Finite Integration Method for a Time-Dependent Heat Source Identification of Inverse Problem

Rama Lesmana

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Author	Mr. Rama Lesmana
Major Program	Applied Mathematics

Major Advisor

. . . .

(Dr. Areena Hazanee)

Co-Advisor

. . . .

(Asst. Prof. Dr. Aniruth Phon-On)

Examining Committee

. . . . Chairperson

(Asst. Prof. Dr. Nifatamah Makaje)

. . . .

(Dr. Areena Hazanee)

. . . .

(Asst. Prof. Dr. Aniruth Phon-On)

. . . .

(Asst. Prof. Dr. Areeyuth Sama-ae)

. . . .

(Dr. Chainarong Kesamoon)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Applied Mathematics.

. . . .

(Assoc. Prof. Dr. Teerapol Srichana)

Dean of Graduate School

This is to certify that the work here submitted is the result of the candidate's own investigation. Due acknowledgement has been made of any assistance received.

.... Signature
(Dr. Areena Hazanee)
Major Advisor

.... Signature
(Mr. Rama Lesmana)
Candidate

I hereby certify that this work has not been accepted in substance for any degree, and is not being currently submitted in candidature for any degree.

. . . . Signature

(Mr. Rama Lesmana)

Candidate

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ผู้เขียน Mr. Rama Lesmana

สาขาวิชา คณิตศาสตร์ประยุกต์

ปีการศึกษา 2559

บทคัดย่อ

วิทยานิพนธ์เรื่องนี้เป็นการศึกษาการหาฟังก์ชันแหล่งกำเนิดที่ขึ้นกับเวลาของสมการความร้อนซึ่งเป็นปัญหาผกผัน ผลเฉลยของปัญหาผกผันนี้มีอยู่จริงและมีเพียงหนึ่งเดียวแต่เป็นผลเฉลยที่ไม่เสถียร สำหรับการศึกษาปัญหาของการคำนวณหาพลังงานความร้อน สำหรับปัญหาที่ต้องการคำนวณหาอุณภูมิของระบบโดยทราบค่าของฟังก์ชันของแหล่งกำเนิดความร้อน (ปัญหามีตัวไม่ทราบค่า 1 ตัว) เราจะเรียกปัญหานี้ว่า ปัญหาตรง หรือ ปัญหาไปข้างหน้า แต่หากเราไม่ทราบฟังก์ชันของแหล่งกำเนิดความร้อนและยังต้องการคำนวณหาอุณภูมิและฟังก์ชันแหล่งความร้อนนี้ (ปัญหามีตัวไม่ทราบค่า 2 ตัว) เราจะเรียกปัญหานี้ว่า ปัญหาผกผัน สำหรับการศึกษาปัญหาผกผันนี้จะมีเทคนิคคือการแปลงระบบของปัญหาผกผันให้เป็นระบบของปัญหาตรงซึ่งหมายความว่าแปลงระบบจากปัญหาที่มีตัวไม่ทราบค่า 2 ตัว ให้เป็นระบบที่มีตัวไม่ทราบค่าเพียงแค่ตัวเดียว ขั้นตอนวิธีที่ได้นำเสนอในการศึกษานี้ไม่เพียงแต่จะง่ายต่อการนำไปใช้ แต่ยังสามารถคำนวณหาผลเฉลยได้อย่างแม่นยำและมีเสถียรอีกด้วย ในการศึกษานี้ เราคาดหวังว่าระเบียบวิธีไฟไนต์อินทิเกรชันจะสามารถนำมาประยุกต์ใช้ในการหาผลเฉลยของปัญหาผกผันที่เราสนใจนี้ได้ และยิ่งไปกว่านี้เราได้ศึกษาการแก้ปัญหาผกผันนี้ด้วยระเบียบวิธีพื้นฐานอย่างระเบียบวิธีผลต่างอันตะเพื่อเป็นการทดสอบเปรียบเทียบผลเฉลยที่ได้กับระเบียบวิธีที่เราคาดหวังไว้ และเนื่องจากปัญหานี้เป็นปัญหาผกผัน ซึ่งเป็นปัญหาที่ตั้งขึ้นอย่างเลว ทำให้ผลเฉลยที่ได้จากปัญหานี้ไม่เสถียร เราจึงทำให้ระบบมีเสถียรภาพด้วยการใช้เทคนิครีกูลาไรเซชันของทิคคอนอฟ และเพื่อตรวจสอบความแม่นยำของระเบียบวิธีที่ศึกษานี้เราจึงได้นำเสนอตัวอย่างเชิงตัวเลขประกอบอีกด้วย

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ABSTRACT

This study investigates an inverse problem of reconstructing a time wise- dependent source function for the heat equation. The solution of the problem is uniquely solvable, yet unstable. The problem of finding the temperature when the heat source function is given, is called forward problem (or direct problem), whereas when the heat source function is unknown then the problem of two unknowns becomes to be called the inverse problem. In this study the inverse source problem is reformulated to be a new direct problem, meaning that the problem for two unknowns is transformed to be the problem for only one unknown. The proposed algorithm is not only easy to be used but also can give an accurate and stable solution. We propose that two kinds of the finite integration method combined with the backward finite difference method can be used to solve the reformulated heat equation. Furthermore, we also carried out the finite difference method for solving the inverse problem, this is in order to test the efficiency of the proposed method. Since the solution is unstable, the instability is overcome by employing the Tikhonov regularization method which is the method for stabilizing the problem. Numerical examples are presented and discussed to verify the accuracy of the proposed computational method.

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Chapter 1

General Introduction

1.1 Introduction and literature review

In the real life, not all problems can be solved directly, the situation is really often reversed, we may not know all information that we need to examine. Because of lack of full information, incomplete system provided, this brings us to consider the inverse problems. The inverse problem occurs mathematically in many branches of research fields such as science, engineering, medical or even economy. At its simplest level, the direct problem or forward problem is to determine the data from the model, whereas, the inverse problem or indirect problem is about to determine the model, parameter/source/initial data/boundary data of model, by considering observed data. Despite the observed data is incomplete and containing error which is unavoidable in practice or reality. This can be concluded that, the inverse problem is always a counterpart of direct problem as expressed as schematic diagram Figure 1.1:

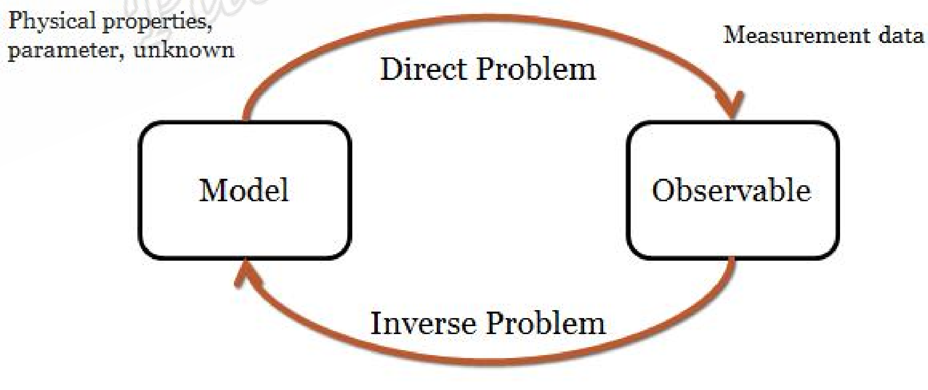


Figure 1.1: Schematic of direct and inverse problems

Generally, in sense of applied mathematical problems, not all problems have solution, and even the problem has the solution it is not guarantee to have only one solution. If the problem has more than one solution, how can we know which one is the correct one? This kind of questions exists when we start considering any mathematical problem. To clarify the problem before seeking the solution, this kind of questions have to be declared. The classification of problem has been defined by J. Hadamard in 1902 that the mathematical problem is well-posed if the following conditions hold,

- Existence: For all (suitable) data, there exists a solution of the problem (in an appropiate sense).
- Uniqueness: For all (suitable) data, the solution is unique.
- Stability: The solution depends continuously on its data (i.e. small perturbations in the input data do not result in large perturbations in the solution).

Based on the above definition, any mathematical problem is ill-posed if one of these conditions is missing. The ill-posed problem normally occurs in inverse problems i.e. this violates the stability of the solution.

Most inverse problems arise from a physical situation modeled by partial differential equation. There are many kinds of inverse problems such as inverse initial value problem, inverse boundary value problem, inverse coefficient identification problem and inverse source problem. The heat equation is a prototypical example of inverse source problems that has significant application in the field of applied sciences such as in mathematical modeling to identify the unknown source function in polution source intensity, melting and freezing process.

In order to understand the idea of inverse source problem, let us consider the direct classical (one-dimension) heat equation with the source term as given by:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t),$$

where

$u(x,t)$ represents temperature as a function of space (x) and time (t),

$\frac{\partial u}{\partial t}(x,t)$ represents the rate of temperature change at a point (x) over time (t),

$\frac{\partial^2 u}{\partial x^2}(x,t)$ is the second order derivative with respect to space (x),

$F(x,t)$ represents a heat source.

For forward heat problem, a typical routine work is to calculate the temperature $u(x,t)$ with a given heat source function $F(x,t)$, satisfying to the boundary and initial conditions. Whereas for the inverse heat source problem. The typical work is not only to estimate the temperature $u(x,t)$ but also to determine the heat source function $F(x,t)$. This inverse study may not be easy to estimate both unknowns $F(x,t)$ and $u(x,t)$ directly as there are more than one unknown in the problem and the conditions are not sufficient to guarantee the unique solution. Hence the additional condition(s) considered as the observed data, or may be called as over determination condition, need to be involved. In practical there are several types of additional conditions such as fixed point temperature, time-average heat flux or integral temperature.

Since 1970s many researchers have been paid attention on inverse source problem for the heat equation see (Cannon, 1967; Prilepko and Solovev, 1988; Malyshev, 1989). Recently, Yan *et al*. (2008) have solved the inverse heat source problem by using the method of fundamental solution (MFS) to discretize the domain step together with the Tikhonov regularization to stabilize the solution. However, The MFS method needs source points given from outside the domain which make the process to be complicated. In order to avoid the complicated procedure as the use of MFS, a few years later, Xiangtuan *et al*. (2011) have also considered the inverse heat source problem by using the finite difference method (FDM) which does not require the source points from the outside the domain.

In the use of the FDM, the discretization step needs to be taken over the domain. Recently, Hazanee *et al*. (2013) have used the boundary element method (BEM) to solve the inverse heat source problem. The main advantage of the BEM is to take the discretization only on the boundary, i.e. this method then uses less number of points than the FDM, saying that only the boundary points to be used. According to those studies, efficiency and difficulty procedures depend on method that we apply, so the method for discretization part is so important and need to be chosen carefully.

In this study, we propose two kinds of the finite integration method (FIM) to solve the inverse source problem. A basic idea of the FIM is to construct a matrix form for discretization the equation of interest. Once FIM with the ordinary linear approximation (OLA) is based on the trapezoidal rule which is numerical integration of using linear function to approximate the integral. The FIM (OLA) was first reconstructed and introduced by Li *et al*. (2013). Therefore, this method has been extensively used for dealing with the direct problem with both ordinary and partial differential equations especially for solving the problem of nonlocal elastic bar under static.

In the same year, the FIM (OLA) and FIM with radial basis function (RBF) was introduced simultaneously by Wen *et al*. (2013). Improvement of FIM is based on taking integration of radial basis interpolation function analytically. Therefore, both FIM (OLA) and FIM (RBF) were proposed to deal with the forward problem especially fractional-order of PDE. They have found that the numerical results obtained by using FIM is better than by using FDM.

The FIM has also been improved to be able to solve various kinds of the differential equations. Recently, Li *et al*. (2016) have improved the FIM by applied the Simpson's rule (SSR) for integration. In this present study, they compared the accuracy of the FIM (OLA) and the present FIM(SSR). Therefore, They have found out that the improved FIM (SSR) is better than FIM (OLA) in term of accuracy.

Along the above mentioned, the inverse heat source problem is very attractive problem and the FIM is also interesting method. Moreover, the FIM is a renew numerical method with a potential applicability to solve the inverse problem. Being an applicable method for solving forward problem brings our curiosity to whether the FIM is capable enough to be applied in inverse problems. Therefore, this study is aimed to propose the FIM to solve the inverse source problem for the heat equation. Furthermore, in order to know the efficiency of the FIM in solving the inverse problems, we also use a classical method which is FDM to deal with the inverse heat source problem and then compare the accuracy of the solution of these two methods.

1.2 The finite difference method (FDM)

Construction of FDM is based on the Taylor series expansion that involves the $n-$ order derivative of a function. We then need to first present the definition of smooth function which is a function that can be taken its any order derivatives everywhere in its domain.

Given a smooth function $f(x)$ on $[a, b]$ and consider mesh size $\Delta x = \frac{b-a}{N}$ and grid points $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, N$. Finite difference approximations are used to approximate the derivatives of f using Taylor series with the reference point at x. Denote $f^{(n)}(x)$ as nth-order derivative of $f(x)$, the Taylor series expansion of $f(x+\Delta x)$ at x can be expressed as

$$f(x+\Delta x) = f(x) + \frac{\Delta x}{1!}f'(x) + \frac{(\Delta x)^2}{2!}f''(x) + \frac{(\Delta x)^3}{3!}f'''(x) + \cdots + \frac{(\Delta x)^n}{n!}f^{(n)}(x) + \cdots \tag{1.1}$$

$$f(x-\Delta x) = f(x) - \frac{\Delta x}{1!}f'(x) + \frac{(\Delta x)^2}{2!}f''(x) - \frac{(\Delta x)^3}{3!}f'''(x) + \cdots - \frac{(\Delta x)^n}{n!}f^{(n)}(x) + \cdots \tag{1.2}$$

From these expansions, three approximation of first-order derivative can be constructed which are forward, backward and central difference. The forward and backward difference are normally used to approximate the first-order derivative $f'(x)$ at the starting point a and the ending point b, respectively, whereas in order to get better accuracy, we use the central difference to approximate first-order derivative $f'(x)$ at interior point $x \in (a,b)$. Furthermore, the FDM can also be used to discretize the heat equation of interest. Here is the following procedures of the FDM.

1.2.1 Forward difference

Consider the Taylor series expansion in (1.1), by rearranging the expansion, we have

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{\Delta x}{2}f''(x) - \frac{(\Delta x)^2}{6}f'''(x) - \cdots - \frac{(\Delta x)^{n-1}}{n!}f^{(n-1)}(x) - \ldots$$

We can approximate the first-order derivative by truncating the second- and higher-order derivative as

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x),$$

where $O(\Delta x)$ represents truncation error.

1.2.2 Backward difference

Consider the Taylor series expansion in (1.2), rearranging the expansion yields

$$f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \frac{\Delta x}{2}f''(x) - \frac{(\Delta x)^2}{6}f'''(x) + \cdots - \frac{(\Delta x)^{n-1}}{n!}f^{(n-1)}(x) + \ldots$$

We can approximate the first-order derivative by truncating the second- and higher-order derivative as

$$f'(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + O(\Delta x).$$

1.2.3 Central difference

In order to obtain better approximation of first-order derivative of f, we can consider the central difference. we then need to subtract (1.1) by (1.2). The approximation of $f'(x)$ becomes

$$f(x + \Delta x) - f(x - \Delta x) = 2\Delta x f'(x) + O(\Delta x)^2,$$

then rearranging the result of subtraction above gives

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(\Delta x)^2.$$

Also, in order to find the second-order derivative of $f(x)$, i.e. $f''(x)$, we need to take the summation of the forward and backward difference, by adding two equations above (1.1) and (1.2), we can get

$$f(x + \Delta x) + f(x - \Delta x) = 2f(x) + (\Delta x)^2 f''(x) + O(\Delta x)^4,$$

then rearranging the result of addition above yields

$$(\Delta x)^2 f''(x) = f(x + \Delta x) - 2f(x) + f(x - \Delta x) + O(\Delta x)^4,$$
$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2} + O(\Delta x)^2.$$

1.3 The finite integration method (FIM)

In this section, let us introduce a renew method for solving PDEs which is proposed to deal with the inverse problems. This is constructed in recently years by employing two kinds of approximation function of FIM which are the linear approximation and the radial basis function.

1.3.1 FIM with ordinary linear approximation (OLA)

Let $f(x)$ be a smooth function on $[a, b]$. Once, we are considering the finite integration method together with the linear approximation to estimate the multi-layer integration of the function. We start with approximating a definite integral of $f(x)$ from a to b; $\int_a^b f(x)dx$, by using trapezoidal rule as the following formula

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_N)],$$

where N is a number of segments in $[a, b]$ with subinterval $\Delta x = \dfrac{b-a}{N}$ and $x_i = a + i\Delta x$ for $i = 0, 1, 2, \ldots, N$. By this integral approximation, we can calculate the definite integral starting from point a to any discreted point x_k for $k = 0, 1, 2, \ldots, N$ as

$$F^{(1)}(x_k) = \int_a^{x_k} f(x)\,dx \approx \Delta x \left[\frac{f(x_0)}{2} + \sum_{i=1}^{k-1} f(x_i) + \frac{f(x_k)}{2}\right].$$

This is called a single layer definite integral. From above, the general form of the single layer definite integral can be rewriten as

$$\int_a^{x_k} f(x)\,dx \approx \sum_{i=0}^{k} a_{ki}^{(1)} f(x_i), \tag{1.3}$$

where $a_{0i}^{(1)} = 0$, $a_{ki}^{(1)} = \begin{cases} \dfrac{\Delta x}{2}, & i = 0, k \\ \Delta x, & i = 1, 2, \ldots, k-1 \end{cases}$, and also the matrix form of integration is expressed as follow:

$$\underline{F}^{(1)} = \boldsymbol{A}^{(1)} \underline{f},$$

where

$$\underline{F}^{(1)} = \left[\int_a^{x_0} f(x)\,dx, \int_a^{x_1} f(x)\,dx, \ldots, \int_a^{x_N} f(x)\,dx\right]^{\intercal},$$

$$\underline{f} = [f(x_0), f(x_1), \ldots, f(x_N)]^{\intercal},$$

$$\boldsymbol{A}^{(1)} = \left[a_{ki}^{(1)}\right] = (\Delta x) \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & \ldots & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & \ldots & 0 & 0 \\ 1/2 & 1 & 1 & 1/2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1/2 & 1 & 1 & 1 & \ldots & 1 & 1/2 \end{bmatrix}_{(N+1)\times(N+1)}.$$

We can also write the double-layer definite integral as

$$F^{(2)}(x_k) = \int_a^{x_k}\int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{k}\sum_{j=0}^{i} a_{ki}^{(1)}a_{ij}^{(1)}f(x_j) = \sum_{i=0}^{k} a_{ki}^{(2)}f(x_i),$$

where $\{y, y_1\}$ is a set of dummy variables, $a_{0i}^{(2)} = 0$ and

$$a_{ki}^{(2)} = \begin{cases} \dfrac{[1+(2k-1)](\Delta x)^2}{4}, & i = 0, \\ (k-i)(\Delta x)^2, & i = 1, 2, 3, \dots, k-1, \\ \dfrac{(\Delta x)^2}{4}, & i = k. \end{cases}$$

The double-layer integral can be also written in a matrix form as

$$\underline{F}^{(2)} = \boldsymbol{A}^{(2)}\underline{f},$$

where the double-layer integration matrix,

$$\boldsymbol{A}^{(2)}=[a_{ki}^{(2)}] = (\Delta x)^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3/4 & 1 & 1/4 & 0 & 0 & \dots & 0 & 0 \\ 5/4 & 2 & 1 & 1/4 & 0 & \dots & 0 & 0 \\ 7/4 & 3 & 2 & 1 & 1/4 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \frac{1+2(N-1)}{4} & N-1 & N-2 & N-3 & \dots & 2 & 1 & 1/4 \end{bmatrix}_{(N+1)\times(N+1)}.$$

In the same way, for higher-layer definite integral, we have

$$F^{(m)}(x_k) = \int_a^{x_k}\int_a^{y_{m-1}} \dots \int_a^{y_1} f(y)dy \dots dy_{m-2}dy_{m-1} = \sum_{i=0}^{k} a_{ki}^{(m)}f(x_i). \quad (1.4)$$

By mathematical induction, the higher-layer integration can be defined as $\boldsymbol{A}^{(k)} = \boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A} = \boldsymbol{A}^k$. Thus, the m-layer integral can be written in a matrix form as

$$\underline{F}^{(m)} = \boldsymbol{A}^{(m)}\underline{f} = \boldsymbol{A}^m\underline{f}.$$

Note that, it is worth to point out that the integral matrix with any order $\boldsymbol{A}^{(m)}$ is lower-triangular matrix.

1.3.2 FIM with the radial basis function (RBF)

The purpose of this section is to develop the accuracy of the approximation technique to construct finite integration matrix. The multi-quadric RBF was introduced by Hardy (1971), for the interpolation of topographical surface. The field variable $u(x)$ in the interval $[a, b]$ can be interpolated over a number of randomly distributed nodes x_i for $i = 0, 1, 2, ..., N$, with $x_0 = a$ and $x_N = b$, as

$$u(x) = \sum_{i=0}^{N} R_i(x, x_i)\alpha_i + \sum_{q=0}^{Q} P_q(x)\beta_q, \tag{1.5}$$

where $R_i(x, x_i)$ is a radial basis function centered at x, $P_q(x)$ is a polynomial basis function, α_i and β_q are any real number as unknown coefficients of $R_i(x, x_i)$ and $P_q(x)$, respectively, for $i = 0, 1, 2, ..., N$ and $q = 0, 1, 2, ..., Q$. The discretization of the interpolation approximation above can be rewritten in matrix form as

$$\underline{u} = \boldsymbol{R}\underline{\alpha} + \boldsymbol{P}\underline{\beta}, \tag{1.6}$$

where

$$\underline{u} = [u(x_0), u(x_1), \dots, u(x_N)]^{\intercal}, \quad \underline{\alpha} = [\alpha_0, \alpha_1, \dots, \alpha_N]^{\intercal}, \quad \underline{\beta} = [\beta_0, \beta_1, \dots, \beta_Q]^{\intercal},$$

$$\boldsymbol{R} = \begin{bmatrix} R_0(x_0, x_0) & R_1(x_0, x_1) & \dots & R_N(x_0, x_N) \\ R_0(x_1, x_0) & R_1(x_1, x_1) & \dots & R_N(x_1, x_N) \\ \dots & \dots & \dots & \dots \\ R_0(x_N, x_0) & R_1(x_N, x_1) & \dots & R_N(x_N, x_N) \end{bmatrix},$$

$$\boldsymbol{P} = \begin{bmatrix} P_0(x_0) & P_1(x_0) & \dots & P_Q(x_0) \\ P_0(x_1) & P_1(x_1) & \dots & P_Q(x_1) \\ \dots & \dots & \dots & \dots \\ P_0(x_N) & P_1(x_N) & \dots & P_Q(x_N) \end{bmatrix}. \tag{1.7}$$

The polynomial term has to satisfy an extra requirement that guarantees unique approximation of a function as follows (Wen *et al.*, 2013)

$$P_q(x_0)\alpha_0 + P_q(x_1)\alpha_1 + P_q(x_2)\alpha_2 + \cdots + P_q(x_N)\alpha_N = 0 \quad \text{for} \quad q = 0, 1, 2, \dots, Q,$$

this can be written in matrix form as

$$\boldsymbol{P}^{\intercal}\underline{\alpha} = 0, \tag{1.8}$$

multiplying $\boldsymbol{R}^{-1}$ through (1.6) gives

$$\boldsymbol{R}^{-1}\underline{u} = \cancel{\boldsymbol{R}^{-1}\boldsymbol{R}}\underline{\alpha} + \boldsymbol{R}^{-1}\boldsymbol{P}\underline{\beta},$$

multiplying $\boldsymbol{P}^{\intercal}$ through the resulted equation above yields

$$\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{u} = \cancel{\boldsymbol{P}^{\intercal}\underline{\alpha}} + \boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\underline{\beta}.$$

Consider the equation (1.8), we have

$$\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{u} = \boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\underline{\beta},$$

multiplying $\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}$ through the resulted equation above gives

$$\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{u} = \cancel{\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}}\underline{\beta}.$$

Thus, we have

$$\underline{\beta} = \left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{u}.$$

From above we have $\boldsymbol{R}^{-1}\underline{u} = \underline{\alpha} + \boldsymbol{R}^{-1}\boldsymbol{P}\underline{\beta}$, then rearranging the equation gives

$$\underline{\alpha} = \boldsymbol{R}^{-1}\underline{u} - \boldsymbol{R}^{-1}\boldsymbol{P}\underline{\beta},$$

substituting β into the equation above yields

$$\underline{\alpha} = \boldsymbol{R}^{-1}\underline{u} - \boldsymbol{R}^{-1}\boldsymbol{P}\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{u},$$

we can be grouping with $\underline{u}$ as a common term, we have

$$\underline{\alpha} = \left[\boldsymbol{R}^{-1} - \boldsymbol{R}^{-1}\boldsymbol{P}\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\right]\underline{u},$$

grouping with $\boldsymbol{R}^{-1}$ as a common term gives

$$\underline{\alpha} = \boldsymbol{R}^{-1}\left[\boldsymbol{I} - \boldsymbol{P}\left(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\right]\underline{u}.$$

Now on, we have

$$\underline{\beta} = (\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\underline{\hat{u}} \quad \text{and} \quad \underline{\alpha} = \boldsymbol{R}^{-1}[\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}]\underline{\hat{u}},$$

where $\boldsymbol{I}$ denotes the identity matrix which satisfying the equation and $\underline{\hat{u}}$ denotes the discrete solution $\underline{\hat{u}} = [\hat{u}_i] = [u(x_i)]$. By substituting coefficients vectors $\underline{\alpha}$ and $\underline{\beta}$ into (1.6), we then obtain

$$u(x) = \left[R(x)\boldsymbol{R^{-1}}(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{P}^{\intercal}\boldsymbol{R^{-1}}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R^{-1}}) + P(x)(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\right]\underline{\hat{u}}.$$

we can also write the system of linear equation based on the above interpolation function in a matrix form

$$\underline{u} = \boldsymbol{X}\underline{\hat{u}}, \tag{1.9}$$

where $\boldsymbol{X} = \left[\boldsymbol{R(x)}\boldsymbol{R^{-1}}(\boldsymbol{I} - \boldsymbol{P}(\boldsymbol{P}^{\intercal}\boldsymbol{R^{-1}}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R^{-1}}) + \boldsymbol{P(x)}(\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\boldsymbol{P})^{-1}\boldsymbol{P}^{\intercal}\boldsymbol{R}^{-1}\right]$.

For convenience, we can describe the system of linear equation (1.9) as expressed as

$$\begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{bmatrix} \approx \begin{bmatrix} b_0(x_0) & b_1(x_0) & b_2(x_0) & \dots & b_N(x_0) \\ b_0(x_1) & b_1(x_1) & b_2(x_1) & \dots & b_N(x_1) \\ b_0(x_2) & b_1(x_2) & b_2(x_2) & \dots & b_N(x_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0(x_N) & b_1(x_N) & b_2(x_N) & \dots & b_N(x_N) \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_N \end{bmatrix}$$

Now on, $u(x_k)$ can be expressed as

$$u(x_k) = \sum_{i=0}^{N} b_i(x_k)\hat{u}_i,$$

where $b_i(x_k)$ is element of matrix $\boldsymbol{X}$ at kth-column and ith-row.

We literally are provided by several kinds of RBF (Hu *et al*, 2005). In this study, the multi-quadric RBF is chosen and written as

$$R_i(x, x_i) = \sqrt{c^2 + (x - x_i)^2} \quad \text{and} \quad P_q(x) = x^q, \tag{1.10}$$

where $c \neq 0$ is a free parameter. Note that, $R_i(x, x_i)$, and $P_q(x)$ are constructed as basis functions, the columns of $\boldsymbol{R}$ and $\boldsymbol{P}$ consequently are linearly independent, this indicates that the matrices $\boldsymbol{R}$ and $\boldsymbol{P}$ are invertible.

The definite integral from a to x can be considered as

$$\int_a^x u(x)dx = \sum_{i=0}^{N} \bar{R}_i(x, x_i)\alpha_i + \sum_{q=0}^{Q} \bar{P}_q(x)\beta_q = \sum_{i=0}^{N} b_i^{(1)}(x)\hat{u}_i, \tag{1.11}$$

where the matrices $\bar{\boldsymbol{R}}$ and $\bar{\boldsymbol{P}}$ are defined as

$$\bar{R}_i(x, x_i) = \frac{x - x_i}{2}\sqrt{c^2 + (x - x_i)^2} + \frac{x_i}{2}\sqrt{c^2 + x_i^2} + \frac{c^2}{2}\ln\frac{x - x_i + \sqrt{c^2 + (x - x_i)^2}}{\sqrt{c^2 + x_i^2} - x_i}$$

$\bar{P}_q(x) = \dfrac{x^{q+1}}{q+1}$ and $b_i^{(1)}$ is the element of single layer integration matrix $\boldsymbol{A}$.

Hence, the finite integration matrix of the first order is $\boldsymbol{A} = \left[b_i^{(1)}(x_k)\right] = \left[b_{ki}^{(1)}\right]$. For multi-layer integration matrix, we have the same property as the ordinary linear approximation as $\boldsymbol{A}^{(m)} = \boldsymbol{A}^m$ (Wen *et al.*, 2013). We now obtain the approximated integration in matrix form, similar to the previous section,

$$\underline{\boldsymbol{F}}^{(m)} = \boldsymbol{A}^{(m)}\underline{\boldsymbol{f}} = \boldsymbol{A}^m\underline{\boldsymbol{f}},$$

where

$$F^{(m)}(x_k) = \int_a^{x_k}\int_a^{y_{m-1}}\cdots\int_a^{y_1} f(y)dy\ldots dy_{m-2}dy_{m-1} = \sum_{i=0}^{k} b_{ki}^{(m)} f(x_i). \tag{1.12}$$

Note that $\boldsymbol{A}$ in RBF and $\boldsymbol{A}$ in OLA are different as $\boldsymbol{A}$ in OLA is a Triangular matrix whereas $\boldsymbol{A}$ in RBF is full matrix.

1.4 Regularization

Inverse problem are mostly ill-posed and its ill-posedness is caused by instability of solution. By means, the small error input to the system causes the large error in the solution. In order to overcome this issues, regularization is normally used. Regularization is aimed to find the stable approximation solution of inverse problem. Consider the system of M linear equations with N unknowns

$$\boldsymbol{X}\underline{\mathrm{g}}^{\delta} = \underline{\mathrm{b}}^{\delta}, \tag{1.13}$$

where $\underline{\mathrm{b}}^{\delta}$ and $\underline{\mathrm{g}}^{\delta}$ are perturbation of right-hand side vector $\underline{\mathrm{b}}$ and the solution of the system (1.13) after the perturbation, respectively. Here we briefly explain a well known regularization method, i.e. the Tikhonov regularization method.

This method is constructed by minimizing the regularized linear least-squares objective function.

$$g_{\lambda}^{\delta} = \min_{\underline{\mathrm{g}}^{\delta}\in\mathbb{R}}\left\{\|\boldsymbol{X}\underline{\mathrm{g}}^{\delta} - \underline{\mathrm{b}}^{\delta}\|^2 + \lambda\left\|\frac{d^2\underline{\mathrm{g}}^{\delta}}{dt^2}\right\|^2\right\}, \tag{1.14}$$

where $\lambda > 0$ is a regularization parameter and the norm $\|.\|$ is defined as the euclidian norm of vector. Evantualy, Tikhonov regularization (see Appendix) gives the solution of minimization (1.14) as

$$\mathrm{g}_\lambda^\delta = \left(\boldsymbol{X}^\intercal \boldsymbol{X} + \lambda \boldsymbol{D}^\intercal \boldsymbol{D}\right)^{-1} \boldsymbol{X}^\intercal \underline{\mathrm{b}}^\delta,$$

where $\mathrm{g}_\lambda^\delta$ is a stable solution of ill-podsed problem (1.13) under regularization parameter λ, and $\boldsymbol{D}$ is a differential regularization matrix order two as defined by

$$\boldsymbol{D} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{bmatrix}_{(N-1)\times(N+1)} .$$

A regularization formula holds regularization parameter λ. Basically, a regularization parameter λ controls the neighborhood properties of the auxiliary problem. Larger values of λ indicates higher stability of the approximate solution but this makes the auxiliary problem being far from the original one. Although values of λ near zero express the auxiliary problem close to the original one, this does not guarantee the stable solution . Hence, suitable regularization has to be chosen carefully with consideration between the conflicting purpose of stability and approximating (Schuster *et al*., 2012). Actually, we are provided many methods to choose the regularization parameter λ such as the discrepancy principle criterion, the generalized cross-validation (GCV) or the L-curve method. Nevertheless in this study, the regularization parameter λ was chosen according to the trial and error, this is because within the range of our research, we would like to provide information that FIM as a renew method can be used to solve an inverse heat source problem. The fruitful idea of this research may be preliminary research for further study that employs the FIM to deal with any inverse problems. This means that we first just consider the simple technique to choose the regularization parameter by using the trial and error.

1.5 Inverse Crime

We may suppose to have time-dependent heat source problem namely $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(t)$ with the over-specified data $g(t)$. Then we would like to solve the problem numerically with a given anaytical solution of $f(t)$. Since finding $f(t)$ needs the knowledge of $g(t)$. By cheating way, we may utilize the analytical solution $f(t)$ to obtain $g(t)$ and then we try to estimate back $f(t)$ by considering its $g(t)$. Shortly, this cheating way is mainly called inverse crime if we make all procedures stuff in the forward problem i.e. method and the number of discretization step similarly as the one used in the inverse problem. The words "inverse crime" is that sampling data in forward problem (from $f(t)$ to $g(t)$) and eventually, the inverse problem (from $g(t)$ to $f(t)$) are done in the same manner. This seems like we have done nothing by just playing around closed circumstances. Moreover, the solution obtained by involving the inverse crime is probably invalid and this may not describe the efficiency of the method.

Hence, we need to avoid the inverse crime because avoiding inverse crime makes valid or realistic results. Literally, avoiding the inverse crime means changing the manner between forward and inverse problems. We have two ways to avoid the inverse crime such as

- We consider numerical procedures differently between forward and inverse problem.
- We make the discretization step in the numerical forward simulation differently as the one used in the inversion.

1.6 Problem statement

In this study, we consider the identification of a time-dependent heat source for the heat equation under the Neuman and additional conditions. Let $T > 0$ be a fixed number and denote

$$D_T = \{(x,t) : 0 < x < 1, 0 < t \leq T\}. \tag{1.15}$$

Consider the problem of finding the time-dependent heat source $f(t)$ and the temperature $u(x,t)$ which satisfy the heat equation, namely

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(t), \quad (x,t) \in D_T, \tag{1.16}$$

with initial data and boundary conditions,

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \tag{1.17}$$

$$\frac{\partial u}{\partial x}(0,t) = s(t), \quad \frac{\partial u}{\partial x}(1,t) = r(t), \quad 0 \leq t \leq T, \tag{1.18}$$

and the additional condition

$$u(x_f,t) = g(t), \quad 0 \leq x_f \leq 1. \tag{1.19}$$

Assume that all given functions satisfy the compatibility conditions

$$\frac{\partial u_0}{\partial x}(0) = s(0), \quad \frac{\partial u_0}{\partial x}(1) = r(0), \quad g(0) = u_0(x_f). \tag{1.20}$$

where $s(t), r(t), g(x)$ are given real valued functions, and can be defined as $s : [0,T] \to \mathbb{R}$, $r : [0,T] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$. The boundary condition (1.18) is called the Neuman boundary condition, this indicates that the heat flux at the starting and end points depand on given functions $s(t)$ and $r(t)$, respectively. Whilst the additional condition (1.19) indicates that the temperature at the specific point x_f and any time t is given by $g(t)$. One thing to note that, for this study, we assume that the inverse problem (1.17)-(1.19) under the above condition (1.20) is uniquely solvable, yet it is still ill-posed as the small error in the input data leading to gain the large error in the solution.

Chapter 2

The Forward Problem for the Heat Equation

We will start this chapter with recalling the forward problem of interest;

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(t), & (x,t) \in D_T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ \frac{\partial u}{\partial x}(0,t) = s(t),\ \frac{\partial u}{\partial x}(1,t) = l(t), & 0 \le t \le T. \end{cases}$$

The main goal of this chapter is to observe the distribution of heat in a given region space $[0,1]$ over time $[0,T]$. The solution of this problem is uniquely solvable with given heat source $f(t)$ and fixed initial and boundary conditions. However, solving the forward problem gives the temperature at any point space x and any point time t which can be archived and used as additional data for dealing with the inverse problems.

2.1 The FDM for solving heat equation

Firstly we divide the domain $[0,1] \times [0,T]$ into several sub-domains N and M of x and t, respectively. Let x_i and t_j be the discretization point of space x and time t, respectively, where $i \in \{0,1,2,\dots,N\}$, $j \in \{0,1,2,\dots,M\}$, and define $x_i = i\Delta x$, $\Delta x = \frac{1}{N}$ and $t_j = j\Delta t$, $\Delta t = \frac{T}{M}$ and we denote that

$$u(x,t) = u(x_i,t_j) =: u_i^j.$$

The first-order derivative of temperature u with respect to t can be approximated by using the forward difference method as

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u_i^{j+1} - u_i^j}{\Delta t},$$

and the second-order derivative of u with respect to x, by using the central difference method, is defined as

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2}.$$

When f^j denotes the discretization of $f(t_j)$ then the equation (1.16) can be discretized as

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + f^j, \tag{2.1}$$

for $i \in \{1, 2, \ldots, N-1\}$ and $j \in \{1, 2, \ldots, M-1\}$. Multiplying Δt through the equation (2.1) yields

$$u_i^{j+1} - u_i^j = \frac{\Delta t}{(\Delta x)^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) + \Delta t f^j, \tag{2.2}$$

Let $h = \dfrac{\Delta t}{(\Delta x)^2}$ and rearranging (2.2) gives

$$u_i^{j+1} = hu_{i+1}^j + (1-2h)u_i^j + hu_{i-1}^j + \Delta t f^j. \tag{2.3}$$

Therefore, the discretization of the initial and boundary conditions in (1.17) and (1.18) become

$$u(x_i, 0) = u_0(x_i) =: u_{0,i}, \ \frac{\partial u}{\partial x}(0, t_j) = s(t_j) =: s^j, \ \text{and} \ \frac{\partial u}{\partial x}(1, t_j) = r(t_j) =: r^j. \tag{2.4}$$

By considering the derivative for the boundary condition (1.18), we have

$$\begin{aligned} \frac{\partial u}{\partial x}(0, t_j) \approx \frac{u_1^j - u_0^j}{\Delta x} = s^j, \quad &\text{then} \quad \frac{\partial u}{\partial x}(1, t_j) \approx \frac{u_N^j - u_{N-1}^j}{\Delta x} = r^j, \\ u_0^j = u_1^j - \Delta x\, s^j, \quad &\text{then} \quad u_N^j = u_{N-1}^j + \Delta x\, r^j. \end{aligned}$$

Consider when $i = 1$,

$$\begin{aligned} u_1^{j+1} &= hu_2^j + (1-2h)u_1^j + hu_0^j + \Delta t f^j, \\ &= (1-h)u_1^j + hu_2^j - h\Delta x\, s^j + \Delta t f^j. \end{aligned}$$

Consider when $i = N-1$,

$$\begin{aligned} u_{N-1}^{j+1} &= hu_N^j + (1-2h)u_{N-1}^j + hu_{N-2}^j + \Delta t f^j, \\ &= hu_{N-2}^j + (1-h)u_{N-1}^j + h\Delta x r^j + \Delta t f^j. \end{aligned}$$

And for others $i \in \{2, 3, 4, \ldots, N-2\}$, the temperature can be calculated by (2.3). Therefore we can construct the system which can be shown as matrix form as follows:

$$\underline{\mathrm{u}}^{j+1} = \boldsymbol{A}\underline{\mathrm{u}}^j + \underline{\mathrm{B}}, \tag{2.5}$$

where

$$
\begin{aligned}
\underline{u}^{j+1} &= \left[u_1^{j+1}, u_2^{j+1}, u_3^{j+1} \ldots, u_{N-1}^{j+1}\right]^T, \quad \underline{u}^j = \left[u_1^j, u_2^j, u_3^j, \ldots, u_{N-1}^j\right]^T, \\
\boldsymbol{A} &= \begin{bmatrix} (1-h) & h & 0 & \ldots & 0 & 0 \\ h & (1-2h) & h & \ldots & 0 & 0 \\ 0 & h & (1-2h) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & (1-2h) & h \\ 0 & 0 & 0 & \ldots & h & (1-h) \end{bmatrix}_{(N-1)\times(N-1)}, \\
\underline{B} &= \left[\Delta t f^j - h\Delta x\, s^j, \Delta t f^j, \Delta t f^j, \ldots, \Delta t f^j, h\Delta x r^j + \Delta t f^j\right]^T.
\end{aligned}
$$

Thus each u_i^{j+1} is calculated by using the known preceding time level u_i^j and the first time step u_i^0 can be obtained by the initial condition $u(x_i, 0) = u_0(x_i)$, which can be calculated by setting $j = 0$ in the equation (2.5). Hence the final time step $\underline{u}^m = [u(x_i, T)]$ can be found afterward.

2.2 The FIM for solving the heat equation

In this section, we consider an alternative numerical method for dealing with the forward problem (1.16) - (1.18). We are using the FIM for discretizing space x and the FDM for discretizing time t. The use of FIM has been constructed by employing two kinds of approximating function which are the ordinary linear approximation (OLA) and the radial basis function (RBF) to solve the forward problem see, Li *et al.* (2013); Yun *et al.* (2015); Li *et al.* (2016). The FIM(OLA) was constructed by applying trapezoidal rule as an integrations tool, whereas the FIM(RBF) was developed by considering the interpolation function that taken its integration analytically as introduced in Section 1.3.

Generally, the way that we apply either FIM(OLA) or FIM(RBF) to the forward problem is quite similar. As we introduced in Chapter 1 for the use of FIM(OLA) and FIM(RBF) for approximating the numerical integrations

$$
F^{(1)}(x_k) = \int_a^{x_k} f(x)\,dx \quad \text{and} \quad F^{(m)} = \int_a^{x_k}\int_a^{y_{m-1}} \cdots \int_a^{y_1} f(y)\,dy\,dy_{m-2}\,dy_{m-1}
$$

as defined for FIM(OLA) in 1.3 and 1.4, respectively, and for FIM(RBF) in 1.11 and 1.12, respectively, it is obviously to see that the similarity of these two techniques is the definitions of integration function, i.e. $a_{ki}^{(1)}$ and $b_{ki}^{(1)}$. Then, for convenience, we denote the interpolating function for FIM as $\phi_{ki}^{(1)}$ for both techniques FIM(OLA) and FIM(RBF).

In order to apply the FIM to the forward problem (1.16)-(1.18), we first discretize the first order derivative. We discretize the first-order derivative with respect to time t by using the backward difference method, the equation (1.16) can be discretized as

$$\frac{u(x,t_j) - u(x,t_{j-1})}{\Delta t} = \frac{\partial^2 u}{\partial x^2}(x,t_j) + f(t_j),$$

rearranging above equation gives

$$u(x,t_j) - \Delta t\,\frac{\partial^2 u}{\partial x^2}(x,t_j) = u(x,t_{j-1}) + \Delta t\, f(t_j). \tag{2.6}$$

Here, we are applying the FIM by taking integration twice with respect to x over the time-discretized heat equation (2.6)

$$\int\!\!\int u(x,t_j)\,dxdx - \Delta t u(x,t_j) = \int\!\!\int u(x,t_{j-1})\,dxdx + \int\!\!\int \Delta t\, f(t_j)\,dxdx.$$

By using the discretization via FIM mentioned in Section 1.4, we have

$$\boldsymbol{A}^2\underline{\mathrm{u}}^j - \Delta t\,\underline{\mathrm{u}}^j = \boldsymbol{A}^2\underline{\mathrm{u}}^{j-1} + \frac{\Delta t\, f^j}{2}\underline{\mathrm{x}}^2 + c_0\underline{\mathrm{x}} + c_1\underline{\mathrm{i}}. \tag{2.7}$$

where c_0 and c_1 are integral constants, $\underline{\mathrm{x}} = [x_0, x_1, \ldots, x_N]^\intercal$, $\underline{\mathrm{x}}^2 = [x_0^2, x_1^2, \ldots, x_N^2]^\intercal$ and $\underline{\mathrm{i}} = [1, 1, \ldots, 1]^\intercal$.

Now consider the boundary condition by taking integration with respect to x once over the time-discritized heat equation (2.6) then we obtain

$$\int u(x,t_j)\,dx - \Delta t\frac{\partial u}{\partial x}(x,t_j) = \int u(x,t_{j-1})\,dx + \int \Delta t\, f(t_j)\,dx.$$

Again by using the discretization via FIM mentioned in Section 1.4, we have

$$\boldsymbol{A}\underline{\mathrm{u}}^j - \Delta t\,\frac{\partial \underline{\mathrm{u}}^j}{\partial x} = \boldsymbol{A}\underline{\mathrm{u}}^{j-1} + \Delta t\, f\underline{\mathrm{x}} + c_0\underline{\mathrm{i}}. \tag{2.8}$$

Considering the above equation at the boundary node $x = 0$ i.e. $x = x_0 = 0$, the equation (2.8) becomes

$$\sum_{i=0}^{0} \phi_{0i}^{(1)} u_i^j - \Delta t\,\frac{\partial u_0^j}{\partial x} = \sum_{i=0}^{0} \phi_{0i}^{(1)} u_i^{j-1} + \cancel{\Delta t\, f^j(0)} + c_0.$$

By the definitions of the integration function $\phi_{0i}^{(1)}$ for FIM(OLA) and FIM(RBF); i.e. $a_{0i}^{(1)}$ and $b_{0i}^{(1)}$, mentioned in Chapter 1 that $\phi_{0i}^{(1)} = a_{0i}^{(1)} = b_{0i}^{(1)} = 0$, therefore

$$c_0 = -\Delta t \frac{\partial u_0^j}{\partial x}.$$

Plugging the boundary condition (1.18), i.e. $\frac{\partial u}{\partial x}(0,t) = s(t)$, at nodes t_j gives

$$c_0 = -\Delta t \, s^j. \tag{2.9}$$

Similary, we consider on the boundary node $x = 1$ i.e. $x = x_N = 1$, the equation (2.8) becomes

$$\sum_{i=0}^{N} \phi_{Ni}^{(1)} u_i^j - \Delta t \frac{\partial u_N^j}{\partial x} = \sum_{i=0}^{N} \phi_{Ni}^{(1)} u_i^{j-1} + \Delta t \, f^j(1) + c_0.$$

And now we apply the boundary condition (1.18); $\frac{\partial u}{\partial x}(1,t_j) = r(t_j) = r^j$ at nodes t_j, we obtain

$$\sum_{i=0}^{N} \phi_{Ni}^{(1)} u_i^j = \Delta t \, r^j + \Delta t \, f^j + \sum_{i=0}^{N} \phi_{Ni}^{(1)} u_i^{j-1} + c_0. \tag{2.10}$$

By combining the equation (2.7) and conditions obtained in (2.9) and (2.10), we can construct a block matrix form of the system as follows:

$$\left[\begin{array}{c|cc} & -x_0 & -1 \\ & -x_1 & -1 \\ \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & -x_2 & -1 \\ & \vdots & \vdots \\ & -x_N & -1 \\ \hline \text{Last row of } \boldsymbol{A} & -1 & 0 \\ \hline 0 \;\; 0 \;\; 0 \;\; \dots \;\; 0 \;\; 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_N^j \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K_1 \\ K_2 \end{bmatrix}, \tag{2.11}$$

where:

$$\begin{aligned} \underline{y} &= \boldsymbol{A}^2 \underline{u}^{j-1} + \tfrac{\Delta t \, f^j}{2} \underline{x}^2, \quad \underline{y} = [y_0, y_1, \dots, y_N]^T, \\ K_1 &= \Delta t \, r^j + \Delta t \, f^j + \textstyle\sum_{i=0}^{N} \phi_{Ni}^{(1)} u_i^{j-1}, \\ K_2 &= -\Delta t \, s^j. \end{aligned}$$

Thus each u_i^j is calculated from the known preceding time level u_i^{j-1} and the first time step u_i^0 can be obtained by the initial condition $u(x_i, 0) = u_0(x_i)$ which can be calculated by setting $j = 1$ in the system (2.11). Hence the final time step $\underline{u}^M = [x_i, T]$ can be found afterward.

2.3 Numerical Example

In this section, we present two benchmark test examples to illustrate the accuracy and efficiency of the proposed method in solving the forward problem and we consider the root mean square error (RMSE) at the middle space domain $x = 0.5$ defined as

$$\text{RMSE}\left(u\left(0.5, t\right)\right) = \sqrt{\frac{1}{M+1}\sum_{i=0}^{M}\left(u_{\text{exact}}\left(0.5, t_i\right) - u_{\text{numerical}}\left(0.5, t_i\right)\right)^2}.$$

2.3.1 Example 1

We consider the forward problem (1.16)-(1.18), with $T = 1$, the input data are given

$$u(x, 0) = u_0(x) = x^2, \quad f(t) = 2\pi\cos(2\pi t),$$
$$\frac{\partial u}{\partial x}(0, t) = s(t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = r(t) = 2,$$

for $(x, t) \in D_t$ (Xiangtuan *et al.*, 2011). In order to test the accuracy, the analytical solution of this forward problem is

$$u(x, t) = x^2 + 2t + \sin 2\pi t, \quad \text{for } (x, t) \in D_T.$$

The outline of using those methods presented in Sections $2.1 - 2.2$ are quite systematic and applicable to any differential equations. However, this is extremely tedious to find solutions with paper and pencil for larger t. This will be lengthy and require considerable effort and patience. However, the technology can be taken for more convenience and time consumption. MATLAB is one such approach of the use of technology to compute the numerical result of this such problem. This enables us to determine the solution by applying methods which require systematic algorithm. This also can simulate graph that displays the agreement between the numerical and analytical solutions.

FDM for solving the forward problem

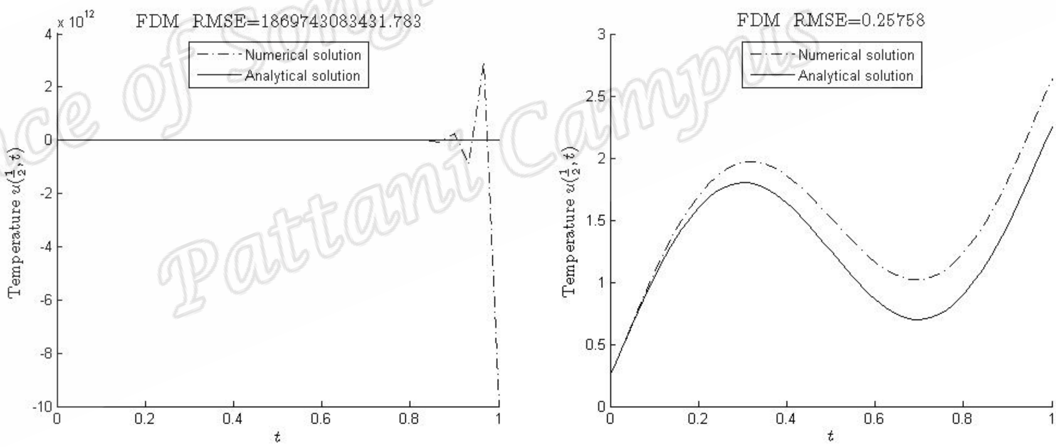
In previous section, we have constructed the system of linear equations in term of general form as shown in (2.5), thus we can simply substitute all given functions to the system and obtain that

$$\underline{\mathrm{u}}^{j+1} = \boldsymbol{A}\underline{\mathrm{u}}^{j} + \underline{\mathrm{B}},$$

where

$$
\begin{aligned}
\underline{\mathrm{u}}^{j+1} &= \left[u_1^{j+1}, u_2^{j+1}, u_3^{j+1} \ldots, u_{N-1}^{j+1}\right]^T, \quad \underline{\mathrm{u}}^{j} = \left[u_1^{j}, u_2^{j}, u_3^{j}, \ldots, u_{N-1}^{j}\right]^T, \\
\boldsymbol{A} &= \begin{bmatrix}
(1-h) & h & 0 & \ldots & 0 & 0 \\
h & (1-2h) & h & \ldots & 0 & 0 \\
0 & h & (1-2h) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (1-2h) & h \\
0 & 0 & 0 & \ldots & h & (1-h)
\end{bmatrix}_{(N-1)\times(N-1)}, \\
\underline{\mathrm{B}} &= \left[\Delta t 2\pi \cos(2\pi t_j), \Delta t 2\pi \cos(2\pi t_j) \ldots, 2h\Delta x + \Delta t 2\pi \cos(2\pi t_j)\right]^T.
\end{aligned}
$$

For the number of discretization, firstly, we perform the number of space and time as $N = 6$ and $M = 30$, respectively. Figure 2.1(a) shows that the performance of FDM for computing the temperature $u(0.5, t)$ is severe inaccurate as can be seen from the figure and RMSE.



(a) $N = 6$, $M = 30$ (b) $N = 6$, $M = 80$

Figure 2.1: The temperature $u\,(0.5, t)$ obtained by FDM with 2.1(a) $N = 6$ and $M = 30$, 2.1(b) $N = 6$ and $M = 80$.

Therefore, the condition for stability has to be required. As it was suggested in Xiangtuan (2010) that the number of discretization under the consideration of stability is stated as

$$h = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$

Then, in our study, we are using this condition as the condition for stability. This is what we can observe by considering the ratio between Δt and Δx^2. If Δx^2 is much smaller than Δt then h may be too big and leads the instability. Then from now, we consider the number of space and time as $N = 6$ and $M = 80$; $h = \frac{9}{20} \leq \frac{1}{2}$, respectively. Figure 2.1(b) mentions that the numerical result is displayed in dash line (--) and the analytical solution is shown in solid line (-). By comparison between Figure 2.1(a) and Figure 2.1(b), This obviously can be seen that the numerical solution obtained by letting $M = 80$ is more accurate than $M = 30$ and also that RMSE for Figure 2.1(b), RMSE $=$ 2.5758E-1 is much more better than Figure 2.1(a), RMSE $=$ 1.87E+12.

FIM(OLA) for solving the forward problem

Similarly, we have constructed the general form of the system of linear equations for using FIM as shown in (2.11). The replacement equations for this problem are easily obtained by considering all given functions, they are

$$c_0 = 0 \quad \text{and} \quad \sum_{i=0}^{N} a_{Ni}^{(1)} u_i^j = 2\Delta t + \Delta t\, f^j + \sum_{i=0}^{N} a_{Ni}^{(1)} u_i^{j-1}.$$

Therefore the equation (2.7) can be rewritten as

$$\boldsymbol{A}^2 \underline{\mathrm{u}}^j - \Delta t\, \underline{\mathrm{u}}^j = \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \frac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2 + c_1 \underline{\mathrm{i}}.$$

By discretizing above we can construct the block matrix as follow

$$\left[\begin{array}{c|c} & -1 \\ & -1 \\ \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & -1 \\ & \vdots \\ & -1 \\ \hline \text{Last row of } \boldsymbol{A} & 0 \end{array}\right] \begin{bmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_N^j \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K \end{bmatrix}, \tag{2.12}$$

where:

$$\begin{aligned} \underline{\mathrm{y}} &= \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \tfrac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2, \quad \underline{\mathrm{y}} = [y_0, y_1, \ldots, y_N]^T, \\ K &= \Delta t\, r^j + \Delta t\, f^j + \textstyle\sum_{i=0}^{N} a_{Ni}^{(1)} u_i^{j-1}, \text{ and } \boldsymbol{A} = \left[a_{ki}^{(1)}\right]. \end{aligned}$$

One thing to note that the integral matrix $\boldsymbol{A}$ obtained by FIM(OLA) is lower-triangular matrix which makes the right-hand side matrix of 2.11 is a kind of sparse matrix; lower-triangular with one column at the last column. This can take advantage for reducing to storage data. For considering the number of discretization, similar to the case of FDM, we firstly perform the number of space and time as $N = 6$ and $M = 30$, respectively. Figure 2.2 shows the analytical and numerical solutions for temperature $u(0.5, t)$ with the RMSE $= 1.262$E-1. This can be seen that the numerical solution obtained by solving equation 2.12 is accurate. This clearly can be seen the good agreement among the analytical and numerical solutions in the both boundaries yet there is a bit gap in the interior points.

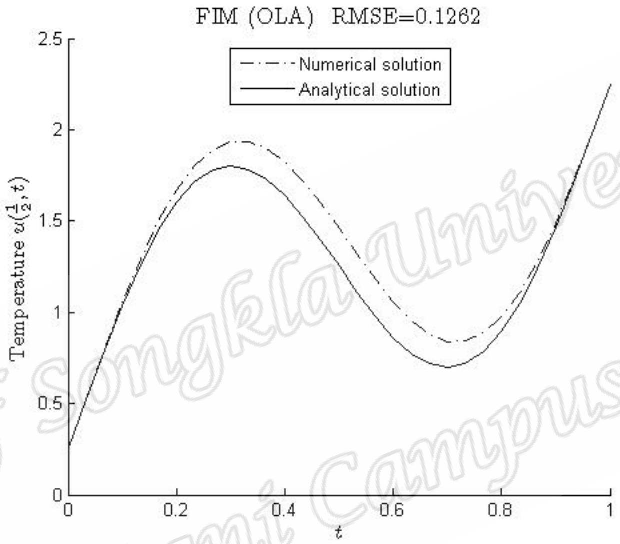


Figure 2.2: The temperature $u\,(0.5, t)$ obtained by FIM(OLA) with $N = 6$ and $M = 30$.

FIM(RBF) for solving the forward problem

In order to apply the FIM(RBF) to the forward problem, we recall the FIM matrix equation (2.11) and take the input function to the problem. Equations (2.9) and (2.10) hold for $s(t) = 0$ and $r(t) = 2$, respectively, These become

$$c_0 = 0 \quad \text{and} \quad \sum_{i=0}^{N} b_{Ni}^{(1)} u_i^j = 2\Delta t + \Delta t\, f^j + \sum_{i=0}^{N} b_{Ni}^{(1)} u_i^{j-1}.$$

Therefore the equation (2.7) can be rewritten as

$$\boldsymbol{A}^2 \underline{\mathrm{u}}^j - \Delta t\, \underline{\mathrm{u}}^j = \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \frac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2 + c_1 \underline{\mathrm{i}}.$$

By considering expressions above, we can construct the block matrix as follow

$$\left[\begin{array}{c|c} & -1 \\ & -1 \\ \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & -1 \\ & \vdots \\ & -1 \\ \hline \text{Last row of } \boldsymbol{A} & 0 \end{array}\right] \begin{bmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_N^j \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K \end{bmatrix}.$$

where:

$$\begin{aligned} \underline{\mathrm{y}} &= \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \tfrac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2, \quad \underline{\mathrm{y}} = [y_0, y_1, \ldots, y_N]^T, \\ K &= \Delta t\, r^j + \Delta t\, f^j + \textstyle\sum_{i=0}^{N} b_{Ni}^{(1)} u_i^{j-1}, \text{ and } \boldsymbol{A} = \left[b_{ki}^{(1)} \right]. \end{aligned}$$

The block matrix above involves the matrix $\boldsymbol{A}$ constructed by RBF, one thing to note that matrix $\boldsymbol{A}$ is a full matrix with the density 98.76% i.e. all zeros at the first row.

The unknowns $u_0^j, \ldots, u_N^j$ and c_1 will be determined by this system of linear equations, while c_0 can be obtained from boundary condition; i.e. $c_0 = 0$. Similarly, We firstly perform the number of space and time as $N = 6$ and $M = 30$, respectively. Figure 2.2 display the behavior temperature $u(0.5, t)$ obtained by FIM(RBF). This can be seen that the insignificant improvement of accuracy from FIM(OLA) can be observed clearly as there is a gap at the interior points and RMSE $= 8.2051$E-2.

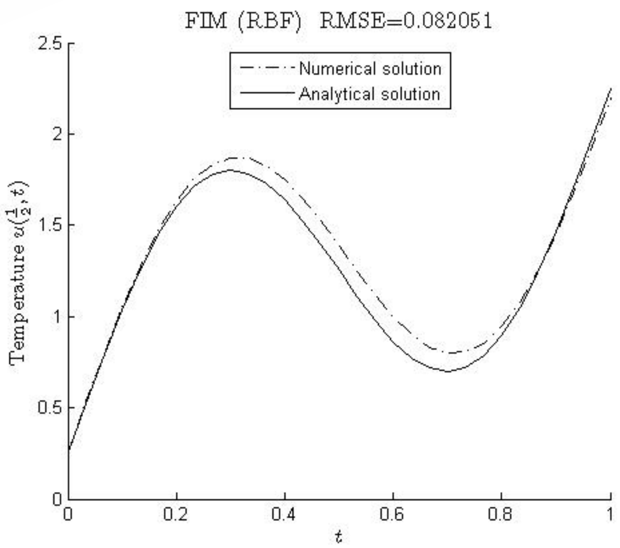


Figure 2.3: The temperature $u\,(0.5, t)$ obtained by FIM(RBF) with $N = 6$ and $M = 30$.

Hereafter is a summary of performance all methods with consideration of the number of space $N = 6$ and $M \in \{30, 80\}$ by using the MATLAB software. Table 2.1 presents the RMSE obtained by using FDM, FIM(OLA) and FIM(RBF). Figure 2.4 display the temperature $u(0.5, t)$ with the number of discretization point $N = 6$ and $M = 80$.

By considering Table 2.1 and Figure 2.4, we observe that the very good agreement between the analytical and numerical solutions obtained by FIM can be recorded and this can be seen that numerical solution obtained by FIM with both OLA and RBF are more accurate than FDM. Note that this comparison can be state in both cases of $M = 30$ and $M = 80$. One thing to note that the numerical result obtained by FDM with $M = 30$ is severely inaccurate, whereas the numerical results obtained by FIM is insignificant accurate.

Table 2.1: RMSE of $u\,(0.5, t)$

N	M	FDM	FIM(OLA)	FIM(RBF)
$N = 6$	$M = 30$	1.8697E+12	1.262E-1	8.2051E-2
$N = 6$	$M = 80$	2.5758E-1	4.78E-2	1.76E-2

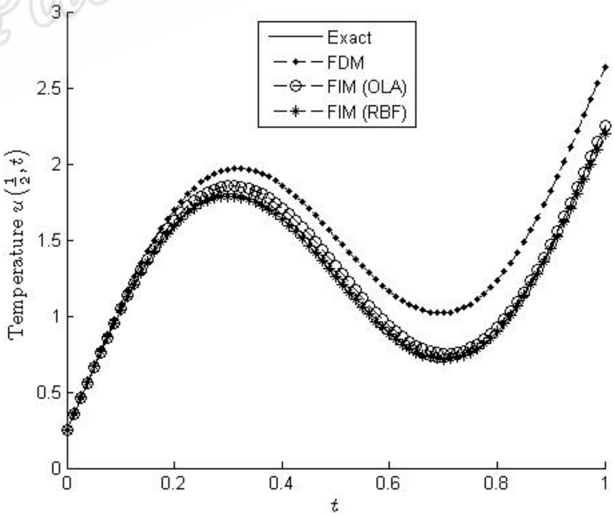


Figure 2.4: The temperature $u\,(0.5, t)$ obtained by all methods with $N = 6$ and $M = 80$

2.3.2 Example 2

In this example, we consider the forward problem (1.16)-(1.18), with $T = 1$ and the input data are given as follows

$$u(x,0) = u_0(x) = x^2,$$
$$\frac{\partial u}{\partial x}(0,t) = s(t) = \frac{\partial u}{\partial x}(1,t) = r(t) = 0,$$
$$f(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{4}, \\ \frac{1}{2}, & \frac{1}{4} \leq t < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

for $(x,t) \in D_T$. Regrettably, we can not observe the accuracy of the solution obtained by all methods as we do not have an analytical solution $u(x,t)$. Unavailableness of the analytical solution of $u(x,t)$ forces to illustrate this numerical example. This is because we need to obtain the over-determination data as additional data of inverse problem in the Chapter 4. Since usually we obtain the additional data by straight way i.e. considering the analytical solution of temperature $u(x,t)$. The clear reason will be explained in Chapter 4. In present numerical example, we directly perform the number point of discretization space and time as $N = 6$ and $M = 80$, respectively, that satisfies with stability condition i.e. $\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$.

FDM for solving the forward problem

In previous section, we have constructed the system of linear equations in term of general form as shown in (2.5), thus we can simply substitute all given functions to the system and obtain

$$\underline{\mathrm{u}}^{j+1} = \boldsymbol{A}\underline{\mathrm{u}}^{j} + \underline{\mathrm{B}},$$

where

$$\underline{\mathrm{u}}^{j+1} = \left[u_1^{j+1}, u_2^{j+1}, u_3^{j+1} \ldots, u_{N-1}^{j+1}\right]^{\mathsf{T}}, \quad \underline{\mathrm{u}}^{j} = \left[u_1^{j}, u_2^{j}, u_3^{j}, \ldots, u_{N-1}^{j}\right]^{\mathsf{T}},$$

$$\boldsymbol{A} = \begin{bmatrix} (1-h) & h & 0 & \ldots & 0 & 0 \\ h & (1-2h) & h & \ldots & 0 & 0 \\ 0 & h & (1-2h) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & (1-2h) & h \\ 0 & 0 & 0 & \ldots & h & (1-h) \end{bmatrix}_{(N-1)\times(N-1)},$$

$$\underline{\mathrm{B}} = [\Delta t f^j, \Delta t f^j, \Delta t f^j, \ldots, \Delta t f^j, \Delta t f^j]^{\mathsf{T}}.$$

FIM(OLA) for solving the forward problem

Similarly, we have constructed the general form of the system of linear equations for using FIM as shown in (2.11). The replacement equations for this problem are easily obtained by considering all given functions, they are

$$c_0 = 0 \quad \text{and} \quad \sum_{i=0}^{N} a_{Ni}^{(1)} u_i^j = \Delta t\, f^j + \sum_{i=0}^{N} a_{Ni}^{(1)} u_i^{j-1}.$$

Therefore the equation (2.7) can be rewritten as

$$\boldsymbol{A}^2 \underline{\mathrm{u}}^j - \Delta t\, \underline{\mathrm{u}}^j = \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \frac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2 + c_1 \underline{\mathrm{i}}.$$

By discretizing above we can construct the block matrix as follow

$$\left[\begin{array}{c|c} & -1 \\ & -1 \\ \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & -1 \\ & \vdots \\ & -1 \\ \hline \text{Last row of } \boldsymbol{A} & 0 \end{array}\right] \begin{bmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_N^j \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K \end{bmatrix}, \tag{2.13}$$

where:

$$\underline{\mathrm{y}} = \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \tfrac{\Delta t f^j}{2} \underline{\mathrm{x}}^2, \quad \underline{\mathrm{y}} = [y_0, y_1, \ldots, y_N]^T,$$

$$K = \Delta t\, f^j + \textstyle\sum_{i=0}^{N} a_{Ni}^{(1)} u_i^{j-1}, \text{ and } \boldsymbol{A} = \left[a_{ki}^{(1)}\right].$$

Note that the integral matrix $\boldsymbol{A}$ obtained by FIM(OLA) is lower-triangular matrix which makes the right-hand side matrix of 2.14 is a kind of sparse matrix; lower-triangular with one column at the last column. This can take advantage for reducing to storage data.

FIM(RBF) for solving the forward problem

In order to apply the FIM(RBF) to the forward problem, we recall the FIM matrix equation (2.11) and take the input function to the problem. Equations (2.9) and (2.10) hold for $s(t) = 0$ and $r(t) = 2$, respectively. These become

$$c_0 = 0 \quad \text{and} \quad \sum_{i=0}^{N} b_{Ni}^{(1)} u_i^j = \Delta t\, f^j + \sum_{i=0}^{N} b_{Ni}^{(1)} u_i^{j-1}.$$

Therefore the equation (2.7) can be rewritten as

$$\boldsymbol{A}^2 \underline{\mathrm{u}}^j - \Delta t\, \underline{\mathrm{u}}^j = \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \frac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2 + c_1 \underline{\mathrm{i}}.$$

By combining above we can construct the block matrix as follow

$$\left[\begin{array}{c|c} & -1 \\ & -1 \\ \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & -1 \\ & \vdots \\ & -1 \\ \hline \text{Last row of } \boldsymbol{A} & 0 \end{array}\right] \begin{bmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_N^j \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K \end{bmatrix}, \tag{2.14}$$

where:

$$\begin{aligned} \underline{\mathrm{y}} &= \boldsymbol{A}^2 \underline{\mathrm{u}}^{j-1} + \tfrac{\Delta t\, f^j}{2} \underline{\mathrm{x}}^2, \quad \underline{\mathrm{y}} = [y_0, y_1, \ldots, y_N]^T, \\ K &= \Delta t\, f^j + \textstyle\sum_{i=0}^{N} b_{Ni}^{(1)} u_i^{j-1}, \text{ and } \boldsymbol{A} = \left[b_{ki}^{(1)} \right]. \end{aligned}$$

The block matrix above involves the matrix $\boldsymbol{A}$ constructed by RBF, one thing to note that matrix $\boldsymbol{A}$ is a full matrix with the density $\frac{M}{M+1}$% i.e. all zeros at the first row. Then the unknowns $u_0^j, \ldots, u_N^j$ and c_1 will be determined by this system, while c_0 can be obtained from boundary condition; i.e. $c_0 = 0$.

Hereafter one can be discussed from the solutions obtained by all methods. From Figure 2.5, This can be seen that the solution obtained by FDM, FIM(OLA) and FIM (RBF) presented in dot line (-.-), circle line (-o-) and star line (-*-), respectively, give different solution $u\,(0.5, t)$. Although there is a gap between them, nevertheless the solutions have a same behavior and even produce a very good agreement at the starting point as we have expected from compatibility condition,

$$g(0) = u_0(0.5) = 2.5.$$

Now on, the three different over-determination data has been provided. In order to consider the fair comparison in inverse problem simulation, we have to choose one of available additional data and indeed we also have to consider the inverse crime introduced in Chapter 1. However, this will be chosen and discussed in Chapter 4.

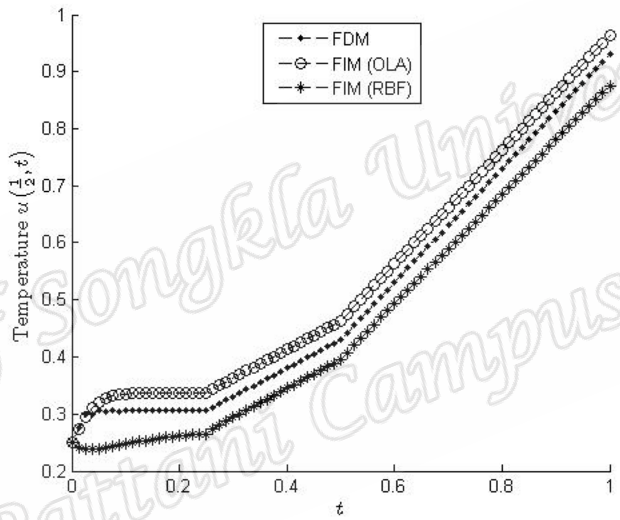


Figure 2.5: The temperature $u\,(0.5, t)$ obtained by all methods with $N = 6$ and $M = 80$

Chapter 3

The Direct Numerical Method

We start this chapter by recalling the inverse problem of interest 1.16-1.19;

$$\begin{cases} u_t(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + f(t), & (x,t) \in D_T, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ \dfrac{\partial u}{\partial x}(0,t) = s(t),\ \dfrac{\partial u}{\partial x}(1,t) = l(t), & 0 \le t \le T, \\ u(x_f,t) = g(t), & 0 \le t \le T. \end{cases}$$

We assume that $s(t), r(t) \in C\left(0,T\right)$, $g(t) \in C^1\left(0,1\right)$ and $u_0(x) \in C^1\left(0,1\right)$. In this chapter, we are presenting the procedure for solving the inverse problem namely the direct numerical method. This method was first introduced in Xiangtuan, (2010) with the use of transformation, differentiation and integration. The first step of this algorithm is about to transform the inverse problem; two unknowns, to be the forward (direct) problem; one unknown. Whereas for the observed data which is the given data with some error is stabilized. Then the numerical solution can be achieved by using the result from the transformed forward problem and the stabilized data.

The first step of the direct numerical method for solving the inverse problem 1.16-1.19 can be started by differentiating the heat equation in 1.16 with respect to x, this yields

$$\frac{\partial^2 u}{\partial x \partial t}(x,t) = \frac{\partial^3 u}{\partial x^3}(x,t). \tag{3.1}$$

As we have mentioned. Let $w(x,t) \in C^1\left([0,1] \times [0,T]\right)$ and

$$w(x,t) = \frac{\partial u}{\partial x}(x,t). \tag{3.2}$$

By definition of the transformation function (3.2), then (3.1) becomes

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \tag{3.3}$$

subject to the initial and boundary conditions;

$$w(x,0) = \frac{\partial u_0}{\partial x}(x), \quad w(0,t) = s(t), \quad w(1,t) = r(t). \tag{3.4}$$

Turning to consider the differential equation 3.1 and taking the finite integration with respect to x on the interval $[x_f, x]$ over (3.1), we then obtain

$$\int_{x_f}^{x} \frac{\partial^2 u}{\partial x \partial t}\, dx = \int_{x_f}^{x} \frac{\partial^3 u}{\partial x^3}\, dx,$$

Applying the second fundamental theorem of calculus (see Appendix) yields

$$\frac{\partial u}{\partial t}(x,t)\Big|_{x_f}^{x} = \frac{\partial^2 u}{\partial x^2}(x,t)\Big|_{x_f}^{x}$$

Substitute the lower and upper limits of integration, we obtain

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x_f,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x_f,t),$$

rearranging the above equation gives

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial u}{\partial t}(x_f,t) - \frac{\partial^2 u}{\partial x^2}(x_f,t),$$

by considering the additional condition; $u(x_f,t) = g(t)$, we have

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial g}{\partial t}(t) - \frac{\partial^2 u}{\partial x^2}(x_f,t). \tag{3.5}$$

Plugging (3.5) into (1.16) gives

$$\cancel{\frac{\partial^2 u}{\partial x^2}(x,t)} + \frac{\partial g}{\partial t}(t) - \frac{\partial^2 u}{\partial x^2}(x_f,t) = \cancel{\frac{\partial^2 u}{\partial x^2}(x,t)} + f(t).$$

Therefore, we can get

$$f(t) = \frac{\partial g}{\partial t}(t) - \frac{\partial^2 u}{\partial x^2}(x_f,t).$$

By $\frac{\partial w}{\partial x}(x_f,t) = \frac{\partial^2 u}{\partial x^2}(x_f,t)$ in 3.2, the time-dependent heat source function becomes

$$f(t) = \frac{\partial g}{\partial t}(t) - \frac{\partial w}{\partial x}(x_f,t). \tag{3.6}$$

As we have reformulated the inverse heat source problem, hereafter is five steps to determine $f(t)$ based on (3.6).

Step 1. *Approximate $w(x,t)$ numerically.* By using the transformation function 3.2, the inverse problem 1.16-1.19 become the following forward problem

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & (x,t) \in D_T, \\ w(x,0) = (u_0)_x(x), & 0 \le x \le 1, \\ w(0,t) = s(t), w(1,t) = r(t), & 0 \le t \le T. \end{cases} \tag{3.7}$$

We then can approximate first-order derivative of heat flux $\dfrac{\partial w}{\partial x}(x,t)$ at any point space x and any time t numerically. Eventually, step 1 above gives $w(x,t)$ at any space x and any time t as archived as the following matrix below,

$$\begin{array}{c} \\ t_0 \\ t_1 \\ \vdots \\ t_{M-1} \\ t_M \end{array} \begin{array}{c} \begin{array}{cccccc} x_0 & \cdots & x_f & x_{f+1} & \cdots & x_N \end{array} \\ \begin{pmatrix} w(x_0,t_0) & \ldots & w(x_f,t_0) & w(x_{f+1},t_0) & \ldots & w(x_N,t_0) \\ w(x_0,t_1) & \ldots & w(x_f,t_1) & w(x_{f+1},t_1) & \ldots & w(x_N,t_1) \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ w(x_0,t_{M-1}) & \ldots & w(x_f,t_{M-1}) & w(x_{f+1},t_{M-1}) & \ldots & w(x_N,t_{M-1}) \\ w(x_0,t_M) & \ldots & w(x_f,t_M) & w(x_{f+1},t_M) & \ldots & w(x_N,t_M) \end{pmatrix} \end{array}.$$

Step 2. *Find the derivative of the heat flux at the specific point x_f; $\dfrac{\partial w}{\partial x}(x_f,t)$.* As we have shown that the reformulated heat source function holds $\dfrac{\partial w}{\partial x}(x_f,t)$. We approximate the derivative $\dfrac{\partial w}{\partial x}(x_f,t)$ by the explicit forward Euler method as expressed as

$$\frac{\partial w}{\partial x}(x_f,t_j) = \frac{w^j_{f+1} - w^j_f}{\Delta x},$$

where $w^j_f = w(x_f,t_j)$ for $j = 0,1,\ldots,M$.

Step 3. *Stabilize the noisy function $g(t)$.* As mentioned in the previous section, Since $g(t)$ is a measured data which probably contains the measurement errors always exist, we denote the noisy data by $g^\delta(t)$. In order to demonstrate this phenomena in reality, we assume the measurement data with the analytical function with noise as follows:

$$g^\delta = g + random(\text{'Normal'}, 0, \sigma, 1, M+1), \tag{3.8}$$

where the $random(\text{'Normal'}, 0, \sigma, 1, M+1)$ is a command in MATLAB generating randomly the variable from normal distribution with mean 0 and standard deviation σ which is taken to be $\sigma = p \times \max_{0 \le t \le T} |g(t)|$, and p is the percentage of the noise to be input. Before approximating the first-order derivative of noisy function $\dfrac{dg^\delta}{dt}(t)$, as we have mentioned before that the inverse problem 1.16-1.19 is ill-posed. Then by adding noise to the system can make the significant errors, we therefore need to stabilized the noisy input function $g^\delta(t)$. We consider the Tikhonov Regularization Method (TRM) as a method for stabilizing of the noisy function. We wish to employ TRM to stabilize the noise function g^δ by minimizing the following functional

$$\min_{g_\lambda^\delta \in \mathbb{R}} \left\{ \left\| \underline{g}_\lambda^\delta - \underline{g}^\delta \right\|^2 + \lambda \left\| \frac{d^2 g^\delta}{dt^2}(t) \right\|^2 \right\}, \tag{3.9}$$

Then, the Tikhonov regularization method gives

$$\underline{g}_\lambda^\delta = [\boldsymbol{I} + \lambda \boldsymbol{D}^\intercal \boldsymbol{D}]^{-1} \underline{g}^\delta. \tag{3.10}$$

where $\boldsymbol{D}$ represents the second-order Tikhonov regularization matrix, $\boldsymbol{D}^\intercal \boldsymbol{D}$ is defined in Section 1.4 and I is an identity matrix.

Step 4. *Find the first-order derivative* $g_\lambda^\delta(t)$. Since the reformulated formula $f(t)$ in 3.6 holds the first-order derivative $\dfrac{dg^\delta}{dt}(t)$, we then employ the finite difference method to approximate the derivative. Up to this point, we have a stable $g_\lambda^\delta(t)$ with corresponding regularization parameter λ. We then can approximate the first-order derivative $\dfrac{dg_\lambda^\delta}{dt}(t)$ by using central finite difference as follows

for $j = 0$, $$\frac{dg_\lambda^\delta}{dt}(t_j) = \frac{g_\lambda^\delta(t_{j+1}) - g_\lambda^\delta(t_j)}{\Delta t},$$

for $j = 1, \ldots, M-1$, $$\frac{dg_\lambda^\delta}{dt}(t_j) = \frac{g_\lambda^\delta(t_{j+1}) - g_\lambda^\delta(t_{j-1})}{2\Delta t},$$

for $j = M$, $$\frac{dg_\lambda^\delta}{dt}(t_j) = \frac{g_\lambda^\delta(t_j) - g_\lambda^\delta(t_{j-1})}{\Delta t}.$$

Step 5. *Compute the heat source* $f(t)$. According to (3.6) the reformulated temperature $w(x,t)$ and the stabilized noisy function $g(t)$ can be obtained by previous steps, respectively. We then simply add those terms to obtain the stabilized heat source $f(t)$ by $f(t) = \dfrac{dg_\lambda^\delta}{dt} - \dfrac{\partial w}{\partial x}(x_f, t)$.

The next chapter we are presenting the procedure for solving the inverse problem 1.16-1.19 by using the either the FIM(OLA), FIM(RBF) and FDM, described in Chapter 2, with the direct numerical method, presented in this chapter. For more convenience, the algorithm for the direct numerical method is summarized as

Step 1: Solve the reformulated forward problem

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & (x,t) \in D_T, \\ w(x,0) = (u_0)_x(x), & 0 \le x \le 1, \\ w(0,t) = s(t), w(1,t) = r(t), & 0 \le t \le T. \end{cases}$$

Step 2: Find $\dfrac{\partial w}{\partial x}(x_f,t)$ by $\dfrac{\partial w}{\partial x}(x_f,t_j) = \dfrac{w_{f+1}^j - w_f^j}{\Delta x}$.

Step 3: Calculate the noisy function by $g^\delta = g + random($'Normal'$, 0, \sigma, 1, M+1)$ and stabilize the noisy function by

$$g_\lambda^\delta = [\boldsymbol{I} + \lambda \boldsymbol{D}^\intercal \boldsymbol{D}]^{-1} g^\delta.$$

Step 4: Find $\dfrac{dg_\lambda^\delta}{dt}(t)$ by following FDM

$$\frac{dg_\lambda^\delta}{dt}(t_j) = \begin{cases} \dfrac{g_\lambda^\delta(t_{j+1}) - g_\lambda^\delta(t_j)}{\Delta t}, & \text{for } j = 0, \\ \dfrac{g_\lambda^\delta(t_{j+1}) - g_\lambda^\delta(t_{j-1})}{2\Delta t}, & \text{for } j = 1, \ldots, M-1, \\ \dfrac{g_\lambda^\delta(t_j) - g_\lambda^\delta(t_{j-1})}{\Delta t}, & \text{for } j = M. \end{cases}$$

Step 5: Find the source function $f(t) = \dfrac{dg_\lambda^\delta}{dt}(t) - \dfrac{\partial w}{\partial x}(x_f,t)$.

Chapter 4

Inverse Problem for The Heat Equation

Time-dependent heat source problem has been transformed into the heat flux equation by using the direct numerical method. From the previous chapter, we have a systematic algorithm to approximate the heat source function. In the step 1, since we have a problem to be dealt, this is in order to make an access to the final formula that has been formulated into the first-order differentiation problem, we wish to employ the FDM, the FIM(OLA) and the FIM(RBF) consecutively for solving a problem in step 1. However, the objective of step 1 is not to determine $f(t)$ directly, but rather to make an access to $w_x(x,t)$ which is held in (3.6).

4.1 The FDM for solving inverse heat source problem

Let us firstly begin with recalling the problem (3.7). Firstly we divide the domain D_T into several sub-domains N and M of x and t, respectively. Let x_i and t_j be the discretization points of space x and time t, respectively, where $i \in \{0,1,2,\dots,N\}$, $j \in \{0,1,2,\dots,M\}$, and define $x_i = i\Delta x$, $\Delta x = \dfrac{1}{N}$ and $t_j = j\Delta t$, $\Delta t = \dfrac{T}{M}$ and we denote that

$$w(x,t) = w(x_i,t_j) =: w_i^j.$$

The first-order derivative of heat flux w with respect to t can be approximated by using the forward difference as

$$w_t(x_i,t_j) = \frac{w_i^{j+1} - w_i^j}{\Delta t},$$

and the second-order derivative of heat flux w with respect to x, by using the central difference, is defined as

$$\frac{\partial^2 w}{\partial x^2}(x_i,t_j) = \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{(\Delta x)^2}.$$

Then the equation (3.7) can be decretized as

$$\frac{w_i^{j+1} - w_i^j}{\Delta t} = \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{(\Delta x)^2}, \tag{4.1}$$

for $i \in \{1, 2, \ldots, N-1\}$ and $j \in \{1, 2, \ldots, M-1\}$. Multiplying Δt through the equation (4.1) yields

$$w_i^{j+1} - w_i^j = \frac{\Delta t}{(\Delta x)^2}\left(w_{i+1}^j - 2w_i^j + w_{i-1}^j\right). \tag{4.2}$$

Let $h = \frac{\Delta t}{(\Delta x)^2}$ and rearranging (4.2) gives

$$w_i^{j+1} = hw_{i+1}^j + (1-2h)w_i^j + hw_{i-1}^j.$$

Therefore, the descretization of the initial and boundary conditions in (3.7) become

$$w(0, t_j) = w_0^j = s^j = s(t_j),\ w(1, t_j) = w_N^j = r^j = r(t_j),\ w(x_i, 0) = (u_0)_x(x_i). \tag{4.3}$$

Consider when $i = 1$,

$$\begin{aligned} w_1^{j+1} &= hw_2^j + (1-2h)w_1^j + hw_0^j \\ &= hw_2^j + (1-2h)w_1^j + hs^j. \end{aligned}$$

Consider when $i = N-1$,

$$\begin{aligned} w_i^{j+1} &= hw_N^j + (1-2h)w_{N-1}^j + hw_{N-2}^j \\ &= hr^j + (1-2h)w_{N-1}^j + hw_{N-2}^j. \end{aligned}$$

We then may construct the system which can be shown as matrix form as following:

$$\underline{\mathrm{w}}^{j+1} = \boldsymbol{A}\underline{\mathrm{w}}^j + \underline{\mathrm{b}}, \tag{4.4}$$

where

$$\begin{aligned} \underline{\mathrm{w}}^{j+1} &= \left[w_1^{j+1}, w_2^{j+1}, w_3^{j+1} \ldots, w_{N-1}^{j+1}\right]^T, \quad \underline{\mathrm{w}}_j = \left[w_1^j, w_2^j, w_3^j, \ldots, w_{N-1}^j\right]^T, \\ \boldsymbol{A} &= \begin{bmatrix} (1-2h) & h & 0 & \ldots & 0 & 0 \\ h & (1-2h) & h & \ldots & 0 & 0 \\ 0 & h & (1-2h) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & h & (1-2h) & h \\ 0 & 0 & \ldots & 0 & h & (1-2h) \end{bmatrix}_{(N-1)\times(N-1)}, \\ \underline{\mathrm{b}} &= \left[hs^j, 0, \ldots, 0, hr^j\right]^T. \end{aligned}$$

Thus each w_i^{j+1} is calculated by using the known preceding time level w_i^j and the first time step w_i^0 can be obtained by the initial condition $w(x_i, 0) = u_{0,x}(x_i)$, which can be calculated by setting $j = 0$ in the equation (4.4). Hence the final time step $\underline{\mathrm{w}}^m = [w(x_i, T)]$ can be found afterward.

4.2 The FIM for solving inverse heat source problem

In order to apply the FIM to problem (3.7), we first descretize the first-order derivative of the heat flux w with respect to time t by using the backward difference method, the equation (1.16) can be discretized as

$$\frac{w(x,t_j) - w(x,t_{j-1})}{\Delta t} = \frac{\partial^2 w}{\partial x^2}(x,t_j),$$

rearranging the equation above gives

$$w(x,t_j) - \Delta t\,\frac{\partial^2 w}{\partial x^2}(x,t_j) = w(x,t_{j-1}). \tag{4.5}$$

Here, we apply the FIM by taking integration twice with respect to x through (4.5)

$$\int\int w(x,t_j)\,dxdx - \Delta t w(x,t_j) = \int\int w(x,t_{j-1})\,dxdx.$$

The matrix form for double layer integration $\boldsymbol{A}^2$ has been constructed in section 1.3, then we apply to the integration equation above, we have

$$\boldsymbol{A}^2\underline{\mathrm{w}}(x,t_j) - \Delta t\,\underline{\mathrm{w}}(x,t_j) = \boldsymbol{A}^2\underline{\mathrm{w}}(x,t_{j-1}) + c_0\underline{\mathrm{x}} + c_1\underline{\mathrm{i}}, \tag{4.6}$$

where c_0 and c_1 are integral constants, $\underline{\mathrm{x}} = [x_0, x_1, x_2, \ldots, x_N]^\intercal$ and $\underline{\mathrm{i}} = [1, 1, \ldots, 1]^\intercal$. Considered the above equation at the boundary node $x = 0$, the equation (4.6) becomes

$$\sum_{i=0}^{0}\phi_{0i}^{(2)} w_i^j - \Delta t\, w_0^j = \sum_{i=0}^{0}\phi_{0i}^{(2)} w_i^{j-1} + \cancel{c_0(0)} + c_1.$$

By the definitions of the integration function $\phi_{0i}^{(1)}$ for FIM(OLA) and FIM(RBF); i.e. $a_{0i}^{(1)}$ and $b_{0i}^{(1)}$, mentioned in Chapter 1 that $\phi_{0i}^{(1)} = a_{0i}^{(1)} = b_{0i}^{(1)} = 0$,

$$c_1 = -\Delta t\, w_0^j.$$

Pluging the boundary condition (3.7), i.e, $w(0,t) = s(t)$ at nodes t_j, we have

$$c_1 = -\Delta t\, s^j. \tag{4.7}$$

Similary, we consider the boundary node $x = 1$. The equation (4.6) becomes

$$\sum_{i=0}^{N} \phi_{Ni}^{(2)} w_i^j - \Delta t\, w_N^j = \sum_{i=0}^{N} \phi_{Ni}^{(2)} w_i^{j-1} + c_0(1) + c_1.$$

And now we apply the boundary condition (3.7), i.e, $w(1,t) =: w_N^j = r(t_j) = r^j$ at nodes t_j. We obtain

$$\sum_{i=0}^{N} \phi_{Ni}^{(2)} w_i^j = \Delta t\, r^j + \sum_{i=0}^{N} \phi_{Ni}^{(2)} w_i^{j-1} + c_0 + c_1. \tag{4.8}$$

By considering the expression above we can construct the block matrix form of the system as follows:

$$\left[\begin{array}{cccccc|cc} & & & & & & -x_0 & -1 \\ & & & & & & -x_1 & -1 \\ & & \boldsymbol{A}^2 - \Delta t \boldsymbol{I} & & & & -x_2 & -1 \\ & & & & & & \vdots & \vdots \\ & & & & & & -x_N & -1 \\ \hline & & \text{Last row of } \boldsymbol{A}^2 & & & & -1 & -1 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} w_0^j \\ w_1^j \\ w_2^j \\ \vdots \\ w_N^j \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \\ K_1 \\ K_2 \end{bmatrix}, \tag{4.9}$$

where

$$\begin{aligned} \underline{\mathrm{y}} &= \boldsymbol{A}^2 \underline{\mathrm{w}}^{j-1}, \quad \underline{\mathrm{y}} = [y_0, y_1, \dots, y_N]^T, \\ K_1 &= \Delta t r^j + \textstyle\sum_{i=0}^{N} \phi_{Ni}^{(2)} w_i^{j-1}, \\ K_2 &= -\Delta t\, s^j. \end{aligned}$$

Thus each w_i^{j+1} is calculated by using the known preceding time level w_i^j and the first time step w_i^0 can be obtained by the initial condition $w(x_i, 0) = u_{0,x}(x_i)$, which can be calculated by setting $j = 0$ in the equation (4.4). Hence the final time step $\underline{\mathrm{w}}^m = [w(x_i, T)]$ can be found afterward.

Eventualy, in step 1 the use of all methods has been explained to solve the problem and they have given their own heat flux $w(x,t)$. This needs to be noticed that the difference of the accuracy obtained by using all methods can be observed and noticed from step 1 only, whereas the rest steps do not effect to the comparison of the methods. Hereafter according to previous chapter, we summarize the brief algorithm for obtaining the heat source function as follows:

- Find $\dfrac{\partial w}{\partial x}(x_f, t)$ by using the finite difference method.
- Use the Tikhonov regularization method to stabilize noisy function $g(t)$.
- Approximate the first-order derivative $g'(t)$ by the central finite difference method.
- Compute $f(t)$ by $f(t) = \dfrac{\partial g}{\partial t}(t) - \dfrac{\partial w}{\partial x}(x_f, t)$.

4.3 Numerical Example

An inverse problem has been formulated systematically and the use of each methods has been explained clearly in the previous section. In this section, we present two benchmark test examples to illustrate the accuracy and efficiency of the proposed method combined with the Tikhonov regularization method. In order to illustrate the accuracy of the numerical solutions we consider the root mean square error (RMSE) expressed as

$$\text{RMSE}\left(f(t)\right) = \sqrt{\frac{1}{M+1}\sum_{i=0}^{M}\left(f_{\text{exact}}(t_i) - f_{\text{numerical}}(t_i)\right)^2}.$$

4.3.1 Example 1

We consider the inverse problem (1.16)-(1.19), with $T = 1$ and the input data are given as

$$u(x,0) = u_0(x) = x^2,$$
$$\frac{\partial u}{\partial x}(0,t) = s(t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = r(t) = 2,$$

and additional condition is $u(0.5,t) = g(t) = \dfrac{1}{4} + 2t + \sin 2\pi t$ for $(x,t) \in D_T$. The analytical solution of this inverse problem is given by

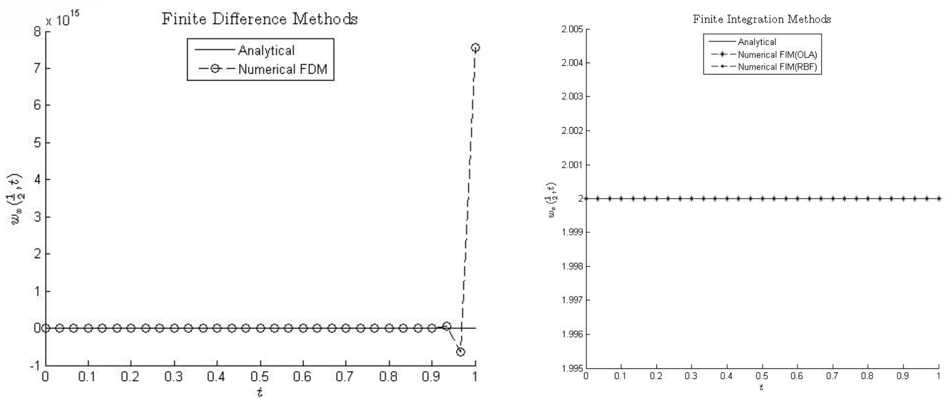
$$f(t) = 2\pi\cos(2\pi t), \text{ and } u(x,t) = x^2 + 2t + \sin 2\pi t, \quad (x,t) \in D_T.$$

In what follows, we simply strict to the algorithm explained in the end of Chapter 3. For the number of discretization, we firstly present the number discritization points of space and time with $N = 10$ and $M = 30$, respectively. In the step 1, we deal with the formulated heat equation, the proposed FIM and FDM play its role in this part. Particularly in the use of FIM (RBF), we have two free parameter and these are chosen $c = \dfrac{1}{N}$ and $Q = 7$ as suggested by Li *et al.* (2013). Furthermore, solving the formulated equation gives $w(x,t)$ that can properly be archived in the matrix below

$$
\begin{array}{c} \\ t_0 \\ t_1 \\ \vdots \\ t_{29} \\ t_{30} \end{array}
\begin{array}{c}
\begin{array}{cccccc} x_0 & \cdots & 0.5 & 0.5+\Delta x & \cdots & x_{10} \end{array} \\
\begin{pmatrix}
w(x_0,t_0) & \dots & w(0.5,t_0) & w(0.5+\Delta x,t_0) & \dots & w(x_{10},t_0) \\
w(x_0,t_1) & \dots & w(0.5,t_1) & w(0.5+\Delta x,t_1) & \dots & w(x_{10},t_1) \\
\vdots & \dots & \vdots & \vdots & \dots & \vdots \\
w(x_0,t_{29}) & \dots & w(0.5,t_{29}) & w(0.5+\Delta x,t_{29}) & \dots & w(x_{10},t_{29}) \\
w(x_0,t_{30}) & \dots & w(0.5,t_{30}) & w(0.5+\Delta x,t_{30}) & \dots & w(x_{10},t_{30})
\end{pmatrix}
\end{array}.
$$

In the step 2, next task concerns taking the first-order derivative of heat flux w with respect to x and corresponding the observed data. The point space to be considered is $x_f = 0.5$, as we read from the observation data, i.e. $g(t) = u(0.5,t)$. Then we need to take the first-order derivative of the heat flux at particular point space $x_f = 0.5$ with respect to x, $\dfrac{\partial w}{\partial x}(0.5,t)$ by using the forward difference method as expressed as

$$\frac{\partial w}{\partial x}(0.5,t) = \frac{w(0.5+\Delta x,t) - w(0.5,t)}{\Delta x}.$$



(a) FDM (b) FIM

Figure 4.1: Numerical results of the first-order derivative of the heat flux $\dfrac{\partial w}{\partial x}(0.5,t)$ obtained by using FDM, FIM (OLA) and FIM (RBF) with $N = 10$ and $M = 30$.

Table 4.1: RMSE of $\dfrac{\partial w}{\partial x}(0.5, t)$ obtained by all methods.

N	M	FDM	FIM(OLA)	FIM(RBF)
$N = 10$	$M = 30$	1.3642E+15	1.7301E-15	2.8813E-15
$N = 6$	$M = 80$	8.8818E-16	7.3244E-15	5.7036E-15

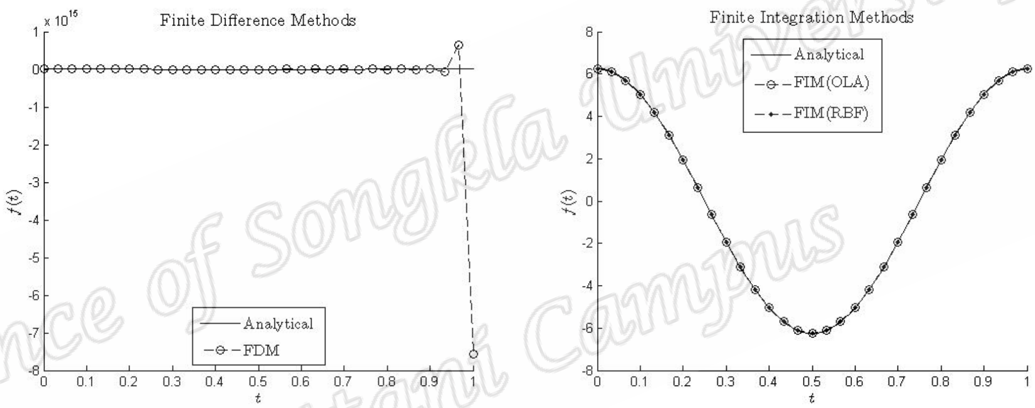
Based on the investigation, FDM does not work well with $N = 10$ and $M = 30$, from Figure 4.1, this clearly can be seen that the high instability occurs at the last two points and produces the severely unstable of $\dfrac{\partial w}{\partial x}(0.5, t)$ as RMSE $=$ 1.3642E+15 as shown in Table 4.1. However, the proposed method approximates $\dfrac{\partial w}{\partial x}(0.5, t)$ perfectly as the RMSE obtained by FIM(OLA) and FIM(RBF) are 1.7301E-15 and 2.8813E-15, respectively. Moreover, the fruitfulness of the method can be seen in Figure 4.1(b), where the numerical solution $\dfrac{\partial w}{\partial x}(0.5, t)$ obtained by FIM can capture the analytical solution properly.

In step 3, we present several cases. We firstly start with the case of the exact data, i.e. no noise to be added, this indicates the regularization part in the step 3 is not needed in this case. Then in step 4, we take first-order derivative of the additional conditional and finaly the solution is obtained by doing step 5. The numerical solution obtained by proposed method are compared with its analytical values in Figure 4.2(b). From Figure 4.2(b) this can be seen that the heat source quantities is very accurate with the same RMSE $3.241E-2$ as shown in Table 4.2, while the numerical heat source $f(t)$ result obtained by FDM is severally unstable and out of expectation as its RMSE $=$ 1.1892E+15 corresponding to $\dfrac{\partial w}{\partial x}(0.5, t)$. Figure 4.2(a) also confirms the highest inaccuracies obtained from last two points. For fair comparison, we have tried to exclude the last two value of numerical solution obtained by FDM from RMSE calculation, yet its RMSE is still high. This confirms that the FDM does not produce the good solution in this current step length.

Table 4.2: RMSE of $f(t)$ obtained by all methods with $N = 10$, $M = 30$ and $p = 0\%$.

Methods	RMSE of $f(t)$
FDM	1.1892E+15
FIM(OLA)	3.241E-2
FIM(RBF)	3.241E-2

Conveniently, since the numerical results obtained by both FIM(OLA) and FIM(RBF) are completely similar, we then can represent the numerical result obtained by using FIM either OLA or RBF in one graph hearafter as displayed in Figure 4.2(b).

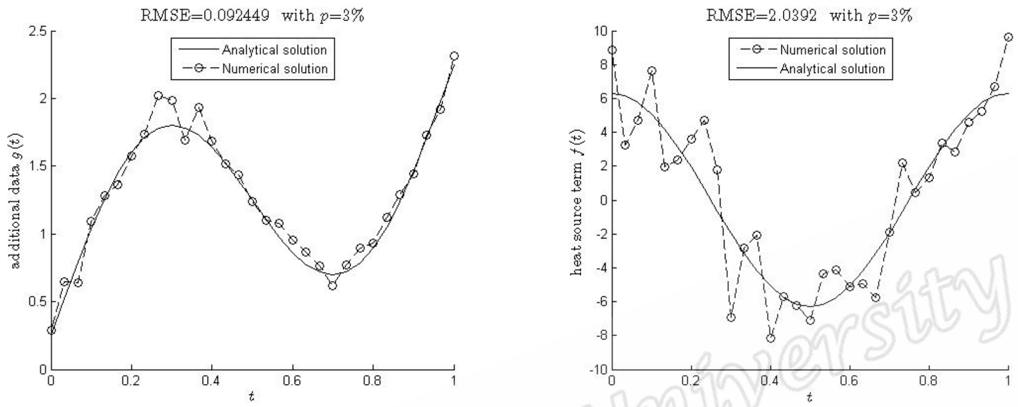


(a) FDM (b) OLA

Figure 4.2: The function $f(t)$ obtained by (a) FDM and (b) FIM, respectively, with $N = 10$, $M = 30$ and $p = 0\%$.

Hereafter, we do not extend to discuss the numerical result obtained by FDM as that is several unstable in spite of no noise input. We furthermore explore the stability of solution obtained by FIM with this current step length. In the case of noisy data, as we have mentioned earlier, most of inverse problem is ill-posed that viloates the stability of the solution. In such cases, we can observe how the solution behaves when we add noise to the over-determination condition (1.19), contaminated the data by (3.8). Then now the specific temperature is perturbed as g^δ and the regularization is needed in this case. After stabilizing the noisy function, the first-order derivative of stable data can be taken in the step 4, heat source data finally can be calculated by doing step 5.

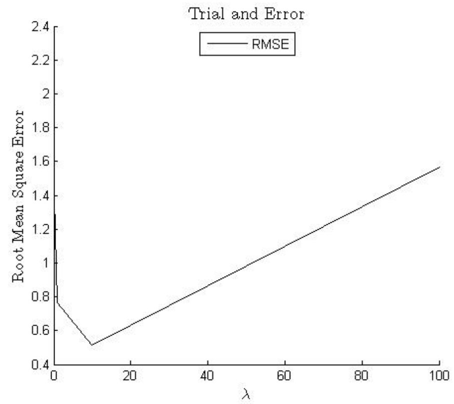
Figure 4.3(b) displays the numerical result of $f(t)$ obtained by using the algorithm introduced in the previous chapter with $p = 3\%$ noisy input and with no regularization, i.e. $\lambda = 0$. From Figure 4.3(b) this can be seen that the numerical solution is unstable since 3% small perturbation in $g(t)$ shown in Figure 4.3(a) causes significant error in the solution.



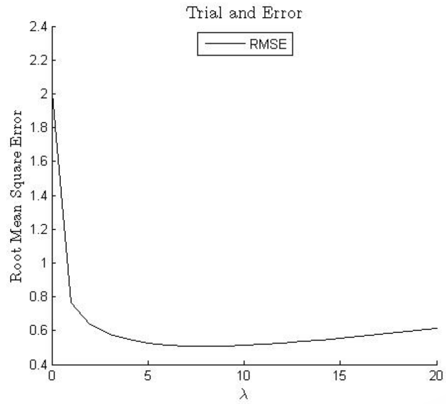
(a) $g^\delta(t)$ (b) $f^\delta(t)$

Figure 4.3: The perturbation function (a) $g^\delta(t)$ and (b) $f^\delta(t)$ with $N = 10$, $M = 30$ and $p = 3\%$.

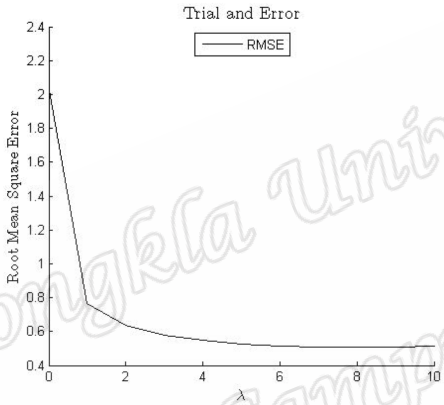
In what follows, Figure 4.4 observes the error by using the trial and error technique. We first start with a huge coverage of λ i.e. $\lambda \in \{10^{-6}, 10^{-5}, 10^{-4}, \ldots, 10, 10^2\}$. From Figure 4.4(a), we can observe that the smallest error can be achieved when the regularization parameter λ is in interval $0 < \lambda < 20$. We consequently minimize the coverage of λ to be $\lambda \in \{0, 1, 2, \ldots, 19, 20\}$. The graph visibly has local minimum as this starts increasing around $\lambda \in (5, 10)$. Then the best solution finally can be obtained when running $\lambda \in (5, 10)$.



(a) $\lambda \in \{10^{-6}, 10^{-5}, 10^{-4}, \ldots, 10, 10^{2}\}$



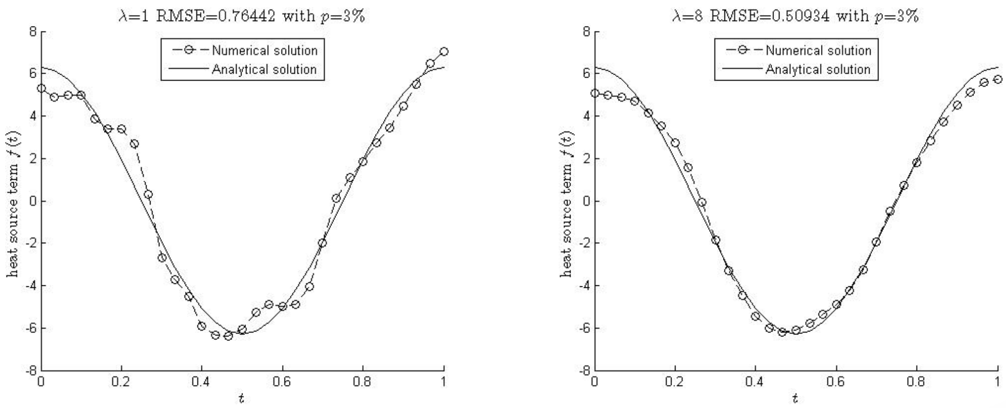
(b) $\lambda \in \{0, 1, 2, \ldots, 19, 20\}$



(c) $\lambda \in \{0, 1, 2, \ldots, 9, 10\}$

Figure 4.4: RMSE of f_{λ}^{δ} with $N = 10$, $M = 30$ and $p = 3\%$.

In addition, the best result for this example was obtained when setting $\lambda = 8$ and that is shown in Figure 4.5 and Figure 4.4. Here can be seen that the interior point of numerical solution i.e. $t \in [0.1, 0.9]$ approximately, is more accurate and stable, whereas the starting and the ending point on $t \in \{0, 1\}$ are getting far away from the exact one. Accordingly, the selection of λ effects to the stability at both boundaries as if λ is getting higher, then the results is getting flat. The inaccuracies at both starting and end points is frequently found elsewhere when using stabilizing technique such the Tikhonov regularization method.



(a) $f^{\delta}(t)$ with $\lambda = 1$ (b) $f^{\delta}(t)$ with $\lambda = 8$

Figure 4.5: The solution $f(t)$ obtained by FIM with noise $p = 3\%$, $\lambda \in \{1, 8\}$, $N = 10$ and $M = 30$.

Finally we extend to explore the stability of the solution by increasing the amount of noise to be plugged in with $p \in \{5, 10\}\%$, the values of the regularization parameter were chosen according to the trial and error as tabulated in Table 4.3. From Table 4.3 this can be seen that suitable regularization parameter λ can achieve the stable and accurate, although the numerical solution with $p \in \{5, 10\}\%$ is not presented.

Table 4.3: RMSE for FIM with noise input $p \in \{3, 5, 10\}\%$, $N = 10$ and $M = 30$.

p	0%	3%			5%		10%	
λ	-	0	1	8	0	14	0	23
RMSE	3.24E-2	2.0392	0.76442	0.50934	3.4003	0.68676	5.5401	0.61956

In summary, FDM does not work well with $N = 10$ and $M = 30$. We apparently have found out that we face the same issue as forward problem that is unstableness. Due to the convenience and fairness of comparison, we again consider the requirement of stability condition i.e. the ratio between Δt and $(\Delta x)^2$ is not more than 0.5. This means that with $N = 10$, we need to set M as not less than 200. Consider the time consumption, since in the computational studies, a running time depends on the step length taken then we can control the consuming time by considering the number of discretization points. Furthermore, in order to avoid a expensiveness of computational cost and without breaking the requirement of stability conditions, we have decided to choose N and M as 6 and 80, respectively.

Similary, we can follow the algorithm to obtain the heat source approximately. In the step 2, all method can solve the reformulated heat equation, then the first-order derivative $\frac{\partial w}{\partial x}(x,t)$ can be approximated by using the forward difference method in the step 2. A set of solution $\frac{\partial w}{\partial x}(x,t)$ obtained by all method is displayed in Figure 4.6. Also, from Tabel 4.1, this clearly can be seen that all methods give very good solution of $\frac{\partial w}{\partial x}(0.5,t)$ with current step length as the RMSE is less then 1E-13.

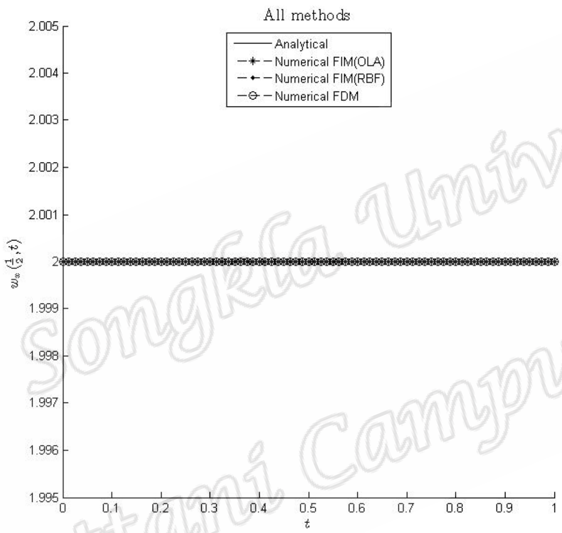


Figure 4.6: The first-order derivative of heat flux obtained by all methods, $\frac{\partial w}{\partial x}(0.5,t)$ with $N = 6$ and $M = 80$.

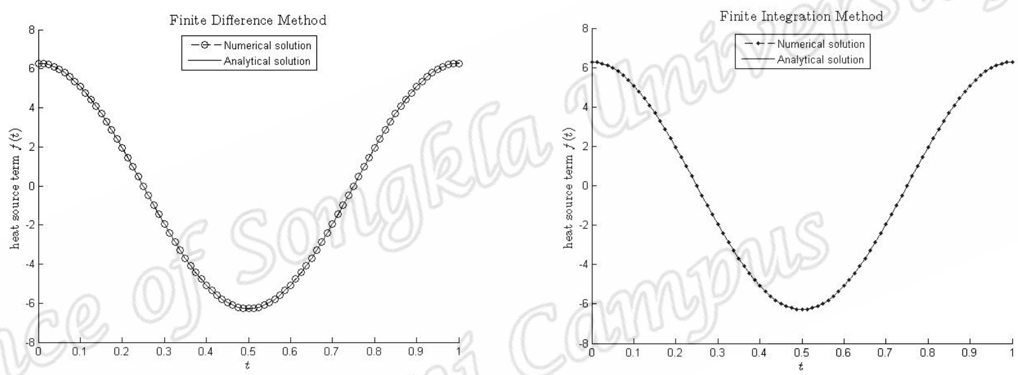
In the step 3, we present several cases. We start with the case of the exact data, i.e. no noise to be input, this indicates the regularization part in the step 3 is not needed in present case. Then in step 4, we take first-order derivative of the additional conditional and finaly the solution is obtained by doing step 5.

Since the formula heat source $f(t)$ holds $\frac{\partial w}{\partial x}(x,t)$ in the step 5 and we have found out that $\frac{\partial w}{\partial x}(x,t)$ obtained by all methods are in a very good agreement, then these seem to remain indistinguishable of $f(t)$ obtained by all methods. This can clearly be seen from Table 4.4 that the numerical solution $f(t)$ obtained by all method are in the same accuracy.

Table 4.4: RMSE of $f(t)$ obtained by all methods with $N = 6$, $M = 80$ and $p = 0\%$.

Methods	RMSE of $f(t)$
FDM	4.5943E-3
FIM	4.5943E-3

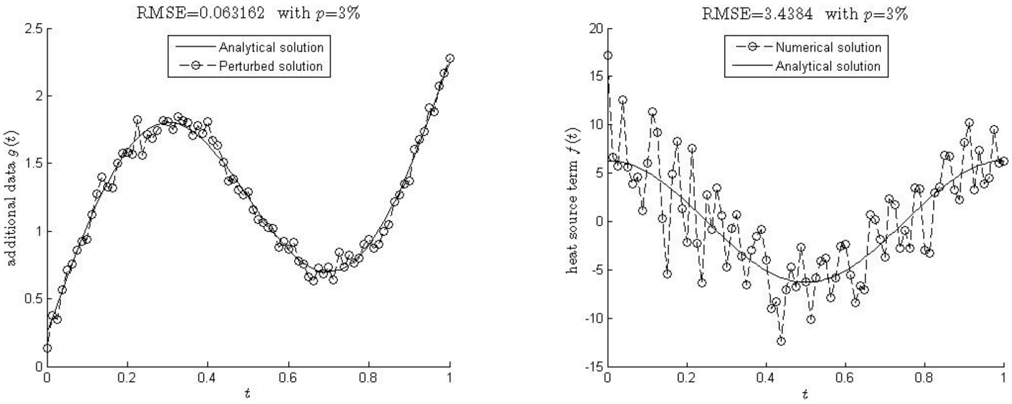
This also is supported by Figure 4.7, without noise to be added, the approximation of $f(t)$ obtained by both FDM and FIM do not have difference. Consequently, the heat source term function $f(t)$ obtained by all methods will be represented by one graph hereafter.



(a) FDM (b) FIM

Figure 4.7: The solution $f(t)$ with $N = 6$, $M = 80$ and $p = 0\%$.

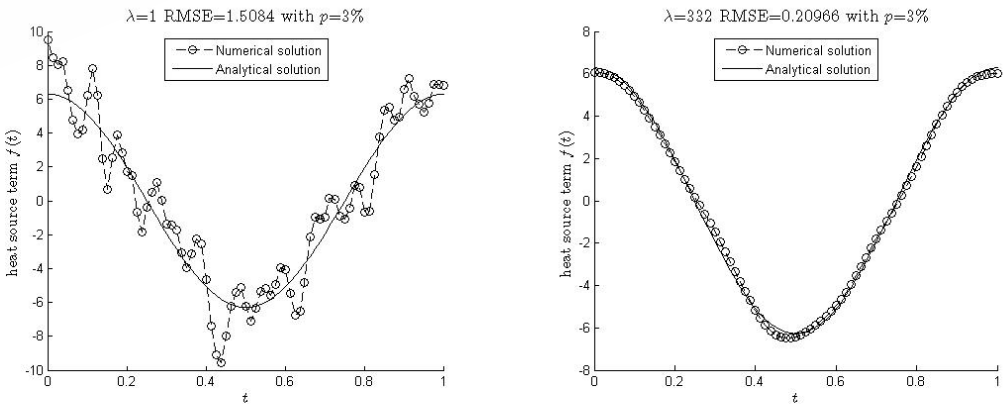
Again, we prefer doing straight way rather than collecting the real data for additional condition. Thus we assume that our purtubed exact solution is our “real” observed data. In the next case, we observed the stability of $f(t)$ with respect to noise in over-determination (1.19). Although from Figure 4.8(a) the perturbation makes just a bit change in additional data $u(0.5, t)$ but Figure 4.8(b) displays that the numerical solution of $f(t)$, with no regularization, is unstable i.e. badly oscillatory. This incident corresponds to the guess that most of inverse problem is ill-posed that viloates the stability of the solution. In such cases, we can observe how the solution behaves when we perturbed $g(t) = u(0.5, t)$ by (3.8) with $p = 3\%$ noisy input as shown 4.8(a), then now the specific temperature is perturbed as g^{δ} as generated with standard deviation given by $\sigma = \frac{9}{4}p$.



(a) $g^\delta(t)$ (b) $f^\delta(t)$

Figure 4.8: The perturbation functions (a) g^δ and (b) f^δ with $N = 6$, $M = 80$ and $p = 3\%$.

In order to recover instability of $f(t)$ when the noise is added, we utilize the Tikhonov regularization of order two. Since the Tikhonov regularization holds the regularization parameter λ to be chosen carefully, the trial and error furthermore is observed to choose the λ. From Figure 4.9(a), This indicates that the regularization parameter λ retrieves the solution from Figure 4.8(b) and carefulness in choosing λ helps us to retrieve the instability of the solution. Finally from Figure 4.9(b), the best solution can be obtained when $\lambda = 332$ based on observation by using the trial and error technique method. Figure 4.9(b) also indicates that the taken discretization step length effect with the accuracy, at the starting and ending points particularly.



(a) $f^\delta_{\lambda_1}(t)$ (b) $f^\delta_{\lambda_2}(t)$

Figure 4.9: The heat source $f(t)$ obtained by all methods with $\lambda \in \{1, 332\}$, $N = 6$ and $M = 80$.

Finally we extend to explore the accuracy and stability of solution by increasing the amount of noise to be plugged in, with $p \in \{5, 10\}\%$, the values of the regularization parameter again are chosen according to the trial and error which can be seen in Table 4.5. In case of noisy function with $p \in \{5, 10\}\%$, we need to inform that a bit instability occurs at the starting and end points only, yet the solution are not displayed in such figure. Then Table 4.5 tabulates the RMSE and regularization parameter that has been chosen appropriately.

Table 4.5: RMSE for all methods with noise input $p \in \{3, 5, 10\}\%$, $N = 6$ and $M = 80$.

p	0%	3%			5%		10%	
λ	-	0	1	332	0	266	0	445
RMSE	4.5943E-3	3.4384	1.5084	0.20966	7.1007	0.48954	12.4189	0.71947

4.3.2 Example 2

We consider the inverse problem (1.16)-(1.19), with $T = 1$ and the input data are given as follows:

$$u(x, 0) = u_0(x) = x^2,$$
$$\frac{\partial u}{\partial x}(0, t) = s(t) = \frac{\partial u}{\partial x}(1, t) = r(t) = 0,$$

In order to observe the accuracy, the analytical solution of this inverse problem has been given as

$$f(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{4}, \\ \frac{1}{2}, & \frac{1}{4} \leq t < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{4.10}$$

In this case, the inverse problem does not have an analytical solution for temperature $u(x, t)$. Hence, the additional condition (1.19) needs to be obtained by solving the forward problem firstly with considering known f in (4.10). Straight way, the additional data has been obtained in Chapter 2. Since we solve the forward problem by using all methods, then we have three options to choose the data as over-specified condition. In order to illustrate a fair comparison, we then decide to choose a solution computed by FDM as additional condition.

One needs to be noticed that, we obtain the additional condition by solving the forward problem with considering the analytical heat source (4.10). In the inversion way, we then approximate back the heat source function by plugging its additional condition. This scheme seems involving the inverse crime that we need to avoid from. We have two ways to avoid the inverse crime introduced in the Chapter 1, furthermore, in this case, we are avoiding the inverse crime by making the discretization in the numerical forward simulation differently as the one used in the inversion i.e. $N = 6$ and $N = 5$ in the forward and inverse problems, respectively, while the number discretization time step is taken stick on $M = 80$.

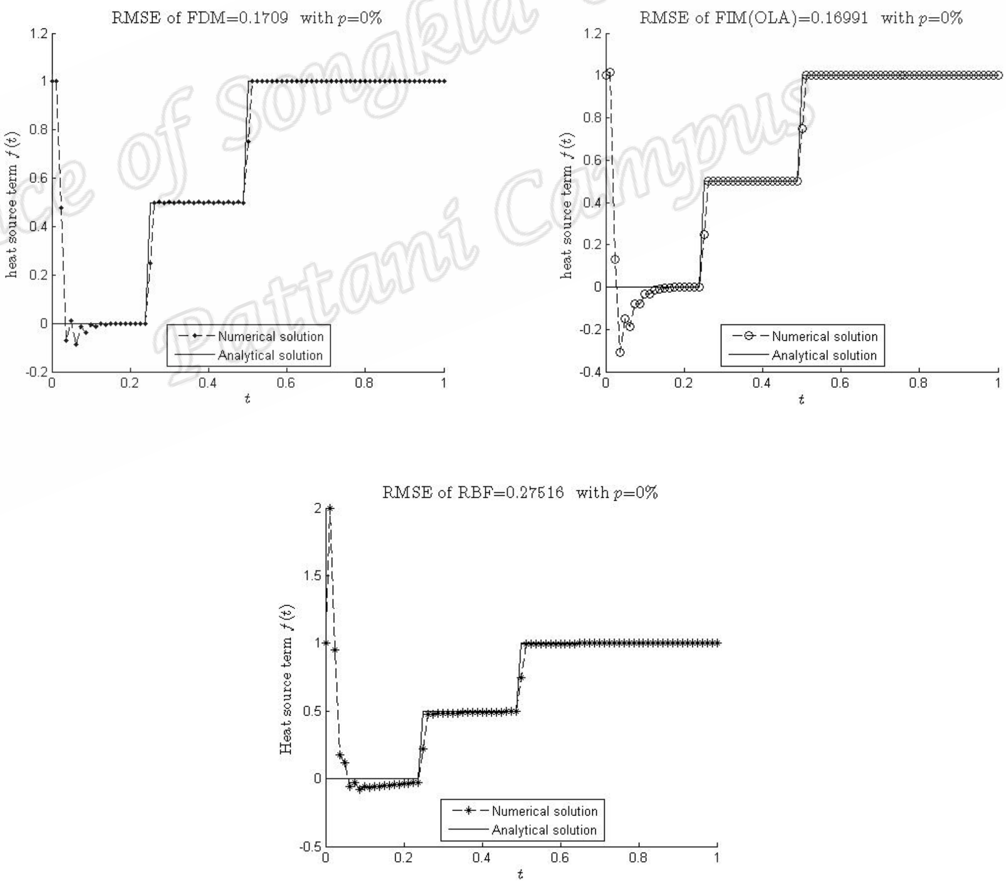


Figure 4.10: The heat source $f(t)$ obtained by all methods with $p = 0\%$, $N = 5$ and $M = 80$.

In what follows, in step 1, each method provides their heat flux $w(x,t)$ as archived in matrix below

$$
\begin{array}{c}
 \\
t_0 \\
t_1 \\
\vdots \\
t_{79} \\
t_{80}
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\quad 0 \quad & \quad 0.2 \quad & \quad 0.4 \quad & \quad 0.6 \quad & \quad 0.8 \quad & \quad 1 \quad
\end{array} \\
\begin{pmatrix}
w(0,t_0) & w(0.2,t_0) & w(0.4,t_0) & w(0.6,t_0) & w(0.8,t_0) & w(1,t_0) \\
w(0,t_1) & w(0.2,t_1) & w(0.4,t_1) & w(0.6,t_1) & w(0.8,t_1) & w(1,t_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
w(0,t_{79}) & w(0.2,t_{79}) & w(0.4,t_{79}) & w(0.6,t_{79}) & w(0.8,t_{79}) & w(1,t_{79}) \\
w(0,t_{80}) & w(0.2,t_{80}) & w(0.4,t_{80}) & w(0.6,t_{80}) & w(0.8,t_{80}) & w(1,t_{80})
\end{pmatrix}
\end{array}
$$

In the step 2, temperature $u(0.5,t)$ has been chosen and considered as additional condition. Hence, we need to take the first-order derivative $\dfrac{\partial w}{\partial x}(0.5,t)$. Since we take the discretization step differently between the forward problem and its inversion, then the way to consider $w(0.5,t)$ is little different. Accordingly, archived $w(x,t)$ above does not contain the heat flux at the specific point space $x_f = 0.5$. Hence, we consequently need to approximate the $w(0.5,t)$ by taking the average value of $w(0.4,t)$ and $w(0.6,t)$ as defined as following formula

$$w(0.5,t) \approx \frac{w(0.4,t) + w(0.6,t)}{2}.$$

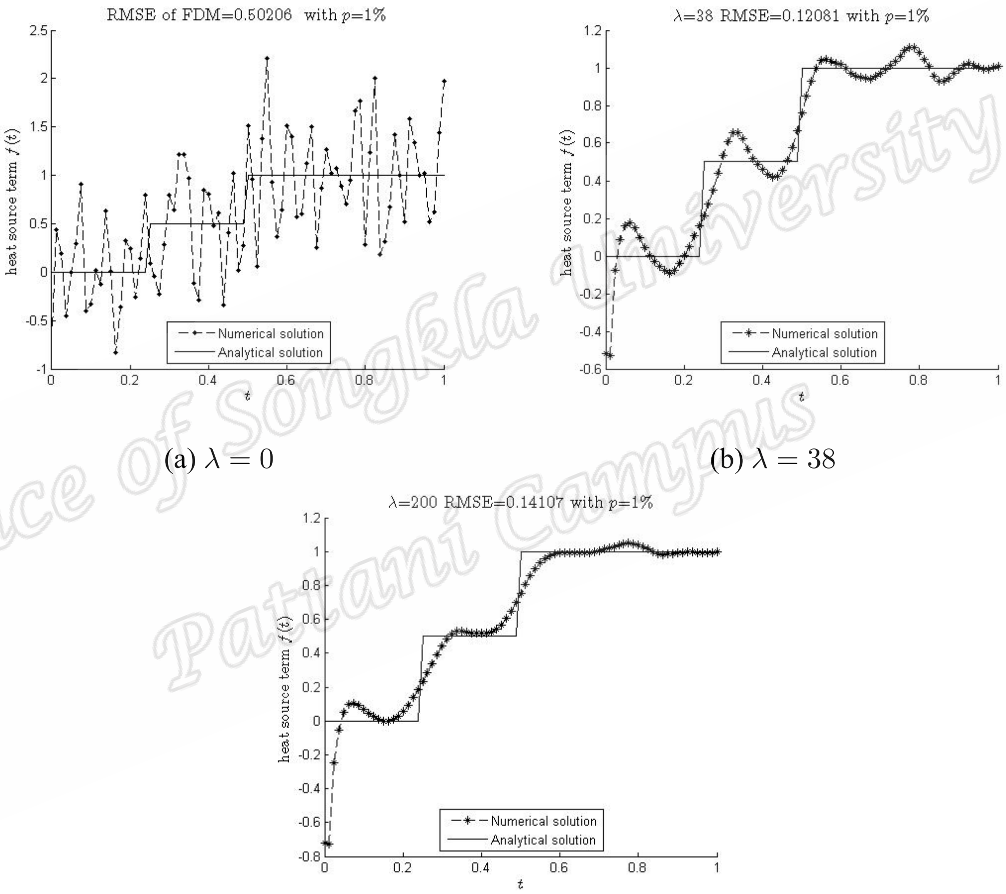
We furthermore have a complete information to simulate the inverse problem without inverse crime.

In the step 3, we again present several cases of noisy level, we first consider the zero noisy data (we assume that the additional data obtained in chapter 2 is exact data). Then in step 4, the first order derivative of additional data need to be taken in order to compute the heat source formula in the step 5.

Figure 4.10 shows the computed solution $f(t)$ obtained by all methods. This is clearly seen that although the noise has not plugged yet, a high oscillatory occurs in the beginning period because the given function is unsatisfied with compatibility condition. Whereas the rest of interior points to ending point have very good agreement with the analytical solution. We actually expect accurate and stable solution from FIM(RBF) even no regularization applied. Unfortunately, the RMSE of FIM(RBF) is the highest among the competitor as shown in Table 4.6.

Table 4.6: The RMSE of $f(t)$ with $p = 0\%$, $N = 5$ and $M = 80$.

Methods	RMSE
FDM	1.709E-1
FIM(OLA)	1.6991E-1
FIM(RBF)	2.7516E-1

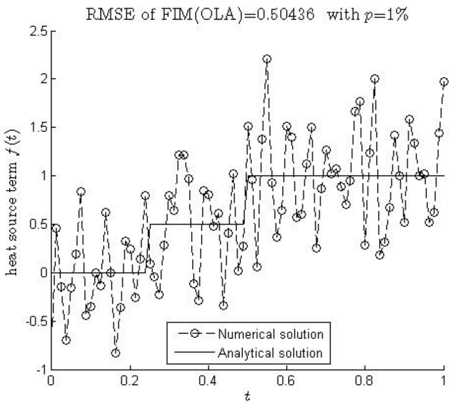


(a) $\lambda = 0$ (b) $\lambda = 38$

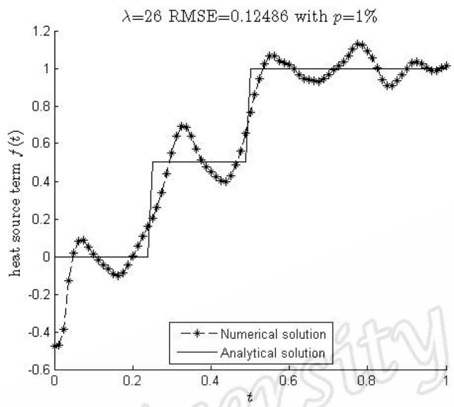
(c) $\lambda = 200$

Figure 4.11: The solution $f(t)$ obtained by FDM with $N = 5$ and $M = 80$.

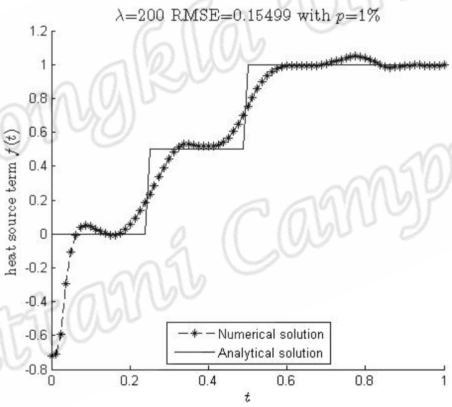
We then observe how the solution behaves when we add noise to the additional condition (1.19), contaminated the data by (3.8) with $p = 1\%$ noisy input. From Figure 4.11(a), 4.12(a) and 4.13(a), the numerical solutions obtained by all methods are poor unstable, because a small error 1% input causes significant error in the solutions. This is what we have expected as the formula heat source $f(t)$ in the step 5 involves the differentiation of noisy function $g(t)$, this means we should deal with an unstable procedure. Now after adding same noise into these systems, the RMSE of FIM(RBF) without regularization i.e. $\lambda = 0$ is still the highest one.



(a) $\lambda = 0$



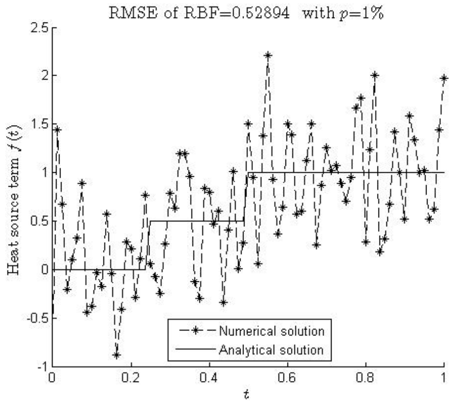
(b) $\lambda = 26$



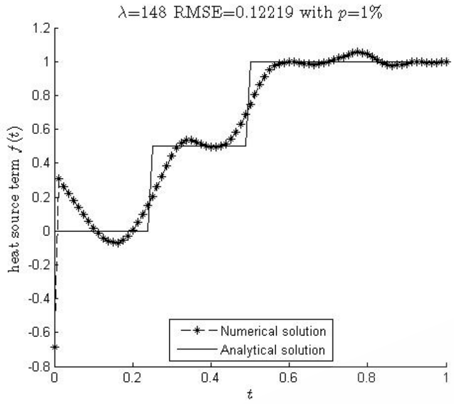
(c) $\lambda = 200$

Figure 4.12: The solution $f(t)$ obtained by FIM(OLA) with $N = 5$ and $M = 80$.

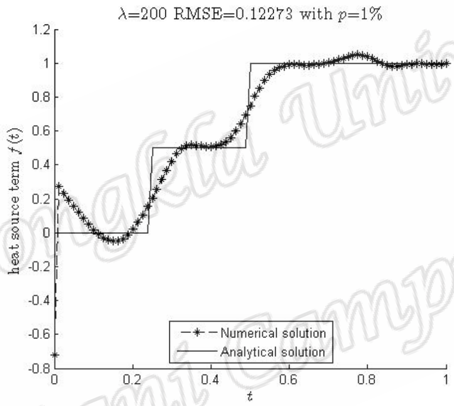
In order to recover instability of $f(t)$ when the noise is added, we utilize the Tikhonov regularization of order two. Since the Tikhonov regularization holds the regularization parameter λ to be chosen carefully, the trial and error technique furthermore is used to observe the λ. From Figures 4.11, 4.12 and 4.13, this can be seen that the TRM retrieves the unstable solution. The least error can be obtained when $\lambda = 38$, $\lambda = 26$ and $\lambda = 148$ for FDM, FIM(OLA) and FIM(RBF), respectively, are applied. Figures 4.11(b), 4.12(b) and 4.13(b) can be investigated that having the least error does not mean being such a good solution as we can see there are fluctuation behavior at the some point. One thing we need to note that, the numerical solution obtained by using FIM(RBF) can produce the least error and smooth solution simultaneously.



(a) $\lambda = 0$



(b) $\lambda = 148$



(c) $\lambda = 200$

Figure 4.13: The solution $f(t)$ obtained by FIM(RBF) with $N = 5$ and $M = 80$.

Whereas in 4.11(b) and 4.12(b), although least error can be produced but there are some instability along the points. Furthermore, Table 4.7 shows that more accurate result is obtained by FIM (RBF), whereas the numerical solution obtained by FDM and FIM(OLA) are not much different as we can see in Table 4.7.

Table 4.7: RMSE of smoother $f(t)$ obtained by all methods with $p = 1\%$, $N = 5$ and $M = 80$.

Methods	$\lambda = 0$	λ_{smooth}
FDM	5.0206E-1	$\lambda = 200,$ 1.4107E-1
FIM(OLA)	5.0436E-1	$\lambda = 200,$ 1.5499E-1
FIM(RBF)	5.2894E-1	$\lambda = 200,$ 1.2273E-1

Finally we explore the stability by increasing the amount of noise to be plugged in with $p \in \{3, 5\}\%$, the values of the regularization parameter were chosen according to the trial and error which can be seen in Table 4.8.

Table 4.8: RMSE for the FDM, FIM(OLA), FIM(RBF) for $p \in \{1, 3, 5\}\%$ noise with $N = 5$ and $M = 80$.

Methods	$p(\%)$	Parameter λ_{best}	RMSE
FDM	1%	$\lambda = 0$	5.0206E-1
		$\lambda = 38$	1.20815E-1
	3%	$\lambda = 0$	1.8713
		$\lambda = 363$	1.7499E-1
	5%	$\lambda = 0$	2.5861
		$\lambda = 343$	2.683E-1
FIM(OLA)	1%	$\lambda = 0$	5.0436E-1
		$\lambda = 26$	1.2486E-1
	3%	$\lambda = 0$	1.87
		$\lambda = 216$	1.7495E-1
	5%	$\lambda = 0$	2.5903
		$\lambda = 286$	2.2741E-1
FIM(RBF)	1%	$\lambda = 0$	5.2894E-1
		$\lambda = 148$	1.2219E-1
	3%	$\lambda = 0$	1.8625
		$\lambda = 5261$	1.5371E-1
	5%	$\lambda = 0$	2.6016
		$\lambda = 6337$	1.5721E-1

In summary, this indicates that a getting higher noise level to be input then a getting higher regularization parameter to be applied, although the behaviour of solution when noise $p \in \{3, 5\}\%$ added is not plotted in a figure. Although, FIM(RBF) produced the highest error when no regularization is imposed whereas this can get the best solution when regularization is imposed. In addition, the high instability in the begining period can not be fixed even though the regularization parameter is applied. This phenomena has been reported by Xiangtuan *et al*. (2013), the instability in the beginning period is caused by unsatisfied compatibility condition, i.e. $\dfrac{\partial u_0}{\partial x}(1) = 2 \neq 0 = r(0)$. We can see that the numerical result is not as good as in the previous examples, but it is in reasonable agreement with analitycal solution (4.10).

Chapter 5

Conclusions and Future Works

5.1 Conclusions

The main objective of this study was to propose the FIM to solve the time-dependent inverse heat source problem with the use of the direct numerical method which is the transformation technique for solving inverse problem. In this study, there are two kinds for approximating integral which are the OLA (Trapezoidal rule) and RBF, presented in Chapter 1. However there is no research/study trying to use the FIM to solve the inverse problem yet. The FIM with OLA and RBF has been first recently in Wen *et al*,. (2013) introduced for solving partial differential equation (the forward problem).

In general of the inverse problems, the additional (or overdetermined) conditions have to be presented and considered to approximate the unknown functions uniquely. Therefore, in this study, the temperature at the specific space x_f and any time t has been considered as the additional information. Generally, the additional data is obtained by doing experiment, real observation or real measurement. This indicates that the additional data probably contains noise unavoidably. Therefore, in computational simulation studies, we assume the perturbed exact data i.e. the given function with noise as the additional data. The problem statement of this study has been presented in Section 1.7, Chapter 1. Before the study of inverse problem case, we first started using the proposed discretization methods; i.e. FIM(OLA), FIM(RBF) and FDM, for solving the forward problem as presented in Chapter 2. the numerical solution obtained by using proposed methods have been produced to be accurate.

To study the inverse problem, we have begun with introducing the direct numerical method in Chapter 3. The main idea of this technique was to transform the inverse problem; two unknowns to be forward problem; one unknown. There were five steps for the direct numerical method. The two first steps were about the reformulating the problem, whereas steps 3 and 4 were about regularized procedure. Since the formula for obtaining the unknown heat source function in step 5 holds the first-order derivative of the additional data and this contained noise unavoidably, then we need to deal with an unstable procedure by employing regularization method introduced in Chapter 1. In this study, a well known regularization method namely the Tikhonov regularization has been employed with order two. Since the TRM holds the regularization parameter, then we have used the trial and error technique to choose the suitable regularization parameter. After stabilizing the noisy functions (additional data), we have taken the first-order derivative by using the central finite difference (the second order accuracy).

The study of inverse problem under the problem statement in section 1.7 have been performed in Chapter 4. The accuracy and stability of the solution have been observed by the root mean square error. Two benchmark numerical examples consisting two cases of unknown functions such as continuous and discontinuous functions have been studied. In first case of study, the continuous heat source function, the numerical solution obtained by the use of either FIM(OLA), FIM(RBF) and FDM together with the direct numerical method has been displayed and compared versus its available analytical solution. Whereas in the case of discontinuous heat source function, where an analytical solution of temperature was not available, we have obtained the additional data by solving the forward problem with the known heat source function; i.e. applying the analytical solution of heat source. We also have set up the step length to be different in the forward problem and the inverse problem case, this is in order to avoid the inverse crime.

In summary of the numerical studies, by taking the number of discretization for space and time to be $N = 10$ and $M = 30$, respectively, the numerical solution without noise, obtained by using FDM has been found to be severely unstable, whereas the numerical results obtained by FIM with both OLA and RBF have been found to have very good agreement between numerical and exact solutions. In order to consider a fair comparison, we have set up the number of discretization for space step to be $N = 6$ and increasing time step to be $M = 80$, this was because of the stability condition under the use of FDM as state that the ratio between the space-step size and the square of time-step size has to be less than a half and this follows that considering the number of step length is a good way to control the consuming time. All numerical solutions for the case of with no noise have been discovered to be accurate and same RMSEs. Since the study of adding noise under the procedure of the direct numerical method was taking part in steps 3-4, whereas steps 1-2 were about the use either the FIM(OLA), FIM(RBF) or FDM to solve the formulated forward problem. Then for this present numerical example, no matter how much noise to be added, all methods produced the same RMSEs. In the case of noise data, this inverse problem was found to be ill-posed as a small error caused a significant error in the solution. Then the TRM with order two has been employed to stabilize the solution. Accordingly, the stable solution was obtained by using this algorithm, the solution has also been found to have a good agreement between the analytical and numerical in the interior points yet not for both boundary points when high noise level was added.

For the Example 2, the discontinuous source function with the case of exact data, we have found that all methods produced high oscillatory numerical solution at the beginning period, this was because a given function was not satisfied with compatibility condition. Whereas the rest of interior points to ending point have very good agreement with the analytical solution. When the noise was input, the numerical heat source solution was severely unstable. Then the TRM with order two has been employed in step 3 of algorithm with the use of trial and error for choosing appropriate regularization parameter. We have found that the numerical solutions obtained by using FDM and FIM (OLA) were not much different, whereas FIM (RBF) could produce better and smooth solution.

Along the explanations above, we finally can conclude that the FIM combined with TRM considering under the algorithm of the direct numerical method can be used to solve the time-dependent heat source problem. The numerical solutions were found to be accurate in the interior points. Furthermore, the use of FIM either with OLA or RBF produced the numerical solution are much better than the FDM in sense of discretization step length as the FDM require to satisfy the stability condition, i.e. no requirement step length needed when using the FIM. However, the accuracy of FIM and FDM has been found not much different.

5.2 Future works

Along our investigation, the FIM (OLA) and FIM (RBF) combined with TRM order two can be used to deal with the inverse problem of finding time-dependent heat source. Although, the good agreement can be seen at interior point only when high noise was input, this commonly appears when using TRM. Somehow, there are many ideas to extend our scope of study, such as

(i) We may consider the FIM to deal with the inverse problem of finding the space-dependent heat source function, as we prefer discretize the space and time by using FIM and FDM, respectively.

(ii) Since FIM (OLA) and FIM (RBF) can be used to deal with inverse problem. As there was introducing in Li *et al*, (2016) for the FIM with the Simpson's Rule, then we can extend to study the inverse problem with the use of the FIM with Simpson’s Rule.

(iii) Since the final formula of heat source holds the first-order derivative of noisy function, then we may consider the smoothing spline technique which is a technique to stabilize the derivative noisy function (Hazanee and Lesnic, 2014) instead of TRM featuring with the central difference method. With hope that, the instability at the both starting and ending points can be fixed.

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Appendix

5th AASIC

Certificate

THIS CERTIFICATE AWARDED TO:

Rama Lesmana

as

Oral Presenter

"at The 5th ASIAN Academic Society International Conference (AASIC)"

Theme : Multidisciplinary Persprective of Local Wisdom: Past, Present, and Future of ASIA

on July 26-27, 2017, Khon Kaen - Thailand.

H.E. Ahmad Rusdi
Ambassador of the Republic of Indonesia to the Kingdom of Thailand

Assoc. Prof. Dr. Kittichai Triratanasirichai
President of Khon Kaen University

Agianto, NS, MNS
Chairman of 5th AASIC

Derivation of Tikhonov Regularization Formula

This method is constructed by minimizing the regularized linear least-squares objective function.

$$\|\boldsymbol{X}\mathrm{g}^{\delta} - \underline{\mathrm{b}}^{\delta}\|^2 + \lambda \left\|\boldsymbol{D}\mathrm{g}^{\delta}\right\|^2 .$$

Reformulating the functional above yields

$$\|\boldsymbol{X}\mathrm{g}^{\delta} - \underline{\mathrm{b}}^{\delta}\|^2 + \left\|\sqrt{\lambda}\boldsymbol{D}\mathrm{g}^{\delta} - 0\right\|^2 ,$$

the reformulated functional above can be written as

$$\left\|\begin{pmatrix} \boldsymbol{X}\mathrm{g}^{\delta} - \underline{\mathrm{b}}^{\delta} \\ \sqrt{\lambda}\boldsymbol{D}\mathrm{g}^{\delta} - 0 \end{pmatrix}\right\| \text{ or } \left\|\begin{pmatrix} \boldsymbol{X}\mathrm{g}^{\delta} \\ \sqrt{\lambda}\boldsymbol{D}\mathrm{g}^{\delta} \end{pmatrix} - \begin{pmatrix} -\underline{\mathrm{b}}^{\delta} \\ 0 \end{pmatrix}\right\| .$$

Grouping the first term with the common term g^{δ} gives

$$\left\|\begin{pmatrix} \boldsymbol{X} \\ \sqrt{\lambda}\boldsymbol{D} \end{pmatrix} \mathrm{g}^{\delta} - \begin{pmatrix} \underline{\mathrm{b}}^{\delta} \\ 0 \end{pmatrix}\right\| .$$

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{X} \\ \sqrt{\lambda}\boldsymbol{D} \end{pmatrix}$ and $y = \begin{pmatrix} \underline{\mathrm{b}}^{\delta} \\ 0 \end{pmatrix}$, we have the tikhonov regularization formula

$$\min_{g^{\delta}\in\mathbb{R}} \left\{ \left\|\boldsymbol{A}\mathrm{g}^{\delta} - y\right\| \right\} .$$

By the definition of euclidian norm $\left\|\boldsymbol{A}\mathrm{g}^{\delta} - y\right\|$, we have $\sum_i \left(\boldsymbol{A}g_i - y_i\right)^2$. Minimizing means taking the derivative and setting its derivative to be equal to zero. Taking derivative of functional above with respect to g yields

$$\frac{\partial}{\partial g}\left\{\sum_i \left(\boldsymbol{A}g_i - y_i\right)^2\right\} = \sum_i \frac{\partial}{\partial g}\left\{\left(\boldsymbol{A}g_i - y_i\right)^2\right\} = 0,$$

we obtain

$$\sum_i \cancel{2}\left(\boldsymbol{A}g_i - y_i\right)\cancel{\boldsymbol{A}} = 0,$$

Applying the linearity of summation gives

$$\sum_i \boldsymbol{A}g_i - \sum_i y_i = 0 \text{ or } \sum_i \boldsymbol{A}g_i = \sum_i y_i .$$

The expression above can be expressed as matrix form as

$$\boldsymbol{A}\underline{\mathrm{g}} = \underline{\mathrm{y}}.$$

By using Gaussian normal equation, we have

$$\boldsymbol{A}^{\intercal}\boldsymbol{A}\underline{\mathrm{g}} = \boldsymbol{A}^{\intercal}\underline{\mathrm{y}}.$$

Since $y = \begin{pmatrix} \underline{\mathrm{b}}^{\delta} \\ 0 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{X} \\ \sqrt{\lambda}\boldsymbol{D} \end{pmatrix}$ then $\boldsymbol{A}^{\intercal} = \begin{pmatrix} \boldsymbol{X}^{\intercal} & \sqrt{\lambda}\boldsymbol{D}^{\intercal} \end{pmatrix}$, the equation above becomes

$$\left(\boldsymbol{X}^{\intercal}\boldsymbol{X} + \lambda\boldsymbol{D}^{\intercal}\boldsymbol{D}\right)\underline{\mathrm{g}}_{\lambda} = \boldsymbol{X}^{\intercal}\underline{\mathrm{b}}.$$

Finally, TRM gives the minimizer the functional above as

$$\underline{\mathrm{g}}_{\lambda} = \left(\boldsymbol{X}^{\intercal}\boldsymbol{X} + \lambda\boldsymbol{D}^{\intercal}\boldsymbol{D}\right)^{-1}\boldsymbol{X}^{\intercal}\underline{\mathrm{b}}.$$

Mathematical Induction of Integration Matrix Form

Recall the approximated definite single integral, we may consider the definite double-layer integral,

$$\begin{aligned} F^{(2)}(x_k) &= \int_a^{x_k}\int_a^{y_1} f(y)dydy_1 \\ &= \int_a^{x_k} g(y)dy \qquad ; \quad g(y_1) = \int_a^{y_1} f(y)\,dy \\ &= \sum_{i=0}^{k} a_{ki}^{(1)} g(x_i) \\ &= \sum_{i=0}^{k}\left(a_{ki}^{(1)}\int_a^{x_i} f(y)\,dy\right). \end{aligned}$$

Then we obtain,

$$F^{(2)}(x_k) = \sum_{i=0}^{k}\left(a_{ki}^{(1)}\sum_{j=0}^{i} a_{ij}^{(1)} f(x_j)\right).$$

Therefore, we have $\int_a^{x_k}\int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{k}\sum_{j=0}^{i} a_{ki}^{(1)} a_{ij}^{(1)} f(x_j)$. Similary as the single-layer, when $k = 0$ we have $\int_a^{x_0}\int_a^{y_1} f(y)dydy_1 = 0$.

When $k = 1$,

$$\int_a^{x_1}\int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{1}\left(a_{1i}^{(1)}\sum_{j=0}^{i}a_{ij}^{(1)}f(x_j)\right) = \cancel{a_{10}^{(1)}\sum_{j=0}^{0}a_{0j}^{(1)}f(x_j)} + a_{11}^{(1)}\sum_{j=0}^{1}a_{1j}^{(1)}f(x_j)$$
$$= a_{11}^{(1)}\left[a_{10}^{(1)}f(x_0) + a_{11}^{(1)}f(x_1)\right] = \frac{1}{2}(\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + \frac{1}{2}(\Delta x)f(x_1)\right]$$
$$= \frac{1}{4}(\Delta x)^2 f(x_0) + \frac{1}{4}(\Delta x)^2 f(x_1).$$

When $k = 2$,

$$\int_a^{x_2}\int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{2}\left(a_{2i}^{(1)}\sum_{j=0}^{i}a_{ij}^{(1)}f(x_j)\right)$$
$$= \cancel{a_{20}^{(1)}\sum_{j=0}^{0}a_{0j}^{(1)}f(x_j)} + a_{21}^{(1)}\sum_{j=0}^{1}a_{1j}^{(1)}f(x_j) + a_{22}^{(1)}\sum_{j=0}^{2}a_{2j}^{(1)}f(x_j)$$
$$= a_{21}^{(1)}\left[a_{10}^{(1)}f(x_0) + a_{11}^{(1)}f(x_1)\right] + a_{22}^{(1)}\left[a_{20}^{(1)}f(x_0) + a_{21}^{(1)}f(x_1) + a_{22}^{(1)}f(x_2)\right]$$
$$= (\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + \frac{1}{2}(\Delta x)f(x_1)\right] + \frac{1}{2}(\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + (\Delta x)f(x_1) + \frac{1}{2}(\Delta x)f(x_2)\right]$$
$$= \left[\frac{1}{2}(\Delta x)^2 + \frac{1}{4}(\Delta x)^2\right]f(x_0) + \left[\frac{1}{2}(\Delta x)^2 + \frac{1}{2}(\Delta x)^2\right]f(x_1) + \frac{1}{4}(\Delta x)^2 f(x_2)$$
$$= \frac{3}{4}(\Delta x)^2 f(x_0) + (\Delta x)^2 f(x_1) + \frac{1}{4}(\Delta x)^2 f(x_2).$$

When $k = 3$,

$$\int_a^{x_3}\int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{3}\left(a_{3i}^{(1)}\sum_{j=0}^{i}a_{ij}^{(1)}f(x_j)\right)$$
$$= \cancel{a_{30}^{(1)}\sum_{j=0}^{0}a_{0j}^{(1)}f(x_j)} + a_{31}^{(1)}\sum_{j=0}^{1}a_{1j}^{(1)}f(x_j) + a_{32}^{(1)}\sum_{j=0}^{2}a_{2j}^{(1)}f(x_j) + a_{33}^{(1)}\sum_{j=0}^{3}a_{3j}^{(1)}f(x_j)$$
$$= a_{31}^{(1)}\left[a_{10}^{(1)}f(x_0) + a_{11}^{(1)}f(x_1)\right] + a_{32}^{(1)}\left[a_{20}^{(1)}f(x_0) + a_{21}^{(1)}f(x_1) + a_{22}^{(1)}f(x_2)\right]$$
$$+ a_{33}^{(1)}\left[a_{30}^{(1)}f(x_0) + a_{31}^{(1)}f(x_1) + a_{32}^{(1)}f(x_2) + a_{33}^{(1)}f(x_3)\right]$$
$$= (\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + \frac{1}{2}(\Delta x)f(x_1)\right] + (\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + (\Delta x)f(x_1) + \frac{1}{2}(\Delta x)f(x_2)\right]$$
$$+ \frac{1}{2}(\Delta x)\left[\frac{1}{2}(\Delta x)f(x_0) + (\Delta x)f(x_1) + (\Delta x)f(x_2) + \frac{1}{2}(\Delta x)f(x_3)\right]$$
$$= \left[\frac{1}{2}(\Delta x)^2 + \frac{1}{2}(\Delta x)^2 + \frac{1}{4}(\Delta x)^2\right]f(x_0) + \left[\frac{1}{2}(\Delta x)^2 + (\Delta x)^2 + \frac{1}{2}(\Delta x)^2\right]f(x_1)$$
$$+ \left[\frac{1}{2}(\Delta x)^2 + \frac{1}{2}(\Delta x)^2\right]f(x_2) + \frac{1}{4}(\Delta x)^2 f(x_3)$$
$$= \frac{5}{4}(\Delta x)^2 f(x_0) + 2(\Delta x)^2 f(x_1) + (\Delta x)^2 f(x_2) + \frac{1}{4}(\Delta x)^2 f(x_3).$$

Then when $k = N$ we consider the double-layer integration over $[a, b]$, i.e $\int_a^b \int_a^{y_1} f(y)dydy_1 = \int_a^{x_n} \int_a^{y_1} f(y)dydy_1$ as follow:

$$\begin{aligned}
\int_a^{x_N} \int_a^{y_1} f(y)\, dydy_1 &= \sum_{i=0}^{N} \left(a_{Ni}^{(1)} \sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right) \\
&= a_{N1}^{(1)} \left[a_{10}^{(1)} f(x_0) + a_{11}^{(1)} f(x_1) \right] + a_{N2}^{(1)} \left[a_{20}^{(1)} f(x_0) + a_{21}^{(1)} f(x_1) + a_{22}^{(1)} f(x_2) \right] \\
&\quad + \cdots + a_{NN}^{(1)} \left[a_{N0}^{(1)} f(x_0) + a_{N1}^{(1)} f(x_1) + a_{N2}^{(1)} f(x_2) + \cdots + a_{NN}^{(1)} f(x_N) \right] \\
&= \left[\frac{1}{2}(N-1) + \frac{1}{4} \right] (\Delta x)^2 f(x_0) + \left[\frac{1}{2} + (N-2) + \frac{1}{2} \right] (\Delta x)^2 f(x_1) \\
&\quad + \left[\frac{1}{2} + (N-3) + \frac{1}{2} \right] (\Delta x)^2 f(x_2) + \cdots + \left[\frac{1}{2} + \frac{1}{2} \right] (\Delta x)^2 f(x_{N-1}) + \frac{1}{4} f(x_N) \\
&= \left(\frac{2(N-1)}{4} \right) (\Delta x)^2 f(x_0) + [N-1] (\Delta x)^2 f(x_1) + [(N-2)] (\Delta x)^2 f(x_2) \\
&\quad + \cdots + (\Delta x)^2 f(x_{N-1}) + \frac{1}{4} f(x_N).
\end{aligned}$$

From above, we can write the general form of the double-layer definite integral as

$$F^{(2)}(x_k) = \int_a^{x_k} \int_a^{y_1} f(y)dydy_1 = \sum_{i=0}^{k} \sum_{j=0}^{i} a_{ki}^{(1)} a_{ij}^{(1)} f(x_j) = \sum_{i=0}^{k} a_{ki}^{(2)} f(x_i),$$

where $\left[a_{0i}^{(2)} \right] = 0$ and $\left[a_{ki}^{(2)} \right] = \begin{cases} \frac{1+2(k-1)}{4} (\Delta x)^2, & i = 0, \\ (k-i)(\Delta x)^2, & i = 1, 2, 3, \ldots, k-1, \\ \frac{(\Delta x)^2}{4}, & i = k. \end{cases}$ The double-layer integral can be also written in a matrix form as

$$\underline{\boldsymbol{F}}^{(2)} = \boldsymbol{A}^{(2)} \underline{\boldsymbol{f}},$$

where the second order integration matrix,

$$\boldsymbol{A}^{(2)} = \left[a_{ki}^{(2)} \right] = (\Delta x)^2 \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & \ldots & 0 & 0 \\ 3/4 & 1 & 1/4 & 0 & \ldots & 0 & 0 \\ 5/4 & 2 & 1 & 1/4 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \frac{1+2(N-1)}{4} & N-1 & N-2 & N-3 & \ldots & 1 & 1/4 \end{bmatrix}_{(N+1)\times(N+1)}.$$

For the convenience of the following analysis, the first order integration matrix is rewritten as $\boldsymbol{A} = \boldsymbol{A}^{(1)}$. And we can observe that $\boldsymbol{A}^{(2)} = \boldsymbol{A} \cdot \boldsymbol{A} = \boldsymbol{A}^2$, this concludes that

$$\underline{\boldsymbol{F}}^{(1)} = \boldsymbol{A}\underline{\boldsymbol{f}} \quad \text{and} \quad \underline{\boldsymbol{F}}^{(2)} = \boldsymbol{A}^{(2)}\underline{\boldsymbol{f}}.$$

Thereafter, we claim to consider a triple-layer integral $F^{(3)}(x) = \int_a^x \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2$, and extend to the m-multi-layer integral $\int_a^{x_k} \int_a^{y_1} \dots \int_a^{y_{m-1}} f(x)dx \dots dy_2dy_1$ along mentioned above, if we can show that $\underline{\boldsymbol{F}}^{(3)} = \boldsymbol{A}^{(3)}\underline{\boldsymbol{f}}$, then $\underline{\boldsymbol{F}}^{(m)} = \boldsymbol{A}^{(m)}\underline{\boldsymbol{f}}$.

Here, we are considering the triple-layer integral $\int_a^{x_k} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2$,

$$\begin{aligned}
F^{(3)}(x_k) &= \int_a^{x_k} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2 \\
F^{(3)}(x_k) &= \int_a^{x_k} \int_a^{y_2} g(y_1)dy_1dy_2 \quad ; g(y_1) = \int_a^{y_1} f(y)dy \\
&= \sum_{i=0}^{k} a_{ki}^{(2)} g(x_i) \\
&= \sum_{i=0}^{k} \left(a_{ki}^{(2)} \int_a^{x_i} f(y)\, dy \right).
\end{aligned}$$

Then we obtain,

$$F^{(3)}(x_k) = \sum_{i=0}^{k} \left(a_{ki}^{(2)} \sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right).$$

Then, we have $\int_a^{x_k} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{k} \left(a_{ki}^{(2)} \sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right)$. Applying double layer integration matrix yields

$$\sum_{i=0}^{k} \left(a_{ki}^{(2)} \sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right) = \sum_{i=0}^{k} \left(\sum_{r=0}^{i} a_{ki}^{(1)} a_{ir}^{(1)} \left(\sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right) \right).$$

Therefore,

$$\int_a^{x_k} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{k} \sum_{r=0}^{i} \sum_{j=0}^{i} a_{ki}^{(1)} a_{ir}^{(1)} a_{ij}^{(1)} f(x_j).$$

Similary as the first and second orders, when $k = 0$ we have $\int_a^{x} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2 = 0$

When $k = 1$,

$$\int_a^{x_1} \int_a^{y_2} \int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{1} \left(a_{1i}^{(2)} \sum_{j=0}^{i} a_{ij}^{(1)} f(x_j) \right) = \cancel{a_{10}^{(2)} \sum_{j=0}^{0} a_{0j}^{(1)} f(x_j)} + a_{11}^{(2)} \sum_{j=0}^{1} a_{1j}^{(1)} f(x_j)$$

$$= a_{11}^{(2)} \left[a_{10}^{(1)} f(x_0) + a_{11}^{(1)} f(x_1) \right] = \frac{1}{4}(\Delta x)^2 \left[\frac{1}{2}(\Delta x) f(x_0) + \frac{1}{2}(\Delta x) f(x_1) \right]$$

$$= \frac{1}{8}(\Delta x)^3 f(x_0) + \frac{1}{8}(\Delta x)^3 f(x_1).$$

When $k=2$,

$$\int_a^{x_2}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{2}\left(a_{2i}^{(2)}\sum_{j=0}^{i}a_{ij}^{(1)}f(x_j)\right)$$

$$= a_{20}^{(2)}\sum_{j=0}^{0}a_{0j}^{(1)}f(x_j) + a_{21}^{(2)}\sum_{j=0}^{1}a_{1j}^{(1)}f(x_j) + a_{22}^{(2)}\sum_{j=0}^{2}a_{2j}^{(1)}f(x_j)$$

$$= a_{21}^{(2)}\left[a_{10}^{(1)}f(x_0)+a_{11}^{(1)}f(x_1)\right] + a_{22}^{(2)}\left[a_{20}^{(1)}f(x_0)+a_{21}^{(1)}f(x_1)+a_{22}^{(1)}f(x_2)\right]$$

$$= (\Delta x)^2\left[\frac{\Delta x}{2}f(x_0)+\frac{\Delta x}{2}f(x_1)\right] + \frac{(\Delta x)^2}{4}\left[\frac{\Delta x}{2}f(x_0)+\Delta x f(x_1)+\frac{\Delta x}{2}f(x_2)\right]$$

$$= \left[\frac{1}{2}(\Delta x)^3+\frac{1}{8}(\Delta x)^3\right]f(x_0) + \left[\frac{1}{2}(\Delta x)^3+\frac{1}{4}(\Delta x)^3\right]f(x_1) + \frac{1}{8}(\Delta x)^3 f(x_2)$$

$$= \frac{5}{8}(\Delta x)^3 f(x_0) + \frac{3}{4}(\Delta x)^3 f(x_1) + \frac{1}{8}(\Delta x)^3 f(x_2).$$

When $k=3$,

$$\int_a^{x_3}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{3}\left(a_{3i}^{(2)}\sum_{j=0}^{i}a_{ij}^{(1)}f(x_j)\right)$$

$$= a_{30}^{(2)}\sum_{j=0}^{0}a_{0j}^{(1)}f(x_j) + a_{31}^{(2)}\sum_{j=0}^{1}a_{1j}^{(1)}f(x_j) + a_{32}^{(2)}\sum_{j=0}^{2}a_{2j}^{(1)}f(x_j) + a_{33}^{(2)}\sum_{j=0}^{3}a_{3j}^{(1)}f(x_j)$$

$$= a_{31}^{(2)}\left[a_{10}^{(1)}f(x_0)+a_{11}^{(1)}f(x_1)\right] + a_{32}^{(2)}\left[a_{20}^{(1)}f(x_0)+a_{21}^{(1)}f(x_1)+a_{22}^{(1)}f(x_2)\right]$$
$$+ a_{33}^{(2)}\left[a_{30}^{(1)}f(x_0)+a_{31}^{(1)}f(x_1)+a_{32}^{(1)}f(x_2)+a_{33}^{(1)}f(x_3)\right]$$

$$= 2(\Delta x)^2\left[\frac{1}{2}(\Delta x)f(x_0)+\frac{1}{2}(\Delta x)f(x_1)\right] + (\Delta x)^2\left[\frac{1}{2}(\Delta x)f(x_0)+(\Delta x)f(x_1)+\frac{1}{2}(\Delta x)f(x_2)\right]$$
$$+ \frac{1}{4}(\Delta x)^2\left[\frac{1}{2}(\Delta x)f(x_0)+(\Delta x)f(x_1)+(\Delta x)f(x_2)+\frac{1}{2}(\Delta x)f(x_3)\right]$$

$$= \left[(\Delta x)^3+\frac{1}{2}(\Delta x)^3+\frac{1}{8}(\Delta x)^3\right]f(x_0) + \left[(\Delta x)^3+(\Delta x)^3+\frac{1}{4}(\Delta x)^3\right]f(x_1)$$
$$+ \left[\frac{1}{2}(\Delta x)^3+\frac{1}{4}(\Delta x)^3\right]f(x_2) + \frac{1}{8}(\Delta x)^3 f(x_3)$$

$$= \frac{13}{8}(\Delta x)^3 f(x_0) + \frac{9}{4}(\Delta x)^3 f(x_1) + \frac{3}{4}(\Delta x)^3 f(x_2) + \frac{1}{8}(\Delta x)^3 f(x_3).$$

Then when $k=N$, this brings us consider the integration over $[a,b]$, i.e

$$\int_a^{b}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2 = \int_a^{x_n}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2,$$

as follow:

$$\int_a^{x_N}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{N}\left(a_{N1}^{(2)}\sum_{j=0}^{i} a_{ij}^{(1)} f(x_j)\right)$$

$$= a_{N1}^{(2)}\left[a_{10}^{(1)} f(x_0) + a_{11}^{(1)} f(x_1)\right] + a_{N2}^{(2)}\left[a_{20}^{(1)} f(x_0) + a_{21}^{(1)} f(x_1) + a_{22}^{(1)} f(x_2)\right]$$

$$+ \cdots + a_{NN}^{(2)}\left[a_{N0}^{(1)} f(x_0) + a_{N1}^{(1)} f(x_1) + a_{N2}^{(1)} f(x_2) + \cdots + a_{NN}^{(1)} f(x_N)\right]$$

$$= \left[\frac{1}{8} + \frac{1}{2} + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) + \cdots + (n-1)\frac{1}{2}\right](\Delta x)^3 f(x_0)$$

$$+ \left[\frac{1}{4} + \frac{1}{2} + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) + \cdots + (2n-3)\frac{1}{2}\right](\Delta x)^3 f(x_1)$$

$$+ \left[\frac{1}{4} + \frac{1}{2} + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) + \cdots + (2n-5)\frac{1}{2}\right](\Delta x)^3 f(x_2)$$

$$+ \cdots + \frac{3}{4}(\Delta x)^3 f(x_{N-1}) + \frac{1}{8}(\Delta x)^3 f(x_N)$$

$$= \frac{1+2N(N-1)}{8}(\Delta x)^3 f(x_0) + \frac{1+2(N-1)^2}{4}(\Delta x)^3 f(x_1)$$

$$+ \frac{1+2(N-2)^2}{4}(\Delta x)^3 f(x_2) + \cdots + \frac{3}{4}(\Delta x)^3 f(x_{N-1}) + \frac{1}{8}(\Delta x)^3 f(x_N).$$

We here obtain the general form of the triple-layer definite integral as

$$F^{(3)}(x_k) = \int_a^{x_k}\int_a^{y_2}\int_a^{y_1} f(y)dydy_1dy_2 = \sum_{i=0}^{k}\sum_{r=0}^{i}\sum_{j=0}^{i} a_{ki}^{(1)} a_{ir}^{(1)} a_{ij}^{(1)} f(x_j) = \sum_{j=0}^{i} a_{ki}^{(3)} f(x_j),$$

where $\left[a_{0i}^{(3)}\right] = 0$ and $\left[a_{ki}^{(3)}\right] = \begin{cases} \frac{1+2k(k-1)}{8}(\Delta x)^3, & i = 0, \\ \frac{1+2(k-i)^2}{4}(\Delta x)^3, & i = 1,2,3,\ldots,k-1, \\ \frac{(\Delta x)^3}{8}, & i = k. \end{cases}$

The triple layer integral can be also written again in a matrix form as

$$\underline{\boldsymbol{F}}^{(3)} = \boldsymbol{A}^{(3)}\underline{\boldsymbol{f}},$$

where the triple order integration matrix,

$$\boldsymbol{A}^3 = a_{ki}^{(3)} = (\Delta x)^3 \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1/8 & 1/8 & 0 & 0 & \ldots & 0 & 0 \\ 5/8 & 3/4 & 1/8 & 0 & \ldots & 0 & 0 \\ 13/8 & 9/4 & 3/4 & 1/8 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \frac{1+2N(N-1)}{8} & \frac{1+2(N-1)^2}{4} & \frac{1+2(N-2)^2}{4} & \frac{1+2(N-3)^2}{4} & \ldots & 3/4 & 1/8 \end{bmatrix}_{(N+1)\times(N+1)}.$$

As we have $\boldsymbol{A}^{(3)}$ defined above, we can observe that $\boldsymbol{A}^{(3)} = \boldsymbol{A}\cdot\boldsymbol{A}\cdot\boldsymbol{A} = \boldsymbol{A}^3$. Therefore we can conclude that

$$\underline{\boldsymbol{F}}^{(3)} = \boldsymbol{A}^{(3)}\underline{\boldsymbol{f}} = \boldsymbol{A}^3\underline{\boldsymbol{f}}.$$

In the same way, for the higher order integration matrix, we have

$$F^{(m)}(x_k) = \int_a^{x_k}\int_a^{y_{m-1}}\dots\int_a^{y_1} f(y)dy\dots dy_{m-2}dy_{m-1} = \sum_{i=0}^{k} a_{ki}^{(m)} f(x_i).$$

Let $P_k : A^{(k)} = AA\dots A = A^k$, since $P_1 : A^{(1)} = A$ and $P_2 : A^{(2)} = AA = A^2$ are true. This brings $P_3 : A^{(3)} = AAA = A^3$ be true as we have shown. By doing the same way we have $P_k : A^{(k)} = AA\dots A = A^k$ be also true.

Thus, the m-layer integral can be written in a matrix form as

$$\boldsymbol{F}^{(m)} = \boldsymbol{A}^{(m)}\boldsymbol{f} = \boldsymbol{A}^m\boldsymbol{f}.$$

Note that, it is worth to point out that the integral matrix with any order $\boldsymbol{A}^{(m)}$ is lower-tringular matrix.

Integration of Radial Basis Function

Indeed, It is not easy to obtain its first integration of RBF analytically as denoted as $\bar{R}_i(x, x_i)$

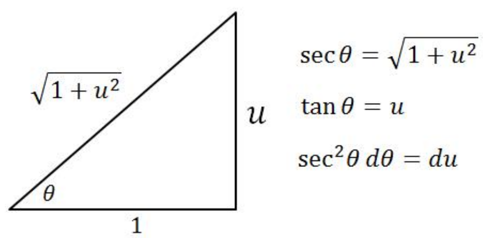
$$\bar{R}_i(x,x_i) = \int R_i(x,x_i)\,dx = \int \sqrt{c^2 + (x-x_i)^2}\,dx$$

$$= \int \sqrt{c^2}\sqrt{1 + \left(\frac{x-x_i}{c}\right)^2}\,dx$$

$$\bar{R}_i(x,x_i) = c\int \sqrt{1 + \left(\frac{x-x_i}{c}\right)^2}\,dx$$

$$= c^2\int \sqrt{1+u^2}\,du \quad \text{where} \quad u = \frac{x-x_i}{c}.$$

Assuming the right triangle below,



Finally, the integration of $R_i(x, x_i)$ is

$$\begin{aligned}\bar{R}_i(x, x_i) = \int R_i(x, x_i)\, dx = c^2 \int \sec\theta \sec^2\theta \, d\theta \\ = c^2 \int \sec^3\theta \, d\theta.\end{aligned}$$

Consider $\int \sec^3\theta\, d\theta$ first and keep the constant c^2,

$$\int \sec^3\theta \, d\theta = \int \sec\theta \sec^2\theta \, d\theta$$

Applying the integration by part yields

$$\begin{aligned}\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\, d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta\end{aligned}$$

Adding $\int \sec^3\theta\, d\theta$ on the both side gives

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

Manipulate the above equation algebraically, this becomes

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta \frac{\tan\theta + \sec\theta}{\tan\theta + \sec\theta}.$$

Let $v = \tan\theta + \sec\theta$, we have

$$\begin{aligned}2\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta + \int \frac{dv}{v} \\ &= \sec\theta\tan\theta + \ln|v| \\ &= \sec\theta\tan\theta + \ln|\tan\theta + \sec\theta|.\end{aligned}$$

Since $\int \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\tan\theta + \sec\theta|\right]$ then now we have

$$c^2 \int \sec^3\theta\, d\theta = c^2 \times \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\tan\theta + \sec\theta|\right].$$

Therefore,

$$\bar{R}(x, x_i) = \int \sqrt{c^2 + (x - x_i)^2}\, dx$$

$$\begin{aligned}
\bar{R}(x, x_i) &= \frac{c^2}{2}\left(\frac{\sqrt{c^2 + (x - x_i)^2}}{c} \times \frac{x - x_i}{c}\right) + \frac{c^2}{2} \times \ln\left|\frac{(x - x_i) + \sqrt{c^2 + (x - x_i)^2}}{c}\right| + C \\
&= \frac{x - x_i}{2}\sqrt{c^2 + (x - x_i)^2} + \frac{c^2}{2} \times \ln\left|\frac{(x - x_i) + \sqrt{c^2 + (x - x_i)^2}}{c}\right| + C.
\end{aligned}$$

Since $\bar{R}(0, x_i) = 0$, then

$$\begin{aligned}
\bar{R}(0, x_i) &= \frac{-x_i}{2}\sqrt{c^2 + x_i^2} + \frac{c^2}{2} \times \ln\left|\frac{\sqrt{c^2 + x_i^2} - x_i}{c}\right| + C \\
0 &= \frac{-x_i}{2}\sqrt{c^2 + x_i^2} + \frac{c^2}{2} \times \ln\left|\frac{\sqrt{c^2 + x_i^2} - x_i}{c}\right| + C \\
C &= \frac{x_i}{2}\sqrt{c^2 + x_i^2} - \frac{c^2}{2} \times \ln\left|\frac{\sqrt{c^2 + x_i^2} - x_i}{c}\right|.
\end{aligned}$$

Subtituting C into $\bar{R}(x, x_i)$ gives

$$\begin{aligned}
\bar{R}(x, x_i) &= \frac{x - x_i}{2}\sqrt{c^2 + (x - x_i)^2} + \frac{c^2}{2} \times \ln\left|\frac{(x - x_i) + \sqrt{c^2 + (x - x_i)^2}}{c}\right| \\
&\quad + \frac{x_i}{2}\sqrt{c^2 + x_i^2} - \frac{c^2}{2} \times \ln\left|\frac{\sqrt{c^2 + x_i^2} - x_i}{c}\right| \\
&= \frac{x - x_i}{2}\sqrt{c^2 + (x - x_i)^2} + \frac{x_i}{2}\sqrt{c^2 + x_i^2} + \frac{c^2}{2} \times \ln\left|\frac{\frac{(x - x_i) + \sqrt{c^2 + (x - x_i)^2}}{\cancel{c}}}{\frac{\sqrt{c^2 + x_i^2} - x_i}{\cancel{c}}}\right|.
\end{aligned}$$

Hence,

$$\bar{R}(x, x_i) = \frac{x - x_i}{2}\sqrt{c^2 + (x - x_i)^2} + \frac{x_i}{2}\sqrt{c^2 + x_i^2} + \frac{c^2}{2}\ln\frac{x - x_i + \sqrt{c^2 + (x - x_i)^2}}{\sqrt{c^2 + x_i^2} - x_i}.$$

The Second Fundamental Theorem of Calculus

If the function $f(x)$ is continuous on the interval $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx = F(a) - F(b),$$

where $F(x)$ is anti-derivative of $f(x)$ on $a \leq x \leq b$.

Vitae

Nama : Rama Lesmana

Student ID : 5820320705

Educational Attainment:

Degree	Name of Institution	Year of Graduation
Bachelor of Education (Mathematics Education)	University of Muhammadiyah Prof. DR. HAMKA	2013

Scholarship Awards during Enrolment:

1. Science and Technology (SAT)-ASEAN scholarship for international students from the Faculty of Science and Technology, Prince of Songkla University, Thailand.
2. Research scholarship from Graduate School, Prince of Songkla University, Thailand.

Seminar:

Rama Lesmana, A Finite Integration Method for Solving Inverse source Problem. International Research Method in Practice, Prince of Songkla University, Thailand. May 2017.

Symposium:

Lesmana, R., Hazanee, A. and Phon-On, A. A Finite Integration Method for Solving Inverse source Problem. The 7th Sci-Tech Grad Symposium and PSU-UMT Joint Seminar, Faculty of Science and Technology, Prince of Songkla University, Thailand. May 2017

Proceeding:

Lesmana, R., Hazanee, A., Phon-On, A. and Saelee, J. 2017. A Finite Integration Method for A Time-Dependent Heat Source Identification of Inverse Problem. The 5th Asian Academic Society International Conference (AASIC). 26-27 July 2017. Khon Kaen, Thailand.