รายงานวิจัยฉบับสมบูรณ์

สมบัติบางประการของฟังก์ชันควอร์ติกทีตา

Certain Properties of Quartic Theta Functions

โดย

ผู้ช่วยศาสตราจารย์ ดร.บุญรอด ยุทธานันท์

โครงการวิจัยนี้ได้รับทุนสนับสนุนจากเงินรายได้มหาวิทยาลัย

มหาวิทยาลัยสงขลานครินทร์

ประจำปีงบประมาณ 2555 รหัสโครงการ SCI550398S

ชื่อโครงการ ภาษาไทย : สมบัติบางประการของฟังก์ชันควอร์ติกทีตา

ภาษาอังกฤษ : Certain Properties of Quartic Theta Functions

คณะผู้ดำเนินการวิจัย

บุญรอด ยุทธานันท์

สถานที่ทำงาน : ภาควิชาคณิตศาสตร์และสถิติ คณะวิทยาศาสตร์ มหาวิทยาลัยสงขลานครินทร์

โทรศัพท์ : 074-288698

สารบัญ

หน้า

กิตติกรรมประกาศ

ผู้วิจัยขอขอบคุณ Prof.Bruce C. Berndt ที่ให้คำแนะนำในการค้นคว้าและให้เอกสารที่เกี่ยวข้องกับทำวิจัย ขอขอบคุณ ผศ.ดร.ศราชัย ก้องศิริวงศ์ ที่เป็นที่ปรึกษาและช่วยเหลือในด้านต่าง ๆ ในการดำเนินการวิจัย และขอขอบคุณมหาวิทยาลัยสงขลานครินทร์ ที่ให้การสนับสนุนงบประมาณ

บุญรอด ยุทธานันท์

ผู้วิจัย

บทคัดย่อ

งานวิจัยนี้ได้ศึกษา Theta functions รูปแบบใหม่ใน Ramanujan's quartic theory ผู้วิจัยได้คิดค้นเอกลักษณ์ของฟังก์ชันนี้ในทำนองเดียวกับ Jacobian theta functions และ Cubic theta functions

Abstract

A new general theta function in Ramanujan's quartic theory is introduced. Some properties analogous to those of classical Jacobian theta functions and cubic theta functions are established here.

บทนำ

โครงงานวิจัยนี้ได้ทำการศึกษาเกี่ยวกับ Theta functions ในการศึกษาคณิตศาสตร์สาขาทฤษฎีจำนวน ฟังก์ชันหนึ่งที่มีความสำคัญมากคือ Theta functions ซึ่งเป็นฟังก์ชันของตัวแปรเชิงซ้อน ฟังก์ชันนี้ถูกค้นพบครั้งแรกในศตวรรษที่ 18 โดย Leonard Euler และได้ถูกศึกษาอย่างกว้างขวางในศตวรรษที่ 19 โดยนักคณิตศาสตร์หลายท่าน เช่น Carl Friedrich Gauss, Carl Gustav Jacob Jacobi เป็นต้น เนื่องจาก Jacobi ได้ศึกษาฟังก์ชันนี้อย่างลึกซึ้งและเป็นระบบ ต่อมาเราจึงนิยมเรียกฟังก์ชันนี้ว่า Jacobi's theta functions (หรือ Classical Theta Functions) ซึ่งมีอยู่ 4 ฟังก์ชันคือ $\theta_1,\theta_2,\theta_3$ และ θ_4 ให้ τ เป็นจำนวนเชิงซ้อนใด ๆ ซึ่งส่วนจินตภาพเป็นจำนวนจริงบวกและเขียน q ในรูป $q=e^{\pi i\tau}$ แล้วเรานิยามว่า

$$\theta_1(z,q):=-i\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}e^{(2n+1)iz}$$

$$\theta_2(z,q):=\sum_{n=0}^{\infty}q^{(n+1/2)^2}e^{(2n+1)iz}$$

$$\theta_3(z,q):=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2niz}$$

$$\theta_4(z,q):=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2niz}$$

ในช่วงต้นศตวรรษที่ 20 Srinivasa Ramanujan นักคณิตศาสตร์ชาวอินเดียได้ค้นพบรูปแบบทั่วไปของ Theta functions ซึ่งมีนิยามดังนี้

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}$$

เราจะเห็นได้ว่า $\theta_1,\theta_2,\theta_3$ และ θ_4 เป็นเพียงกรณีพิเศษของฟังก์ชัน f เท่านั้น จากการค้นพบนี้ทำให้องค์ความรู้เกี่ยวกับฟังก์ชันนี้ได้พัฒนามากยิ่งขึ้นไปอีก ฟังก์ชันนี้ได้มีการนำไปใช้ในหลายสาขาวิชาของคณิตศาสตร์ เช่น q-Series, Modular Equations, Partition Theory, Elliptic Functions, Algebraic Geometry, Complex Analysis, Quadratic Forms เป็นต้น และยังได้มีการนำฟังก์ชันนี้ไปประยุกต์ใช้ในสาขาวิชาอื่น ๆ อีกด้วย เช่น Soliton Theory, Quantum Field Theory, Models of Lattice Gases, Particle Physics เป็นต้น ในปี ค.ศ. 1989 Jonathan M. Borwein และ Peter B. Borwein ได้ค้นพบฟังก์ชันที่มีลักษณะคล้ายกับ Jacobi's theta functions แต่มีความซับซ้อนมากขึ้น ต่อมาเรียกว่า Cubic theta functions (หรือ two-dimesional theta functions) มีนิยามดังต่อไปนี้

$$a(q):=\sum_{m,n=-\infty}^{\infty}q^{n^2+nm+m^2}$$

$$b(q):=\sum_{m,n=-\infty}^{\infty}q^{n^2+nm+m^2}\omega^{n-m}$$

$$c(q):=\sum_{m,n=-\infty}^{\infty}q^{\left(m+\frac{1}{3}\right)^2+\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)+\left(n+\frac{1}{3}\right)^2}$$

เมื่อ $\omega = e^{2\pi i/3}$ ซึ่งในภายหลังสมบัติต่าง ๆ ของฟังก์ชันนี้ได้ถูกคิดค้นขึ้นมามากมายโดยนักคณิตศาสตร์หลายท่าน เมื่อเร็ว ๆ นี้ในปี 2009 Daniel Schultz ได้ค้นพบฟังก์ชันใหม่ที่เรียกว่า Quartic theta functions จากการศึกษา Jacobi's theta functions และ Cubic theta functions บน Riemann surfaces ฟังก์ชันดังกล่าวมีนิยามดังนี้

$$\vartheta(q_1, q_2, x_1, x_2, x_3) := \sum_{a,b,c=-\infty}^{\infty} q_1^{\frac{1}{2}(a+c)^2} q_2^{\frac{1}{2}(a+b)^2+\frac{1}{2}(b+c)^2} x_1^a x_2^b x_3^c$$

โดยที่ $|q_1|<1, |q_2|<1$ ฟังก์ชันใหม่ที่เพิ่งค้นพบนี้ยังไม่ได้เป็นที่รู้จักกันกว้างขวางนัก ผู้วิจัยเกิดความสนใจและได้ศึกษาฟังก์ชันนี้ ซึ่งสามารถคิดค้นสมบัติใหม่ ๆ ของฟังก์ชันนี้เพิ่มเติมได้ในทำนองเดียวกับ Jacobi's Theta Functions และ Cubic Theta Functions ซึ่งจะทำให้องค์ความรู้เกี่ยวกับ Quartic Theta Functions นี้กว้างขวางมากยิ่งขึ้น และคาดหวังว่าในอนาคตอาจจะมีการนำเอาความรู้เกี่ยวกับฟังก์ชันนี้ไปประยุกต์ใช้ในสาขาวิชาอื่นอีกด้วย

วัตถุประสงค์

1. ศึกษาคิดค้นเอกลักษณ์และสมบัติต่าง ๆ ของ Quartic theta functions ในทำนองเดียวกับ Jacobi's theta functions และ Cubic theta functions
2. สร้างความสัมพันธ์ระหว่าง Quartic theta functions และ Cubic theta functions
3. การประยุกต์ใช้เอกลักษณ์ของ Quartic theta functions
4. สร้างความสัมพันธ์ระหว่าง Quartic theta functions กับฟังก์ชันอื่น ๆ ที่ใช้กันอย่างแพร่หลายในคณิตศาสตร์

ผลการวิจัย

ผู้วิจัยได้ศึกษา Quartic theta function ซึ่งมีนิยามดังนี้

$$\vartheta(q_2,q_4,x_1,x_2,x_3):=\sum_{a,b,c=-\infty}^{\infty} q_2^{\frac{1}{2}(a+c)^2} q_4^{\frac{1}{2}(a+b)^2+\frac{1}{2}(b+c)^2} x_1^a x_2^b x_3^c$$

โดยที่

$$q_2:=\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-m_2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;m_2)}\right),\quad q_4:=\exp\left(-\pi\sqrt{2}\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-m_4)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;m_4)}\right)$$

โดยที่ $0<m_2<1, 0<m_4<1$ และ ${}_2F_1(a,b;c;z)$ คือ Ordinary hypergeometric function

ทฤษฎีบทที่กล่าวถึงต่อไปนี้จะขอละการพิสูจน์ไว้ สำหรับการพิสูจน์สามารถดูได้จากบทความที่ตีพิมพ์แล้วในภาคผนวก ทฤษฎีบทแรกจะกล่าวถึงสมบัติพื้นฐานของฟังก์ชัน $\vartheta(q_1,q_2,x_1,x_2,x_3)$

ทฤษฎีบท 1

1. $\vartheta(q_2,q_4,x_1,x_2,x_3)=\vartheta(q_2,q_4,x_3,x_2,x_1)$
2. $\vartheta(q_2,q_4,x_1,x_2,x_3)=\vartheta(q_2,q_4,x_1^{-1},x_2^{-1},x_3^{-1})$
3.

$$\begin{aligned}&2\vartheta(q_2,q_4,x_1,x_2,x_3)\\&=f\left(\left(\frac{x_1x_3}{x_2}\right)^{\frac{1}{2}}q_2^{\frac{1}{2}},\left(\frac{x_1x_3}{x_2}\right)^{-\frac{1}{2}}q_2^{\frac{1}{2}}\right)f\left(\left(\frac{x_1x_2}{x_3}\right)^{\frac{1}{2}}q_4^{\frac{1}{2}},\left(\frac{x_1x_2}{x_3}\right)^{-\frac{1}{2}}q_4^{\frac{1}{2}}\right)f\left(\left(\frac{x_2x_3}{x_1}\right)^{\frac{1}{2}}q_4^{\frac{1}{2}},\left(\frac{x_2x_3}{x_1}\right)^{-\frac{1}{2}}q_4^{\frac{1}{2}}\right)\\&\quad+f\left(-\left(\frac{x_1x_3}{x_2}\right)^{\frac{1}{2}}q_2^{\frac{1}{2}},-\left(\frac{x_1x_3}{x_2}\right)^{-\frac{1}{2}}q_2^{\frac{1}{2}}\right)f\left(-\left(\frac{x_1x_2}{x_3}\right)^{\frac{1}{2}}q_4^{\frac{1}{2}},-\left(\frac{x_1x_2}{x_3}\right)^{-\frac{1}{2}}q_4^{\frac{1}{2}}\right)f\left(-\left(\frac{x_2x_3}{x_1}\right)^{\frac{1}{2}}q_4^{\frac{1}{2}},-\left(\frac{x_2x_3}{x_1}\right)^{-\frac{1}{2}}q_4^{\frac{1}{2}}\right)\end{aligned}$$

   เมื่อ $f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}$ คือ Ramanujan's general theta function

4. ถ้า p,q,r เป็นจำนวนคู่แล้ว

$$\begin{aligned}&\vartheta(q_2,q_4,x_1,x_2,x_3)\\&=x_1^{p/2}x_2^{q/2}x_3^{r/2}q_2^{(p+r)^2/8}q_4^{(p+q)^2/8+(q+r)^2/8}\vartheta(q_2,q_4,q_2^{(p+r)/2}q_4^{(p+q)/2}x_1,q_4^{(p+2q+r)/2}x_2,q_2^{(p+r)/2}q_4^{(q+r)/2}x_3)\end{aligned}$$

5. ถ้า m,n,k,p,q,r เป็นจำนวนคู่โดยที่ $\frac{1}{2}(mp+nq+kr)\not\equiv 0\pmod 4$ แล้ว

$$\vartheta\left(q_{2,}q_4,i^m q_2^{(p+r)/4}q_4^{(p+q)/4},i^n q_4^{(p+2q+r)/4},i^k q_2^{(p+r)/4}q_4^{(q+r)/4}\right)=0$$

6. ถ้า $\operatorname{Re} s, \operatorname{Re} t > 0$ แล้ว

$$\vartheta(e^{-\pi s}, e^{-\pi t}, e^{ia}, e^{ib}, e^{ic})$$
$$= \frac{\sqrt{2}}{t\sqrt{s}} e^{-(a-b+c)^2/8\pi s - \left((a-c)^2+b^2\right)/4\pi t} \vartheta\left(e^{-\pi/s}, e^{-2\pi/t}, e^{(a-b+c)/2s+b/t}, e^{(a+b-c)/t}, e^{(a-b+c)/2s+(a-c)/t}\right)$$

7. ให้ $m \in \{0,1,2,3\}$ และ n เป็นจำนวนเต็ม จะได้ว่า

$$\vartheta\left(q_2, q_4, i^m (q_2q_4)^{n/2}, i^{2m} q_4^n, i^{3m} (q_2q_4)^{n/2}\right)$$
$$= \begin{cases} q_2^{-(n^2-1)/8} q_4^{-(n^2-1)/4} \vartheta\left(q_2, q_4, (q_2q_4)^{1/2}, q_4, (q_2q_4)^{1/2}\right), & m = 0 \text{ and } n \equiv 1 \pmod 2 \\ q_2^{-n^2/8} q_4^{-n^2/4} \vartheta(q_2, q_4, 1, 1, 1), & m = 0 \text{ and } n \equiv 0 \pmod 4 \\ q_2^{-(n^2-4)/8} q_4^{-(n^2-4)/4} \vartheta\left(q_2, q_4, q_2q_4, q_4^2, q_2q_4\right), & m = 0 \text{ and } n \equiv 2 \pmod 4 \\ i^{n/2} q_2^{-n^2/8} q_4^{-n^2/4} \vartheta\left(q_2, q_4, i, i^2, i^3\right), & m = 1,3 \text{ and } n \equiv 0 \pmod 4 \\ q_2^{-n^2/8} q_4^{-n^2/4} \vartheta\left(q_2, q_4, i^2, i^4, i^6\right), & m = 2 \text{ and } n \equiv 0 \pmod 4 \\ q_2^{-(n^2-4)/8} q_4^{-(n^2-4)/4} \vartheta\left(q_2, q_4, i^2 q_2q_4, i^4 q_4^2, i^6 q_2q_4\right), & m = 2 \text{ and } n \equiv 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

เนื่องจากฟังก์ชันมีความซับซ้อนของพารามิเตอร์ ผู้วิจัยจึงทำการศึกษากรณีเฉพาะของฟังก์ชัน ϑ ดังต่อไปนี้ กำหนดให้

$$\vartheta_1(q_2, q_4) := \vartheta(q_2, q_4, 1, 1, 1)$$
$$\vartheta_2(q_2, q_4) := \vartheta(q_2, q_4, i, i^2, i^3)$$
$$\vartheta_3(q_2, q_4) := \vartheta(q_2, q_4, i^2, i^4, i^6)$$
$$\vartheta_4(q_2, q_4) := q_2^{1/8} q_4^{1/4} \vartheta\left(q_2, q_4, (q_2q_4)^{1/2}, q_4, (q_2q_4)^{1/2}\right)$$
$$\vartheta_5(q_2, q_4) := q_2^{1/2} q_4 \vartheta(q_2, q_4, q_2q_4, q_4^2, q_2q_4)$$
$$\vartheta_6(q_2, q_4) := q_2^{1/2} q_4 \vartheta(q_2, q_4, i^2 q_2q_4, i^4 q_4^2, i^6 q_2q_4)$$

จากความสัมพันธ์ระหว่างฟังก์ชัน ϑ กับฟังก์ชัน f ในทฤษฎีบท 1 ข้อ 3 ผู้วิจัยได้ใช้เอกลักษณ์ของฟังก์ชัน f ซึ่งคิดค้นโดย Srinivasa Ramanujan มาพิสูจน์เอกลักษณ์ในทฤษฎีบทต่อไปนี้

ทฤษฎีบท 2

$$\vartheta_1(q_2,q_4)=\frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(q_4^{1/2})+\varphi(-q_2^{1/2})\varphi^2(-q_4^{1/2})\right)$$
$$\vartheta_2(q_2,q_4)=\varphi(-q_2^2)\varphi(q_4^{1/2})\varphi(-q_4^{1/2})$$
$$\vartheta_3(q_2,q_4)=\frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(-q_4^{1/2})+\varphi(-q_2^{1/2})\varphi^2(q_4^{1/2})\right)$$
$$\vartheta_4(q_2,q_4)=4\psi(q_2)\psi^2(q_4)$$
$$\vartheta_5(q_2,q_4)=\frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(q_4^{1/2})-\varphi(-q_2^{1/2})\varphi^2(-q_4^{1/2})\right)$$
$$\vartheta_6(q_2,q_4)=\frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(-q_4^{1/2})-\varphi(-q_2^{1/2})\varphi^2(q_4^{1/2})\right)$$

โดยที่ $\varphi(q)=f(q,q)$ และ $\psi(q)=f(q,q^3)$ เมื่อ f คือฟังก์ชันที่นิยามไว้ในทฤษฎีบท 1 ข้อ 3 ซึ่งสองฟังก์ชันนี้เป็นกรณีเฉพาะของฟังก์ชัน f ที่รู้จักกันอย่างกว้างขวาง และจากการประยุกต์ใช้ความสัมพันธ์ของ φ และ ψ ทำให้เราได้บทแทรกต่อไปนี้

บทแทรก 3

$$\vartheta_1(q_2,q_4)=\frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1+\sqrt{m_2m_4}+\sqrt{(1-m_2)(1-m_4)}\right)^{1/2}$$
$$\vartheta_2(q_2,q_4)=\sqrt{z_2}z_4(1-m_2)^{1/8}(1-m_4)^{1/4}$$
$$\vartheta_3(q_2,q_4)=\frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1-\sqrt{m_2m_4}+\sqrt{(1-m_2)(1-m_4)}\right)^{1/2}$$
$$\vartheta_4(q_2,q_4)=\sqrt{z_2}z_4m_2^{1/8}m_4^{1/4}$$
$$\vartheta_5(q_2,q_4)=\frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1+\sqrt{m_2m_4}-\sqrt{(1-m_2)(1-m_4)}\right)^{1/2}$$
$$\vartheta_6(q_2,q_4)=\frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1-\sqrt{m_2m_4}-\sqrt{(1-m_2)(1-m_4)}\right)^{1/2}$$

เมื่อ $z_2:={}_2F_1(\frac{1}{2},\frac{1}{2};1;m_2)$ และ $z_4:={}_2F_1(\frac{1}{2},\frac{1}{2};1;m_4)$

บทแทรก 4

$$\vartheta_1^2(q_2,q_4)+\vartheta_6^2(q_2,q_4)=\vartheta_3^2(q_2,q_4)+\vartheta_5^2(q_2,q_4)$$
$$\vartheta_1^2(q_2,q_4)-\vartheta_6^2(q_2,q_4)=\vartheta_2^2(q_2^{1/2},q_4)+2\vartheta_4^2(q_2^2,q_4)$$
$$\vartheta_3^2(q_2,q_4)-\vartheta_5^2(q_2,q_4)=\vartheta_2^2(q_2^{1/2},q_4)-2\vartheta_4^2(q_2^2,q_4)$$

บทแทรก 5 ถ้าให้ $m = m_1 = m_2$ จะได้ว่า

$$\vartheta_1(q_2,q_4) = \sqrt{z_2}z_4 \qquad \vartheta_4(q_2,q_4) = \sqrt{z_2}z_4 m^{3/8}$$
$$\vartheta_2(q_2,q_4) = \sqrt{z_2}z_4(1-m)^{3/8} \qquad \vartheta_5(q_2,q_4) = \sqrt{z_2}z_4 m^{1/2}$$
$$\vartheta_3(q_2,q_4) = \sqrt{z_2}z_4(1-m)^{1/2} \qquad \vartheta_6(q_2,q_4) = 0$$

บทแทรก 6 ถ้าให้ $m = m_1 = m_2$ จะได้ว่า

$$\vartheta_3^2(q_2,q_4) + \vartheta_5^2(q_2,q_4) = \vartheta_1^2(q_2,q_4)$$
$$\vartheta_2^{8/3}(q_2,q_4) + \vartheta_4^{8/3}(q_2,q_4) = \vartheta_1^{8/3}(q_2,q_4)$$
$$\vartheta_2^8(q_2,q_4) = \vartheta_1^2(q_2,q_4)\vartheta_3^6(q_2,q_4)$$
$$\vartheta_4^8(q_2,q_4) = \vartheta_1^2(q_2,q_4)\vartheta_5^6(q_2,q_4)$$
$$\vartheta_1(q_2^2,q_4^2) - \vartheta_5(q_2^2,q_4^2) = \vartheta_3(q_2,q_4)$$
$$\vartheta_3(q_2^2,q_4^2) + \vartheta_6(q_2^2,q_4^2) = \vartheta_1^{1/2}(q_2,q_4)\vartheta_3^{1/2}(q_2,q_4)$$
$$\vartheta_1(q_2^{1/2},q_4^{1/2}) - \vartheta_3(q_2^{1/2},q_4^{1/2}) = 2\sqrt{2}\vartheta_5(q_2,q_4)$$
$$\vartheta_5(q_2^{1/2},q_4^{1/2}) - \vartheta_6(q_2^{1/2},q_4^{1/2}) = 2\sqrt{2}\vartheta_1^{1/2}(q_2,q_4)\vartheta_5^{1/2}(q_2,q_4)$$

จากสูตรข้างต้น ผู้วิจัยพบว่าถ้าเราคำนวณค่าของฟังก์ชัน ϑ สำหรับบางค่า m_2 และ m_4 ในทฤษฎีบท 2 จะได้ความสัมพันธ์ระหว่าง ϑ กับ Γ (Gamma function) ดังต่อไปนี้

ทฤษฎีบท 3

1. ถ้า $m_2 = m_4 = \dfrac{1}{2}$ แล้ว

$$\vartheta_1(e^{-\pi}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$$
$$\vartheta_2(e^{-\pi}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{2^{3/8}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$$
$$\vartheta_3(e^{-\pi}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{2^{1/2}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},$$
$$\vartheta_4(e^{-\pi}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{2^{3/8}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$$
$$\vartheta_5(e^{-\pi}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{2^{1/2}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}$$
$$\vartheta_6(e^{-\pi}, e^{-\pi\sqrt{2}}) = 0$$

2. ถ้า $m_2 = \frac{2-\sqrt{3}}{4}$ และ $m_4 = \frac{1}{2}$ แล้ว

$$\vartheta_1(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}(\sqrt{3}+1)}{3^{3/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

$$\vartheta_2(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}\left(\sqrt{3}+1\right)^{1/4}}{2^{1/8}3^{3/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

$$\vartheta_3(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}3^{1/8}}{\sqrt{2}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

$$\vartheta_4(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}\left(\sqrt{3}-1\right)^{1/4}}{2^{1/8}3^{3/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

$$\vartheta_5(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{3^{3/8}\sqrt{2}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

$$\vartheta_6(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}(\sqrt{3}-1)}{3^{3/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

3. ถ้า $m_2 = \frac{2+\sqrt{3}}{4}$ และ $m_4 = \frac{1}{2}$ แล้ว

$$\vartheta_1(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}(\sqrt{3}+1)}{3^{1/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_2(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}\left(\sqrt{3}-1\right)^{1/4}}{2^{1/8}3^{1/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_3(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{2^{1/8}3^{1/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_4(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}\left(\sqrt{3}+1\right)^{1/4}}{2^{1/8}3^{1/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_5(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}3^{3/8}}{\sqrt{2}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_6(e^{-\pi/\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}(\sqrt{3}-1)}{3^{1/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}$$

4. ถ้า $m_2 = \frac{2}{\sqrt{2}+1}$ และ $m_4 = \frac{8}{9}$ แล้ว

$$\vartheta_1(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+14}+4\right)^{1/2}}{2^{9/8}\Gamma(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

$$\vartheta_2(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}}{2^{5/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

$$\vartheta_3(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+14}-4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

$$\vartheta_4(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{2}+1\right)^{1/8} 2^{1/4}}{\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

$$\vartheta_5(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+2}+4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

$$\vartheta_6(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+2}-4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}$$

เอกสารอ้างอิง

1. B.C. Berndt, *Ramanujan's Notebooks, Part II*, Springer-Verlag, New York, 1989.
2. B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.
3. B.C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.
4. S. Bhargava, *Unification of the cubic analogues of the Jacobian theta-function*, J. Math. Anal. Appl. 193 (2) (1995) 543–558.
5. J.M. Borwein, P.B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. 323 (2) (1991) 691–701.
6. M.D. Hirschhorn, F. Garvan, J. Borwein, *Cubic analogues of the Jacobian theta function* $\theta(z, q)$, Canad. J. Math. 45 (4) (1993) 673–694.
7. S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
8. S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
9. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
10. D. Schultz, *Cubic theta functions*, Adv. Math. 248 (2013) 618–697.

ภาคผนวก

1. ข้อคิดเห็น/ข้อเสนอแนะในส่วนที่ไม่สามารถดำเนินการวิจัยได้ตามวัตถุประสงค์

งานวิจัยนี้ได้วางแผนการดำเนินงานตลอดโครงการไว้ 4 หัวข้อ คือ

1. สร้างเอกลักษณ์ของ Quartic theta functions
2. สร้างความสัมพันธ์ระหว่าง Quartic theta functions และ Cubic theta functions
3. ประยุกต์เอกลักษณ์ของ Quartic theta functions เพื่อใช้สร้างเอกลักษณ์ใหม่ใน Partition Theory
4. หาความสัมพันธ์ระหว่าง Quartic theta functions กับฟังก์ชันอื่นๆ

ส่วนที่ไม่สามารถดำเนินการวิจัยได้ตามวัตถุประสงค์มีอยู่ 2 หัวข้อ คือ

1. หัวข้อที่ 2 การสร้างความสัมพันธ์ระหว่าง Quartic theta functions และ Cubic theta functions นั้น ผู้วิจัยยังไม่สามารถทำงานวิจัยในหัวข้อนี้ได้ตามเป้าหมายที่วางไว้ เพราะฟังก์ชันที่ผู้วิจัยศึกษามีความสัมพันธ์กับ ${}_2F_1(\frac{1}{2},\frac{1}{2};1;m)$ และ ${}_2F_1(\frac{1}{4},\frac{3}{4};1;m)$ ส่วน Cubic theta functions นั้นมีความสัมพันธ์กับ ${}_2F_1(\frac{1}{3},\frac{2}{3};1;m)$ แต่ความสัมพันธ์ของ Ordinary hypergeometric function เหล่านี้ยังมีน้อย จึงทำให้ไม่สามารถสร้างความสัมพันธ์ระหว่าง Quartic theta functions และ Cubic theta functions ได้
2. หัวข้อที่ 3 ประยุกต์เอกลักษณ์ของ Quartic theta functions เพื่อใช้สร้างเอกลักษณ์ใหม่ใน Partition Theory ไม่สามารถดำเนินได้ เนื่องจากเวลาในการทำวิจัยไม่เพียงพอ

2. บทความที่ตีพิมพ์แล้ว

งานวิจัยนี้ได้ตีพิมพ์ในวารสาร Journal of Mathematical Analysis and Applications ซึ่งอยู่ในฐาน ISI (Q1) และมี impact factor เท่ากับ 1.119 ซึ่งเปลี่ยนไปจากที่กล่าวไว้ในแบบเสนอโครงการวิจัย โดยผู้วิจัยได้ระบุว่าจะตีพิมพ์ผลงานในวารสาร The Ramanujan Journal ซึ่งอยู่ในฐาน ISI (Q3) และมี impact factor เท่ากับ 0.507 ผู้วิจัยได้แนบบทความที่ตีพิมพ์แล้วมาด้วย

J. Math. Anal. Appl. 430 (2015) 205–230

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Ramanujan's alternative quartic theory of theta functions

Boonrod Yuttanan

Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University, Hatyai, Songkhla 90112, Thailand

ARTICLE INFO

Article history:
Received 13 January 2015
Available online 30 April 2015
Submitted by B.C. Berndt

Keywords:
Jacobian theta functions
Cubic theta functions
Quartic theta functions

ABSTRACT

A new general theta function in Ramanujan's quartic theory is introduced. Some properties analogous to those of classical Jacobian theta functions and cubic theta functions are established here.

1. Introduction

In the classical notation, Jacobi's theta functions are given by

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

and

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

Jacobi's well-known identity is

$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q). \tag{1}$$

E-mail address: boonrod.y@psu.ac.th.

http://dx.doi.org/10.1016/j.jmaa.2015.04.091
0022-247X/

In 1989, J.M. Borwein and P.B. Borwein introduced three elegant functions analogous to Jacobi's theta functions, namely, for $\omega = e^{2\pi i/3}$,

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2},$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}\omega^{n-m},$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2+n+m+1/3},$$

and in 1991 [5], they showed that

$$a(q)^3 = b(q)^3 + c(q)^3. \tag{2}$$

In 1993, M. Hirschhorn, F. Garvan and J.M. Borwein [6] introduced, for $\omega = e^{2\pi i/3}$,

$$a(q,z) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}z^{n-m},$$

$$a'(q,z) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}z^{n},$$

$$b(q,z) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}\omega^{n-m}z^{m},$$

and

$$c(q,z) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}q^{n+m}z^{n-m},$$

which are generalizations of $a(q)$, $b(q)$ and $c(q)$. They also gave proofs of several identities by employing Jacobi's triple product identity. In 1995, S. Bhargava [4] introduced

$$a(q,\zeta,x) = \sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}\zeta^{n+m}z^{n-m},$$

a generalization of the Hirschhorn–Garvan–Borwein cubic analogues and established some properties of $a(q,\zeta,x)$. Recently, another generalization of cubic analogues in the form

$$\sum_{m,n=-\infty}^{\infty} q^{n^2+nm+m^2}z_1^m z_2^n$$

was discovered by Daniel Schultz [10] and was used to extend the previous work of cubic theta functions.

In this paper, we study a function

$$\vartheta(q_2,q_4,x_1,x_2,x_3) := \sum_{a,b,c=-\infty}^{\infty} q_2^{(a+c)^2/2}q_4^{(a+b)^2/2+(b+c)^2/2}x_1^a x_2^b x_3^c, \tag{3}$$

where, for $0 < \alpha_2, \alpha_4 < 1$,

$$q_2 := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_2\right)}\right)$$

and

$$q_4 := \exp\left(-\pi\sqrt{2} \frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\alpha_4\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \alpha_4\right)}\right),$$

which we call the general theta function in Ramanujan's quartic theory. This function was first studied by Daniel Schultz for the case $\alpha_2 = \alpha_4$ in his unpublished work. However, not too much work has been done for this function. We now will establish some of its new identities analogous to those of classical Jacobian theta functions and cubic theta functions. In Section 3, we show that $\vartheta(q_2, q_4, x_1, x_2, x_3)$ can be written as a sum of products of Ramanujan's general theta function and we also present its functional equations. In Section 4, we derive several identities and some of them are analogous to (1) and (2). Moreover, in the same section, explicit evaluations of this function are given.

2. Preliminaries

In his notebooks and his lost notebook [1–3,7–9], Srinivasa Ramanujan developed numerous mathematical results involving theta functions. We will use those results to study our quartic theta function. We now provide some definitions and preliminary results. As customary and throughout this paper, we assume that $|q| < 1$. For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{4}$$

The three most important special cases of $f(a, b)$ [2, p. 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Ramanujan recorded several identities for $f(a, b)$, $\varphi(q)$, and $\psi(q)$. The following lemma provides such identities.

Lemma 2.1. *(See [2, p. 34].) We have*

$$f(a,b) = f(b,a), \tag{5}$$

$$f(1,a) = 2f(a,a^3), \tag{6}$$

$$f(-1,a) = 0, \tag{7}$$

and if n is an integer, then

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{8}$$

Lemma 2.2. *(See [2, pp. 39–40].) We have*

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{9}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \tag{10}$$

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \tag{11}$$

The complete elliptic integral of the first kind $K(k)$ is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right),$$

where $0 < k < 1$ is called the modulus of K and where ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$ denotes an ordinary hypergeometric function with $|x| < 1$. The complementary modulus k' is defined by $k' := \sqrt{1-k^2}$. Let K and K' denote complete elliptic integrals of the first kind associated with the moduli k and k', respectively. If $q = \exp(-\pi K'/K)$, then one of the fundamental properties of elliptic functions affirms that [2, p. 101]

$$\varphi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right). \tag{12}$$

Ramanujan also recorded several formulas for φ, ψ, and f at different arguments in terms of $\alpha := k^2$, q, and $z := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ by using (12). The following lemmas provide such formulas. First, we give evaluations for φ.

Lemma 2.3. *(See [2, p. 122].) If α, q, and z are defined as above, then*

$$\varphi(q) = \sqrt{z}, \tag{13}$$

$$\varphi(q^2) = \sqrt{z}\left(\frac{1+\sqrt{1-\alpha}}{2}\right)^{1/2}, \tag{14}$$

$$\varphi(q^4) = \frac{1}{2}\sqrt{z}\left(1+(1-\alpha)^{1/4}\right), \tag{15}$$

$$\varphi(q^{1/2}) = \sqrt{z}\left(1+\sqrt{\alpha}\right)^{1/2}, \tag{16}$$

$$\varphi(-q) = \sqrt{z}(1-\alpha)^{1/4}, \tag{17}$$

$$\varphi(-q^2) = \sqrt{z}(1-\alpha)^{1/8}, \tag{18}$$

$$\varphi(-q^{1/2}) = \sqrt{z}\left(1-\sqrt{\alpha}\right)^{1/2}. \tag{19}$$

Next, the following are evaluations for ψ.

Lemma 2.4. *(See [2, p. 123].) In the notation above, we have*

$$\psi(q) = \sqrt{\frac{z}{2}}\left(\alpha q^{-1}\right)^{1/8}, \tag{20}$$

$$\psi(q^2) = \frac{1}{2}\sqrt{z}\left(\alpha q^{-1}\right)^{1/4}, \tag{21}$$

$$\psi(q^{1/2}) = \sqrt{z}\left(\frac{1+\sqrt{\alpha}}{2}\right)^{1/4}\left(\alpha q^{-1}\right)^{1/16}, \tag{22}$$

$$\psi(-q^{1/2}) = \sqrt{z}\left(\frac{1-\sqrt{\alpha}}{2}\right)^{1/4}\left(\alpha q^{-1}\right)^{1/16}. \tag{23}$$

3. Basic properties

Recall that a quartic theta function is given by (3) and Ramanujan's general theta function $f(a,b)$ is defined in (4).

Theorem 3.1. *We have*

(i)

$$\vartheta(q_2,q_4,x_1,x_2,x_3) = \vartheta(q_2,q_4,x_3,x_2,x_1),$$

(ii)

$$\vartheta(q_2,q_4,x_1,x_2,x_3) = \vartheta(q_2,q_4,x_1^{-1},x_2^{-1},x_3^{-1}),$$

(iii)

$$\begin{aligned}
&2\vartheta(q_2,q_4,x_1,x_2,x_3)\\
&= f\left(\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) f\left(\left(\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, \left(\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right)\\
&\quad \times f\left(\left(\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, \left(\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right)\\
&\quad + f\left(-\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, -\left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) f\left(-\left(\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, -\left(\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right)\\
&\quad \times f\left(-\left(\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, -\left(\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right),
\end{aligned}$$

(iv) *for any integers p, q and r such that they are all even,*

$$\begin{aligned}
&\vartheta(q_2,q_4,x_1,x_2,x_3)\\
&= x_1^{p/2}x_2^{q/2}x_3^{r/2}q_2^{(p+r)^2/8}q_4^{(p+q)^2/8+(q+r)^2/8}\\
&\quad \times \vartheta(q_2,q_4,q_2^{(p+r)/2}q_4^{(p+q)/2}x_1, q_4^{(p+2q+r)/2}x_2, q_2^{(p+r)/2}q_4^{(q+r)/2}x_3),
\end{aligned}$$

(v) *if m, n, k, p, q and r are even integers such that $\frac{1}{2}(mp+nq+kr) \not\equiv 0 \pmod 4$, then*

$$\vartheta\left(q_2,q_4,i^m q_2^{(p+r)/4}q_4^{(p+q)/4}, i^n q_4^{(p+2q+r)/4}, i^k q_2^{(p+r)/4}q_4^{(q+r)/4}\right) = 0,$$

and

(vi) *if* $\operatorname{Re} s, \operatorname{Re} t > 0$, *then*

$$\begin{aligned}&\vartheta(e^{-\pi s}, e^{-\pi t}, e^{ia}, e^{ib}, e^{ic}) \\ &= \frac{\sqrt{2}}{t\sqrt{s}} e^{-(a-b+c)^2/8\pi s - ((a-c)^2+b^2)/4\pi t} \\ &\quad \times \vartheta\left(e^{-\pi/s}, e^{-2\pi/t}, e^{(a-b+c)/2s+b/t}, e^{(a+b-c)/t}, e^{(a-b+c)/2s+(a-c)/t}\right).\end{aligned}$$

Proof of (i) and (ii). These are obvious from the definition of $\vartheta(q_2, q_4, x_1, x_2, x_3)$. $\square$

Proof of (iii). We observe that

$$\begin{aligned}2\vartheta(q_2, q_4, x_1, x_2, x_3) = &\sum_{\substack{a,b,c=-\infty \\ \text{same parity}}}^{\infty} q_2^{(a+c)^2/8} q_4^{(a+b)^2/8+(b+c)^2/8} x_1^{a/2} x_2^{b/2} x_3^{c/2} \\ &+ \sum_{\substack{a,b,c=-\infty \\ \text{same parity}}}^{\infty} (-1)^{a+b+c} q_2^{(a+c)^2/8} q_4^{(a+b)^2/8+(b+c)^2/8} x_1^{a/2} x_2^{b/2} x_3^{c/2}. \qquad (24)\end{aligned}$$

For the first sum on the right hand side of (24), change variables by letting $a + b = 2p$, $b + c = 2q$ and $a + c = 2r$. Then

$$\begin{aligned}&\sum_{\substack{a,b,c=-\infty \\ \text{same parity}}}^{\infty} q_2^{(a+c)^2/8} q_4^{(a+b)^2/8+(b+c)^2/8} x_1^{a/2} x_2^{b/2} x_3^{c/2} \\ &= \sum_{p,q,r=-\infty}^{\infty} q_2^{r^2/2} q_4^{p^2/2+q^2/2} x_1^{(p-q+r)/2} x_2^{(p+q-r)/2} x_3^{(-p+q+r)/2} \\ &= \left(\sum_{r=-\infty}^{\infty} q_2^{r^2/2} x_1^{r/2} x_2^{-r/2} x_3^{r/2}\right)\left(\sum_{p=-\infty}^{\infty} q_4^{p^2/2} x_1^{p/2} x_2^{p/2} x_3^{-p/2}\right)\left(\sum_{q=-\infty}^{\infty} q_4^{q^2/2} x_1^{-q/2} x_2^{q/2} x_3^{q/2}\right) \\ &= f\left(\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) f\left(\left(\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, \left(\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right) \\ &\quad \times f\left(\left(\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, \left(\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right),\end{aligned}$$

where the last equality follows from the definition of $f(a, b)$. Similarly, the second sum on the right hand side of (24) becomes

$$\begin{aligned}&\sum_{\substack{a,b,c=-\infty \\ \text{same parity}}}^{\infty} (-1)^{a+b+c} q_2^{(a+c)^2/8} q_4^{(a+b)^2/8+(b+c)^2/8} x_1^{a/2} x_2^{b/2} x_3^{c/2} \\ &= f\left(-\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, -\left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) f\left(-\left(\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, -\left(\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right) \\ &\quad \times f\left(-\left(\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, -\left(\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right).\end{aligned}$$

Hence this finishes the proof of (iii). $\square$

Proof of (iv). Let p, q and r be integers with the same parity. We will utilize (8) to each theta function in (iii). For the first theta function of (iii), we see that by (8) with $n = (p + r)/2$,

$$
\begin{aligned}
& f\left(\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) \\
&= \left(\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}\right)^{(p+r)(p+r+2)/8} \left(\left(\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right)^{(p+r)(p+r-2)/8} \\
&\quad \times f\left(\left(\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{(p+r+1)/2}, \left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{(-p-r+1)/2}\right) \\
&= \left(\frac{x_1x_3}{x_2}\right)^{(p+r)/4} q_2^{(p+r)^2/8} f\left(\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right).
\end{aligned}
$$

Similarly, we use (8) with $n = (p + q)/2$, $(q + r)/2$, $(p + r)/2$, $(p + q)/2$, and $(q + r)/2$ to the other five theta functions of (iii), respectively. Then we obtain

$$
\begin{aligned}
& 2\vartheta(q_2, q_4, x_1, x_2, x_3) \\
&= \left(\frac{x_1x_3}{x_2}\right)^{(p+r)/4} q_2^{(p+r)^2/8} f\left(\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) \\
&\quad \times \left(\frac{x_1x_2}{x_3}\right)^{(p+q)/4} q_4^{(p+q)^2/8} f\left(\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, \left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right) \\
&\quad \times \left(\frac{x_2x_3}{x_1}\right)^{(q+r)/4} q_4^{(q+r)^2/8} f\left(\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, \left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right) \\
&\quad + (-1)^{(p+r)^2/4} \left(\frac{x_1x_3}{x_2}\right)^{(p+r)/4} q_2^{(p+r)^2/8} f\left(-\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, -\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) \\
&\quad \times (-1)^{(p+q)^2/4} \left(\frac{x_1x_2}{x_3}\right)^{(p+q)/4} q_4^{(p+q)^2/8} f\left(-\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, -\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right) \\
&\quad \times (-1)^{(q+r)^2/4} \left(\frac{x_2x_3}{x_1}\right)^{(q+r)/4} q_4^{(q+r)^2/8} f\left(-\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, -\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right) \\
&= x_1^{p/2} x_2^{q/2} x_3^{r/2} q_2^{(p+r)^2/8} q_4^{(p+q)^2/8+(q+r)^2/8} \\
&\quad \times \Bigg\{ f\left(\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, \left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right) \\
&\quad \times f\left(\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{1/2} q_4^{1/2}, \left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{-1/2} q_4^{1/2}\right) \\
&\quad \times f\left(\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{1/2} q_4^{1/2}, \left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{-1/2} q_4^{1/2}\right) \\
&\quad + f\left(-\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{1/2} q_2^{1/2}, -\left(q_2^{p+r}\frac{x_1x_3}{x_2}\right)^{-1/2} q_2^{1/2}\right)
\end{aligned}
$$

$$\times f\left(-\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{1/2}q_4^{1/2}, -\left(q_4^{p+q}\frac{x_1x_2}{x_3}\right)^{-1/2}q_4^{1/2}\right)$$
$$\times f\left(-\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{1/2}q_4^{1/2}, -\left(q_4^{q+r}\frac{x_2x_3}{x_1}\right)^{-1/2}q_4^{1/2}\right)\Bigg\}.$$

After employing part (iii) with x_1, x_2 and x_3 replaced by $q_2^{(p+r)/2}q_4^{(p+q)/2}x_1$, $q_4^{(p+2q+r)/2}x_2$ and $q_2^{(p+r)/2}q_4^{(q+r)/2}x_3$, respectively, we achieve the proposed formula. □

Proof of (v). By (ii) and (iv), respectively, this yields

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i^m q_2^{(p+r)/4}q_4^{(p+q)/4}, i^n q_4^{(p+2q+r)/4}, i^k q_2^{(p+r)/4}q_4^{(q+r)/4}\right)\\
&= \vartheta\left(q_2, q_4, i^{-m} q_2^{-(p+r)/4}q_4^{-(p+q)/4}, i^{-n} q_4^{-(p+2q+r)/4}, i^{-k} q_2^{-(p+r)/4}q_4^{-(q+r)/4}\right)\\
&= i^{-(mp+nq+kr)/2}q_2^{-p(p+r)/8}q_4^{-p(p+q)/8}q_4^{-q(p+2q+r)/8}q_2^{-r(p+r)/8}q_4^{-r(q+r)/8}q_2^{(p+r)^2/8}\\
&\quad\times q_4^{(p+q)^2/8+(q+r)^2/8}\vartheta\left(q_2, q_4, i^m q_2^{(p+r)/4}q_4^{(p+q)/4}, i^n q_4^{(p+2q+r)/4}, i^k q_2^{(p+r)/4}q_4^{(q+r)/4}\right)\\
&= i^{-(mp+nq+kr)/2}\vartheta\left(q_2, q_4, i^m q_2^{(p+r)/4}q_4^{(p+q)/4}, i^n q_4^{(p+2q+r)/4}, i^k q_2^{(p+r)/4}q_4^{(q+r)/4}\right).
\end{aligned}$$

Since $\frac{1}{2}(mp+nq+kr) \not\equiv 0 \pmod 4$, the desired result follows immediately. □

Proof of (vi). From (iii) with $q_2 = e^{-\pi s}$, $q_4 = e^{-\pi t}$, $x_1 = e^{ia}$, $x_2 = e^{ib}$, and $x_3 = e^{ic}$, we find that

$$\begin{aligned}
&2\vartheta(e^{-\pi s}, e^{-\pi t}, e^{ia}, e^{ib}, e^{ic})\\
&= f\left(e^{i(a-b+c)/2-\pi s/2}, e^{-i(a-b+c)/2-\pi s/2}\right) f\left(e^{i(a+b-c)/2-\pi t/2}, e^{-i(a+b-c)/2-\pi t/2}\right)\\
&\quad\times f\left(e^{i(-a+b+c)/2-\pi t/2}, e^{-i(-a+b+c)/2-\pi t/2}\right)\\
&\quad+ f\left(-e^{i(a-b+c)/2-\pi s/2}, -e^{-i(a-b+c)/2-\pi s/2}\right) f\left(-e^{i(a+b-c)/2-\pi t/2}, -e^{-i(a+b-c)/2-\pi t/2}\right)\\
&\quad\times f\left(-e^{i(-a+b+c)/2-\pi t/2}, -e^{-i(-a+b+c)/2-\pi t/2}\right).
\end{aligned}\tag{25}$$

Before proceeding further, we will establish the following identity. By the definition of $f(a, b)$,

$$\begin{aligned}
&f(a,b)f(c,d)f(e,g) + f(-a,-b)f(-c,-d)f(-e,-g)\\
&= \sum_{m,n,k=-\infty}^{\infty} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}e^{k(k+1)/2}g^{k(k-1)/2}\\
&\quad+ \sum_{m,n,k=-\infty}^{\infty} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}e^{k(k+1)/2}g^{k(k-1)/2}(-1)^{m^2+n^2+k^2}\\
&= 2\sum_{m\text{ even}}\sum_{n\text{ even}}\sum_{k\text{ even}} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}e^{k(k+1)/2}g^{k(k-1)/2}\\
&\quad+ 2\sum_{m\text{ odd}}\sum_{n\text{ odd}}\sum_{k\text{ even}} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}e^{k(k+1)/2}g^{k(k-1)/2}\\
&\quad+ 2\sum_{m\text{ odd}}\sum_{n\text{ even}}\sum_{k\text{ odd}} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}e^{k(k+1)/2}g^{k(k-1)/2}
\end{aligned}$$

$$+ 2 \sum_{m \text{ even}} \sum_{n \text{ odd}} \sum_{k \text{ odd}} a^{m(m+1)/2} b^{m(m-1)/2} c^{n(n+1)/2} d^{n(n-1)/2} e^{k(k+1)/2} g^{k(k-1)/2}$$

$$= 2f(a^3b, ab^3)f(c^3d, cd^3)f(e^3g, eg^3) + 2acf(a^5b^3, \frac{b}{a})f(c^5d^3, \frac{d}{c})f(e^3g, eg^3)$$

$$+ 2aef(a^5b^3, \frac{b}{a})f(c^3d, cd^3)f(e^5g^3, \frac{g}{e}) + 2cef(a^3b, ab^3)f(c^5d^3, \frac{d}{c})f(e^5g^3, \frac{g}{e}). \tag{26}$$

Utilize (26) to (25) and obtain

$$\begin{aligned}
&\vartheta(e^{-\pi s}, e^{-\pi t}, e^{ia}, e^{ib}, e^{ic}) \\
&\quad = f\left(e^{i(a-b+c)-2\pi s}, e^{-i(a-b+c)-2\pi s}\right) \\
&\qquad \times \left\{ f\left(e^{i(a+b-c)-2\pi t}, e^{-i(a+b-c)-2\pi t}\right) f\left(e^{i(-a+b+c)-2\pi t}, e^{-i(-a+b+c)-2\pi t}\right) \right. \\
&\qquad \left. + e^{ib-\pi t} f\left(e^{i(a+b-c)-4\pi t}, e^{-i(a+b-c)}\right) f\left(e^{i(-a+b+c)-4\pi t}, e^{-i(-a+b+c)}\right) \right\} \\
&\qquad + f\left(e^{i(a-b+c)-4\pi s}, e^{-i(a-b+c)}\right) \\
&\qquad \times \left\{ e^{ic-(\pi s+\pi t)/2} f\left(e^{i(a+b-c)-2\pi t}, e^{-i(a+b-c)-2\pi t}\right) f\left(e^{i(-a+b+c)-4\pi t}, e^{-i(-a+b+c)}\right) \right. \\
&\qquad \left. + e^{ia-(\pi s+\pi t)/2} f\left(e^{i(a+b-c)-4\pi t}, e^{-i(a+b-c)}\right) f\left(e^{i(-a+b+c)-2\pi t}, e^{-i(-a+b+c)-2\pi t}\right) \right\}.
\end{aligned} \tag{27}$$

Recall that from Entry 20 in [2, p. 36], if $\alpha\beta = \pi$, $\mathrm{Re}(\alpha^2) > 0$, and n is any complex number, then

$$\sqrt{\alpha} f\left(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}\right) = e^{n^2/4}\sqrt{\beta} f\left(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}\right). \tag{28}$$

Take $\alpha = \sqrt{2\pi t}$, $\beta = \sqrt{\pi/2t}$ and $n = i\theta/\sqrt{2\pi t}$ in (28) to obtain

$$f(e^{i\theta-2\pi t}, e^{-i\theta-2\pi t}) = \frac{1}{\sqrt{2t}} e^{-\theta^2/8\pi t} f\left(e^{-\theta/2t-\pi/2t}, e^{\theta/2t-\pi/2t}\right) \tag{29}$$

and take $\alpha = \sqrt{2\pi t}$, $\beta = \sqrt{\pi/2t}$ and $n = -\sqrt{2\pi t} + i\theta/\sqrt{2\pi t}$ in (28) to obtain

$$f(e^{i\theta-4\pi t}, e^{-i\theta}) = \frac{1}{\sqrt{2t}} e^{\pi t/2-i\theta/2-\theta^2/8\pi t} f\left(-e^{-\theta/2t-\pi/2t}, -e^{\theta/2t-\pi/2t}\right). \tag{30}$$

Applying (29) and (30) to (27) yields

$$\begin{aligned}
&\vartheta(e^{-\pi s}, e^{-\pi t}, e^{ia}, e^{ib}, e^{ic}) \\
&\quad = \frac{1}{\sqrt{2s}} e^{-(a-b+c)^2/8\pi s} f\left(e^{-(a-b+c)/2s-\pi/2s}, e^{(a-b+c)/2s-\pi/2s}\right) \\
&\qquad \times \left\{ \frac{1}{\sqrt{2t}} e^{-(a+b-c)^2/8\pi t} f\left(e^{-(a+b-c)/2t-\pi/2t}, e^{(a+b-c)/2t-\pi/2t}\right) \right. \\
&\qquad \times \frac{1}{\sqrt{2t}} e^{-(-a+b+c)^2/8\pi t} f\left(e^{-(-a+b+c)/2t-\pi/2t}, e^{(-a+b+c)/2t-\pi/2t}\right) \\
&\qquad + e^{ib-\pi t} \frac{1}{\sqrt{2t}} e^{\pi t/2-i(a+b-c)/2-(a+b-c)^2/8\pi t} f\left(-e^{-(a+b-c)/2t-\pi/2t}, -e^{(a+b-c)/2t-\pi/2t}\right) \\
&\qquad \left. \times \frac{1}{\sqrt{2t}} e^{\pi t/2-i(-a+b+c)/2-(-a+b+c)^2/8\pi t} f\left(-e^{-(-a+b+c)/2t-\pi/2t}, -e^{(-a+b+c)/2t-\pi/2t}\right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\sqrt{2s}}e^{\pi s/2-i(a-b+c)/2-(a-b+c)^2/8\pi s}f\left(-e^{-(a-b+c)/2s-\pi/2s},-e^{(a-b+c)/2s-\pi/2s}\right)\\
&\times\left\{e^{ic-(\pi s+\pi t)/2}\frac{1}{\sqrt{2t}}e^{-(a+b-c)^2/8\pi t}f\left(e^{-(a+b-c)/2t-\pi/2t},e^{(a+b-c)/2t-\pi/2t}\right)\right\}\\
&\times\frac{1}{\sqrt{2t}}e^{\pi t/2-i(-a+b+c)/2-(-a+b+c)^2/8\pi t}f\left(-e^{-(-a+b+c)/2t-\pi/2t},-e^{(-a+b+c)/2t-\pi/2t}\right)\\
&+e^{ia-(\pi s+\pi t)/2}\frac{1}{\sqrt{2t}}e^{\pi t/2-i(a+b-c)/2-(a+b-c)^2/8\pi t}f\left(-e^{-(a+b-c)/2t-\pi/2t},-e^{(a+b-c)/2t-\pi/2t}\right)\\
&\times\frac{1}{\sqrt{2t}}e^{-(-a+b+c)^2/8\pi t}f\left(e^{-(-a+b+c)/2t-\pi/2t},e^{(-a+b+c)/2t-\pi/2t}\right)\Big\}\\
=&\frac{1}{2t\sqrt{2s}}e^{-(a-b+c)^2/8\pi s-((a-c)^2+b^2)/4\pi t}\Big\{f\left(e^{-(a-b+c)/2s-\pi/2s},e^{(a-b+c)/2s-\pi/2s}\right)\\
&\times\Big\{f\left(e^{-(a+b-c)/2t-\pi/2t},e^{(a+b-c)/2t-\pi/2t}\right)f\left(e^{-(-a+b+c)/2t-\pi/2t},e^{(-a+b+c)/2t-\pi/2t}\right)\\
&+f\left(-e^{-(a+b-c)/2t-\pi/2t},-e^{(a+b-c)/2t-\pi/2t}\right)f\left(-e^{-(-a+b+c)/2t-\pi/2t},-e^{(-a+b+c)/2t-\pi/2t}\right)\Big\}\\
&+f\left(-e^{-(a-b+c)/2s-\pi/2s},-e^{(a-b+c)/2s-\pi/2s}\right)\\
&\times\Big\{f\left(e^{-(a+b-c)/2t-\pi/2t},e^{(a+b-c)/2t-\pi/2t}\right)f\left(-e^{-(-a+b+c)/2t-\pi/2t},-e^{-(-a+b+c)/2t-\pi/2t}\right)\\
&+f\left(-e^{-(a+b-c)/2t-\pi/2t},-e^{(a+b-c)/2t-\pi/2t}\right)f\left(e^{-(-a+b+c)/2t-\pi/2t},e^{(-a+b+c)/2t-\pi/2t}\right)\Big\}\Big\}.
\end{aligned}
$$

Now by Entry 29 in [2, p. 45], if $ab=cd$, then

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc). \tag{31}$$

By (31), we deduce that

$$
\begin{aligned}
&\vartheta(e^{-\pi s},e^{-\pi t},e^{ia},e^{ib},e^{ic})\\
&=\frac{1}{t\sqrt{2s}}e^{-(a-b+c)^2/8\pi s-((a-c)^2+b^2)/4\pi t}\left\{f\left(e^{(a-b+c)/2s-\pi/2s},e^{-(a-b+c)/2s-\pi/2s}\right)\right.\\
&\quad\times f\left(e^{b/t-\pi/t},e^{-b/t-\pi/t}\right)f\left(e^{(a-c)/t-\pi/t},e^{-(a-c)/t-\pi/t}\right)\\
&\quad+f\left(-e^{(a-b+c)/2s-\pi/2s},-e^{-(a-b+c)/2s-\pi/2s}\right)\\
&\quad\left.\times f\left(-e^{b/t-\pi/t},-e^{-b/t-\pi/t}\right)f\left(-e^{(a-c)/t-\pi/t},-e^{-(a-c)/t-\pi/t}\right)\right\}\\
&=\frac{\sqrt{2}}{t\sqrt{s}}e^{-(a-b+c)^2/8\pi s-((a-c)^2+b^2)/4\pi t}\\
&\quad\times\vartheta\left(e^{-\pi/s},e^{-2\pi/t},e^{(a-b+c)/2s+b/t},e^{(a+b-c)/t},e^{(a-b+c)/2s+(a-c)/t}\right).
\end{aligned}
$$

The proof is complete. □

4. Evaluations of quartic theta functions

In this section, we shall evaluate some values of quartic theta functions by using hypergeometric functions. Now, we will give explicit values for the function ϑ at arguments $x_1=i^m(q_2q_4)^{n/2}$, $x_2=i^{2m}q_4^n$ and $x_3=i^{3m}(q_2q_4)^{n/2}$, where $m\in\{0,1,2,3\}$ and n is an integer.

Theorem 4.1. *For $m \in \{0,1,2,3\}$ and an integer n, we have*

$$\vartheta\left(q_2, q_4, i^m (q_2q_4)^{n/2}, i^{2m} q_4^n, i^{3m} (q_2q_4)^{n/2}\right)$$
$$= \begin{cases} q_2^{-(n^2-1)/8} q_4^{-(n^2-1)/4} \vartheta\left(q_2, q_4, (q_2q_4)^{1/2}, q_4, (q_2q_4)^{1/2}\right), & \text{if } m = 0 \text{ and } n \equiv 1 \pmod 2, \\ q_2^{-n^2/8} q_4^{-n^2/4} \vartheta(q_2, q_4, 1, 1, 1), & \text{if } m = 0 \text{ and } n \equiv 0 \pmod 4, \\ q_2^{-(n^2-4)/8} q_4^{-(n^2-4)/4} \vartheta\left(q_2, q_4, q_2q_4, q_4^2, q_2q_4\right), & \text{if } m = 0 \text{ and } n \equiv 2 \pmod 4, \\ i^{n/2} q_2^{-n^2/8} q_4^{-n^2/4} \vartheta\left(q_2, q_4, i, i^2, i^3\right), & \text{if } m = 1,3 \text{ and } n \equiv 0 \pmod 4, \\ q_2^{-n^2/8} q_4^{-n^2/4} \vartheta\left(q_2, q_4, i^2, i^4, i^6\right), & \text{if } m = 2 \text{ and } n \equiv 0 \pmod 4, \\ q_2^{-(n^2-4)/8} q_4^{-(n^2-4)/4} \vartheta\left(q_2, q_4, i^2 q_2q_4, i^4 q_4^2, i^6 q_2q_4\right), & \text{if } m = 2 \text{ and } n \equiv 2 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1(iii), we obtain

$$\begin{aligned} &\vartheta\left(q_2, q_4, i^m (q_2q_4)^{n/2}, i^{2m} q_4^n, i^{3m} (q_2q_4)^{n/2}\right) \\ &= \frac{1}{2}\left\{ f\left(i^m q_2^{(n+1)/2}, (-i)^m q_2^{(-n+1)/2}\right) f\left(q_4^{(n+1)/2}, q_4^{(-n+1)/2}\right) f\left(i^{2m} q_4^{(n+1)/2}, i^{2m} q_4^{(-n+1)/2}\right) \right. \\ &\quad + f\left(-i^m q_2^{(n+1)/2}, -(-i)^m q_2^{(-n+1)/2}\right) f\left(-q_4^{(n+1)/2}, -q_4^{(-n+1)/2}\right) \\ &\quad \left. \times f\left(-i^{2m} q_4^{(n+1)/2}, -i^{2m} q_4^{(-n+1)/2}\right) \right\}. \end{aligned} \tag{32}$$

We divide the proof into many cases as follows.

Case 1. $m = 0$.

Case 1.1. $n \equiv 1 \pmod 2$.

Eq. (32) becomes

$$\begin{aligned} &\vartheta\left(q_2, q_4, (q_2q_4)^{(2n+1)/2}, q_4^{2n+1}, (q_2q_4)^{(2n+1)/2}\right) \\ &= \frac{1}{2}\left\{ f\left(q_2^{n+1}, q_2^{-n}\right) f^2\left(q_4^{n+1}, q_4^{-n}\right) + f\left(-q_2^{n+1}, -q_2^{-n}\right) f^2\left(-q_4^{n+1}, -q_4^{-n}\right) \right\}. \end{aligned} \tag{33}$$

Note that by (8) and (7), respectively,

$$f(-q^{n+1}, -q^n) = \left(-q^{n+1}\right)^{n(n-1)/2} \left(-q^n\right)^{n(n+1)/2} f(-q, -1) = 0. \tag{34}$$

Also, by (8) again, we have

$$f(q^{n+1}, q^{-n}) = q^{-n} f(q^n, q^{-(n-1)}).$$

Hence we arrive at

$$\begin{aligned} &\vartheta\left(q_2, q_4, (q_2q_4)^{(2n+1)/2}, q_4^{2n+1}, (q_2q_4)^{(2n+1)/2}\right) \\ &= \frac{1}{2} q_2^{-n} q_4^{-2n} f\left(q_2^n, q_2^{-(n-1)}\right) f^2\left(q_4^n, q_4^{-(n-1)}\right). \end{aligned} \tag{35}$$

Then by (33) with n replaced by $n-1$ and (34), it follows that

$$\vartheta\left(q_2, q_4, (q_2q_4)^{(2n-1)/2}, q_4^{2n-1}, (q_2q_4)^{(2n-1)/2}\right) = \frac{1}{2} f\left(q_2^n, q_2^{-(n-1)}\right) f^2\left(q_4^n, q_4^{-(n-1)}\right). \tag{36}$$

Substituting (36) into (35), we find that

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{(2n+1)/2}, q_4^{2n+1}, (q_2q_4)^{(2n+1)/2}\right)\\&\quad = q_2^{-n}q_4^{-2n}\vartheta\left(q_2, q_4, (q_2q_4)^{(2n-1)/2}, q_4^{2n-1}, (q_2q_4)^{(2n-1)/2}\right).\end{aligned}$$

Iterate this n times to deduce the desired result. This case is proved.

Case 1.2. $n \equiv 0 \pmod 4$.

Eq. (32) becomes

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{2n}, q_4^{4n}, (q_2q_4)^{2n}\right)\\&\quad = \frac{1}{2}\left\{f\left(q_2^{2n+1/2}, q_2^{-2n+1/2}\right)f^2\left(q_4^{2n+1/2}, q_4^{-2n+1/2}\right)\right.\\&\qquad \left.+ f\left(-q_2^{2n+1/2}, -q_2^{-2n+1/2}\right)f^2\left(-q_4^{2n+1/2}, -q_4^{-2n+1/2}\right)\right\}.\end{aligned} \tag{37}$$

By (8),

$$f\left(\pm q^{2n+1/2}, \pm q^{-2n+1/2}\right) = q^{-4n+2}f\left(\pm q^{2n-3/2}, \pm q^{-2n+5/2}\right).$$

Then

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{2n}, q_4^{4n}, (q_2q_4)^{2n}\right)\\&\quad = \frac{1}{2}q_2^{-4n+2}q_4^{-8n+4}\left\{f\left(q_2^{2n-3/2}, q_2^{-2n+5/2}\right)f^2\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right)\right.\\&\qquad \left.+ f\left(-q_2^{2n-3/2}, -q_2^{-2n+5/2}\right)f^2\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right)\right\}.\end{aligned} \tag{38}$$

By (37) with n replaced by $n-1$, it follows that

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{2n-2}, q_4^{4n-4}, (q_2q_4)^{2n-2}\right)\\&\quad = \frac{1}{2}\left\{f\left(q_2^{2n-3/2}, q_2^{-2n+5/2}\right)f^2\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right)\right.\\&\qquad \left.+ f\left(-q_2^{2n-3/2}, -q_2^{-2n+5/2}\right)f^2\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right)\right\}.\end{aligned} \tag{39}$$

Substituting (39) into (38), we see that

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{2n}, q_4^{4n}, (q_2q_4)^{2n}\right)\\&\quad = q_2^{-4n+2}q_4^{-8n+4}\vartheta\left(q_2, q_4, (q_2q_4)^{2n-2}, q_4^{4n-4}, (q_2q_4)^{2n-2}\right).\end{aligned}$$

Iterate this n times to obtain the desired result. We finish the proof.

Case 1.3. $n \equiv 2 \pmod 4$.

From (32), we have

$$\begin{aligned}&\vartheta\left(q_2, q_4, (q_2q_4)^{2n+1}, q_4^{4n+2}, (q_2q_4)^{2n+1}\right)\\&\quad = \frac{1}{2}\left\{f\left(q_2^{2n+3/2}, q_2^{-2n-1/2}\right)f^2\left(q_4^{2n+3/2}, q_4^{-2n-1/2}\right)\right.\\&\qquad \left.+ f\left(-q_2^{2n+3/2}, -q_2^{-2n-1/2}\right)f^2\left(-q_4^{2n+3/2}, -q_4^{-2n-1/2}\right)\right\}.\end{aligned} \tag{40}$$

By (8),

$$f\left(\pm q^{2n+3/2}, \pm q^{-2n-1/2}\right) = q^{-4n} f\left(\pm q^{2n-1/2}, \pm q^{-2n+3/2}\right).$$

Then

$$\begin{aligned}
&\vartheta\left(q_2, q_4, (q_2q_4)^{2n+1}, q_4^{4n+2}, (q_2q_4)^{2n+1}\right) \\
&= \frac{1}{2} q_2^{-4n} q_4^{-8n} \left\{ f\left(q_2^{2n-1/2}, q_2^{-2n+3/2}\right) f^2\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right) \right. \\
&\quad \left. + f\left(-q_2^{2n-1/2}, -q_2^{-2n+3/2}\right) f^2\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right) \right\}. \qquad (41)
\end{aligned}$$

By (40) with n replaced by $n-1$, we deduce that

$$\begin{aligned}
&\vartheta\left(q_2, q_4, (q_2q_4)^{2n-1}, q_4^{4n-2}, (q_2q_4)^{2n-1}\right) \\
&= \frac{1}{2} \left\{ f\left(q_2^{2n-1/2}, q_2^{-2n+3/2}\right) f^2\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right) \right. \\
&\quad \left. + f\left(-q_2^{2n-1/2}, -q_2^{-2n+3/2}\right) f^2\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right) \right\}. \qquad (42)
\end{aligned}$$

Substituting (42) into (41), we conclude that

$$\vartheta\left(q_2, q_4, (q_2q_4)^{2n+1}, q_4^{4n+2}, (q_2q_4)^{2n+1}\right) = q_2^{-4n} q_4^{-8n} \vartheta\left(q_2, q_4, (q_2q_4)^{2n-1}, q_4^{4n-2}, (q_2q_4)^{2n-1}\right).$$

Iterate this identity n times to complete the proof.

Case 2. $m = 1$.

Case 2.1. $n \equiv 1 \pmod 2$.

Eq. (32) becomes

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i(q_2q_4)^{(2n+1)/2}, i^2 q_4^{2n+1}, i^3 (q_2q_4)^{(2n+1)/2}\right) \\
&= \frac{1}{2} \left\{ f\left(iq_2^{n+1}, -iq_2^{-n}\right) f\left(q_4^{n+1}, q_4^{-n}\right) f\left(-q_4^{n+1}, -q_4^{-n}\right) \right. \\
&\quad \left. + f\left(-iq_2^{n+1}, -q_2^{-n}\right) f\left(-q_4^{n+1}, -q_4^{-n}\right) f\left(q_4^{n+1}, q_4^{-n}\right) \right\}.
\end{aligned}$$

By (34), it follows immediately that

$$\vartheta\left(q_2, q_4, i(q_2q_4)^{(2n+1)/2}, i^2 q_4^{2n+1}, i^3 (q_2q_4)^{(2n+1)/2}\right) = 0.$$

The proof is complete.

Case 2.2. $n \equiv 0 \pmod 4$.

We have

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i(q_2q_4)^{2n}, i^2 q_4^{4n}, i^3 (q_2q_4)^{2n}\right) \\
&= \frac{1}{2} \left\{ f\left(iq_2^{2n+1/2}, -iq_2^{-2n+1/2}\right) f\left(q_4^{2n+1/2}, q_4^{-2n+1/2}\right) f\left(-q_4^{2n+1/2}, -q_4^{-2n+1/2}\right) \right. \\
&\quad \left. + f\left(-iq_2^{2n+1/2}, iq_2^{-2n+1/2}\right) f\left(-q_4^{2n+1/2}, -q_4^{-2n+1/2}\right) f\left(q_4^{2n+1/2}, q_4^{-2n+1/2}\right) \right\}. \qquad (43)
\end{aligned}$$

Note that if $ab = 1$, then by (8),

$$f\left(aq^{2n+1/2}, bq^{-2n+1/2}\right) = b^2 q^{-4n+2} f\left(aq^{2n-3/2}, bq^{-2n+5/2}\right).$$

Then

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n}, i^2 q_4^{4n}, i^3\left(q_2q_4\right)^{2n}\right) \\
&= \frac{1}{2} i^2 q_2^{-4n+2} q_4^{-8n+4} \\
&\times \left\{ f\left(iq_2^{2n-3/2}, -iq_2^{-2n+5/2}\right) f\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right) f\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right) \right. \\
&\left. + f\left(-iq_2^{2n-3/2}, iq_2^{-2n+5/2}\right) f\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right) f\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right) \right\}. \qquad (44)
\end{aligned}$$

By (43) with n replaced by $n-1$, we find that

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n-2}, i^2 q_4^{4n-4}, i^3\left(q_2q_4\right)^{2n-2}\right) \\
&= \frac{1}{2} \left\{ f\left(iq_2^{2n-3/2}, -iq_2^{-2n+5/2}\right) f\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right) f\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right) \right. \\
&\left. + f\left(-iq_2^{2n-3/2}, iq_2^{-2n+5/2}\right) f\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right) f\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right) \right\}. \qquad (45)
\end{aligned}$$

Substitute (45) into (44) to obtain

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n}, i^2 q_4^{4n}, i^3\left(q_2q_4\right)^{2n}\right) \\
&= i^2 q_2^{-4n+2} q_4^{-8n+4} \vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n-2}, i^2 q_4^{4n-4}, i^3\left(q_2q_4\right)^{2n-2}\right).
\end{aligned}$$

After iterating this identity n times, we complete the proof.

Case 2.3. $n \equiv 2 \pmod 4$.

We see that

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n+1}, i^2 q_4^{4n+2}, i^3\left(q_2q_4\right)^{2n+1}\right) \\
&= \frac{1}{2} \left\{ f\left(iq_2^{2n+3/2}, -iq_2^{-2n-1/2}\right) f\left(q_4^{2n+3/2}, q_4^{-2n-1/2}\right) f\left(-q_4^{2n+3/2}, -q_4^{-2n-1/2}\right) \right. \\
&\left. + f\left(-iq_2^{2n+3/2}, iq_2^{-2n-1/2}\right) f\left(-q_4^{2n+3/2}, -q_4^{-2n-1/2}\right) f\left(q_4^{2n+3/2}, q_4^{-2n-1/2}\right) \right\}. \qquad (46)
\end{aligned}$$

By (8), if $ab = 1$, then

$$f\left(aq^{2n+3/2}, bq^{-2n-1/2}\right) = b^2 q^{-4n} f\left(aq^{2n-1/2}, bq^{-2n+3/2}\right).$$

Then

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n+1}, i^2 q_4^{4n+2}, i^3\left(q_2q_4\right)^{2n+1}\right) \\
&= \frac{1}{2} q_2^{-4n} q_4^{-8n} \left\{ f\left(iq_2^{2n-1/2}, -iq_2^{-2n+3/2}\right) f\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right) f\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right) \right. \\
&\left. + f\left(-iq_2^{2n-1/2}, iq_2^{-2n+3/2}\right) f\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right) f\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right) \right\}. \qquad (47)
\end{aligned}$$

By (46) with n replaced by $n-1$, we find that

$$\begin{aligned}&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n-1}, i^2q_4^{4n-2}, i^3\left(q_2q_4\right)^{2n-1}\right)\\&\quad=\frac{1}{2}\left\{f\left(iq_2^{2n-1/2}, -iq_2^{-2n+3/2}\right)f\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right)f\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right)\right.\\&\qquad\left.+f\left(-iq_2^{2n-1/2}, iq_2^{-2n+3/2}\right)f\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right)f\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right)\right\}.\end{aligned}\tag{48}$$

Substitute (48) into (47) to obtain

$$\begin{aligned}&\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n+1}, i^2q_4^{4n+2}, i^3\left(q_2q_4\right)^{2n+1}\right)\\&\quad=q_2^{-4n}q_4^{-8n}\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n-1}, i^2q_4^{4n-2}, i^3\left(q_2q_4\right)^{2n-1}\right).\end{aligned}$$

Iterate this identity n times to obtain

$$\vartheta\left(q_2, q_4, i\left(q_2q_4\right)^{2n+1}, i^2q_4^{4n+2}, i^3\left(q_2q_4\right)^{2n+1}\right)=q_2^{-2n^2}q_4^{-4n^2}\vartheta\left(q_2, q_4, iq_2q_4, i^2q_4^2, i^3q_2q_4\right).$$

By Theorem 3.1(ii), (iv) and (i), respectively, we find that

$$\begin{aligned}\vartheta\left(q_2, q_4, iq_2q_4, i^2q_4^2, i^3q_2q_4\right)&=\vartheta\left(q_2, q_4, (iq_2q_4)^{-1}, (i^2q_4^2)^{-1}, (i^3q_2q_4)^{-1}\right)\\&=i^{-6}\vartheta\left(q_2, q_4, i^3q_2q_4, i^2q_4^2, iq_2q_4\right)\\&=-\vartheta\left(q_2, q_4, iq_2q_4, i^2q_4^2, i^3q_2q_4\right).\end{aligned}$$

Then $\vartheta\left(q_2, q_4, iq_2q_4, i^2q_4^2, i^3q_2q_4\right)=0$ which completes the proof.

Case 3. $m=2$.

Case 3.1. $n\equiv 1 \pmod 2$.

Eq. (32) becomes

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2\left(q_2q_4\right)^{(2n+1)/2}, i^4q_4^{2n+1}, i^6\left(q_2q_4\right)^{(2n+1)/2}\right)\\&\quad=\frac{1}{2}\left\{f\left(-q_2^{n+1}, -q_2^{-n}\right)f^2\left(q_4^{n+1}, q_4^{-n}\right)+f\left(q_2^{n+1}, q_2^{-n}\right)f^2\left(-q_4^{n+1}, -q_4^{-n}\right)\right\}.\end{aligned}$$

By (34), it follows immediately that

$$\vartheta\left(q_2, q_4, i^2\left(q_2q_4\right)^{(2n+1)/2}, i^4q_4^{2n+1}, i^6\left(q_2q_4\right)^{(2n+1)/2}\right)=0.$$

The proof is complete.

Case 3.2. $n\equiv 0 \pmod 4$.

Eq. (32) becomes

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2\left(q_2q_4\right)^{2n}, i^4q_4^{4n}, i^6\left(q_2q_4\right)^{2n}\right)\\&\quad=\frac{1}{2}\left\{f\left(-q_2^{2n+1/2}, -q_2^{-2n+1/2}\right)f^2\left(q_4^{2n+1/2}, q_4^{-2n+1/2}\right)\right.\\&\qquad\left.+f\left(q_2^{2n+1/2}, q_2^{-2n+1/2}\right)f^2\left(-q_4^{2n+1/2}, -q_4^{-2n+1/2}\right)\right\}.\end{aligned}\tag{49}$$

Note that by (8),

$$f\left(\pm q^{2n+1/2}, \pm q^{-2n+1/2}\right) = q^{-4n+2} f\left(\pm q^{2n-3/2}, \pm q^{-2n+5/2}\right).$$

Then

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n}, i^4 q_4^{4n}, i^6 (q_2q_4)^{2n}\right)\\ &= \frac{1}{2} q_2^{-4n+2} q_4^{-8n+4}\left\{f\left(-q_2^{2n-3/2}, -q_2^{-2n+5/2}\right) f^2\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right)\right.\\ &\quad\left.+ f\left(q_2^{2n-3/2}, q_2^{-2n+5/2}\right) f^2\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right)\right\}. \end{aligned} \tag{50}$$

By (49) with n replaced by $n+1$, it follows that

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n-2}, i^4 q_4^{4n-4}, i^6 (q_2q_4)^{2n-2}\right)\\ &= \frac{1}{2}\left\{f\left(-q_2^{2n-3/2}, -q_2^{-2n+5/2}\right) f^2\left(q_4^{2n-3/2}, q_4^{-2n+5/2}\right)\right.\\ &\quad\left.+ f\left(q_2^{2n-3/2}, q_2^{-2n+5/2}\right) f^2\left(-q_4^{2n-3/2}, -q_4^{-2n+5/2}\right)\right\}. \end{aligned} \tag{51}$$

Substituting (51) into (50), we conclude that

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n}, i^4 q_4^{4n}, i^6 (q_2q_4)^{2n}\right)\\ &= q_2^{-4n+2} q_4^{-8n+4}\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n-2}, i^4 q_4^{4n-4}, i^6 (q_2q_4)^{2n-2}\right). \end{aligned}$$

After iterating above identity n times, this yields the desired result.

Case 3.3. $n \equiv 2 \pmod 4$.

From (32), we have

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n+1}, i^4 q_4^{4n+2}, i^6 (q_2q_4)^{2n+1}\right)\\ &= \frac{1}{2}\left\{f\left(-q_2^{2n+3/2}, -q_2^{-2n-1/2}\right) f^2\left(q_4^{2n+3/2}, q_4^{-2n-1/2}\right)\right.\\ &\quad\left.+ f\left(q_2^{2n+3/2}, q_2^{-2n-1/2}\right) f^2\left(-q_4^{2n+3/2}, -q_4^{-2n-1/2}\right)\right\}. \end{aligned} \tag{52}$$

By (8),

$$f\left(\pm q^{2n+3/2}, \pm q^{-2n-1/2}\right) = q^{-4n} f\left(\pm q^{2n-1/2}, \pm q^{-2n+3/2}\right).$$

Then

$$\begin{aligned}&\vartheta\left(q_2, q_4, i^2 (q_2q_4)^{2n+1}, i^4 q_4^{4n+2}, i^6 (q_2q_4)^{2n+1}\right)\\ &= \frac{1}{2} q_2^{-4n} q_4^{-8n}\left\{f\left(-q_2^{2n-1/2}, -q_2^{-2n+3/2}\right) f^2\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right)\right.\\ &\quad\left.+ f\left(q_2^{2n-1/2}, q_2^{-2n+3/2}\right) f^2\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right)\right\}. \end{aligned} \tag{53}$$

By (52) with n replaced by $n+1$, we see that

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i^2(q_2q_4)^{2n-1}, i^4q_4^{4n-2}, i^6(q_2q_4)^{2n-1}\right)\\
&=\frac{1}{2}\left\{f\left(-q_2^{2n-1/2}, -q_2^{-2n+3/2}\right)f^2\left(q_4^{2n-1/2}, q_4^{-2n+3/2}\right)\right.\\
&\quad\left.+\, f\left(q_2^{2n-1/2}, q_2^{-2n+3/2}\right)f^2\left(-q_4^{2n-1/2}, -q_4^{-2n+3/2}\right)\right\}. \qquad (54)
\end{aligned}$$

Substitute (54) into (53) to obtain

$$\begin{aligned}
&\vartheta\left(q_2, q_4, i^2(q_2q_4)^{2n+1}, i^4q_4^{4n+2}, i^6(q_2q_4)^{2n+1}\right)\\
&=q_2^{-4n}q_4^{-8n}\vartheta\left(q_2, q_4, i^2(q_2q_4)^{2n-1}, i^4q_4^{4n-2}, i^6(q_2q_4)^{2n-1}\right).
\end{aligned}$$

Iterate above formula n times to finish the proof.

Case 4. $m=3$.

We observe that

$$\begin{aligned}
\vartheta\left(q_2, q_4, i^3(q_2q_4)^{n/2}, i^6q_4^n, i^9(q_2q_4)^{n/2}\right) &= \vartheta\left(q_2, q_4, i^3(q_2q_4)^{n/2}, i^2q_4^n, i(q_2q_4)^{n/2}\right)\\
&= \vartheta\left(q_2, q_4, i(q_2q_4)^{n/2}, i^2q_4^n, i^3(q_2q_4)^{n/2}\right),
\end{aligned}$$

where the last equality is deduced by Theorem 3.1(i). Hence this case is the same as the case $m=1$. □

Like Ramanujan's general theta function $f(a,b)$ mentioned in Section 2, we now consider some special cases of the function ϑ as follows. For our convenience, write

$$\begin{aligned}
\vartheta_1(q_2,q_4) &:= \vartheta(q_2,q_4,1,1,1),\\
\vartheta_2(q_2,q_4) &:= \vartheta(q_2,q_4,i,i^2,i^3),\\
\vartheta_3(q_2,q_4) &:= \vartheta(q_2,q_4,i^2,i^4,i^6),\\
\vartheta_4(q_2,q_4) &:= q_2^{1/8}q_4^{1/4}\vartheta\left(q_2,q_4,(q_2q_4)^{1/2},q_4,(q_2q_4)^{1/2}\right),\\
\vartheta_5(q_2,q_4) &:= q_2^{1/2}q_4\vartheta(q_2,q_4,q_2q_4,q_4^2,q_2q_4),\\
\vartheta_6(q_2,q_4) &:= q_2^{1/2}q_4\vartheta(q_2,q_4,i^2q_2q_4,i^4q_4^2,i^6q_2q_4).
\end{aligned}$$

Next, we will express the functions ϑ_1, ϑ_2, ϑ_3, ϑ_4, ϑ_5 and ϑ_6 in terms of φ and ψ.

Theorem 4.2. *We have*

$$\vartheta_1(q_2,q_4) = \frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(q_4^{1/2}) + \varphi(-q_2^{1/2})\varphi^2(-q_4^{1/2})\right), \qquad (55)$$

$$\vartheta_2(q_2,q_4) = \varphi(-q_2^2)\varphi(q_4^{1/2})\varphi(-q_4^{1/2}), \qquad (56)$$

$$\vartheta_3(q_2,q_4) = \frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(-q_4^{1/2}) + \varphi(-q_2^{1/2})\varphi^2(q_4^{1/2})\right), \qquad (57)$$

$$\vartheta_4(q_2,q_4) = 4q_2^{1/8}q_4^{1/4}\psi(q_2)\psi^2(q_4), \qquad (58)$$

$$\vartheta_5(q_2,q_4) = \frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(q_4^{1/2}) - \varphi(-q_2^{1/2})\varphi^2(-q_4^{1/2})\right), \qquad (59)$$

$$\vartheta_6(q_2,q_4) = \frac{1}{2}\left(\varphi(q_2^{1/2})\varphi^2(-q_4^{1/2}) - \varphi(-q_2^{1/2})\varphi^2(q_4^{1/2})\right). \qquad (60)$$

Proof. We employ Theorem 3.1(iii) and use the notation in Section 2. The first and third equalities come straightforwardly. For the second and forth identities, we also utilize Eqs. (6) and (7). For the last two formulas, we use (8) with $n = -1$. The proof is complete. □

Before proceeding further, we will establish some formulas for φ and ψ in terms of α_2, α_4, q_2, q_4, $z_2 := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_2\right)$ and $z_4 := {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \alpha_4\right)$. We can rewrite (13)–(23) in terms of α_2, q_2 and z_2 as

$$\varphi(q_2) = \sqrt{z_2}, \tag{61}$$

$$\varphi(q_2^2) = \sqrt{z_2}\left(\frac{1+\sqrt{1-\alpha_2}}{2}\right)^{1/2}, \tag{62}$$

$$\varphi(q_2^4) = \frac{1}{2}\sqrt{z_2}\left(1 + (1-\alpha_2)^{1/4}\right), \tag{63}$$

$$\varphi(q_2^{1/2}) = \sqrt{z_2}\,(1+\sqrt{\alpha_2}\,)^{1/2}, \tag{64}$$

$$\varphi(-q_2) = \sqrt{z_2}(1-\alpha_2)^{1/4}, \tag{65}$$

$$\varphi(-q_2^2) = \sqrt{z_2}(1-\alpha_2)^{1/8}, \tag{66}$$

$$\varphi(-q_2^{1/2}) = \sqrt{z_2}\,(1-\sqrt{\alpha_2}\,)^{1/2}, \tag{67}$$

$$\psi(q_2) = \sqrt{\frac{z_2}{2}\alpha_2^{1/8} q_2^{-1/8}}, \tag{68}$$

$$\psi(q_2^2) = \frac{\sqrt{z_2}}{2}\alpha_2^{1/4} q_2^{-1/4}, \tag{69}$$

$$\psi(q_2^{1/2}) = \sqrt{z_2}\left(\frac{1+\sqrt{\alpha_2}}{2}\right)^{1/4}\left(\alpha_2 q_2^{-1}\right)^{1/16}, \tag{70}$$

$$\psi(-q_2^{1/2}) = \sqrt{z_2}\left(\frac{1-\sqrt{\alpha_2}}{2}\right)^{1/4}\left(\alpha_2 q_2^{-1}\right)^{1/16}, \tag{71}$$

respectively. In [3, p. 146], there is a procedure for producing formulas in the theory of signature 4 from formulas in the classical theory. Suppose that we have a formula

$$\Omega\left(\alpha_2, q_2, z_2\right) = 0.$$

Then we deduce the formula

$$\Omega\left(\frac{2\sqrt{\alpha_4}}{1+\sqrt{\alpha_4}}, q_4^{1/2}, z_4\,(1+\sqrt{\alpha_4}\,)^{1/2}\right) = 0.$$

By (61), (62), (65), (66) and (69) together with the above procedure, this yields

$$\varphi(q_4^{1/2}) = \sqrt{z_4}\,(1+\sqrt{\alpha_4}\,)^{1/4}, \tag{72}$$

$$\varphi(q_4) = \sqrt{z_4}\left(\frac{1+\sqrt{1-\alpha_4}}{2}\right)^{1/4}, \tag{73}$$

$$\varphi(-q_4^{1/2}) = \sqrt{z_4}\,(1-\sqrt{\alpha_4}\,)^{1/4}, \tag{74}$$

$$\varphi(-q_4) = \sqrt{z_4}\,(1-\alpha_4)^{1/8}, \tag{75}$$

and

$$\psi(q_4) = \frac{\sqrt{z_4}}{2}\,(2\sqrt{\alpha_4}\,)^{1/4}\,q_4^{-1/8}, \tag{76}$$

respectively.

With above formulas, we can write ϑ_1, ϑ_2, ϑ_3, ϑ_4, ϑ_5 and ϑ_6 in terms of α_2, α_4, z_2 and z_4 as analogues of Lemma 2.3 and Lemma 2.4.

Corollary 4.3. *We have*

$$\begin{aligned}
\vartheta_1(q_2, q_4) &= \frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1 + \sqrt{\alpha_2\alpha_4} + \sqrt{(1-\alpha_2)\,(1-\alpha_4)}\right)^{1/2},\\
\vartheta_2(q_2, q_4) &= \sqrt{z_2}z_4(1-\alpha_2)^{1/8}\,(1-\alpha_4)^{1/4},\\
\vartheta_3(q_2, q_4) &= \frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1 - \sqrt{\alpha_2\alpha_4} + \sqrt{(1-\alpha_2)\,(1-\alpha_4)}\right)^{1/2},\\
\vartheta_4(q_2, q_4) &= \sqrt{z_2}z_4\alpha_2^{1/8}\alpha_4^{1/4},\\
\vartheta_5(q_2, q_4) &= \frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1 + \sqrt{\alpha_2\alpha_4} - \sqrt{(1-\alpha_2)\,(1-\alpha_4)}\right)^{1/2},\\
\vartheta_6(q_2, q_4) &= \frac{\sqrt{z_2}z_4}{\sqrt{2}}\left(1 - \sqrt{\alpha_2\alpha_4} - \sqrt{(1-\alpha_2)\,(1-\alpha_4)}\right)^{1/2}.
\end{aligned}$$

Proof. We will utilize Eqs. (64), (66)–(68) and (72)–(76) to Theorem 4.2 together with the identity

$$\sqrt{1+\sqrt{a}}\sqrt{1+\sqrt{b}} \pm \sqrt{1-\sqrt{a}}\sqrt{1-\sqrt{b}} = \sqrt{2}\left(1 + \sqrt{ab} \pm \sqrt{(1-a)(1-b)}\right)^{1/2},$$

where a, b are any numbers in $[0, 1]$. □

The next corollary is an analogue of (1) and (2).

Corollary 4.4. *We have*

(i)

$$\vartheta_1^2(q_2, q_4) + \vartheta_6^2(q_2, q_4) = \vartheta_3^2(q_2, q_4) + \vartheta_5^2(q_2, q_4),$$

(ii)

$$\vartheta_1^2(q_2, q_4) - \vartheta_6^2(q_2, q_4) = \vartheta_2^2(q_2^{1/2}, q_4) + 2\vartheta_4^2(q_2^2, q_4),$$

(iii)

$$\vartheta_3^2(q_2, q_4) - \vartheta_5^2(q_2, q_4) = \vartheta_2^2(q_2^{1/2}, q_4) - 2\vartheta_4^2(q_2^2, q_4).$$

Proof of (i). The identity follows straightforwardly from Corollary 4.3. □

Proof of (ii). By Corollary 4.3, it follows that

$$\vartheta_1^2(q_2, q_4) - \vartheta_6^2(q_2, q_4) = z_2z_4^2\left(\sqrt{\alpha_2\alpha_4} + \sqrt{(1-\alpha_2)\,(1-\alpha_4)}\right).$$

On the other hand, by (56) and (58), we find that

$$\vartheta_2^2(q_2^{1/2}, q_4) + 2\vartheta_4^2(q_2^2, q_4) = \varphi^2(-q_2)\varphi^2(q_4^{1/2})\varphi^2(-q_4^{1/2}) + 32q_2^{1/2}q_4^{1/2}\psi^2(q_2^2)\psi^4(q_4).$$

We utilize (65), (69)–(74), and (76) to obtain

$$\vartheta_2^2(q_2^{1/2}, q_4) + 2\vartheta_4^2(q_2^2, q_4) = z_2 z_4^2\sqrt{(1-\alpha_2)(1-\alpha_4)} + z_2 z_4^2\sqrt{\alpha_2\alpha_4}.$$

This finishes the proof. □

Proof of (iii). By Corollary 4.3, we deduce that

$$\vartheta_3^2(q_2, q_4) - \vartheta_5^2(q_2, q_4) = z_2 z_4^2\left(\sqrt{(1-\alpha_2)(1-\alpha_4)} - \sqrt{\alpha_2\alpha_4}\right).$$

On the other hand, by (56) and (58), we see that

$$\vartheta_2^2(q_2^{1/2}, q_4) - 2\vartheta_4^2(q_2^2, q_4) = \varphi^2(-q_2)\varphi^2(q_4^{1/2})\varphi^2(-q_4^{1/2}) - 32q_2^{1/2}q_4^{1/2}\psi^2(q_2^2)\psi^4(q_4).$$

Employing (65), (69)–(74), and (76), we conclude that

$$\vartheta_2^2(q_2^{1/2}, q_4) + 2\vartheta_4^2(q_2^2, q_4) = z_2 z_4^2\sqrt{(1-\alpha_2)(1-\alpha_4)} - z_2 z_4^2\sqrt{\alpha_2\alpha_4}.$$

We complete the proof. □

Corollary 4.5. *If* $\alpha := \alpha_2 = \alpha_4$, *then*

$$\begin{aligned}
\vartheta_1(q_2, q_4) &= \sqrt{z_2}z_4, \\
\vartheta_2(q_2, q_4) &= \sqrt{z_2}z_4(1-\alpha)^{3/8}, \\
\vartheta_3(q_2, q_4) &= \sqrt{z_2}z_4(1-\alpha)^{1/2}, \\
\vartheta_4(q_2, q_4) &= \sqrt{z_2}z_4\alpha^{3/8}, \\
\vartheta_5(q_2, q_4) &= \sqrt{z_2}z_4\alpha^{1/2}, \\
\vartheta_6(q_2, q_4) &= 0.
\end{aligned}$$

Proof. The proposed formulas follow readily from Corollary 4.3. □

Corollary 4.6. *If* $\alpha := \alpha_2 = \alpha_4$, *then*

(i)

$$\vartheta_3^2(q_2, q_4) + \vartheta_5^2(q_2, q_4) = \vartheta_1^2(q_2, q_4),$$

(ii)

$$\vartheta_2^{8/3}(q_2, q_4) + \vartheta_4^{8/3}(q_2, q_4) = \vartheta_1^{8/3}(q_2, q_4),$$

(iii)

$$\vartheta_2^8(q_2, q_4) = \vartheta_1^2(q_2, q_4)\vartheta_3^6(q_2, q_4),$$

(iv)

$$\vartheta_4^8(q_2, q_4) = \vartheta_1^2(q_2, q_4)\vartheta_5^6(q_2, q_4),$$

(v)

$$\vartheta_1(q_2^2, q_4^2) - \vartheta_5(q_2^2, q_4^2) = \vartheta_3(q_2, q_4),$$

(vi)

$$\vartheta_3(q_2^2, q_4^2) + \vartheta_6(q_2^2, q_4^2) = \vartheta_1^{1/2}(q_2, q_4)\vartheta_3^{1/2}(q_2, q_4),$$

(vii)

$$\vartheta_1(q_2^{1/2}, q_4^{1/2}) - \vartheta_3(q_2^{1/2}, q_4^{1/2}) = 2\sqrt{2}\vartheta_5(q_2, q_4),$$

(viii)

$$\vartheta_5(q_2^{1/2}, q_4^{1/2}) - \vartheta_6(q_2^{1/2}, q_4^{1/2}) = 2\sqrt{2}\vartheta_1^{1/2}(q_2, q_4)\vartheta_5^{1/2}(q_2, q_4).$$

Proof of (i)–(iv). The identities (i)–(iv) come straightforwardly from Corollary 4.5. □

Proof of (v). By (55) and (59), we have

$$\vartheta_1(q_2^2, q_4^2) - \vartheta_5(q_2^2, q_4^2) = \varphi(-q_2)\varphi^2(-q_4).$$

Employing (65) and (75) yields

$$\begin{aligned}\varphi(-q_2)\varphi^2(-q_4) &= \sqrt{z_2}z_4(1-\alpha)^{1/2}\\ &= \vartheta_3(q_2, q_4).\end{aligned}$$

The last equality follows from (57). We complete the proof. □

Proof of (vi). By (57) and (60), we find that

$$\vartheta_3(q_2^2, q_4^2) + \vartheta_6(q_2^2, q_4^2) = \varphi(q_2)\varphi^2(-q_4).$$

Utilizing (61) and (75), we deduce that

$$\begin{aligned}\varphi(q_2)\varphi^2(-q_4) &= \sqrt{z_2}z_4\,(1-\alpha)^{1/4}\\ &= \vartheta_1^{1/2}(q_2, q_4)\vartheta_3^{1/2}(q_2, q_4).\end{aligned}$$

This completes the proof. □

Proof of (vii). Using (55) and (57) along with (10) and (11), it follows that

$$\begin{aligned}\vartheta_1(q_2^{1/2}, q_4^{1/2}) - \vartheta_3(q_2^{1/2}, q_4^{1/2}) &= \frac{1}{2}\left(\varphi(q_2^{1/4}) - \varphi(-q_2^{1/4})\right)\left(\varphi^2(q_4^{1/4}) - \varphi^2(-q_4^{1/4})\right)\\ &= \frac{1}{2}\left(4q_2^{1/4}\psi(q_2^2)\right)\left(8q_4^{1/4}\psi^2(q_4)\right).\end{aligned}$$

By (69) and (76), we obtain

$$\frac{1}{2}\left(4q_2^{1/4}\psi(q_2^2)\right)\left(8q_4^{1/4}\psi^2(q_4)\right) = 2\sqrt{2}\sqrt{z_2}z_4\alpha^{1/2} = 2\sqrt{2}\vartheta_5(q_2,q_4).$$

We finish the proof. □

Proof of (viii). Employing (55) and (57) together with (9) and (11), we find that

$$\vartheta_5(q_2^{1/2},q_4^{1/2}) - \vartheta_6(q_2^{1/2},q_4^{1/2}) = \frac{1}{2}\left(\varphi(q_2^{1/4})+\varphi(-q_2^{1/4})\right)\left(\varphi^2(q_4^{1/4})-\varphi^2(-q_4^{1/4})\right) = \frac{1}{2}\left(2\varphi(q_2)\right)\left(8q_4^{1/4}\psi^2(q_4)\right).$$

Using (69) and (76) yields

$$\frac{1}{2}\left(2\varphi(q_2)\right)\left(8q_4^{1/4}\psi^2(q_4)\right) = 2\sqrt{2}\sqrt{z_2}z_4\alpha^{1/4} = 2\sqrt{2}\vartheta_1^{1/2}(q_2,q_4)\vartheta_5^{1/2}(q_2,q_4).$$

The proof is complete. □

Next, we will give explicit evaluations of quartic theta functions for some values of α_2 and α_4. Before proceeding further, we will establish some values of hypergeometric functions.

Lemma 4.7. *We have*

(i)

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})},$$

(ii)

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})},$$

(iii)

$${}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},$$

(iv)

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2-\sqrt{3}}{4}\right) = \frac{2\cdot 3^{1/4}\sqrt{\pi}}{3\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})},$$

(v)

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2+\sqrt{3}}{4}\right) = \frac{2\cdot 3^{3/4}\sqrt{\pi}}{3\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})},$$

(vi)

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2}{\sqrt{2}+1}\right)=\frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}, $$

(vii)

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)=\frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}, $$

(viii)

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{8}{9}\right)=\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}\sqrt{\frac{3}{2}}, $$

(ix)

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{1}{9}\right)=\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}\sqrt{\frac{3}{4}}. $$

Proof. It is well known that Gauss's second summation formula is

$$ {}_2F_1\left(a,b;\frac{1+a+b}{2};\frac{1}{2}\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})}. \tag{77} $$

Parts (i), (ii) and (iii) follow from (77) with $(a,b)=(\frac{1}{2},\frac{1}{2})$, $(a,b)=(\frac{1}{3},\frac{2}{3})$ and $(a,b)=(\frac{1}{4},\frac{3}{4})$ in (77), respectively.

Part (iv) arises from Theorem 5.6 in [3, p. 112],

$$ (1+x+x^2)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{x^3(2+x)}{1+2x}\right)=\sqrt{1+2x}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27x^2(1+x)^2}{4(1+x+x^2)^3}\right) $$

with $x=\frac{\sqrt{3}-1}{2}$ and part (ii).

To establish part (v), use Corollary 5.7 in [3, p. 113],

$$ (1+x+x^2)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{x^3(2+x)}{1+2x}\right)=\sqrt{3+6x}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\frac{27x^2(1+x)^2}{4(1+x+x^2)^3}\right) $$

with $x=\frac{\sqrt{3}-1}{2}$ and part (ii).

Setting $x=\frac{1}{\sqrt{2}}$ in Theorem 9.1 from [3, p. 145],

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2x}{1+x}\right)=\sqrt{1+x}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x^2\right), $$

we easily deduce part (vi) with the help of part (iii).

Part (vii) is an immediate consequence of Theorem 9.2 in [3, p. 145],

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1-x}{1+x}\right)=\sqrt{\frac{1+x}{2}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-x^2\right), $$

with $x=\frac{1}{\sqrt{2}}$ and part (iii).

To prove part (viii), we employ Entry 33(iv) of Chapter 11 in [1, p. 95],

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{4x}{(1+x)^2}\right)=\sqrt{1+x}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right) $$

with $x=\frac{1}{2}$ and part (i).

Part (ix) follows on using Theorem 9.4 in [3, p. 146],

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x^2\right)=\frac{1}{\sqrt{1+3x}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-\left(\frac{1-x}{1+3x}\right)^2\right) $$

with $x=\frac{1}{3}$ and part (viii). □

Some values of the function ϑ are given in the next theorem analogously to that of φ and ψ in [3, Chapter 35]. We record three cases here; further evaluations involving a hypergeometric function can be obtained by similar reasoning.

Theorem 4.8. *We have*

(i)

$$ \begin{aligned} \vartheta_1(e^{-\pi},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},\\ \vartheta_2(e^{-\pi},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}}{2^{3/8}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},\\ \vartheta_3(e^{-\pi},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}}{2^{1/2}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},\\ \vartheta_4(e^{-\pi},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}}{2^{3/8}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},\\ \vartheta_5(e^{-\pi},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}}{2^{1/2}\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})},\\ \vartheta_6(e^{-\pi},e^{-\pi\sqrt{2}})&=0, \end{aligned} $$

(ii)

$$ \begin{aligned} \vartheta_1(e^{-\pi\sqrt{3}},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}(\sqrt{3}+1)}{3^{3/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},\\ \vartheta_2(e^{-\pi\sqrt{3}},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}\left(\sqrt{3}+1\right)^{1/4}}{2^{1/8}3^{3/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},\\ \vartheta_3(e^{-\pi\sqrt{3}},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}3^{1/8}}{\sqrt{2}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},\\ \vartheta_4(e^{-\pi\sqrt{3}},e^{-\pi\sqrt{2}})&=\frac{\pi^{3/4}\left(\sqrt{3}-1\right)^{1/4}}{2^{1/8}3^{3/8}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}}, \end{aligned} $$

$$\vartheta_5(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}}{3^{3/8}\sqrt{2}\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

$$\vartheta_6(e^{-\pi\sqrt{3}}, e^{-\pi\sqrt{2}}) = \frac{\pi^{3/4}(\sqrt{3}-1)}{3^{3/8}2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})\sqrt{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}},$$

(iii)

$$\vartheta_1(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+14}+4\right)^{1/2}}{2^{9/8}\Gamma(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}},$$

$$\vartheta_2(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}}{2^{5/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}},$$

$$\vartheta_3(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+14}-4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}},$$

$$\vartheta_4(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{2}+1\right)^{1/8}2^{1/4}}{\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}},$$

$$\vartheta_5(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+2}+4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}},$$

$$\vartheta_6(e^{-\pi/\sqrt{2}}, e^{-\pi}) = \frac{\pi^{3/4}\left(\sqrt{10\sqrt{2}+2}-4\right)^{1/2}}{2^{9/8}\Gamma^2(\frac{3}{4})\sqrt{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}}.$$

Proof of (i). Putting $\alpha_2 = \alpha_4 = \frac{1}{2}$ in the definitions of z_2, z_4, q_2 and q_4 together with parts (i) and (iii) of Lemma 4.7, it follows that

$$z_2 = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}, \qquad z_4 = \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}, \qquad q_2 = e^{-\pi} \qquad \text{and} \qquad q_4 = e^{-\pi\sqrt{2}}.$$

Taking $\alpha = \frac{1}{2}$ in Corollary 4.5, the results follow immediately. □

Proof of (ii). Letting $\alpha_2 = \frac{2-\sqrt{3}}{4}$, $\alpha_4 = \frac{1}{2}$ in the definitions of z_2, z_4, q_2, q_4 and utilizing parts (iii), (iv) and (v) of Lemma 4.7, we find that

$$z_2 = \frac{2\cdot 3^{1/4}\sqrt{\pi}}{3\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}, \qquad z_4 = \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}, \qquad q_2 = e^{-\pi\sqrt{3}} \qquad \text{and} \qquad q_4 = e^{-\pi\sqrt{2}}.$$

The proposed results follow readily from Corollary 4.3. □

Proof of (iii). Taking $\alpha_2 = \frac{2}{\sqrt{2}+1}$, $\alpha_4 = \frac{8}{9}$ in the definitions of z_2, z_4, q_2, q_4 together with parts (vi), (vii), (viii) and (ix) of Lemma 4.7, we deduce that

$$z_2 = \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}}}, \qquad z_4 = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}\sqrt{\frac{3}{2}}, \qquad q_2 = e^{-\pi/\sqrt{2}} \qquad \text{and} \qquad q_4 = e^{-\pi}.$$

Employing Corollary 4.3 and simplifying, we arrive at the desired results. □

Acknowledgments

The author wishes to thank Bruce C. Berndt and Sarachai Kongsiriwong for useful suggestions and encouragement. The author is supported by a grant from Prince of Songkla University, contract No. SCI550398S.

References

[1] B.C. Berndt, Ramanujan's Notebooks, Part II, Springer-Verlag, New York, 1989.
[2] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
[3] B.C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1998.
[4] S. Bhargava, Unification of the cubic analogues of the Jacobian theta-function, J. Math. Anal. Appl. 193 (2) (1995) 543–558.
[5] J.M. Borwein, P.B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, Trans. Amer. Math. Soc. 323 (2) (1991) 691–701.
[6] M.D. Hirschhorn, F. Garvan, J. Borwein, Cubic analogues of the Jacobian theta function $\vartheta(z, q)$, Canad. J. Math. 45 (4) (1993) 673–694.
[7] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.
[8] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[9] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[10] D. Schultz, Cubic theta functions, Adv. Math. 248 (2013) 618–697.