= (|K_p^e| - 1) \cdot |A_n|. \end{aligned}$$

Therefore, $|A_{pn}| = (|K_p^e| - 1) \cdot |A_n|$. □

From the previous theorem and Theorem 4.1, we will show a new relation of cardinality of child $p-$ary set A_{p^kn} and A_{pn} stated as the following corollary.

Corollary 4.3. *For all child p-ary sets A_{p^kn}, where $k = 1, 2, 3, \ldots$ and $K_p^e = \{0, 2, 4, \ldots, p-1\}$. Then*

$$|A_{p^kn}| = \left(|K_p^e|^k - |K_p^e|^{k-1}\right) \cdot |A_n|.$$

Proof. Follows from Theorems 4.1 and 4.2. □

Acknowledgements I would like to thank my advisor Aniruth Phon-On and Saofee Busaman for their guidances and comments throughout this research. I also wish to thank faculty of Science and Technology, Prince of Songkla university, Pattani campus, for financial supports.

References

[1] F. Chovanec, Cantor sets, *Sci. Military J.*, **1**(2010), 5-11.

[2] R. L. Devaney, *A First Course in Chaotic Dynamical Systems*, Addison Wesley Publishing Company, 1992.

[3] S. G. Krantz, *Foundations of Analysis*, CRC Press, 2015.

[4] J. N. McDonald, *A Course in Real Analysis*, (2nd ed.), Academic Press, 1999.

[5] D. R. Nelson, The cantor set-a brief introduction. [Online] Available: http://wwwmpa.mpa-garching.mpg.de/ dnelson/storage/dnelson.cantor-set.pdf, (July 11, 2014).

[6] A. Phon-On, Rational points in Cantor p-ary sets. [Online] Available: http://www.m-hikari.com/ijma/ijma-2013/ijma-29-32-2013/phononIJMA29-32-2013.pdf, (August 23, 2014).

(2) Elements in spawning p-ary sets A_n^p

Table 1: All elements in spawning 5-ary set A_6^5

$\frac{1}{6} = 0.\overline{04}$	$\frac{5}{6} = 0.\overline{40}$

Table 2: All elements in spawning 5-ary set A_{12}^5

$\frac{1}{12} = 0.\overline{02}$	$\frac{11}{12} = 0.\overline{42}$
$\frac{5}{12} = 0.\overline{20}$	$\frac{7}{12} = 0.\overline{24}$

Table 3: All elements in spawning 5-ary set A_{26}^5

$\frac{1}{26} = 0.\overline{0044}$	$\frac{25}{26} = 0.\overline{4400}$
$\frac{3}{26} = 0.\overline{0242}$	$\frac{23}{26} = 0.\overline{4202}$
$\frac{5}{26} = 0.\overline{0440}$	$\frac{21}{26} = 0.\overline{4004}$
$\frac{11}{26} = 0.\overline{2024}$	$\frac{15}{26} = 0.\overline{2420}$

Table 4: All elements in spawning 5-ary set A_{62}^5

$\frac{1}{62} = 0.\overline{002}$	$\frac{61}{62} = 0.\overline{442}$
$\frac{5}{62} = 0.\overline{020}$	$\frac{57}{62} = 0.\overline{424}$
$\frac{25}{62} = 0.\overline{200}$	$\frac{37}{62} = 0.\overline{244}$
$\frac{7}{62} = 0.\overline{024}$	$\frac{55}{62} = 0.\overline{420}$
$\frac{35}{62} = 0.\overline{240}$	$\frac{27}{62} = 0.\overline{204}$
$\frac{51}{62} = 0.\overline{402}$	$\frac{11}{62} = 0.\overline{042}$

Table 5: All elements in spawning 5-ary set A^5_{126}

$\frac{1}{126} = 0.\overline{000444}$	$\frac{125}{126} = 0.\overline{000444}$
$\frac{5}{126} = 0.\overline{004440}$	$\frac{121}{126} = 0.\overline{440004}$
$\frac{25}{126} = 0.\overline{044400}$	$\frac{101}{126} = 0.\overline{400044}$
$\frac{11}{126} = 0.\overline{020424}$	$\frac{115}{126} = 0.\overline{424020}$
$\frac{55}{126} = 0.\overline{204240}$	$\frac{71}{126} = 0.\overline{240204}$
$\frac{23}{126} = 0.\overline{042402}$	$\frac{103}{126} = 0.\overline{402042}$
$\frac{13}{126} = 0.\overline{022422}$	$\frac{113}{126} = 0.\overline{422022}$
$\frac{65}{126} = 0.\overline{224220}$	$\frac{61}{126} = 0.\overline{220224}$
$\frac{73}{126} = 0.\overline{242202}$	$\frac{53}{126} = 0.\overline{202242}$

Table 6: All elements in spawning 5-ary set A^5_{312}

$\frac{1}{312} = 0.\overline{0002}$	$\frac{311}{312} = 0.\overline{4442}$	$\frac{11}{312} = 0.\overline{0042}$	$\frac{301}{312} = 0.\overline{4402}$
$\frac{5}{312} = 0.\overline{0020}$	$\frac{307}{312} = 0.\overline{4424}$	$\frac{55}{312} = 0.\overline{0420}$	$\frac{257}{312} = 0.\overline{4024}$
$\frac{25}{312} = 0.\overline{0200}$	$\frac{287}{312} = 0.\overline{4244}$	$\frac{275}{312} = 0.\overline{4200}$	$\frac{37}{312} = 0.\overline{0244}$
$\frac{125}{312} = 0.\overline{2000}$	$\frac{187}{312} = 0.\overline{2444}$	$\frac{127}{312} = 0.\overline{2004}$	$\frac{185}{312} = 0.\overline{2440}$
$\frac{7}{312} = 0.\overline{0024}$	$\frac{305}{312} = 0.\overline{4420}$	$\frac{31}{312} = 0.\overline{0222}$	$\frac{281}{312} = 0.\overline{4222}$
$\frac{35}{312} = 0.\overline{0402}$	$\frac{277}{312} = 0.\overline{4204}$	$\frac{155}{312} = 0.\overline{2220}$	$\frac{157}{312} = 0.\overline{2224}$
$\frac{175}{312} = 0.\overline{2400}$	$\frac{137}{312} = 0.\overline{2044}$	$\frac{151}{312} = 0.\overline{2202}$	$\frac{161}{312} = 0.\overline{2242}$
$\frac{251}{312} = 0.\overline{4002}$	$\frac{61}{312} = 0.\overline{0442}$	$\frac{131}{312} = 0.\overline{2022}$	$\frac{181}{312} = 0.\overline{2422}$

Table 7: All elements in spawning 7-ary set A^7_3

$\frac{1}{3} = 0.\overline{2}$	$\frac{2}{3} = 0.\overline{4}$

Table 8: All elements in spawning 7-ary set A_8^7

$\frac{1}{8} = 0.\overline{06}$	$\frac{7}{8} = 0.\overline{60}$
$\frac{3}{8} = 0.\overline{24}$	$\frac{5}{8} = 0.\overline{42}$

Table 9: All elements in spawning 7-ary set A_{24}^7

$\frac{1}{24} = 0.\overline{02}$	$\frac{23}{24} = 0.\overline{64}$
$\frac{7}{24} = 0.\overline{20}$	$\frac{17}{24} = 0.\overline{46}$
$\frac{3}{24} = 0.\overline{06}$	$\frac{21}{24} = 0.\overline{60}$

Table 10: All elements in spawning 7-ary set A_{50}^7

$\frac{1}{50} = 0.\overline{0066}$	$\frac{49}{50} = 0.\overline{6600}$
$\frac{3}{50} = 0.\overline{0264}$	$\frac{47}{50} = 0.\overline{6402}$
$\frac{7}{50} = 0.\overline{0660}$	$\frac{43}{50} = 0.\overline{6006}$
$\frac{17}{50} = 0.\overline{2244}$	$\frac{33}{50} = 0.\overline{4422}$
$\frac{19}{50} = 0.\overline{2442}$	$\frac{31}{50} = 0.\overline{4224}$
$\frac{21}{50} = 0.\overline{2640}$	$\frac{29}{50} = 0.\overline{4026}$

Table 11: All elements in spawning 7-ary set A^7_{171}

$\frac{1}{171} = 0.\overline{002}$	$\frac{170}{171} = 0.\overline{664}$	$\frac{10}{171} = 0.\overline{026}$	$\frac{161}{171} = 0.\overline{640}$
$\frac{7}{171} = 0.\overline{020}$	$\frac{164}{171} = 0.\overline{646}$	$\frac{70}{171} = 0.\overline{260}$	$\frac{101}{171} = 0.\overline{406}$
$\frac{49}{171} = 0.\overline{200}$	$\frac{122}{171} = 0.\overline{466}$	$\frac{148}{171} = 0.\overline{602}$	$\frac{23}{171} = 0.\overline{064}$
$\frac{2}{171} = 0.\overline{004}$	$\frac{169}{171} = 0.\overline{662}$	$\frac{15}{171} = 0.\overline{042}$	$\frac{156}{171} = 0.\overline{624}$
$\frac{14}{171} = 0.\overline{040}$	$\frac{157}{171} = 0.\overline{626}$	$\frac{105}{171} = 0.\overline{420}$	$\frac{66}{171} = 0.\overline{246}$
$\frac{98}{171} = 0.\overline{400}$	$\frac{73}{171} = 0.\overline{266}$	$\frac{51}{171} = 0.\overline{204}$	$\frac{120}{171} = 0.\overline{462}$
$\frac{3}{171} = 0.\overline{006}$	$\frac{168}{171} = 0.\overline{660}$	$\frac{16}{171} = 0.\overline{044}$	$\frac{155}{171} = 0.\overline{622}$
$\frac{21}{171} = 0.\overline{060}$	$\frac{150}{171} = 0.\overline{606}$	$\frac{112}{171} = 0.\overline{440}$	$\frac{59}{171} = 0.\overline{226}$
$\frac{147}{171} = 0.\overline{600}$	$\frac{24}{171} = 0.\overline{066}$	$\frac{100}{171} = 0.\overline{404}$	$\frac{71}{171} = 0.\overline{262}$
$\frac{8}{171} = 0.\overline{022}$	$\frac{163}{171} = 0.\overline{644}$	$\frac{17}{171} = 0.\overline{046}$	$\frac{154}{171} = 0.\overline{620}$
$\frac{56}{171} = 0.\overline{220}$	$\frac{115}{171} = 0.\overline{446}$	$\frac{119}{171} = 0.\overline{460}$	$\frac{52}{171} = 0.\overline{206}$
$\frac{50}{171} = 0.\overline{202}$	$\frac{121}{171} = 0.\overline{464}$	$\frac{149}{171} = 0.\overline{604}$	$\frac{22}{171} = 0.\overline{062}$
$\frac{9}{171} = 0.\overline{024}$	$\frac{162}{171} = 0.\overline{642}$	$\frac{58}{171} = 0.\overline{224}$	$\frac{113}{171} = 0.\overline{442}$
$\frac{63}{171} = 0.\overline{240}$	$\frac{108}{171} = 0.\overline{426}$	$\frac{64}{171} = 0.\overline{242}$	$\frac{107}{171} = 0.\overline{424}$
$\frac{99}{171} = 0.\overline{402}$	$\frac{72}{171} = 0.\overline{264}$	$\frac{106}{171} = 0.\overline{422}$	$\frac{65}{171} = 0.\overline{244}$

Table 12: All elements in spawning 11-ary set A^{11}_5

$\frac{1}{5} = 0.\overline{2}$	$\frac{4}{5} = 0.\overline{8}$
$\frac{2}{5} = 0.\overline{4}$	$\frac{3}{5} = 0.\overline{6}$

Table 13: All elements in spawning 11-ary set A^{11}_{12}

$\frac{1}{12} = 0.\overline{0A}$	$\frac{11}{12} = 0.\overline{A0}$
$\frac{3}{12} = 0.\overline{28}$	$\frac{9}{12} = 0.\overline{82}$
$\frac{5}{12} = 0.\overline{46}$	$\frac{7}{12} = 0.\overline{64}$

Table 14: All elements in spawning 11-ary set A_{60}^{11}

$\frac{1}{60}$	$0.\overline{02}$	$\frac{59}{60}$	$0.\overline{A8}$
$\frac{11}{60}$	$0.\overline{20}$	$\frac{49}{60}$	$0.\overline{8A}$
$\frac{13}{60}$	$0.\overline{24}$	$\frac{47}{60}$	$0.\overline{86}$
$\frac{23}{60}$	$0.\overline{42}$	$\frac{37}{60}$	$0.\overline{68}$

Table 15: All elements in spawning 11-ary set A_{122}^{11}

$\frac{1}{122} = 0.\overline{00AA}$	$\frac{121}{122} = 0.\overline{AA00}$	$\frac{53}{122} = 0.\overline{4862}$	$\frac{69}{122} = 0.\overline{6248}$
$\frac{3}{122} = 0.\overline{02A8}$	$\frac{119}{122} = 0.\overline{A802}$	$\frac{29}{122} = 0.\overline{2684}$	$\frac{93}{122} = 0.\overline{8426}$
$\frac{5}{122} = 0.\overline{04A6}$	$\frac{117}{122} = 0.\overline{A604}$	$\frac{31}{122} = 0.\overline{2882}$	$\frac{91}{122} = 0.\overline{8228}$
$\frac{7}{122} = 0.\overline{06A4}$	$\frac{115}{122} = 0.\overline{A406}$	$\frac{33}{122} = 0.\overline{2A80}$	$\frac{89}{122} = 0.\overline{802A}$
$\frac{9}{122} = 0.\overline{08A2}$	$\frac{113}{122} = 0.\overline{A208}$	$\frac{45}{122} = 0.\overline{406A}$	$\frac{77}{122} = 0.\overline{6A40}$
$\frac{11}{122} = 0.\overline{0AA0}$	$\frac{111}{122} = 0.\overline{A00A}$	$\frac{47}{122} = 0.\overline{4268}$	$\frac{75}{122} = 0.\overline{6842}$
$\frac{23}{122} = 0.\overline{208A}$	$\frac{99}{122} = 0.\overline{8A20}$	$\frac{49}{122} = 0.\overline{4466}$	$\frac{73}{122} = 0.\overline{6644}$
$\frac{25}{122} = 0.\overline{2288}$	$\frac{97}{122} = 0.\overline{8822}$	$\frac{51}{122} = 0.\overline{4664}$	$\frac{71}{122} = 0.\overline{6446}$
$\frac{27}{122} = 0.\overline{2486}$	$\frac{95}{122} = 0.\overline{8624}$	$\frac{55}{122} = 0.\overline{4A60}$	$\frac{67}{122} = 0.\overline{604A}$

Table 16: All elements in spawning 11-ary set A_{665}^{11}

$\frac{1}{665} = 0.\overline{002}$	$\frac{664}{665} = 0.\overline{AA8}$	$\frac{157}{665} = 0.\overline{266}$	$\frac{508}{665} = 0.\overline{844}$
$\frac{2}{665} = 0.\overline{004}$	$\frac{663}{665} = 0.\overline{AA6}$	$\frac{158}{665} = 0.\overline{268}$	$\frac{507}{665} = 0.\overline{842}$
$\frac{3}{665} = 0.\overline{006}$	$\frac{662}{665} = 0.\overline{AA4}$	$\frac{159}{665} = 0.\overline{26A}$	$\frac{506}{665} = 0.\overline{840}$
$\frac{4}{665} = 0.\overline{008}$	$\frac{661}{665} = 0.\overline{AA2}$	$\frac{166}{665} = 0.\overline{282}$	$\frac{499}{665} = 0.\overline{828}$
$\frac{11}{665} = 0.\overline{020}$	$\frac{654}{665} = 0.\overline{A8A}$	$\frac{167}{665} = 0.\overline{284}$	$\frac{498}{665} = 0.\overline{826}$
$\frac{12}{665} = 0.\overline{022}$	$\frac{653}{665} = 0.\overline{A88}$	$\frac{169}{665} = 0.\overline{288}$	$\frac{496}{665} = 0.\overline{822}$
$\frac{13}{665} = 0.\overline{024}$	$\frac{652}{665} = 0.\overline{A86}$	$\frac{176}{665} = 0.\overline{2A0}$	$\frac{489}{665} = 0.\overline{80A}$

$\frac{16}{665} = 0.\overline{02A}$	$\frac{649}{665} = 0.\overline{A80}$	$\frac{177}{665} = 0.\overline{2A2}$	$\frac{488}{665} = 0.\overline{808}$
$\frac{22}{665} = 0.\overline{040}$	$\frac{643}{665} = 0.\overline{A6A}$	$\frac{178}{665} = 0.\overline{2A4}$	$\frac{487}{665} = 0.\overline{806}$
$\frac{23}{665} = 0.\overline{042}$	$\frac{642}{665} = 0.\overline{A68}$	$\frac{179}{665} = 0.\overline{2A6}$	$\frac{486}{665} = 0.\overline{804}$
$\frac{24}{665} = 0.\overline{044}$	$\frac{641}{665} = 0.\overline{A66}$	$\frac{181}{665} = 0.\overline{2AA}$	$\frac{484}{665} = 0.\overline{800}$
$\frac{26}{665} = 0.\overline{048}$	$\frac{639}{665} = 0.\overline{A62}$	$\frac{242}{665} = 0.\overline{400}$	$\frac{423}{665} = 0.\overline{6AA}$
$\frac{27}{665} = 0.\overline{04A}$	$\frac{638}{665} = 0.\overline{A60}$	$\frac{243}{665} = 0.\overline{402}$	$\frac{422}{665} = 0.\overline{6A8}$
$\frac{33}{665} = 0.\overline{060}$	$\frac{632}{665} = 0.\overline{A4A}$	$\frac{244}{665} = 0.\overline{404}$	$\frac{421}{665} = 0.\overline{6A6}$
$\frac{34}{665} = 0.\overline{062}$	$\frac{631}{665} = 0.\overline{A48}$	$\frac{246}{665} = 0.\overline{408}$	$\frac{419}{665} = 0.\overline{6A2}$
$\frac{36}{665} = 0.\overline{066}$	$\frac{629}{665} = 0.\overline{A44}$	$\frac{253}{665} = 0.\overline{420}$	$\frac{412}{665} = 0.\overline{68A}$
$\frac{37}{665} = 0.\overline{068}$	$\frac{628}{665} = 0.\overline{A42}$	$\frac{254}{665} = 0.\overline{422}$	$\frac{411}{665} = 0.\overline{688}$
$\frac{44}{665} = 0.\overline{080}$	$\frac{621}{665} = 0.\overline{A2A}$	$\frac{256}{665} = 0.\overline{426}$	$\frac{409}{665} = 0.\overline{684}$
$\frac{46}{665} = 0.\overline{084}$	$\frac{619}{665} = 0.\overline{A26}$	$\frac{257}{665} = 0.\overline{428}$	$\frac{408}{665} = 0.\overline{682}$
$\frac{47}{665} = 0.\overline{086}$	$\frac{618}{665} = 0.\overline{A24}$	$\frac{258}{665} = 0.\overline{42A}$	$\frac{407}{665} = 0.\overline{680}$
$\frac{48}{665} = 0.\overline{088}$	$\frac{617}{665} = 0.\overline{A22}$	$\frac{264}{665} = 0.\overline{440}$	$\frac{401}{665} = 0.\overline{66A}$
$\frac{58}{665} = 0.\overline{0A6}$	$\frac{607}{665} = 0.\overline{A04}$	$\frac{267}{665} = 0.\overline{446}$	$\frac{398}{665} = 0.\overline{664}$
$\frac{59}{665} = 0.\overline{0A8}$	$\frac{606}{665} = 0.\overline{A02}$	$\frac{268}{665} = 0.\overline{448}$	$\frac{397}{665} = 0.\overline{662}$
$\frac{121}{665} = 0.\overline{200}$	$\frac{544}{665} = 0.\overline{8AA}$	$\frac{269}{665} = 0.\overline{44A}$	$\frac{396}{665} = 0.\overline{660}$
$\frac{122}{665} = 0.\overline{202}$	$\frac{543}{665} = 0.\overline{8A8}$	$\frac{276}{665} = 0.\overline{462}$	$\frac{389}{665} = 0.\overline{648}$
$\frac{123}{665} = 0.\overline{204}$	$\frac{542}{665} = 0.\overline{8A6}$	$\frac{277}{665} = 0.\overline{464}$	$\frac{388}{665} = 0.\overline{646}$
$\frac{124}{665} = 0.\overline{206}$	$\frac{541}{665} = 0.\overline{8A4}$	$\frac{278}{665} = 0.\overline{466}$	$\frac{387}{665} = 0.\overline{644}$
$\frac{132}{665} = 0.\overline{220}$	$\frac{533}{665} = 0.\overline{88A}$	$\frac{279}{665} = 0.\overline{468}$	$\frac{386}{665} = 0.\overline{642}$
$\frac{134}{665} = 0.\overline{224}$	$\frac{531}{665} = 0.\overline{886}$	$\frac{286}{665} = 0.\overline{480}$	$\frac{379}{665} = 0.\overline{62A}$
$\frac{136}{665} = 0.\overline{228}$	$\frac{529}{665} = 0.\overline{882}$	$\frac{288}{665} = 0.\overline{484}$	$\frac{377}{665} = 0.\overline{626}$
$\frac{137}{665} = 0.\overline{22A}$	$\frac{528}{665} = 0.\overline{880}$	$\frac{289}{665} = 0.\overline{486}$	$\frac{376}{665} = 0.\overline{624}$
$\frac{143}{665} = 0.\overline{240}$	$\frac{522}{665} = 0.\overline{86A}$	$\frac{291}{665} = 0.\overline{48A}$	$\frac{374}{665} = 0.\overline{620}$
$\frac{144}{665} = 0.\overline{242}$	$\frac{521}{665} = 0.\overline{868}$	$\frac{297}{665} = 0.\overline{4A0}$	$\frac{368}{665} = 0.\overline{60A}$
$\frac{146}{665} = 0.\overline{246}$	$\frac{519}{665} = 0.\overline{864}$	$\frac{298}{665} = 0.\overline{4A2}$	$\frac{367}{665} = 0.\overline{608}$

$\frac{148}{665} = 0.\overline{24A}$	$\frac{517}{665} = 0.\overline{860}$	$\frac{299}{665} = 0.\overline{4A4}$	$\frac{366}{665} = 0.\overline{606}$
$\frac{156}{665} = 0.\overline{264}$	$\frac{509}{665} = 0.\overline{846}$	$\frac{302}{665} = 0.\overline{4AA}$	$\frac{363}{665} = 0.\overline{600}$

Table 17: All elements in spawning 13-ary set A_6^{13}

$\frac{1}{6} = 0.\overline{2}$	$\frac{5}{6} = 0.\overline{A}$

Table 18: All elements in spawning 13-ary set A_{14}^{13}

$\frac{1}{14} = 0.\overline{0C}$	$\frac{13}{14} = 0.\overline{C0}$
$\frac{3}{14} = 0.\overline{2A}$	$\frac{11}{14} = 0.\overline{A2}$
$\frac{5}{14} = 0.\overline{48}$	$\frac{9}{14} = 0.\overline{84}$

Table 19: All elements in spawning 13-ary set A_{84}^{13}

$\frac{1}{84} = 0.\overline{02}$	$\frac{83}{84} = 0.\overline{CA}$	$\frac{19}{84} = 0.\overline{2C}$	$\frac{65}{84} = 0.\overline{A0}$
$\frac{5}{84} = 0.\overline{0A}$	$\frac{79}{84} = 0.\overline{C2}$	$\frac{29}{84} = 0.\overline{28}$	$\frac{55}{84} = 0.\overline{86}$
$\frac{13}{84} = 0.\overline{20}$	$\frac{71}{84} = 0.\overline{AC}$	$\frac{31}{84} = 0.\overline{28}$	$\frac{53}{84} = 0.\overline{82}$
$\frac{17}{84} = 0.\overline{28}$	$\frac{67}{84} = 0.\overline{A4}$	$\frac{41}{84} = 0.\overline{28}$	$\frac{43}{84} = 0.\overline{68}$

Table 20: All elements in spawning 13-ary set A^{13}_{170}

$\frac{1}{170} = 0.\overline{00CC}$	$\frac{169}{170} = 0.\overline{CC00}$	$\frac{37}{170} = 0.\overline{2AA2}$	$\frac{133}{170} = 0.\overline{A22A}$
$\frac{3}{170} = 0.\overline{02CA}$	$\frac{167}{170} = 0.\overline{CA02}$	$\frac{39}{170} = 0.\overline{2CA0}$	$\frac{131}{170} = 0.\overline{A02C}$
$\frac{7}{170} = 0.\overline{06C6}$	$\frac{163}{170} = 0.\overline{C606}$	$\frac{53}{170} = 0.\overline{408C}$	$\frac{117}{170} = 0.\overline{8C40}$
$\frac{9}{170} = 0.\overline{08C4}$	$\frac{161}{170} = 0.\overline{C408}$	$\frac{57}{170} = 0.\overline{4488}$	$\frac{113}{170} = 0.\overline{8844}$
$\frac{11}{170} = 0.\overline{0AC2}$	$\frac{159}{170} = 0.\overline{C20A}$	$\frac{59}{170} = 0.\overline{4686}$	$\frac{111}{170} = 0.\overline{8646}$
$\frac{13}{170} = 0.\overline{0CC0}$	$\frac{157}{170} = 0.\overline{C00C}$	$\frac{61}{170} = 0.\overline{4884}$	$\frac{109}{170} = 0.\overline{8448}$
$\frac{27}{170} = 0.\overline{20AC}$	$\frac{143}{170} = 0.\overline{AC20}$	$\frac{63}{170} = 0.\overline{4A82}$	$\frac{107}{170} = 0.\overline{824A}$
$\frac{29}{170} = 0.\overline{22AA}$	$\frac{141}{170} = 0.\overline{AA22}$	$\frac{79}{170} = 0.\overline{606C}$	$\frac{91}{170} = 0.\overline{6C60}$
$\frac{31}{170} = 0.\overline{24A8}$	$\frac{139}{170} = 0.\overline{A824}$	$\frac{81}{170} = 0.\overline{626A}$	$\frac{89}{170} = 0.\overline{6A62}$
$\frac{33}{170} = 0.\overline{26A6}$	$\frac{137}{170} = 0.\overline{A626}$	$\frac{83}{170} = 0.\overline{6468}$	$\frac{87}{170} = 0.\overline{6864}$

(3) Elements in child p-ary sets $A^{p}_{p^k n}$

Table 21: All elements in child 5-ary set A^{5}_{10}

$\frac{1}{10} = 0.0\overline{2}$	$\frac{9}{10} = 0.4\overline{2}$

Table 22: All elements in child 5-ary set A^{5}_{30}

$\frac{1}{30} = 0.0\overline{04}$	$\frac{29}{30} = 0.4\overline{40}$
$\frac{13}{30} = 0.2\overline{04}$	$\frac{17}{30} = 0.2\overline{40}$

Table 23: All elements in child 5-ary set A^{5}_{50}

$\frac{1}{50} = 0.00\overline{2}$	$\frac{49}{50} = 0.44\overline{2}$
$\frac{9}{50} = 0.04\overline{2}$	$\frac{41}{50} = 0.40\overline{2}$
$\frac{21}{50} = 0.20\overline{2}$	$\frac{29}{50} = 0.24\overline{2}$

Table 24: All elements in child 5-ary set A^5_{60}

$\frac{1}{60} = 0.0\overline{02}$	$\frac{59}{60} = 0.4\overline{42}$
$\frac{7}{60} = 0.0\overline{24}$	$\frac{53}{60} = 0.4\overline{20}$
$\frac{11}{60} = 0.0\overline{42}$	$\frac{49}{60} = 0.4\overline{02}$
$\frac{29}{60} = 0.2\overline{20}$	$\frac{31}{60} = 0.2\overline{24}$

Table 25: All elements in child 5-ary set A^5_{130}

$\frac{1}{130} = 0.0\overline{0044}$	$\frac{129}{130} = 0.4\overline{4400}$
$\frac{3}{130} = 0.0\overline{0242}$	$\frac{127}{130} = 0.4\overline{4202}$
$\frac{11}{130} = 0.0\overline{2024}$	$\frac{119}{130} = 0.4\overline{2420}$
$\frac{21}{130} = 0.0\overline{4004}$	$\frac{109}{130} = 0.4\overline{0440}$
$\frac{23}{130} = 0.0\overline{4202}$	$\frac{107}{130} = 0.4\overline{0242}$
$\frac{53}{130} = 0.2\overline{0044}$	$\frac{77}{130} = 0.2\overline{4400}$
$\frac{57}{130} = 0.2\overline{0440}$	$\frac{73}{130} = 0.2\overline{4004}$
$\frac{63}{130} = 0.2\overline{2024}$	$\frac{67}{130} = 0.2\overline{2420}$

Table 26: All elements in child 5-ary set A^5_{150}

$\frac{1}{150} = 0.00\overline{04}$	$\frac{149}{150} = 0.44\overline{40}$
$\frac{13}{150} = 0.02\overline{04}$	$\frac{137}{150} = 0.42\overline{40}$
$\frac{17}{150} = 0.02\overline{40}$	$\frac{133}{150} = 0.42\overline{04}$
$\frac{29}{150} = 0.04\overline{40}$	$\frac{121}{150} = 0.40\overline{04}$
$\frac{61}{150} = 0.20\overline{04}$	$\frac{89}{150} = 0.24\overline{40}$
$\frac{73}{150} = 0.22\overline{04}$	$\frac{77}{150} = 0.22\overline{40}$

Table 27: All elements in child 5-ary set A^5_{300}

$\frac{1}{300} = 0.00\overline{02}$	$\frac{299}{300} = 0.44\overline{42}$	$\frac{53}{300} = 0.04\overline{20}$	$\frac{247}{150} = 0.40\overline{24}$
$\frac{7}{300} = 0.00\overline{24}$	$\frac{293}{300} = 0.44\overline{20}$	$\frac{59}{300} = 0.04\overline{42}$	$\frac{241}{150} = 0.40\overline{02}$
$\frac{11}{300} = 0.00\overline{42}$	$\frac{289}{300} = 0.44\overline{02}$	$\frac{121}{300} = 0.20\overline{02}$	$\frac{179}{150} = 0.24\overline{42}$
$\frac{29}{300} = 0.02\overline{20}$	$\frac{271}{300} = 0.42\overline{24}$	$\frac{127}{300} = 0.20\overline{24}$	$\frac{173}{150} = 0.24\overline{20}$
$\frac{31}{300} = 0.02\overline{24}$	$\frac{269}{300} = 0.42\overline{20}$	$\frac{131}{300} = 0.20\overline{42}$	$\frac{169}{150} = 0.24\overline{02}$
$\frac{49}{300} = 0.04\overline{02}$	$\frac{251}{300} = 0.40\overline{42}$	$\frac{149}{300} = 0.22\overline{20}$	$\frac{151}{150} = 0.22\overline{24}$

Table 28: All elements in child 7-ary set A^7_{21}

$\frac{1}{21} = 0.0\overline{2}$	$\frac{20}{21} = 0.6\overline{4}$
$\frac{2}{21} = 0.0\overline{4}$	$\frac{19}{21} = 0.6\overline{2}$
$\frac{8}{21} = 0.2\overline{4}$	$\frac{13}{21} = 0.4\overline{2}$

Table 29: All elements in child 7-ary set A^7_{56}

$\frac{1}{56} = 0.0\overline{06}$	$\frac{55}{56} = 0.6\overline{60}$
$\frac{3}{56} = 0.0\overline{24}$	$\frac{53}{56} = 0.6\overline{42}$
$\frac{5}{56} = 0.0\overline{42}$	$\frac{51}{56} = 0.6\overline{24}$
$\frac{17}{56} = 0.2\overline{06}$	$\frac{39}{56} = 0.4\overline{60}$
$\frac{19}{56} = 0.2\overline{24}$	$\frac{37}{56} = 0.4\overline{42}$
$\frac{23}{56} = 0.2\overline{60}$	$\frac{33}{56} = 0.4\overline{60}$

Table 30: All elements in child 7-ary set A^7_{147}

$\frac{1}{147} = 0.00\overline{2}$	$\frac{146}{147} = 0.66\overline{4}$	$\frac{43}{147} = 0.20\overline{2}$	$\frac{104}{21} = 0.46\overline{4}$
$\frac{2}{147} = 0.00\overline{4}$	$\frac{145}{147} = 0.66\overline{2}$	$\frac{44}{147} = 0.20\overline{4}$	$\frac{103}{21} = 0.46\overline{2}$
$\frac{8}{147} = 0.02\overline{4}$	$\frac{139}{147} = 0.64\overline{2}$	$\frac{50}{147} = 0.22\overline{4}$	$\frac{97}{21} = 0.44\overline{2}$
$\frac{13}{147} = 0.04\overline{2}$	$\frac{134}{147} = 0.62\overline{4}$	$\frac{55}{147} = 0.24\overline{2}$	$\frac{92}{21} = 0.42\overline{4}$
$\frac{19}{147} = 0.06\overline{2}$	$\frac{128}{147} = 0.60\overline{4}$	$\frac{61}{147} = 0.26\overline{2}$	$\frac{86}{21} = 0.40\overline{4}$
$\frac{20}{147} = 0.06\overline{4}$	$\frac{127}{147} = 0.60\overline{6}$	$\frac{62}{147} = 0.26\overline{4}$	$\frac{85}{21} = 0.40\overline{2}$

Table 31: All elements in child 7-ary set A^7_{168}

$\frac{1}{168} = 0.0\overline{02}$	$\frac{167}{168} = 0.6\overline{64}$
$\frac{17}{168} = 0.0\overline{46}$	$\frac{151}{168} = 0.6\overline{20}$
$\frac{23}{168} = 0.0\overline{64}$	$\frac{145}{168} = 0.6\overline{02}$
$\frac{55}{168} = 0.2\overline{20}$	$\frac{113}{168} = 0.4\overline{46}$
$\frac{65}{168} = 0.2\overline{46}$	$\frac{103}{168} = 0.4\overline{20}$
$\frac{71}{168} = 0.2\overline{64}$	$\frac{97}{168} = 0.4\overline{02}$

Table 32: All elements in child 7-ary set A^7_{350}

$\frac{1}{350} = 0.0\overline{0066}$	$\frac{349}{350} = 0.6\overline{6600}$	$\frac{101}{350} = 0.2\overline{0066}$	$\frac{249}{350} = 0.4\overline{6600}$
$\frac{3}{350} = 0.0\overline{0264}$	$\frac{347}{350} = 0.6\overline{6402}$	$\frac{103}{350} = 0.2\overline{0264}$	$\frac{247}{350} = 0.4\overline{6402}$
$\frac{17}{350} = 0.0\overline{2244}$	$\frac{333}{350} = 0.6\overline{4422}$	$\frac{107}{350} = 0.2\overline{0660}$	$\frac{243}{350} = 0.4\overline{6006}$
$\frac{19}{350} = 0.0\overline{2442}$	$\frac{331}{350} = 0.6\overline{4224}$	$\frac{117}{350} = 0.2\overline{2244}$	$\frac{233}{350} = 0.4\overline{4422}$
$\frac{29}{350} = 0.0\overline{4026}$	$\frac{321}{350} = 0.6\overline{2640}$	$\frac{121}{350} = 0.2\overline{2640}$	$\frac{229}{350} = 0.4\overline{4026}$
$\frac{31}{350} = 0.0\overline{4224}$	$\frac{319}{350} = 0.6\overline{2442}$	$\frac{129}{350} = 0.2\overline{4026}$	$\frac{221}{350} = 0.4\overline{2640}$
$\frac{33}{350} = 0.0\overline{4422}$	$\frac{317}{350} = 0.6\overline{2244}$	$\frac{131}{350} = 0.2\overline{4224}$	$\frac{219}{350} = 0.4\overline{2442}$
$\frac{43}{350} = 0.0\overline{6006}$	$\frac{307}{350} = 0.6\overline{0660}$	$\frac{143}{350} = 0.2\overline{6006}$	$\frac{207}{350} = 0.4\overline{0660}$
$\frac{47}{350} = 0.0\overline{6402}$	$\frac{303}{350} = 0.6\overline{0264}$	$\frac{149}{350} = 0.2\overline{6600}$	$\frac{201}{350} = 0.4\overline{0066}$

Table 33: All elements in child 7-ary set A_{392}^{7}

$\frac{1}{392} = 0.00\overline{06}$	$\frac{391}{392} = 0.66\overline{00}$	$\frac{113}{392} = 0.20\overline{06}$	$\frac{279}{392} = 0.46\overline{60}$
$\frac{3}{392} = 0.00\overline{24}$	$\frac{389}{392} = 0.66\overline{42}$	$\frac{115}{392} = 0.20\overline{24}$	$\frac{277}{392} = 0.46\overline{42}$
$\frac{5}{392} = 0.00\overline{42}$	$\frac{387}{392} = 0.66\overline{24}$	$\frac{117}{392} = 0.20\overline{42}$	$\frac{275}{392} = 0.46\overline{24}$
$\frac{17}{392} = 0.02\overline{06}$	$\frac{375}{392} = 0.64\overline{60}$	$\frac{129}{392} = 0.22\overline{06}$	$\frac{263}{392} = 0.44\overline{60}$
$\frac{19}{392} = 0.02\overline{24}$	$\frac{373}{392} = 0.64\overline{42}$	$\frac{131}{392} = 0.22\overline{24}$	$\frac{261}{392} = 0.44\overline{42}$
$\frac{23}{392} = 0.02\overline{60}$	$\frac{369}{392} = 0.64\overline{06}$	$\frac{135}{392} = 0.22\overline{60}$	$\frac{257}{392} = 0.44\overline{06}$
$\frac{33}{392} = 0.04\overline{06}$	$\frac{359}{392} = 0.62\overline{60}$	$\frac{145}{392} = 0.24\overline{06}$	$\frac{247}{392} = 0.42\overline{60}$
$\frac{37}{392} = 0.04\overline{42}$	$\frac{355}{392} = 0.62\overline{24}$	$\frac{149}{392} = 0.24\overline{42}$	$\frac{243}{392} = 0.42\overline{24}$
$\frac{39}{392} = 0.04\overline{60}$	$\frac{353}{392} = 0.62\overline{06}$	$\frac{151}{392} = 0.24\overline{60}$	$\frac{241}{392} = 0.42\overline{06}$
$\frac{51}{392} = 0.06\overline{24}$	$\frac{341}{392} = 0.60\overline{42}$	$\frac{163}{392} = 0.26\overline{24}$	$\frac{229}{392} = 0.40\overline{42}$
$\frac{53}{392} = 0.06\overline{42}$	$\frac{339}{392} = 0.60\overline{24}$	$\frac{165}{392} = 0.26\overline{42}$	$\frac{227}{392} = 0.40\overline{24}$
$\frac{55}{392} = 0.06\overline{60}$	$\frac{337}{392} = 0.60\overline{06}$	$\frac{167}{392} = 0.26\overline{60}$	$\frac{225}{392} = 0.40\overline{06}$

Table 34: All elements in child 11-ary set A_{55}^{11}

$\frac{1}{55} = 0.0\overline{2}$	$\frac{54}{55} = 0.A\overline{8}$	$\frac{13}{55} = 0.2\overline{6}$	$\frac{42}{55} = 0.8\overline{4}$
$\frac{2}{55} = 0.0\overline{4}$	$\frac{53}{55} = 0.A\overline{6}$	$\frac{14}{55} = 0.2\overline{8}$	$\frac{41}{55} = 0.8\overline{2}$
$\frac{3}{55} = 0.0\overline{6}$	$\frac{52}{55} = 0.A\overline{4}$	$\frac{21}{55} = 0.4\overline{2}$	$\frac{34}{55} = 0.6\overline{8}$
$\frac{4}{55} = 0.0\overline{8}$	$\frac{51}{55} = 0.A\overline{2}$	$\frac{23}{55} = 0.4\overline{6}$	$\frac{32}{55} = 0.6\overline{4}$
$\frac{12}{55} = 0.2\overline{4}$	$\frac{43}{55} = 0.8\overline{6}$	$\frac{24}{55} = 0.4\overline{8}$	$\frac{31}{55} = 0.6\overline{2}$

Table 35: All elements in child 11-ary set A^{11}_{132}

$\frac{1}{132} = 0.0\overline{0A}$	$\frac{131}{132} = 0.A\overline{A0}$	$\frac{31}{132} = 0.2\overline{64}$	$\frac{101}{132} = 0.8\overline{46}$
$\frac{5}{132} = 0.0\overline{46}$	$\frac{127}{132} = 0.A\overline{64}$	$\frac{35}{132} = 0.2\overline{A0}$	$\frac{97}{132} = 0.8\overline{0A}$
$\frac{7}{132} = 0.0\overline{64}$	$\frac{125}{132} = 0.A\overline{46}$	$\frac{49}{132} = 0.4\overline{0A}$	$\frac{83}{132} = 0.6\overline{A0}$
$\frac{25}{132} = 0.2\overline{0A}$	$\frac{107}{132} = 0.8\overline{A0}$	$\frac{53}{132} = 0.4\overline{46}$	$\frac{79}{132} = 0.6\overline{64}$
$\frac{29}{132} = 0.2\overline{46}$	$\frac{103}{132} = 0.8\overline{64}$	$\frac{59}{132} = 0.4\overline{A0}$	$\frac{73}{132} = 0.6\overline{0A}$

Table 36: All elements in child 11-ary set A^{11}_{605}

$\frac{1}{605} = 0.00\overline{2}$	$\frac{604}{605} = 0.AA\overline{8}$	$\frac{141}{605} = 0.26\overline{2}$	$\frac{464}{605} = 0.84\overline{8}$
$\frac{2}{605} = 0.00\overline{4}$	$\frac{603}{605} = 0.AA\overline{6}$	$\frac{142}{605} = 0.26\overline{4}$	$\frac{463}{605} = 0.84\overline{6}$
$\frac{3}{605} = 0.00\overline{6}$	$\frac{602}{605} = 0.AA\overline{4}$	$\frac{144}{605} = 0.26\overline{8}$	$\frac{461}{605} = 0.84\overline{2}$
$\frac{4}{605} = 0.00\overline{8}$	$\frac{601}{605} = 0.AA\overline{2}$	$\frac{151}{605} = 0.28\overline{2}$	$\frac{454}{605} = 0.82\overline{8}$
$\frac{12}{605} = 0.02\overline{4}$	$\frac{593}{605} = 0.A8\overline{6}$	$\frac{152}{605} = 0.28\overline{4}$	$\frac{453}{605} = 0.82\overline{6}$
$\frac{13}{605} = 0.02\overline{6}$	$\frac{592}{605} = 0.A8\overline{4}$	$\frac{153}{605} = 0.28\overline{6}$	$\frac{452}{605} = 0.82\overline{4}$
$\frac{14}{605} = 0.02\overline{8}$	$\frac{591}{605} = 0.A8\overline{2}$	$\frac{161}{605} = 0.2A\overline{2}$	$\frac{444}{605} = 0.80\overline{8}$
$\frac{21}{605} = 0.04\overline{2}$	$\frac{584}{605} = 0.A6\overline{8}$	$\frac{162}{605} = 0.2A\overline{4}$	$\frac{443}{605} = 0.80\overline{6}$
$\frac{23}{605} = 0.04\overline{6}$	$\frac{582}{605} = 0.A6\overline{4}$	$\frac{163}{605} = 0.2A\overline{6}$	$\frac{442}{605} = 0.80\overline{4}$
$\frac{24}{605} = 0.04\overline{8}$	$\frac{581}{605} = 0.A6\overline{2}$	$\frac{164}{605} = 0.2A\overline{8}$	$\frac{441}{605} = 0.80\overline{2}$
$\frac{31}{605} = 0.06\overline{2}$	$\frac{574}{605} = 0.A4\overline{8}$	$\frac{221}{605} = 0.40\overline{2}$	$\frac{384}{605} = 0.6A\overline{8}$
$\frac{32}{605} = 0.06\overline{4}$	$\frac{573}{605} = 0.A4\overline{6}$	$\frac{222}{605} = 0.40\overline{4}$	$\frac{383}{605} = 0.6A\overline{6}$
$\frac{34}{605} = 0.06\overline{8}$	$\frac{571}{605} = 0.A4\overline{2}$	$\frac{223}{605} = 0.40\overline{6}$	$\frac{382}{605} = 0.6A\overline{4}$
$\frac{41}{605} = 0.08\overline{2}$	$\frac{564}{605} = 0.A2\overline{8}$	$\frac{224}{605} = 0.40\overline{8}$	$\frac{381}{605} = 0.6A\overline{2}$
$\frac{42}{605} = 0.08\overline{4}$	$\frac{563}{605} = 0.A2\overline{6}$	$\frac{232}{605} = 0.42\overline{4}$	$\frac{384}{605} = 0.68\overline{6}$
$\frac{43}{605} = 0.08\overline{6}$	$\frac{562}{605} = 0.A2\overline{4}$	$\frac{233}{605} = 0.42\overline{6}$	$\frac{383}{605} = 0.68\overline{4}$
$\frac{51}{605} = 0.0A\overline{2}$	$\frac{554}{605} = 0.A0\overline{8}$	$\frac{234}{605} = 0.42\overline{8}$	$\frac{382}{605} = 0.68\overline{2}$
$\frac{52}{605} = 0.0A\overline{4}$	$\frac{553}{605} = 0.A0\overline{6}$	$\frac{241}{605} = 0.44\overline{2}$	$\frac{364}{605} = 0.66\overline{8}$

$\frac{53}{605} = 0.0A\overline{6}$	$\frac{552}{605} = 0.A0\overline{4}$	$\frac{243}{605} = 0.44\overline{6}$	$\frac{362}{605} = 0.66\overline{4}$
$\frac{54}{605} = 0.0A\overline{8}$	$\frac{551}{605} = 0.A0\overline{2}$	$\frac{244}{605} = 0.44\overline{8}$	$\frac{361}{605} = 0.66\overline{2}$
$\frac{111}{605} = 0.20\overline{2}$	$\frac{494}{605} = 0.8A\overline{8}$	$\frac{251}{605} = 0.46\overline{2}$	$\frac{354}{605} = 0.64\overline{8}$
$\frac{112}{605} = 0.20\overline{4}$	$\frac{493}{605} = 0.8A\overline{6}$	$\frac{252}{605} = 0.46\overline{4}$	$\frac{353}{605} = 0.64\overline{6}$
$\frac{113}{605} = 0.20\overline{6}$	$\frac{492}{605} = 0.8A\overline{4}$	$\frac{254}{605} = 0.46\overline{8}$	$\frac{351}{605} = 0.64\overline{2}$
$\frac{114}{605} = 0.20\overline{8}$	$\frac{491}{605} = 0.8A\overline{2}$	$\frac{261}{605} = 0.48\overline{2}$	$\frac{344}{605} = 0.62\overline{8}$
$\frac{122}{605} = 0.22\overline{4}$	$\frac{483}{605} = 0.88\overline{6}$	$\frac{262}{605} = 0.48\overline{4}$	$\frac{343}{605} = 0.62\overline{6}$
$\frac{123}{605} = 0.22\overline{6}$	$\frac{482}{605} = 0.88\overline{4}$	$\frac{263}{605} = 0.48\overline{6}$	$\frac{342}{605} = 0.62\overline{4}$
$\frac{124}{605} = 0.22\overline{8}$	$\frac{481}{605} = 0.88\overline{2}$	$\frac{271}{605} = 0.4A\overline{2}$	$\frac{334}{605} = 0.60\overline{8}$
$\frac{131}{605} = 0.24\overline{2}$	$\frac{474}{605} = 0.86\overline{8}$	$\frac{272}{605} = 0.4A\overline{4}$	$\frac{333}{605} = 0.60\overline{6}$
$\frac{133}{605} = 0.24\overline{6}$	$\frac{472}{605} = 0.86\overline{4}$	$\frac{273}{605} = 0.4A\overline{6}$	$\frac{332}{605} = 0.60\overline{4}$
$\frac{134}{605} = 0.24\overline{8}$	$\frac{471}{605} = 0.86\overline{2}$	$\frac{274}{605} = 0.4A\overline{8}$	$\frac{331}{605} = 0.60\overline{2}$

Table 37: All elements in child 11-ary set A^{11}_{660}

$\frac{1}{660} = 0.0\overline{02}$	$\frac{659}{660} = 0.A\overline{A8}$	$\frac{167}{660} = 0.2\overline{86}$	$\frac{493}{660} = 0.8\overline{24}$
$\frac{13}{660} = 0.0\overline{24}$	$\frac{647}{660} = 0.A\overline{86}$	$\frac{169}{660} = 0.2\overline{8A}$	$\frac{491}{660} = 0.8\overline{20}$
$\frac{23}{660} = 0.0\overline{42}$	$\frac{637}{660} = 0.A\overline{68}$	$\frac{179}{660} = 0.2\overline{A8}$	$\frac{481}{660} = 0.8\overline{02}$
$\frac{37}{660} = 0.0\overline{68}$	$\frac{623}{660} = 0.A\overline{42}$	$\frac{241}{660} = 0.4\overline{02}$	$\frac{419}{660} = 0.6\overline{A8}$
$\frac{47}{660} = 0.0\overline{86}$	$\frac{613}{660} = 0.A\overline{24}$	$\frac{251}{660} = 0.4\overline{20}$	$\frac{409}{660} = 0.6\overline{8A}$
$\frac{49}{660} = 0.0\overline{8A}$	$\frac{611}{660} = 0.A\overline{20}$	$\frac{263}{660} = 0.4\overline{42}$	$\frac{397}{660} = 0.6\overline{68}$
$\frac{59}{660} = 0.0\overline{A8}$	$\frac{601}{660} = 0.A\overline{02}$	$\frac{277}{660} = 0.4\overline{68}$	$\frac{383}{660} = 0.6\overline{42}$
$\frac{131}{660} = 0.2\overline{20}$	$\frac{529}{660} = 0.8\overline{8A}$	$\frac{287}{660} = 0.4\overline{86}$	$\frac{373}{660} = 0.6\overline{24}$
$\frac{133}{660} = 0.2\overline{24}$	$\frac{527}{660} = 0.8\overline{86}$	$\frac{361}{660} = 0.6\overline{02}$	$\frac{299}{660} = 0.4\overline{A8}$
$\frac{157}{660} = 0.2\overline{68}$	$\frac{503}{660} = 0.8\overline{42}$	$\frac{371}{660} = 0.6\overline{20}$	$\frac{289}{660} = 0.4\overline{8A}$

Table 38: All elements in child 13-ary set A_{78}^{13}

$\frac{1}{78} = 0.0\overline{2}$	$\frac{77}{78} = 0.C\overline{A}$
$\frac{5}{78} = 0.0\overline{A}$	$\frac{73}{78} = 0.C\overline{2}$
$\frac{17}{78} = 0.2\overline{A}$	$\frac{61}{78} = 0.A\overline{2}$
$\frac{25}{78} = 0.4\overline{2}$	$\frac{53}{78} = 0.8\overline{A}$
$\frac{29}{78} = 0.4\overline{A}$	$\frac{49}{78} = 0.8\overline{2}$
$\frac{37}{78} = 0.6\overline{2}$	$\frac{41}{78} = 0.6\overline{A}$

Table 39: All elements in child 13-ary set A_{182}^{13}

$\frac{1}{182} = 0.0\overline{0C}$	$\frac{181}{182} = 0.C\overline{C0}$	$\frac{41}{182} = 0.2\overline{C0}$	$\frac{141}{182} = 0.A\overline{0C}$
$\frac{3}{182} = 0.0\overline{2A}$	$\frac{179}{182} = 0.C\overline{A2}$	$\frac{57}{182} = 0.4\overline{0C}$	$\frac{125}{182} = 0.8\overline{C0}$
$\frac{5}{182} = 0.0\overline{48}$	$\frac{177}{182} = 0.C\overline{84}$	$\frac{59}{182} = 0.4\overline{2A}$	$\frac{123}{182} = 0.8\overline{A2}$
$\frac{9}{182} = 0.0\overline{84}$	$\frac{173}{182} = 0.C\overline{48}$	$\frac{61}{182} = 0.4\overline{48}$	$\frac{121}{182} = 0.8\overline{84}$
$\frac{11}{182} = 0.0\overline{A2}$	$\frac{171}{182} = 0.C\overline{2A}$	$\frac{67}{182} = 0.4\overline{A2}$	$\frac{115}{182} = 0.8\overline{2A}$
$\frac{29}{182} = 0.2\overline{0C}$	$\frac{153}{182} = 0.A\overline{C0}$	$\frac{69}{182} = 0.4\overline{C0}$	$\frac{113}{182} = 0.8\overline{0C}$
$\frac{31}{182} = 0.2\overline{2A}$	$\frac{151}{182} = 0.A\overline{A2}$	$\frac{85}{182} = 0.6\overline{04}$	$\frac{97}{182} = 0.6\overline{C0}$
$\frac{33}{182} = 0.2\overline{48}$	$\frac{149}{182} = 0.A\overline{84}$	$\frac{87}{182} = 0.6\overline{2A}$	$\frac{95}{182} = 0.6\overline{A2}$
$\frac{37}{182} = 0.2\overline{84}$	$\frac{145}{182} = 0.A\overline{48}$	$\frac{89}{182} = 0.6\overline{48}$	$\frac{93}{182} = 0.6\overline{84}$

Table 40: All elements in child 13-ary set A_{1014}^{13}

$\frac{1}{1014} = 0.00\overline{2}$	$\frac{1013}{1014} = 0.CC\overline{A}$	$\frac{217}{1014} = 0.2A\overline{2}$	$\frac{797}{1014} = 0.A2\overline{A}$
$\frac{5}{1014} = 0.00\overline{A}$	$\frac{1009}{1014} = 0.CC\overline{2}$	$\frac{229}{1014} = 0.2C\overline{2}$	$\frac{797}{1014} = 0.A0\overline{A}$
$\frac{17}{1014} = 0.02\overline{A}$	$\frac{997}{1014} = 0.CA\overline{2}$	$\frac{233}{1014} = 0.2C\overline{A}$	$\frac{781}{1014} = 0.A0\overline{2}$
$\frac{25}{1014} = 0.04\overline{2}$	$\frac{989}{1014} = 0.C8\overline{A}$	$\frac{313}{1014} = 0.40\overline{2}$	$\frac{701}{1014} = 0.8C\overline{A}$
$\frac{29}{1014} = 0.04\overline{A}$	$\frac{985}{1014} = 0.C8\overline{2}$	$\frac{317}{1014} = 0.40\overline{A}$	$\frac{697}{1014} = 0.8C\overline{2}$

$\frac{37}{1014} = 0.06\overline{2}$	$\frac{977}{1014} = 0.C6\overline{A}$	$\frac{329}{1014} = 0.42\overline{A}$	$\frac{685}{1014} = 0.8A\overline{2}$
$\frac{41}{1014} = 0.06\overline{A}$	$\frac{973}{1014} = 0.C6\overline{2}$	$\frac{337}{1014} = 0.44\overline{2}$	$\frac{677}{1014} = 0.88\overline{A}$
$\frac{49}{1014} = 0.08\overline{2}$	$\frac{965}{1014} = 0.C4\overline{A}$	$\frac{341}{1014} = 0.44\overline{A}$	$\frac{673}{1014} = 0.88\overline{2}$
$\frac{53}{1014} = 0.08\overline{A}$	$\frac{961}{1014} = 0.C4\overline{2}$	$\frac{349}{1014} = 0.46\overline{2}$	$\frac{665}{1014} = 0.86\overline{A}$
$\frac{61}{1014} = 0.0A\overline{2}$	$\frac{953}{1014} = 0.C2\overline{A}$	$\frac{353}{1014} = 0.46\overline{A}$	$\frac{661}{1014} = 0.86\overline{2}$
$\frac{73}{1014} = 0.0C\overline{2}$	$\frac{941}{1014} = 0.C0\overline{A}$	$\frac{361}{1014} = 0.48\overline{2}$	$\frac{653}{1014} = 0.84\overline{A}$
$\frac{77}{1014} = 0.0C\overline{A}$	$\frac{937}{1014} = 0.C0\overline{2}$	$\frac{365}{1014} = 0.48\overline{A}$	$\frac{649}{1014} = 0.84\overline{2}$
$\frac{157}{1014} = 0.20\overline{2}$	$\frac{857}{1014} = 0.AC\overline{A}$	$\frac{373}{1014} = 0.4A\overline{2}$	$\frac{641}{1014} = 0.82\overline{A}$
$\frac{161}{1014} = 0.20\overline{A}$	$\frac{853}{1014} = 0.AC\overline{2}$	$\frac{385}{1014} = 0.4C\overline{2}$	$\frac{629}{1014} = 0.80\overline{A}$
$\frac{173}{1014} = 0.22\overline{A}$	$\frac{841}{1014} = 0.AA\overline{2}$	$\frac{389}{1014} = 0.4C\overline{A}$	$\frac{625}{1014} = 0.80\overline{2}$
$\frac{181}{1014} = 0.24\overline{2}$	$\frac{833}{1014} = 0.A8\overline{A}$	$\frac{469}{1014} = 0.60\overline{2}$	$\frac{545}{1014} = 0.6C\overline{A}$
$\frac{185}{1014} = 0.24\overline{A}$	$\frac{829}{1014} = 0.A8\overline{2}$	$\frac{473}{1014} = 0.60\overline{A}$	$\frac{541}{1014} = 0.6C\overline{2}$
$\frac{193}{1014} = 0.26\overline{2}$	$\frac{821}{1014} = 0.A6\overline{A}$	$\frac{485}{1014} = 0.62\overline{A}$	$\frac{529}{1014} = 0.6A\overline{2}$
$\frac{197}{1014} = 0.26\overline{A}$	$\frac{817}{1014} = 0.A6\overline{2}$	$\frac{493}{1014} = 0.64\overline{2}$	$\frac{521}{1014} = 0.68\overline{A}$
$\frac{205}{1014} = 0.28\overline{2}$	$\frac{809}{1014} = 0.A4\overline{A}$	$\frac{497}{1014} = 0.64\overline{A}$	$\frac{517}{1014} = 0.68\overline{2}$
$\frac{209}{1014} = 0.28\overline{A}$	$\frac{805}{1014} = 0.A4\overline{2}$	$\frac{505}{1014} = 0.66\overline{2}$	$\frac{509}{1014} = 0.66\overline{A}$

(4) Characteristic of elements in partitions

Table 41: Characteristic of elements in each partition of the set A_{62}^{5}

P_1	P_2
$1 \equiv 1 \cdot 5^0 \pmod{62}$	$7 \equiv 7 \cdot 5^0 \pmod{62}$
$5 \equiv 1 \cdot 5^1 \pmod{62}$	$35 \equiv 7 \cdot 5^1 \pmod{62}$
$25 \equiv 1 \cdot 5^2 \pmod{62}$	$51 \equiv 7 \cdot 5^2 \pmod{62}$
$61 \equiv -1 \cdot 5^3 \pmod{62}$	$55 \equiv -7 \cdot 5^3 \pmod{62}$
$57 \equiv -1 \cdot 5^4 \pmod{62}$	$27 \equiv -7 \cdot 5^4 \pmod{62}$
$37 \equiv -1 \cdot 5^5 \pmod{62}$	$11 \equiv -7 \cdot 5^5 \pmod{62}$

Table 42: Characteristic of elements in each partition of the set A_{126}^{5}

P_1	P_2	P_3
$1 \equiv 1 \cdot 5^0 \pmod{126}$	$11 \equiv 11 \cdot 5^0 \pmod{126}$	$13 \equiv 13 \cdot 5^0 \pmod{126}$
$5 \equiv 1 \cdot 5^1 \pmod{126}$	$55 \equiv 11 \cdot 5^1 \pmod{126}$	$65 \equiv 13 \cdot 5^1 \pmod{126}$
$25 \equiv 1 \cdot 5^2 \pmod{126}$	$23 \equiv 11 \cdot 5^2 \pmod{126}$	$73 \equiv 13 \cdot 5^2 \pmod{126}$
$125 \equiv 1 \cdot 5^3 \pmod{126}$	$115 \equiv 11 \cdot 5^3 \pmod{126}$	$113 \equiv 13 \cdot 5^3 \pmod{126}$
$121 \equiv 1 \cdot 5^4 \pmod{126}$	$71 \equiv 11 \cdot 5^4 \pmod{126}$	$61 \equiv 13 \cdot 5^4 \pmod{126}$
$101 \equiv 1 \cdot 5^5 \pmod{126}$	$103 \equiv 11 \cdot 5^5 \pmod{126}$	$53 \equiv 13 \cdot 5^5 \pmod{126}$

Table 43: Characteristic of elements in each partition of the set A_{50}^{7}

P_1	P_2	P_3
$1 \equiv 1 \cdot 7^0 \pmod{50}$	$3 \equiv 3 \cdot 7^0 \pmod{50}$	$17 \equiv 17 \cdot 7^0 \pmod{50}$
$7 \equiv 1 \cdot 7^1 \pmod{50}$	$21 \equiv 3 \cdot 7^1 \pmod{50}$	$19 \equiv 17 \cdot 7^1 \pmod{50}$
$49 \equiv 1 \cdot 7^2 \pmod{50}$	$47 \equiv 3 \cdot 7^2 \pmod{50}$	$33 \equiv 17 \cdot 7^2 \pmod{50}$
$43 \equiv 1 \cdot 7^3 \pmod{50}$	$29 \equiv 3 \cdot 7^3 \pmod{50}$	$31 \equiv 17 \cdot 7^3 \pmod{50}$

Table 44: Characteristic of elements in each partition of the set A^7_{171}

P_1	P_2
$1 \equiv 1 \cdot 7^0 \pmod{171}$	$2 \equiv 2 \cdot 7^0 \pmod{171}$
$7 \equiv 1 \cdot 7^1 \pmod{171}$	$14 \equiv 2 \cdot 7^1 \pmod{171}$
$49 \equiv 1 \cdot 7^2 \pmod{171}$	$98 \equiv 2 \cdot 7^2 \pmod{171}$
$170 \equiv -1 \cdot 7^3 \pmod{171}$	$169 \equiv -2 \cdot 7^3 \pmod{171}$
$164 \equiv -1 \cdot 7^4 \pmod{171}$	$157 \equiv -2 \cdot 7^4 \pmod{171}$
$122 \equiv -1 \cdot 7^5 \pmod{171}$	$73 \equiv -2 \cdot 7^5 \pmod{171}$
P_3	P_4
$3 \equiv 3 \cdot 7^0 \pmod{171}$	$8 \equiv 8 \cdot 7^0 \pmod{171}$
$21 \equiv 3 \cdot 7^1 \pmod{171}$	$56 \equiv 8 \cdot 7^1 \pmod{171}$
$147 \equiv 3 \cdot 7^2 \pmod{171}$	$50 \equiv 8 \cdot 7^2 \pmod{171}$
$168 \equiv -3 \cdot 7^3 \pmod{171}$	$163 \equiv -8 \cdot 7^3 \pmod{171}$
$150 \equiv -3 \cdot 7^4 \pmod{171}$	$115 \equiv -8 \cdot 7^4 \pmod{171}$
$24 \equiv -3 \cdot 7^5 \pmod{171}$	$121 \equiv -8 \cdot 7^5 \pmod{171}$
P_5	P_6
$9 \equiv 9 \cdot 7^0 \pmod{171}$	$10 \equiv 10 \cdot 7^0 \pmod{171}$
$63 \equiv 9 \cdot 7^1 \pmod{171}$	$70 \equiv 10 \cdot 7^1 \pmod{171}$
$99 \equiv 9 \cdot 7^2 \pmod{171}$	$148 \equiv 10 \cdot 7^2 \pmod{171}$
$162 \equiv -9 \cdot 7^3 \pmod{171}$	$161 \equiv -10 \cdot 7^3 \pmod{171}$
$108 \equiv -9 \cdot 7^4 \pmod{171}$	$101 \equiv -10 \cdot 7^4 \pmod{171}$
$72 \equiv -9 \cdot 7^5 \pmod{171}$	$23 \equiv -10 \cdot 7^5 \pmod{171}$
P_7	P_8
$15 \equiv 15 \cdot 7^0 \pmod{171}$	$16 \equiv 16 \cdot 7^0 \pmod{171}$
$105 \equiv 15 \cdot 7^1 \pmod{171}$	$112 \equiv 16 \cdot 7^1 \pmod{171}$

$51 \equiv 15 \cdot 7^2 \pmod{171}$	$100 \equiv 16 \cdot 7^2 \pmod{171}$
$156 \equiv -15 \cdot 7^3 \pmod{171}$	$155 \equiv -16 \cdot 7^3 \pmod{171}$
$66 \equiv -15 \cdot 7^4 \pmod{171}$	$59 \equiv -16 \cdot 7^4 \pmod{171}$
$120 \equiv -15 \cdot 7^5 \pmod{171}$	$71 \equiv -16 \cdot 7^5 \pmod{171}$
P_9	P_{10}
$17 \equiv 17 \cdot 7^0 \pmod{171}$	$58 \equiv 58 \cdot 7^0 \pmod{171}$
$119 \equiv 17 \cdot 7^1 \pmod{171}$	$64 \equiv 58 \cdot 7^1 \pmod{171}$
$149 \equiv 17 \cdot 7^2 \pmod{171}$	$106 \equiv 58 \cdot 7^2 \pmod{171}$
$154 \equiv -17 \cdot 7^3 \pmod{171}$	$113 \equiv -58 \cdot 7^3 \pmod{171}$
$52 \equiv -17 \cdot 7^4 \pmod{171}$	$107 \equiv -58 \cdot 7^4 \pmod{171}$
$22 \equiv -17 \cdot 7^5 \pmod{171}$	$65 \equiv -58 \cdot 7^5 \pmod{171}$

Table 45: Characteristic of elements in each partition of the set A_{12}^{11}

P_1	P_2	P_3
$1 \equiv 1 \cdot 11^0 \pmod{12}$	$3 \equiv 3 \cdot 11^0 \pmod{12}$	$5 \equiv 5 \cdot 11^0 \pmod{12}$
$11 \equiv 1 \cdot 11^1 \pmod{12}$	$9 \equiv 3 \cdot 11^1 \pmod{12}$	$7 \equiv 5 \cdot 11^1 \pmod{12}$

Table 46: Characteristic of elements in each partition of the set A_{665}^{11}

P_1	P_2
$1 \equiv 1 \cdot 11^0 \pmod{665}$	$2 \equiv 2 \cdot 11^0 \pmod{665}$
$11 \equiv 1 \cdot 11^1 \pmod{665}$	$22 \equiv 2 \cdot 11^1 \pmod{665}$
$121 \equiv 1 \cdot 11^2 \pmod{665}$	$242 \equiv 2 \cdot 11^2 \pmod{665}$
$664 \equiv -1 \cdot 11^3 \pmod{665}$	$663 \equiv -2 \cdot 11^3 \pmod{665}$
$654 \equiv -1 \cdot 11^4 \pmod{665}$	$643 \equiv -2 \cdot 11^4 \pmod{665}$

$544 \equiv -1 \cdot 11^5 \pmod{665}$	$423 \equiv -2 \cdot 11^5 \pmod{665}$
P_3	P_4
$3 \equiv 3 \cdot 11^0 \pmod{665}$	$4 \equiv 4 \cdot 11^0 \pmod{665}$
$33 \equiv 3 \cdot 11^1 \pmod{665}$	$44 \equiv 4 \cdot 11^1 \pmod{665}$
$363 \equiv 3 \cdot 11^2 \pmod{665}$	$484 \equiv 4 \cdot 11^2 \pmod{665}$
$662 \equiv -3 \cdot 11^3 \pmod{665}$	$661 \equiv -4 \cdot 11^3 \pmod{665}$
$632 \equiv -3 \cdot 11^4 \pmod{665}$	$621 \equiv -4 \cdot 11^4 \pmod{665}$
$302 \equiv -3 \cdot 11^5 \pmod{665}$	$181 \equiv -4 \cdot 11^5 \pmod{665}$
P_5	P_6
$12 \equiv 12 \cdot 11^0 \pmod{665}$	$13 \equiv 13 \cdot 11^0 \pmod{665}$
$132 \equiv 12 \cdot 11^1 \pmod{665}$	$143 \equiv 13 \cdot 11^1 \pmod{665}$
$122 \equiv 12 \cdot 11^2 \pmod{665}$	$243 \equiv 13 \cdot 11^2 \pmod{665}$
$653 \equiv -12 \cdot 11^3 \pmod{665}$	$652 \equiv -13 \cdot 11^3 \pmod{665}$
$533 \equiv -12 \cdot 11^4 \pmod{665}$	$522 \equiv -13 \cdot 11^4 \pmod{665}$
$543 \equiv -12 \cdot 11^5 \pmod{665}$	$422 \equiv -13 \cdot 11^5 \pmod{665}$
P_7	P_8
$16 \equiv 16 \cdot 11^0 \pmod{665}$	$23 \equiv 23 \cdot 11^0 \pmod{665}$
$176 \equiv 16 \cdot 11^1 \pmod{665}$	$253 \equiv 23 \cdot 11^1 \pmod{665}$
$606 \equiv 16 \cdot 11^2 \pmod{665}$	$123 \equiv 23 \cdot 11^2 \pmod{665}$
$649 \equiv -16 \cdot 11^3 \pmod{665}$	$642 \equiv -23 \cdot 11^3 \pmod{665}$
$489 \equiv -16 \cdot 11^4 \pmod{665}$	$412 \equiv -23 \cdot 11^4 \pmod{665}$
$59 \equiv -16 \cdot 11^5 \pmod{665}$	$542 \equiv -23 \cdot 11^5 \pmod{665}$
P_9	P_{10}
$24 \equiv 24 \cdot 11^0 \pmod{665}$	$26 \equiv 26 \cdot 11^0 \pmod{665}$
$264 \equiv 24 \cdot 11^1 \pmod{665}$	$286 \equiv 26 \cdot 11^1 \pmod{665}$
$244 \equiv 24 \cdot 11^2 \pmod{665}$	$486 \equiv 26 \cdot 11^2 \pmod{665}$
$641 \equiv -24 \cdot 11^3 \pmod{665}$	$639 \equiv -26 \cdot 11^3 \pmod{665}$

$401 \equiv -24 \cdot 11^4 \pmod{665}$	$379 \equiv -26 \cdot 11^4 \pmod{665}$
$421 \equiv -24 \cdot 11^5 \pmod{665}$	$179 \equiv -26 \cdot 11^5 \pmod{665}$
P_{11}	P_{12}
$27 \equiv 27 \cdot 11^0 \pmod{665}$	$34 \equiv 34 \cdot 11^0 \pmod{665}$
$297 \equiv 27 \cdot 11^1 \pmod{665}$	$374 \equiv 34 \cdot 11^1 \pmod{665}$
$607 \equiv 27 \cdot 11^2 \pmod{665}$	$124 \equiv 34 \cdot 11^2 \pmod{665}$
$638 \equiv -27 \cdot 11^3 \pmod{665}$	$631 \equiv -34 \cdot 11^3 \pmod{665}$
$368 \equiv -27 \cdot 11^4 \pmod{665}$	$291 \equiv -34 \cdot 11^4 \pmod{665}$
$58 \equiv -27 \cdot 11^5 \pmod{665}$	$541 \equiv -34 \cdot 11^5 \pmod{665}$
P_{13}	P_{14}
$36 \equiv 36 \cdot 11^0 \pmod{665}$	$37 \equiv 37 \cdot 11^0 \pmod{665}$
$396 \equiv 36 \cdot 11^1 \pmod{665}$	$407 \equiv 37 \cdot 11^1 \pmod{665}$
$366 \equiv 36 \cdot 11^2 \pmod{665}$	$487 \equiv 37 \cdot 11^2 \pmod{665}$
$629 \equiv -36 \cdot 11^3 \pmod{665}$	$628 \equiv -37 \cdot 11^3 \pmod{665}$
$269 \equiv -36 \cdot 11^4 \pmod{665}$	$258 \equiv -37 \cdot 11^4 \pmod{665}$
$299 \equiv -36 \cdot 11^5 \pmod{665}$	$178 \equiv -37 \cdot 11^5 \pmod{665}$
P_{15}	P_{16}
$46 \equiv 46 \cdot 11^0 \pmod{665}$	$47 \equiv 47 \cdot 11^0 \pmod{665}$
$506 \equiv 46 \cdot 11^1 \pmod{665}$	$517 \equiv 47 \cdot 11^1 \pmod{665}$
$246 \equiv 46 \cdot 11^2 \pmod{665}$	$367 \equiv 47 \cdot 11^2 \pmod{665}$
$619 \equiv -46 \cdot 11^3 \pmod{665}$	$618 \equiv -47 \cdot 11^3 \pmod{665}$
$159 \equiv -46 \cdot 11^4 \pmod{665}$	$148 \equiv -47 \cdot 11^4 \pmod{665}$
$419 \equiv -46 \cdot 11^5 \pmod{665}$	$298 \equiv -47 \cdot 11^5 \pmod{665}$
P_{17}	P_{18}
$48 \equiv 48 \cdot 11^0 \pmod{665}$	$134 \equiv 134 \cdot 11^0 \pmod{665}$
$528 \equiv 48 \cdot 11^1 \pmod{665}$	$144 \equiv 134 \cdot 11^1 \pmod{665}$
$488 \equiv 48 \cdot 11^2 \pmod{665}$	$254 \equiv 134 \cdot 11^2 \pmod{665}$

$617 \equiv -48 \cdot 11^3 \pmod{665}$	$531 \equiv -134 \cdot 11^3 \pmod{665}$
$137 \equiv -48 \cdot 11^4 \pmod{665}$	$521 \equiv -134 \cdot 11^4 \pmod{665}$
$177 \equiv -48 \cdot 11^5 \pmod{665}$	$411 \equiv -134 \cdot 11^5 \pmod{665}$
P_{19}	P_{20}
$136 \equiv 136 \cdot 11^0 \pmod{665}$	$146 \equiv 146 \cdot 11^0 \pmod{665}$
$166 \equiv 136 \cdot 11^1 \pmod{665}$	$276 \equiv 146 \cdot 11^1 \pmod{665}$
$496 \equiv 136 \cdot 11^2 \pmod{665}$	$376 \equiv 146 \cdot 11^2 \pmod{665}$
$529 \equiv -136 \cdot 11^3 \pmod{665}$	$519 \equiv -146 \cdot 11^3 \pmod{665}$
$499 \equiv -136 \cdot 11^4 \pmod{665}$	$389 \equiv -146 \cdot 11^4 \pmod{665}$
$169 \equiv -136 \cdot 11^5 \pmod{665}$	$289 \equiv -146 \cdot 11^5 \pmod{665}$
P_{21}	P_{22}
$156 \equiv 134 \cdot 11^0 \pmod{665}$	$157 \equiv 157 \cdot 11^0 \pmod{665}$
$386 \equiv 156 \cdot 11^1 \pmod{665}$	$397 \equiv 157 \cdot 11^1 \pmod{665}$
$256 \equiv 156 \cdot 11^2 \pmod{665}$	$377 \equiv 157 \cdot 11^2 \pmod{665}$
$509 \equiv -156 \cdot 11^3 \pmod{665}$	$508 \equiv -157 \cdot 11^3 \pmod{665}$
$279 \equiv -156 \cdot 11^4 \pmod{665}$	$268 \equiv -157 \cdot 11^4 \pmod{665}$
$409 \equiv -156 \cdot 11^5 \pmod{665}$	$288 \equiv -157 \cdot 11^5 \pmod{665}$
P_{23}	P_{24}
$158 \equiv 158 \cdot 11^0 \pmod{665}$	$267 \equiv 267 \cdot 11^0 \pmod{665}$
$408 \equiv 158 \cdot 11^1 \pmod{665}$	$277 \equiv 267 \cdot 11^1 \pmod{665}$
$498 \equiv 158 \cdot 11^2 \pmod{665}$	$387 \equiv 267 \cdot 11^2 \pmod{665}$
$507 \equiv -158 \cdot 11^3 \pmod{665}$	$398 \equiv -267 \cdot 11^3 \pmod{665}$
$257 \equiv -158 \cdot 11^4 \pmod{665}$	$388 \equiv -267 \cdot 11^4 \pmod{665}$
$167 \equiv -158 \cdot 11^5 \pmod{665}$	$278 \equiv -267 \cdot 11^5 \pmod{665}$

Table 47: Characteristic of elements in each partition of the set A_{84}^{13}

P_1	P_2
$1 \equiv 1 \cdot 13^0 \pmod{84}$	$5 \equiv 5 \cdot 13^0 \pmod{84}$
$13 \equiv 1 \cdot 13^1 \pmod{84}$	$65 \equiv 5 \cdot 13^1 \pmod{84}$
$83 \equiv -1 \cdot 13^2 \pmod{84}$	$79 \equiv -5 \cdot 13^2 \pmod{84}$
$71 \equiv -1 \cdot 13^3 \pmod{84}$	$19 \equiv -5 \cdot 13^3 \pmod{84}$
P_3	P_4
$17 \equiv 17 \cdot 13^0 \pmod{84}$	$29 \equiv 29 \cdot 13^0 \pmod{84}$
$53 \equiv 17 \cdot 13^1 \pmod{84}$	$41 \equiv 29 \cdot 13^1 \pmod{84}$
$67 \equiv -17 \cdot 13^2 \pmod{84}$	$55 \equiv -29 \cdot 13^2 \pmod{84}$
$31 \equiv -17 \cdot 13^3 \pmod{84}$	$43 \equiv -29 \cdot 13^3 \pmod{84}$

Table 48: Characteristic of elements in each partition of the set A_{170}^{13}

P_1	P_2
$1 \equiv 1 \cdot 13^0 \pmod{170}$	$3 \equiv 3 \cdot 13^0 \pmod{170}$
$13 \equiv 1 \cdot 13^1 \pmod{170}$	$39 \equiv 3 \cdot 13^1 \pmod{170}$
$169 \equiv 1 \cdot 13^2 \pmod{170}$	$167 \equiv 3 \cdot 13^2 \pmod{170}$
$157 \equiv 1 \cdot 13^3 \pmod{170}$	$131 \equiv 3 \cdot 13^3 \pmod{170}$
P_3	P_4
$7 \equiv 7 \cdot 13^0 \pmod{170}$	$9 \equiv 9 \cdot 13^0 \pmod{170}$
$91 \equiv 7 \cdot 13^1 \pmod{170}$	$117 \equiv 9 \cdot 13^1 \pmod{170}$
$163 \equiv 7 \cdot 13^2 \pmod{170}$	$161 \equiv 9 \cdot 13^2 \pmod{170}$
$79 \equiv 7 \cdot 13^3 \pmod{170}$	$53 \equiv 9 \cdot 13^3 \pmod{170}$
P_5	P_6
$11 \equiv 11 \cdot 13^0 \pmod{170}$	$29 \equiv 29 \cdot 13^0 \pmod{170}$

$143 \equiv 11 \cdot 13^1 \pmod{170}$	$37 \equiv 29 \cdot 13^1 \pmod{170}$
$159 \equiv 11 \cdot 13^2 \pmod{170}$	$141 \equiv 29 \cdot 13^2 \pmod{170}$
$27 \equiv 11 \cdot 13^3 \pmod{170}$	$133 \equiv 29 \cdot 13^3 \pmod{170}$
P_7	P_8
$31 \equiv 31 \cdot 13^0 \pmod{170}$	$33 \equiv 33 \cdot 13^0 \pmod{170}$
$63 \equiv 31 \cdot 13^1 \pmod{170}$	$89 \equiv 33 \cdot 13^1 \pmod{170}$
$139 \equiv 31 \cdot 13^2 \pmod{170}$	$137 \equiv 33 \cdot 13^2 \pmod{170}$
$107 \equiv 31 \cdot 13^3 \pmod{170}$	$81 \equiv 33 \cdot 13^3 \pmod{170}$
P_9	P_{10}
$57 \equiv 57 \cdot 13^0 \pmod{170}$	$59 \equiv 59 \cdot 13^0 \pmod{170}$
$61 \equiv 57 \cdot 13^1 \pmod{170}$	$87 \equiv 59 \cdot 13^1 \pmod{170}$
$113 \equiv 57 \cdot 13^2 \pmod{170}$	$111 \equiv 59 \cdot 13^2 \pmod{170}$
$109 \equiv 57 \cdot 13^3 \pmod{170}$	$83 \equiv 59 \cdot 13^3 \pmod{170}$

Vitae

Name: Mr.Riduan Waema

Student ID: 5620320705

Educational Attainment:

Degree	Name of Institution	Year of Graduation
B.Sc.Ed. (Mathematics)	Prince of Songkla University, Pattani Campus, Thailand	2013

Scholarship

- Faculty of Science and Technology's scholarship for graduate students.
- PSU.GS. Financial Support for Thesis.
- Centre of Excellence in Mathematics Scholarship (CEM).

Work-Position and Address

The leader of The 5^{th} Sci-Tech Grad Symposium, Faculty of Science and Technology, Prince of Songkla University, Pattani campus.

List of Proceeding

Waema R., Phon-On A. and Busaman S. 2016. Cardinality of Child p-ary Sets in Cantor p-ary Sets. Proceeding of Annual Meeting in Mathematics 2016 and Annual Pure and Applied Mathematics Conference 2016: 23-25 May 2016, Chula longkorn University, Thailand.