Ternary Semigroups and Ordered Ternary Semigroups

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ชื่อวิทยานิพนธ์ กึ่งกรุปเทอนารีและกึ่งกรุปเทอนารีอันดับ

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บทคัดย่อ

ให้ T เป็นเซตไม่ว่าง $\cdot$ เป็นตัวดำเนินการไตรวิภาคบนเซต T และ $\leq$ เป็นอันดับบางส่วนบนเซต T เราเรียก $(T,\cdot)$ ว่ากึ่งกรุปไตรวิภาคถ้าสำหรับทุกสมาชิก x_1,x_2,x_3, x_4,x_5 ในเซต T มีสมบัติ

$$(x_1x_2x_3)x_4x_5 = x_1(x_2x_3x_4)x_5 = x_1x_2(x_3x_4x_5)$$

เราเรียก $(T,\cdot,\leq)$ ว่ากึ่งกรุปไตรวิภาคอันดับ ถ้า T เป็นกึ่งกรุปไตรวิภาคและสำหรับทุกสมาชิก x_1,x_2,x_3,x_4 ในเซต T ถ้า $x_1 \leq x_2$ แล้ว

$$x_1x_3x_4 \leq x_2x_3x_4, x_3x_1x_4 \leq x_3x_2x_4 \text{ และ } x_3x_4x_1 \leq x_3x_4x_2$$

ในวิทยานิพนธ์นี้ เราสร้างทฤษฎีบทสมสัณฐานสำหรับกึ่งกรุปไตรวิภาคอันดับ และศึกษาเกี่ยวกับไอดีลวิภัชนัย ฟิลเตอร์วิภัชนัยของกึ่งกรุปไตรวิภาคอันดับ นอกจากนี้เรายังศึกษาเกี่ยวกับไบ-ไอดีลหยาบ ไบ-ไอดีลวิภัชนัยและไบ-ไอดีลวิภัชนัยหยาบของกึ่งกรุปไตรวิภาคอีกด้วย

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ABSTRACT

Let T be a nonempty set, $\cdot$ a ternary operation $T \times T \times T \to T$ and $\leq$ a partially order on T. Then $(T, \cdot)$ is called a ternary semigroup if for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$(x_1x_2x_3)x_4x_5 = x_1(x_2x_3x_4)x_5 = x_1x_2(x_3x_4x_5)$$

and $(T, \cdot, \leq)$ is an ordered ternary semigroup if T is a ternary semigroup and for all $x_1, x_2, x_3, x_4 \in T, x_1 \leq x_2$ implies

$$x_1x_3x_4 \leq x_2x_3x_4, x_3x_1x_4 \leq x_3x_2x_4 \text{ and } x_3x_4x_1 \leq x_3x_4x_2.$$

In this thesis, we give isomorphism theorems for ordered ternary semigroups, and we study fuzzy ideal and fuzzy filter of ordered ternary semigroups.

Moreover, we study rough bi-ideals, fuzzy bi-ideals and rough fuzzy bi-ideals of ternary semigroups.

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Sompob Saelee

CONTENTS

CHAPTER 1

Introductions and Preliminaries

The formal definition of a ternary algebraic structure was given by Lehmer (Lehmer, 1932), but earlier such structure was studied by Kasner (Kasner, 1904) and Prüfer (Prüfer, 1924). Lehmer investigated certain triple systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was known to Banach (cf. Los, 1955) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not neccessarily reduce to a semigroup. Many classical notions of semigroups have been extended to ternary semigroups. The notion of fuzzy sets was introduced by Zadeh in 1965 (Zadeh, 1965). Many classical notions of fuzzy subset of semigroups have been extended to ternary semigroups and ordered ternary semigroups.

In this thesis, we give isomorphism theorems of ordered ternary semigroups. We study fuzzy ideal and fuzzy filter of ordered ternary semigroups.

Moreover, we study rough bi-ideals, fuzzy bi-ideals and rough fuzzy bi-ideals of ternary semigroups.

1.1 Semigroups

We will use the notation and terminology of Howie (Howie, 1976) to introduce the notion of a semigroup as follows :

Definition 1. Let S be a nonempty set and $*$ a binary operation on S. $(S, *)$ is called a *semigroup* if $*$ is associative, i.e.,

$$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in S$$

Example 1.1. $(\mathbb{N}, +)$ and $(\mathbb{R}, \times)$ are semigroups.

Example 1.2. $(\mathbb{Z}, -)$ is not a semigroup since for $a, b, c \in \mathbb{Z}$ such that $c \neq 0$, we have

$$a - (b - c) = a - b + c \neq a - b - c = (a - b) - c.$$

Definition 2. Let S be a semigroup. A nonempty subset T of S is called a *sub-semigroup* of S if $ab \in T$ for all $a, b \in T$.

Definition 3. Let A be a nonempty set. An arbitrary subset of $A \times A$ is called a *relation* on A.

Definition 4. Let S be a semigroup. A relation ρ on S is called an *equivalence relation* on S if

(1) $a\rho a$ for all $a \in S$ (reflexive),

(2) $a\rho b$ implies $b\rho a$ for all $a, b \in S$ (symmetric),

(3) $a\rho b$ and $b\rho c$ imply $a\rho c$ for all $a, b, c \in S$ (transitive).

We will use the notation and terminology of Howie (Howie, 1976) to introduce congruences and isomorphism theorems for semigroups as follows :

Definition 5. Let S be a semigroup. An equivalence relation ρ on S is called a *right congruence* on S if

$$(a, b) \in \rho \text{ implies } (at, bt) \in \rho \quad \text{for all } a, b, t \in S,$$

and an equivalence relation ρ on S is called a *left congruence* on S if

$$(a, b) \in \rho \text{ implies } (ta, tb) \in \rho \quad \text{for all } a, b, t \in S.$$

An equivalence relation ρ on S is called a *congruence* on S if ρ is both a right and left congruence on S.

Example 1.3. Let ρ be a relation on a semigroup $(\mathbb{N}, +)$ defined by

$$a\rho b \Leftrightarrow 4|a - b \quad \text{for all } a, b \in \mathbb{N}.$$

We have ρ is a congruence on $\mathbb{N}$.

Let S be a semigroup. For $a \in S$, the ρ-congruence class containing a is denoted by $a\rho$.

Definition 6. Let S be a semigroup and ρ a congruence on S. We define

$$S/\rho = \{a\rho \mid a \in S\}.$$

Theorem 1. *Let S be a semigroup and ρ a congruence on S. For $a\rho$, $b\rho \in S/\rho$, let $(a\rho)(b\rho) = (ab)\rho$. Then S/ρ is a semigroup.*

Definition 7. Let ρ be a congruence on a semigroup S. The semigroup S/ρ is called a *quotient semigroup of S by a congruence ρ.*

Definition 8. Let S be a semigroup. A subsemigroup A of S is called

(1) a *left ideal* of S if $SA \subseteq A$,

(2) a *right ideal* of S if $AS \subseteq A$,

(3) an *ideal* of S if A is both a left and right ideal of S.

Example 1.4. Let $\mathbb{Z}$ be the set of all integers and $M_2(\mathbb{Z})$ the set of all 2×2 matrices over $\mathbb{Z}$. We have known that $M_2(\mathbb{Z})$ is a semigroup under the usual multiplication. Let

$$L = \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in \mathbb{Z}\} \text{ and } R = \{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{Z}\}.$$

Then L is a left ideal of $M_2(\mathbb{Z})$ and R is a right ideal of $M_2(\mathbb{Z})$.

Definition 9. Let S be a semigroup. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$.

Example 1.5. Let $S = [0, 1]$. Then S is a semigroup under usual multiplication. Let $B = [0, \frac{1}{2}]$. Then B is a subsemigroup of S. We have that $BSB \subseteq B = [0, \frac{1}{4}] \subseteq B$. Therefore B is a bi-ideal of S.

Example 1.6. Let $\mathbb{N}$ be the set of all positive integers. Then $\mathbb{N}$ is a semigroup under the usual multiplication. Let $B = 2\mathbb{N}$. Thus $B\mathbb{N}B = 4\mathbb{N} \subseteq 2\mathbb{N} = B$. Hence B is a bi-ideal of $\mathbb{N}$.

Definition 10. Let S and T be semigroups. The mapping $\phi : S \to T$ is called a *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$.

Example 1.7. Let $\mathbb{R}$ be a semigroup of the set of all real numbers under the usual addition. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(a) = 2a$ for all $a \in \mathbb{R}$. Let $a, b \in \mathbb{R}$. We have

$$\begin{aligned}\phi(a+b) &= 2(a+b)\\ &= 2a+2b\\ &= \phi(a)+\phi(b).\end{aligned}$$

Hence ϕ is a homomorphism.

Definition 11. Let S and T be semigroups. The mapping $\phi : S \to T$ is called an *isomorphism* if ϕ is a bijective homomorphism.

Theorem 2. *The following statements are true.*

(1) Let ρ be a congruence on a semigroup S. The mapping $\rho^{\#} : S \to S/\rho$ defined by

$$\rho^{\#}(x) = x\rho \textit{ for all } \quad x \in S$$

is a homomorphism.

(2) Let S and T be semigroups. If $\phi : S \to T$ is a homomorphism, then the relation

$$ker\ \phi = \phi^{-1} \circ \phi = \{(x, y) \in S \times S \mid \phi(x) = \phi(y)\}$$

is a congruence on S and there is a monomorphism $\alpha : S/ker\ \phi \to T$ such that $ran\ \alpha = ran\ \phi$ and the diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "(ker\phi)^{\#}"'] & T \\ S/ker\phi \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Theorem 3. *(First Isomorphism Theorem) Let S and T be semigroups. If $\phi : S \to T$ is a homomorphism, then $S/ker\ \phi \cong ran\ \phi$.*

Theorem 4. *Let ρ be a congruence on a semigroup S. If $\phi : S \to T$ is a homomorphism such that $\rho \subseteq ker\ \phi$, then there is a unique homomorphism $\beta : S/\rho \to T$ such that $ran\ \beta = ran\ \phi$ and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^{\#}"'] & T \\ S/\rho \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes.

Let ρ and σ be congruences on a semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on S/ρ by

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}.$$

The following theorem holds.

Theorem 5. *(Third Isomorphism Theorem) Let ρ and σ be congruences on a semigroup S such that $\rho \subseteq \sigma$. The following statements hold.*

(1) σ/ρ is a congruence on S/ρ.

(2) $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

1.2 Ternary semigroups

In 1932, Lehmer (Lehmer, 1932) introduced the definition of a ternary semigroup as follows :

Definition 12. A nonempty set T is called a *ternary semigroup* if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$(x_1x_2x_3)x_4x_5 = x_1(x_2x_3x_4)x_5 = x_1x_2(x_3x_4x_5).$$

We can see that any semigroup can be reduced to a ternary semigroup. Banach showed that a ternary semigroup does not neccessarily reduce to a semigroup by this example

Example 1.8. $T = \{-i, 0, i\}$ is a ternary semigroup while T is not a semigroup under the multiplication over complex numbers.

Example 1.9. $T = \mathbb{Z}^-$ is a ternary semigroup while T is not a semigroup under the multiplication over integers.

Definition 13. A nonempty subset S of T is called a *ternary subsemigroup* if $S^3 \subseteq S$.

Sioson (Sioson,1965) studied ideal theory in ternary semigroups. Now we give the definition of ideals of ternary semigroups.

Definition 14. Let A be a nonempty subset of a ternary semigroup T. Then A is called

(1) a *left ideal* of T if $TTA \subseteq A$,

(2) a *right ideal* of T if $ATT \subseteq A$,

(3) a *lateral ideal* of T if $TAT \subseteq A$,

If A is a left, right and lateral ideal of T, A is called an *ideal* of T.

Definition 15. A ternary subsemigroup B of a ternary semigroup T is called a *bi-ideal* of T if $BTBTB \subseteq B$.

Kar and Maity (Kar and Maity, 2007) gave the definition of congruences and isomorphism theorems of ternary semigroups as follows:

Definition 16. An equivalence relation ρ on a ternary semigroup T is called

(1) a *left congruence* if $a\rho b$ then $(sta)\rho(stb)$ for all $a, b, s, t \in T$,

(2) a *right congruence* if $a\rho b$ then $(ast)\rho(bst)$ for all $a, b, s, t \in T$,

(3) a *lateral congruence* if $a\rho b$ then $(sat)\rho(sbt)$ for all $a, b, s, t \in T$,

(4) a *congruence* if ρ is a left, right and lateral congruence.

Proposition 6. *Let T be a ternary semigroup and ρ a congruence on T. Then $T/\rho = \{a\rho \mid a \in T\}$ is a ternary semigroup under the ternary operation defined by $(a\rho)(b\rho)(c\rho) = (abc)\rho$ for all $a, b, c \in T$.*

Definition 17. Let ρ be a congruence on a ternary semigroup T. The ternary semigroup T/ρ is called a *quotient ternary semigroup of T by a congruence ρ.*

Definition 18. Let S and T be ternary semigroups. A mapping ϕ from S to T is called a *homomorphism* from S to T if $\phi(abc) = \phi(a)\phi(b)\phi(c)$ for all $a, b, c \in S$.

Theorem 7. *Let ρ be a congruence on a ternary semigroup T. Then the mapping $f^{\#} : T \to T/\rho$ given by $f^{\#}(x) = x\rho$ is a homomorphism.*

Theorem 8. *Let S and T be ternary semigroups. If $\phi : S \to T$ is a homomorphism, then the relation*

$$ker\ \phi = \phi^{-1} \circ \phi = \{(x, y) \in S \times S \mid \phi(x) = \phi(y)\}$$

is a congruence on S and there is a homomorphism $\alpha : S/ker\ \phi \to T$ such that $ran\ \alpha = ran\ \phi$ and the diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "(ker\phi)^{\#}"'] & T \\ S/ker\phi \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Theorem 9. *Let S and T be ternary semigroups. Let ρ be a congruence on a semigroup S. If $\phi : S \to T$ is a homomorphism such that $\rho \subseteq ker\ \phi$, then there is a unique homomorphism $\beta : S/\rho \to T$ such that $ran\ \beta = ran\ \phi$ and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^{\#}"'] & T \\ S/\rho \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes.

1.3 Ordered semigroups

First, we review some definitions and theorems of order which are basic knowledge for order semigroups and isomorphism theorems for ordered semigroups.

Definition 19. Let S be a nonempty set and $\leq$ a relation on S. We call $\leq$ is an *order* on S if

(1) $\forall a \in S$, $a \leq a$ (reflexive),

(2) $\forall a, b \in S$ if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetric),

(3) $\forall a, b, c \in S$ if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive).

Example 1.10. We have that $\leq$ is an order on $\mathbb{R}$.

Example 1.11. Let X be any set. We have that $\subseteq$ is an order on $P(X)$.

Definition 20. If $\leq$ is an order on a nonempty set S, then $(S, \leq)$ is called a *partially ordered set*.

Example 1.12. By Example 1.10 and 1.11, we have that $(\mathbb{R}, \leq)$ and $(P(X), \subseteq)$ are partially ordered sets.

Definition 21. Let S be a nonempty set, $\cdot$ a binary operation on S and $\leq$ a relation on S. We call $(S, \cdot, \leq)$ is an *ordered semigroup* if

(1) $(S, \cdot)$ is a semigroup,

(2) $(S, \leq)$ is a partially ordered set,

(3) for all $a, b, c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$.

Example 1.13. $(\mathbb{N}, +, \leq)$ is an ordered semigroup.

Example 1.14. $(P(X), \cup, \subseteq)$ is an ordered semigroup.

Definition 22. Let $(S, \cdot, \leq)$ be an ordered semigroup. A relation ρ on S is called a *pseudo-order* on S if

(1) $\leq \subseteq \rho$,

(2) for all $a, b \in S$, $(a, b) \in \rho$ and $(b, c) \in \rho$ imply $(a, c) \in \rho$,

(3) for all $a, b, c \in S$, $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$.

Definition 23. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $f : S \to T$ a mapping from S to T. A mapping f is called

(1) *isotone* if $x, y \in S$, $x \leq_S y$ implies $f(x) \leq_T f(y)$,

(2) *reverse isotone* if $x, y \in S$, $f(x) \leq_T f(y)$ implies $x \leq_S y$.

Remark 1. Every reverse isotone mapping is 1-1.

Definition 24. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $f : S \to T$ a mapping from S to T. A mapping f is called a *homomorphism* if

(1) f is isotone;

(2) $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in S$.

Definition 25. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups. A mapping $f : S \to T$ is called an *isomorphism* if f is an onto homomorphism which is also reverse isotone.

In 1995, Kehayopulu and Tsingelis (Kehayopulu and Tsingelis, 1995) have given isomorphism theorems for ordered semigroups.

If ρ is a pseudo-order on S, let $\bar{\rho}$ be a relation on S defined by

$$\bar{\rho} = \{(a, b) \in S \times S \mid (a, b) \in \rho \text{ and } (b, a) \in \rho\}.$$

Proposition 10. *Let $(S, \cdot, \leq)$ be an ordered semigroup and ρ a pseudo-order on S. Then $\bar{\rho}$ is a congruence on S.*

Let S be an ordered semigroup and ρ a pseudo-order on S. By the Proposition 1.9, we have that $\bar{\rho}$ is a congruence on S. Then the set $S/\bar{\rho} = \{a\bar{\rho} \mid a \in S\}$ with multiplication $(a\bar{\rho}) \cdot (b\bar{\rho}) = (ab)\bar{\rho}$ is a semigroup and an order $\preceq_\rho$ defined by

$$a\rho \preceq_\rho b\rho \Leftrightarrow \text{there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } x \leq y.$$

Proposition 11. *Let S be an ordered semigroup and ρ a pseudo-order on S. The following statements hold.*

(1) For $a, b \in S$, $a\bar{\rho} \preceq_\rho b\bar{\rho}$ if and only if $(a, b) \in \rho$.

(2) $\preceq_\rho$ is an order on $S/\bar{\rho}$.

Proposition 12. *Let $(S, \cdot, \leq)$ be an ordered semigroup and ρ a pseudo-order on S. Then $S/\bar{\rho}$ is an ordered semigroup.*

Let $\rho^{\#}$ be a homomorphism of S onto $S/\bar{\rho}$ defined by $\rho^{\#} : S \to S/\bar{\rho}$ such that $\rho^{\#}(a) = a\bar{\rho}$ for all $a \in S$.

Proposition 13. *Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. Define the relation $\tilde{\phi}$ on S by*

$$\tilde{\phi} = \{(a, b) \in S \times S \mid \phi(a) \leq_T \phi(b)\}.$$

Then $\tilde{\phi}$ is a pseudo-order on S.

Theorem 14. *Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. If ρ is a pseudo-order on S such that $\rho \subseteq \tilde{\phi}$, then the mapping $\psi : S/\bar{\rho} \to T$ defined by $\psi(a\bar{\rho}) = \phi(a)$ is a unique homomorphism of $S/\bar{\rho}$ into T such that $ran\ \psi = ran\ \phi$ and the diagram*

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\rho^{\#}"'] & T \\
S/\bar{\rho} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes (i.e., $\psi \circ \rho^{\#} = \phi$).

Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. Define $ker\ \phi = \{(a, b) \in S \times S \mid \phi(a) = \phi(b)\}$. It is easy to see that $ker\phi$ is a congruence on S. We have

$$\begin{aligned}
(a, b) \in ker\ \phi &\Leftrightarrow \phi(a) = \phi(b) \\
&\Leftrightarrow \phi(a) \leq_T \phi(b)\ and\ \phi(b) \leq_T \phi(a) \\
&\Leftrightarrow (a, b) \in \tilde{\phi}\ and\ (b, a) \in \tilde{\phi} \\
&\Leftrightarrow (a, b) \in \bar{\tilde{\phi}}.
\end{aligned}$$

So $ker\ \phi = \bar{\bar{\phi}}$. The following theorem holds.

Theorem 15. *(First Isomorphism Theorem) Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered semigroups and $\phi : S \to T$ a homomorphism. Then $S/ker\ \phi \cong ran\ \phi$.*

Let ρ and σ be pseudo-orders on an ordered semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on $S/\bar{\rho}$ by

$$\sigma/\rho = \{(x\bar{\rho}, y\bar{\rho}) \in S/\bar{\rho} \times S/\bar{\rho} \mid (x, y) \in \sigma\}.$$

Theorem 16. *(Third Isomorphism Theorem) Let $(S, \cdot, \leq_S)$ be an ordered semigroup, ρ and σ pseudo-orders on S such that $\rho \subseteq \sigma$. The following statements hold:*

(1) σ/ρ is a pseudo-order on $S/\bar{\rho}$.

(2) $(S/\bar{\rho})/(\overline{\sigma/\rho}) \cong S/\bar{\sigma}$.

1.4 Ordered ternary semigroups

Definition 26. A partially ordered ternary semigroup T is called an *ordered ternary semigroup* if for all $x_1, x_2, x_3, x_4 \in T$,

$$x_1 \leq x_2 \text{ implies } x_1x_3x_4 \leq x_2x_3x_4, x_3x_1x_4 \leq x_3x_2x_4 \text{ and } x_3x_4x_1 \leq x_3x_4x_2.$$

Example 1.15. $(\mathbb{Z}, \cdot, \leq)$ is an ordered ternary semigroup.

Let T be an ordered ternary semigroup. For nonempty subsets A, B and C of T, let $ABC = \{abc | a \in A, b \in B, \text{ and } c \in C\}$. For a nonempty subset A

of T, we note $(A] = \{t \in T | t \leq h$ for some $h \in A\}$. For any nonempty subsets A and B of T, if $A \subseteq B$, we have $(A] \subseteq (B]$ and $(A \cup B] \subseteq (A] \cup (B]$.

Definition 27. A nonempty subset S of T is called an *ordered ternary subsemigroup* of T if $(S] \subseteq S$ and $S^3 \subseteq S$.

Definition 28. Let A be a nonempty subset of an ordered ternary semigroup T. Then A is called

(1) a *left ideal* of T if $(A] \subseteq A$ and $TTA \subseteq A$,

(2) a *right ideal* of T if $(A] \subseteq A$ and $ATT \subseteq A$,

(3) a *lateral ideal* of T if $(A] \subseteq A$ and $TAT \subseteq A$,

If A is a left, right and lateral ideal of T, A is call an *ideal* of T.

Definition 29. An ordered ternary subsemigroup B of an ordered ternary semigroup T is called a *bi-ideal* of T if $(B] \subseteq B$ and $BTBTB \subseteq B$.

CHAPTER 2

Isomorphism theorems for ordered ternary semigroups

In 1995, Kehayopulu and Tsingelis (Kehayopulu and Tsingelis, 1995) have given two isomorphism theorems for ordered semigroups. Pseudo-order played an important role in concepts of congruences and quotient of ordered semigroups. In this chapter, we give some properties of isomorphisms for ordered ternary semigroups.

Let S be a ternary semigroup and ρ a congruence on S, by proposition 1.6, we have that S/ρ is a ternary semigroup. The following question is natural : If $(S,\cdot,\leq)$ is an ordered ternary semigroup and ρ is a congruence on S, then is the set S/ρ an ordered ternary semigroup ? A probable order on S/ρ could be the relation $\preceq_\rho$ on S/ρ defined by a mean of the order $\leq$ on S, that is

$$a\rho \preceq_\rho b\rho \Leftrightarrow \text{there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } x \leq y.$$

But this relation is not an order, in general. We prove it in the following example.

Example 2.1. We consider the ordered ternary semigroup $S = \{a, b, c, d, e\}$ defined by the ternary multiplication and the order $\leq$ below:

a	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

b	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

c	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

d	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

e	a	b	c	d	e
a	a	e	c	d	e
b	a	e	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

and $\leq = \{(a,a),(a,d),(b,b),(c,c),(c,e),(d,d),(e,e)\}$. It is easy to see that S is an ordered ternary semigroup.

Let ρ be the congruence on S defined as follows:

$$\rho = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,e),(e,a),(c,d),(d,c)\}.$$

Let $\preceq_\rho$ be an order on S/ρ defined by means of the order $\leq$ on S, that is

$$a\rho \preceq_\rho b\rho \Leftrightarrow \text{there exist } x \in a\rho \text{ and } y \in b\rho \text{ such that } x \leq y.$$

We have $a\rho = \{a,e\}, b\rho = \{b\}$ and $c\rho = \{c,d\}$. Also we have $a\rho \preceq_\rho c\rho$ and $c\rho \preceq_\rho a\rho$ but $a\rho \neq c\rho$. Thus $\preceq_\rho$ is not an order relation on S/ρ. $\square$

The following question arises : Is there a congruence ρ on an ordered ternary semigroup S for which S/ρ is an ordered ternary semigroup ? This leads us to the concept of pseudo-orders.

Now we study pseudo-orders and isomorphism theorems in ordered ternary semigroups analogously to pseudo-orders and isomorphism theorems in ordered semigroups considered by Kehayopulu and Tsingelis.

Definition 30. Let $(S,\cdot,\leq)$ be an ordered ternary semigroup. A relation ρ is called a *pseudo-order* if

(1) $\leq \subseteq \rho$,

(2) for all $a,b,c \in S, (a,b) \in \rho$ and $(b,c) \in \rho$ imply $(a,c) \in \rho$ and

(3) for all $a,b \in S, (a,b) \in \rho$ implies $(ast,bst),(sat,sbt),(sta,stb) \in \rho$ for all $s,t \in S$.

If ρ is a pseudo-order on S, let $\overline{\rho}$ be a relation on S defined by $\bar{\rho} = \{(a,b) \in S \times S \mid (a,b) \in \rho$ and $(b,a) \in \rho\}$.

Proposition 17. *Let $(S, \cdot, \leq)$ be an ordered ternary semigroup and ρ a pseudo-order on S. Then $\overline{\rho}$ is a congruence on S.*

Proof. Let $a \in S$. Since $(a,a) \in \leq$ and $\leq \subseteq \rho$, $(a,a) \in \rho$. Then $(a,a) \in \overline{\rho}$. Next, let $a, b \in S$ such that $(a,b) \in \overline{\rho}$. Then $(a,b) \in \rho$ and $(b,a) \in \rho$. This implies that $(b,a) \in \overline{\rho}$. To show that ρ is transitive, let $a, b, c \in S$ such that $(a,b), (b,c) \in \overline{\rho}$. Then $(a,b), (b,a), (b,c), (c,b) \in \rho$. Thus $(a,c), (c,a) \in \rho$. Hence $(a,c) \in \overline{\rho}$. Finally, let $a, b \in S$ such that $(a,b) \in \overline{\rho}$. Then $(a,b), (b,a) \in \rho$. Then $(sta, stb), (sat, sbt), (ast, bst), (stb, sta), (sbt, sat),\ (bst, ast) \in \rho$ for all $s, t \in S$. Therefore $(ast, bst), (sat, sbt), (sta, stb) \in \overline{\rho}$ for all $s, t \in S$. $\square$

Let $(S, \cdot, \leq)$ be an ordered ternary semigroup and ρ a pseudo-order on S. By Proposition 2.1, we have that $\overline{\rho}$ is a congruence on S. Then $S/\overline{\rho}$ is a ternary semigroup. Next, for each $a\overline{\rho}, b\overline{\rho} \in S/\overline{\rho}$, define the order $\preceq_{\overline{\rho}}$ on $S/\overline{\rho}$ by

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Leftrightarrow \text{there exist } x \in a\overline{\rho} \text{ and } y \in b\overline{\rho} \text{ such that } (x,y) \in \rho.$$

Proposition 18. *Let $(S, \cdot, \leq)$ be an ordered ternary semigroup and ρ a pseudo-order on S. Then*

(1) For $a, b \in S, a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ if and only if $(a,b) \in \rho$.

(2) $\preceq_{\overline{\rho}}$ is an order on $S/\overline{\rho}$.

Proof. (1) If $(a,b) \in \rho$, then clearly, $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$. Conversely, assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$. Then there exist $x \in a\overline{\rho}$ and $y \in b\overline{\rho}$ such that $(x,y) \in \rho$. Since $(x,a) \in \overline{\rho}$ and $(y,b) \in \overline{\rho}$, $(x,a), (a,x), (b,y), (y,b) \in \rho$. Since $(a,x), (x,y), (y,b) \in \rho, (a,b) \in \rho$.

(2) Let $a, b, c \in S$. Since $(a,a) \in \leq \subseteq \rho$, $a\overline{\rho} \preceq_{\overline{\rho}} a\overline{\rho}$. Assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ and $b\overline{\rho} \preceq_{\overline{\rho}} a\overline{\rho}$. By (i), $(a,b) \in \rho$ and $(b,a) \in \rho$. Then $(a,b) \in \overline{\rho}$. So $a\overline{\rho} = b\overline{\rho}$. Finally, assume $a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho}$ and $b\overline{\rho} \preceq_{\overline{\rho}} c\overline{\rho}$. By (i), $(a,b) \in \rho$ and $(b,c) \in \rho$. Therefore $(a,c) \in \rho$. By (i), $a\overline{\rho} \preceq_{\overline{\rho}} c\overline{\rho}$. Hence $\preceq_{\overline{\rho}}$ is an order on $S/\overline{\rho}$. $\square$

Let $x, y \in S$ such that $x\overline{\rho} \preceq_{\overline{\rho}} y\overline{\rho}$. Then there exist $a \in x\overline{\rho}$ and $b \in y\overline{\rho}$ such that $(a, b) \in \rho$. Thus $(x, a) \in \overline{\rho}$ and $(y, b) \in \overline{\rho}$. Then $(x, a), (a, x), (y, b), (b, y) \in \rho$. Let $s, t \in S$. Therefore

$$(xst, ast), (ast, xst), (yst, bst), (bst, yst) \in \rho.$$

Thus $(xst, ast), (yst, bst) \in \overline{\rho}$. So $(xst)\overline{\rho} = (ast)\overline{\rho}, (yst)\overline{\rho} = (bst)\overline{\rho}$. Since $(a, b) \in \rho, (ast, bst) \in \rho$. Hence $(xst)\overline{\rho} \preceq_{\overline{\rho}} (yst)\overline{\rho}$, that is, $(x\overline{\rho})(s\overline{\rho})(t\overline{\rho}) \preceq_{\overline{\rho}} (y\overline{\rho})(s\overline{\rho})(t\overline{\rho})$. Similarly, $(s\overline{\rho})(x\overline{\rho})(t\overline{\rho}) \preceq_{\overline{\rho}} (s\overline{\rho})(y\overline{\rho})(t\overline{\rho})$ and $(s\overline{\rho})(t\overline{\rho})(x\overline{\rho}) \preceq_{\overline{\rho}} (s\overline{\rho})(t\overline{\rho})(y\overline{\rho})$. Then $S/\overline{\rho}$ is an ordered ternary semigroup. Then the following proposition holds.

Proposition 19. *Let $(S, \cdot, \leq)$ be an ordered ternary semigroup and ρ a pseudo-order on S. Then $S/\overline{\rho}$ is an ordered ternary semigroup.*

Definition 31. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a mapping from S into T. A mapping ϕ is called

(1) *isotone* if for $x, y \in S, x \leq_S y$ implies $\phi(x) \leq_T \phi(y)$,

(2) *reverse isotone* if $x, y \in S, \phi(x) \leq_T \phi(y)$ implies $x \leq_S y$.

Definition 32. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a mapping from S into T. A mapping ϕ is called an *ordered ternary semigroup homomorphism* or *homomorphism* if ϕ is isotone and satisfies $\phi(abc) = \phi(a)\phi(b)\phi(c)$ for all $a, b, c \in S$.

Definition 33. Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a mapping from S into T. A mapping ϕ is called an *isomorphism* if it is a homomorphism, onto and reverse isotone.

Proposition 20. *Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a homomorphism. Define the relation $\widetilde{\phi}$ on S by*

$$\widetilde{\phi} = \{(a, b) \in S \times S \mid \phi(a) \leq_T \phi(b)\}.$$

Then $\widetilde{\phi}$ is a pseudo-order on S.

Proof. Let $(a, b) \in \leq_S$. Since $a \leq_S b$ and ϕ is isotone, $\phi(a) \leq_T \phi(b)$. Then $(a, b) \in \widetilde{\phi}$. Next, let $a, b, c \in S$ such that $(a, b), (b, c) \in \widetilde{\phi}$. So $\phi(a) \leq_T \phi(b), \phi(b) \leq_T \phi(c)$. Then $\phi(a) \leq_T \phi(c)$. This implies $(a, c) \in \widetilde{\phi}$. Finally, let $a, b, s, t \in S$. Assume $(a, b) \in \widetilde{\phi}$. Since $\phi(a) \leq_T \phi(b), \phi$ is a homomorphism and T is an ordered ternary semigroup,

$$\phi(ast) = \phi(a)\phi(s)\phi(t) \leq_T \phi(b)\phi(s)\phi(t) = \phi(bst).$$

Then $(ast, bst) \in \widetilde{\phi}$. Similarly, $(sat, sbt), (sta, bst) \in \widetilde{\phi}$. Hence $\widetilde{\phi}$ is a pseudo-order on S. □

Theorem 21. *Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a homomorphism. If ρ is a pseudo-order on S such that $\rho \subseteq \widetilde{\phi}$, then the mapping $\varphi : S/\overline{\rho} \to T$ defined by $\varphi(a\overline{\rho}) = \phi(a)$ is a unique homomorphism of $S/\overline{\rho}$ into T such that ran $\varphi =$ ran ϕ and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^\sharp"'] & T \\ S/\overline{\rho} \arrow[ur, "\varphi"] & \end{tikzcd}$$

commutes (i.e, $\varphi \circ \rho^\sharp = \phi$) where the mapping $\rho^\sharp : S \to S/\overline{\rho}$ defined by $\rho^\sharp(a) = a\overline{\rho}$ for all $a \in S$.

Proof. Define $\varphi : S/\overline{\rho} \to T$ by $\varphi(a\overline{\rho}) = \phi(a)$ for all $a \in S$. We have φ is well-defined since for all $a, b \in S$,

$$\begin{aligned} a\overline{\rho} = b\overline{\rho} &\Rightarrow (a, b) \in \overline{\rho} \\ &\Rightarrow (a, b), (b, a) \in \rho \\ &\Rightarrow (a, b), (b, a) \in \widetilde{\phi} \\ &\Rightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a) \\ &\Rightarrow \phi(a) = \phi(b) \end{aligned}$$

Let $a, b, c \in S$. We have

$$\varphi(a\overline{\rho}b\overline{\rho}c\overline{\rho}) = \varphi((abc)\overline{\rho}) = \phi(abc) = \phi(a)\phi(c)\phi(b) = \varphi(a\overline{\rho})\varphi(b\overline{\rho})\varphi(b\overline{\rho})$$

and

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Rightarrow (a,b) \in \rho \subseteq \widetilde{\phi} \Rightarrow \phi(a) \leq_T \phi(b).$$

Therefore φ is a homomorphism. For each $a \in S$, we have

$$(\varphi \circ \rho^\sharp)(a) = \varphi(\rho^\sharp(a)) = \varphi(a\overline{\rho}) = \phi(a).$$

Then $\varphi \circ \rho^\sharp = \phi$. Next, let $\psi : S/\overline{\rho} \to T$ be any homomorphism such that $\psi \circ \rho^\sharp = \phi$. For all $a \in S$, we have

$$\psi(a\overline{\rho}) = \psi(\rho^\sharp(a)) = (\psi \circ \rho^\sharp)(a) = \phi(a) = \varphi(a\overline{\rho}),$$

so $\psi = \varphi$. Finally, we have ran $\varphi = \{\varphi(a\overline{\rho}) \mid a \in S\} = \{\phi(a) \mid a \in S\} =$ ran ϕ. Hence the theorem is proved. $\square$

Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a homomorphism. Define $ker\ \phi = \{(a,b) \in S \times S \mid \phi(a) = \phi(b)\}$. It is easy to see that $ker\ \phi$ is a congruence on S. We have

$$\begin{aligned}(a,b) \in ker\ \phi &\Leftrightarrow \phi(a) = \phi(b)\\ &\Leftrightarrow \phi(a) \leq_T \phi(b) \text{ and } \phi(b) \leq_T \phi(a)\\ &\Leftrightarrow (a,b) \in \widetilde{\phi} \text{ and } (b,a) \in \widetilde{\phi}\\ &\Leftrightarrow (a,b) \in \overline{\widetilde{\phi}}.\end{aligned}$$

So $ker\ \phi = \overline{\widetilde{\phi}}$. Then the following corollary holds.

Corollary 22. *(First Isomorphism Theorem for Ordered Ternary Semigroups) Let $(S, \cdot, \leq_S)$ and $(T, *, \leq_T)$ be ordered ternary semigroups and $\phi : S \to T$ a homomorphism. Then $S/ker\ \phi \cong ran\ \phi$.*

Proof. We apply the first part of Theorem 2.5 for $\rho = \widetilde{\phi}$ and $ker\ \phi = \overline{\widetilde{\phi}}$. Then the mapping $\varphi : S/ker\ \phi \to T$ defined by $\varphi(aker\ \phi) = \phi(a)$ is a homomorphism. To show that φ is reverse isotone, let $a, b \in S$ such that $\phi(a) \leq_T \phi(b)$. Then $(a,b) \in \widetilde{\phi}$. Since $\widetilde{\phi}$ is a pseudo-order on S, by Proposition 2.2(1), $aker\ \phi \preceq_{\overline{\widetilde{\phi}}} bker\ \phi$. So φ is reverse isotone. Hence φ is an isomorphism. $\square$

Let ρ and σ be pseudo-orders on an ordered ternary semigroup S with $\rho \subseteq \sigma$. Define the relation σ/ρ on $S/\bar{\rho}$ by

$$\sigma/\rho = \{(x\bar{\rho}, y\bar{\rho}) \in S/\bar{\rho} \times S/\bar{\rho} \mid (x,y) \in \sigma\}.$$

Theorem 23. *(Third Isomorphism Theorem for Ordered Ternary Semigroups) Let $(S, \cdot, \leq_S)$ be an ordered ternary semigroup, ρ and σ be pseudo-orders on S such that $\rho \subseteq \sigma$. The following statements are true.*

(1) σ/ρ is a pseudo-order on $S/\overline{\rho}$,

(2) $(S/\overline{\rho})/(\overline{\sigma/\rho}) \cong S/\overline{\sigma}$.

Proof. (1) Let $(a\overline{\rho}, b\overline{\rho}) \in \preceq_{\overline{\rho}}$. Then $(a,b) \in \rho$, it implies $(a,b) \in \sigma$. So $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$. Therefore $\preceq_{\overline{\rho}} \subseteq \sigma/\rho$. Next, let $a,b,c \in S$ such that $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$ and $(b\overline{\rho}, c\overline{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$ and $(b,c) \in \sigma$, so $(a,c) \in \sigma$. Therefore $(a\overline{\rho}, c\overline{\rho}) \in \sigma/\rho$. Finally, let $a,b,s,t \in S$. Assume $(a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho$. Then $(a,b) \in \sigma$, thus $(ast, bst) \in \sigma$. So $((ast)\overline{\rho}, (bst)\overline{\rho}) \in \sigma/\rho$. Hence $(a\overline{\rho}s\overline{\rho}t\overline{\rho}, b\overline{\rho}s\overline{\rho}t\overline{\rho}) \in \sigma/\rho$. Similarly, $(s\overline{\rho}a\overline{\rho}t\overline{\rho}, s\overline{\rho}b\overline{\rho}t\overline{\rho}), (s\overline{\rho}t\overline{\rho}a\overline{\rho}, s\overline{\rho}t\overline{\rho}b\overline{\rho}) \in \sigma/\rho$.

(2) Define $\phi : S/\overline{\rho} \to S/\overline{\sigma}$ by

$$\phi(a\overline{\rho}) = a\overline{\sigma} \text{ for all } a \in S.$$

We have ϕ is well-defined since for all $a,b \in S$,

$$a\overline{\rho} = b\overline{\rho} \Rightarrow (a,b) \in \overline{\rho} \Rightarrow (a,b),(b,a) \in \rho \subseteq \sigma \Rightarrow (a,b) \in \overline{\sigma} \Rightarrow a\overline{\sigma} = b\overline{\sigma}.$$

Next, let $a,b,c \in S$. We have

$$\phi(a\overline{\rho}b\overline{\rho}c\overline{\rho}) = \phi((abc)\overline{\rho}) = (abc)\overline{\sigma} = a\overline{\sigma}b\overline{\sigma}c\overline{\sigma} = \phi(a\overline{\rho})\phi(b\overline{\rho})\phi(c\overline{\rho})$$

and

$$a\overline{\rho} \preceq_{\overline{\rho}} b\overline{\rho} \Rightarrow (a,b) \in \rho \Rightarrow (a,b) \in \sigma \Rightarrow a\overline{\sigma} \preceq_{\overline{\sigma}} b\overline{\sigma}.$$

Hence ϕ is a homomorphism.

By the definition of $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \{(a\overline{\rho}, b\overline{\rho}) \in S/\overline{\rho} \times S/\overline{\rho} \mid \phi(a\overline{\rho}) \preceq_{\overline{\sigma}} \phi(b\overline{\rho})\}.$$

Thus

$$(a\overline{\rho}, b\overline{\rho}) \in \widetilde{\phi} \Leftrightarrow \phi(a\overline{\rho}) \preceq_{\overline{\sigma}} \phi(b\overline{\rho}) \Leftrightarrow a\overline{\sigma} \preceq_{\overline{\sigma}} b\overline{\sigma} \Leftrightarrow (a, b) \in \sigma \Leftrightarrow (a\overline{\rho}, b\overline{\rho}) \in \sigma/\rho.$$

Then $\widetilde{\phi} = \sigma/\rho$, so $ker\ \phi = \overline{\widetilde{\phi}} = \overline{\sigma/\rho}$. It is easy to show that ran $\phi = S/\overline{\sigma}$. By Corollary 2.6, $(S/\overline{\rho})/(\overline{\sigma/\rho}) \cong S/\overline{\sigma}$. $\square$

CHAPTER 3

Fuzzy ideals and fuzzy filters of ordered ternary semigroups

The notion of fuzzy sets was introduced by Zadeh (Zadeh, 1965). Several researchs were conducted on the generalizations of the notion of fuzzy sets. Fuzzy semigroups have been first considered by Kuroki (Kuroki, 1981, 1991 and 1993) and fuzzy ordered semigroups by Kehayopulu and Tsingelis (Kehayopulu, 1990, Kehayopulu and Tsingelis, 1999 and 2002, Kehayopulu, Xie and Tsingelis, 2001). In 2008, Shabir and Khan (Shabir and Khan, 2008) studied fuzzy filters in ordered semigroups .

In this chapter, we study fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups.

Definition 34. Let T be an ordered ternary semigroup. A function f from T to the unit interval $[0, 1]$ is called a *fuzzy subset* of T.

The ordered ternary semigroup T itself is a fuzzy subset of T such that $T(x) = 1$ for all $x \in T$, denoted also by T.

Definition 35. Let A be a nonempty subset of ordered ternary semigroup T. The *characteristic function* f_A of A is a fuzzy subset of T defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 36. Let T be an ordered ternary semigroup and f a fuzzy subset of T. The fuzzy subset f' defined by $f'(x) = 1 - f(x)$ for all $x \in T$ is called the *complement* of f in T.

3.1 Fuzzy ideals of ordered ternary semigroups

Now we define fuzzy ordered ternary subsemigroups, fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals and fuzzy ideals of ordered ternary semigroups.

Definition 37. Let T be a ordered ternary semigroup. A fuzzy subset f of T is called

(1) a *fuzzy ordered ternary subsemigroup* of T if $x \leq y$ implies $f(x) \geq f(y)$ and $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$,

(2) a *fuzzy left ideal* of T if $x \leq y$ implies $f(x) \geq f(y)$ and $f(xyz) \geq f(z)$ for all $x, y, z \in T$,

(3) a *fuzzy right ideal* of T if $x \leq y$ implies $f(x) \geq f(y)$ and $f(xyz) \geq f(x)$ for all $x, y, z \in T$,

(4) a *fuzzy lateral ideal* of T if $x \leq y$ implies $f(x) \geq f(y)$ and $f(xyz) \geq f(y)$ for all $x, y, z \in T$,

(5) a *fuzzy ideal* of T if $x \leq y$ implies $f(x) \geq f(y)$ and $f(xyz) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Lemma 24. *Let T be an ordered ternary semigroup and A a nonempty subset of T. Then $(A] \subseteq A$ if and only if $x \leq y$ implies $f_A(x) \geq f_A(y)$.*

Proof. Assume that $(A] \subseteq A$. Let $x, y \in T$ such that $x \leq y$.
Case 1 : $y \notin A$. Then $f_A(y) = 0 \leq f_A(x)$.
Case 2 : $y \in A$. Since $(A] \subseteq A, x \in A$. So $f_A(x) = 1 \geq f_A(y)$.

Conversely, let $x \in (A]$. Then there exists $y \in A$ such that $x \leq y$. So $f_A(x) \geq f_A(y) = 1$. This implies $x \in A$. □

Now we characterize ordered ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) of ordered ternary semigroups in terms of fuzzy ordered ternary subsemigroups (fuzzy left ideals, fuzzy right ideals, fuzzy lateral ideals, fuzzy ideals).

Theorem 25. *Let T be an ordered ternary semigroup and A a nonempty subset of T. The following statements are true.*

(1) A is an ordered ternary subsemigroup of T if and only if f_A is a fuzzy ordered ternary subsemigroup of T.

(2) A is a left ideal (right ideal, lateral ideal, ideal) of T if and only if f_A is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T.

Proof. (1) Assume that A is a ordered ternary subsemigroup of T. By Lemma 3.1, $x \leq y$ implies $f_A(x) \geq f_A(y)$. Next, let $x, y, z \in T$.

Case 1 : $x, y, z \in A$. Since A is an ordered ternary subsemigroup of T, $xyz \in A$. Therefore $f_A(xyz) = 1 \geq \min\{f_A(x), f_A(y), f_A(z)\}$.

Case 2 : $x \notin A$ or $y \notin A$ or $z \notin A$. Thus $f_A(x) = 0$ or $f_A(y) = 0$ or $f_A(z) = 0$. Hence $\min\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xyz)$.

Conversely, assume that f_A is a fuzzy ordered ternary subsemigroup of T. By Lemma 3.1, $(A] \subseteq A$. Next, let $x, y, z \in A$. So $f_A(x) = f_A(y) = f_A(z) = 1$. Since f_A is a fuzzy ordered ternary subsemigroup of T, $f_A(xyz) \geq \min\{f_A(x), f_A(y), f_A(z)\}$ $= 1$. Then $xyz \in A$.

(2) Assume that A is a left ideal of T. By Lemma 3.1, we have that $x \leq y$ implies $f_A(x) \geq f_A(y)$. Next, let $x, y, z \in T$.

Case 1 : $z \in A$. Since A is a left ideal of T, $xyz \in A$. Then $f_A(xyz) = 1$. Therefore $f_A(xyz) \geq f_A(z)$.

Case 2 : $z \notin A$. So $f_A(z) = 0$. Hence $f_A(xyz) \geq f_A(z)$.

Conversely, assume that f_A is a fuzzy left ideal of T. By Lemma 3.1, $(A] \subseteq A$. Next, let $x, y \in T$ and $z \in A$. Since f_A is a fuzzy left ideal of T and $z \in A$, $f_A(xyz) \geq f_A(z) = 1$. So $xyz \in A$.

The other parts of (2) can be proved in a similar way. □

Definition 38. Let T be an ordered ternary semigroup. A nonempty subset S of T is called a *prime subset* of T if for all $x, y, z \in T, xyz \in S$ implies $x \in S$ or $y \in S$ or $z \in S$.

Definition 39. Let T be an ordered ternary semigroup. A nonempty subset S of T is called

(1) a *prime ordered ternary subsemigroup* of T if S is a prime subset and ordered ternary subsemigroup of T,

(2) a *prime left ideal* of T if S is a prime subset and left ideal of T,

(3) a *prime right ideal* of T if S is a prime subset and right ideal of T,

(4) a *prime lateral ideal* of T if S is a prime subset and lateral ideal of T,

(5) a *prime ideal* of T if S is a prime subset and ideal of T.

Definition 40. Let T be an ordered ternary semigroup. A fuzzy subset f of T is called a *prime fuzzy subset* of T if $f(xyz) \leq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Definition 41. Let T be an ordered ternary semigroup. A fuzzy ordered ternary subsemigroup f of T is called

(1) a *prime fuzzy ordered ternary subsemigroup* of T if f is a prime fuzzy subset of T and fuzzy ordered ternary subsemigroup,

(2) a *prime fuzzy left ideal* of T if f is a prime fuzzy subset and fuzzy left ideal of T,

(3) a *prime fuzzy right ideal* of T if f is a prime fuzzy subset and fuzzy right ideal of T,

(4) a *prime fuzzy lateral ideal* of T if f is a prime fuzzy subset and fuzzy lateral ideal of T,

(5) a *prime fuzzy ideal* of T if f is a prime fuzzy subset and fuzzy ideal of T.

Theorem 26. *Let T be an ordered ternary semigroup and A a nonempty subset of T. The following statements are true.*

(1) A is a prime subset of T if and only if f_A is a prime fuzzy subset of T.

(2) A is a prime ordered ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T if and only if f_A is a prime fuzzy ordered ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T.

Proof. (1) Let A be a prime subset of T and $x, y, z \in T$.

Case 1 : $xyz \in A$. Since A is a prime subset of T, $x \in A$ or $y \in A$ or $z \in A$. So $\max\{f_A(x), f_A(y), f_A(z)\} = 1 \geq f_A(xyz)$.

Case 2 : $xyz \notin A$. So $f_A(xyz) = 0 \leq \max\{f_A(x), f_A(y), f_A(z)\}$.

Conversely, let $x, y, z \in T$ such that $xyz \in A$. Thus $f_A(xyz) = 1$. Since f_A is prime, $\max\{f_A(x), f_A(y), f_A(z)\} = 1$. Then $f_A(x) = 1$ or $f_A(y) = 1$ or $f_A(z) = 1$. Hence $x \in A$ or $y \in A$ or $z \in A$.

(2) follows from (1) and Theorem 3.2. □

Definition 42. Let f be a fuzzy subset of an ordered ternary semigroup T. For any $t \in [0, 1]$, the set

$$f_t = \{x \in T \mid f(x) \geq t\} \text{ and } f_t^s = \{x \in T \mid f(x) > t\}$$

are called a *t-levelset* and a *t-strong levelset* of f, respectively.

Theorem 27. *Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are true.*

(1) f is a fuzzy ordered ternary subsemigroup of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is an ordered ternary subsemigroup of T.

(2) f is a fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a left ideal (right ideal, lateral ideal, ideal) of T.

Proof. (1) Assume that f is a fuzzy ordered ternary subsemigroup of T. Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x \in (f_t]$. Then there exists $y \in f_t$ such that $x \leq y$. Thus

$f(x) \geq f(y) \geq t$. Hence $x \in f_t$. Next, let $x, y, z \in f_t$. Then $f(x), f(y), f(z) \geq t$. Thus $\min\{f(x), f(y), f(z)\} \geq t$. Since f is a fuzzy ordered ternary subsemigroup of T, $f(xyz) \geq t$. Hence $xyz \in f_t$.

Conversely, assume for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is an ordered ternary subsemigroup of T. Let $x, y \in T$ such that $x \leq y$. Choose $t = f(y)$. Thus $y \in f_t$. This implies $x \in f_t$. Then $f(x) \geq t = f(y)$. Next, let $x, y, z \in T$. Choose $t = \min\{f(x), f(y), f(z)\}$. Then $f(x), f(y),\ f(z) \geq t$. Thus $x, y, z \in f_t$. Since f_t is an ordered ternary subsemigroup of T, $xyz \in f_t$. Therefore $f(xyz) \geq t = \min\{f(x), f(y), f(z)\}$.

(2) Assume that f is a fuzzy left ideal of T. Let $t \in [0, 1]$. Suppose that $f_t \neq \emptyset$. Let $x \in (f_t]$. Then there exists $y \in f_t$ such that $x \leq y$. Thus $f(x) \geq f(y) \geq t$. Next, let $x, y, z \in T$ and $z \in f_t$. So $f(xyz) \geq f(z) \geq t$. Therefore $xyz \in f_t$.

Conversely, assume for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a left ideal of T. Let $x, y \in T$ such that $x \leq y$. Choose $t = f(y)$. Thus $y \in f_t$. This implies $x \in f_t$. Then $f(x) \geq t = f(y)$. Next, let $x, y, z \in T$. Choose $t = f(z)$. Thus $z \in f_t$, this implies $f_t \neq \emptyset$. By assumption, we have f_t is a left ideal of T. So $xyz \in f_t$. Therefore $f(xyz) \geq t$. So $f(xyz) \geq f(z)$.

The other parts of (2) can be proved in a similar way. □

Theorem 28. *Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are true.*

(1) f is a prime fuzzy subset of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime subset of T.

(2) f is a prime fuzzy ordered ternary subsemigroup (prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime ordered ternary subsemigroup (prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T.

Proof. (1) Assume that f is a prime fuzzy subset of T. Let $t \in [0, 1]$. Suppose that $f_t \neq \emptyset$. Let $x, y, z \in T$ such that $xyz \in f_t$. Thus $f(xyz) \geq t$. Since f is prime,

$f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. Hence $x \in f_t$ or $y \in f_t$ or $z \in f_t$.

Conversely, let $x, y, z \in T$. Choose $t = f(xyz)$. Thus $xyz \in f_t$. Since f_t is prime, $x \in f_t$ or $y \in f_t$ or $z \in f_t$. Then $f(x) \geq t$ or $f(y) \geq t$ or $f(z) \geq t$. Therefore $\max\{f(x), f(y), f(z)\} \geq t = f(xyz)$.

(2) follows from (1) and Theorem 3.4. □

Theorem 29. *Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a fuzzy ordered ternary subsemigroup (fuzzy left ideal, fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is an ordered ternary subsemigroup (left ideal, right ideal, lateral ideal, ideal) of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.4. □

Theorem 30. *Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a prime fuzzy subset (prime fuzzy ordered ternary subsemigroup, prime fuzzy left ideal, prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T if and only if for all $t \in [0,1]$, if $f_t^s \neq \emptyset$, then f_t^s is a prime subset (prime ordered ternary subsemigroup, prime left ideal, prime right ideal, prime lateral ideal, prime ideal) of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.5. □

3.2 Fuzzy filters of ordered ternary semigroups

Now we study left filters, right filters, lateral filters, filters, fuzzy left filters, fuzzy right filters, fuzzy lateral filters and fuzzy filters of ordered ternary semigroups.

Definition 43. Let T be an ordered ternary semigroup. A nonempty subset F of T is called

(1) a *left filter* of T if (i) $F^3 \subseteq F$, (ii) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (iii) for all $x, y, z \in T, xyz \in F$ implies $z \in F$,

(2) a *right filter* of T if (i) $F^3 \subseteq F$, (ii) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (iii) for all $x, y, z \in T, xyz \in F$ implies $x \in F$,

(3) a *lateral filter* of T if (i) $F^3 \subseteq F$, (ii) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (iii) for all $x, y, z \in T, xyz \in F$ implies $y \in F$,

(4) a *filter* of T if (i) $F^3 \subseteq F$, (ii) for all $x, y \in T, x \leq y$ and $x \in F$ imply $y \in F$ and (iii) for all $x, y, z \in T, xyz \in F$ implies $x, y, z \in F$.

Definition 44. Let T be an ordered ternary semigroup. A fuzzy subset f of T is called

(1) a *fuzzy left filter* of T if for all $x, y, z \in T$ (i) $x \leq y$ implies $f(x) \leq f(y)$, (ii) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (iii)$f(xyz) \leq f(z)$,

(2) a *fuzzy right filter* of T if for all $x, y, z \in T$ (i) $x \leq y$ implies $f(x) \leq f(y)$, (ii) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (iii)$f(xyz) \leq f(x)$,

(3) a *fuzzy lateral filter* of T if for all $x, y, z \in T$ (i) $x \leq y$ implies $f(x) \leq f(y)$, (ii) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and (iii)$f(xyz) \leq f(y)$,

(4) a *fuzzy filter* of T if for all $x, y, z \in T$ (i) $x \leq y$ implies $f(x) \leq f(y)$, (ii) $f(xyz) = \min\{f(x), f(y), f(z)\}$.

We also characterize left filters (right filters, lateral filters, filters) of ordered ternary semigroups in terms of fuzzy left filters (fuzzy right filters, fuzzy lateral filters, fuzzy filters).

Theorem 31. *Let F be a nonempty subset of an ordered ternary semigroup T. Then F is a left filter (right filter, lateral filter, filter) of T if and only if the characteristic function f_F of F is a fuzzy left filter (right filter, lateral filter, filter) of T.*

Proof. Assume that F is a left filter of T. Let $x, y \in T$ such that $x \leq y$.
Case 1: $x \notin F$. Then $f_F(x) = 0$. Then $f_F(x) \leq f_F(y)$.

Case 2: $x \in F$. Since $x \leq y$ and F is a left filter of $T, y \in F$. Thus $f_F(y) = 1$. Hence $f_F(x) \leq f_F(y)$. Next, let $x, y, z \in T$.

Case 1: $x, y, z \in F$. Then $xyz \in F$. Hence $f_F(xyz) = 1$. Therefore $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\}$.

Case 2: $x \notin F$ or $y \notin F$ or $z \notin F$. So $f_F(x) = 0$ or $f_F(y) = 0$ or $f_F(z) = 0$. This implies $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\}$.

Finally, let $x, y, z \in T$.

Case 1: $xyz \in F$. Since F is a left filter of T and $xyz \in F, z \in F$. So $f_F(z) = 1$. Therefore $f_F(xyz) \leq f_F(z)$.

Case 2: $xyz \notin F$. Then $f_F(xyz) = 0$. Therefore $f_F(xyz) \leq f(z)$.

Conversely, assume f_F is a fuzzy left filter of T. Let $x, y, z \in F$. Then $f_F(x) = f_F(y) = f_F(z) = 1$. Thus $f_F(xyz) \geq \min\{f_F(x), f_F(y), f_F(z)\} = 1$. Hence $xyz \in F$. Next, let $x, y \in T$. Assume $x \leq y$ and $x \in F$. Then $f_F(x) \leq f_F(y)$ and $f_F(x) = 1$. Thus $f_F(y) = 1$, this implies $y \in F$. Finally, let $x, y, z \in T$ such that $xyz \in F$. So $f_F(xyz) = 1$. Since f_F is a fuzzy left filter of T, then $f_F(z) \geq f_F(xyz)$. This implies $f_F(z) = 1$. So $z \in F$.

The other parts can be proved in a similar way. □

Lemma 32. *Let f be a fuzzy subset of an ordered ternary semigroup T. The following statements are equivalent.*

(1) $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$ *for all* $x, y, z \in T$.

(2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ *for all* $x, y, z \in T$.

Proof. Let $x, y, z \in T$. Assume that $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$.
Since $f(xyz) = 1 - f'(xyz)$,

$$\begin{aligned} f(xyz) &\geq 1 - \max\{f'(x), f'(y), f'(z)\} \\ &\geq min\{1 - f'(x), 1 - f'(y), 1 - f'(z)\} \\ &\geq min\{f(x), f(y), f(z)\} \end{aligned}$$

Conversely, the proof is similar. □

Theorem 33. *Let f be a fuzzy subset of an ordered ternary semigroup T. Then f is a fuzzy left filter (fuzzy right filter, fuzzy lateral filter, fuzzy filter) of T if and only if the complement f' of f is a prime fuzzy left ideal (prime fuzzy right ideal, prime fuzzy lateral ideal, prime fuzzy ideal) of T.*

Proof. Assume f is a fuzzy left filter of T. Let $x, y \in T$ such that $x \leq y$. Since f is a fuzzy left filter of $T, f(x) \leq f(y)$. This implies $f'(x) \geq f'(y)$. Next, let $x, y, z \in T$. Since f is a fuzzy left filter of $T, f(xyz) \leq f(z)$. Thus $f'(xyz) \geq f'(z)$. Finally, let $x, y, z \in T$. Since f is a fuzzy left filter of $T, f(xyz) \geq \min\{f(x), f(y), f(z)\}$. By Lemma 3.9, $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$.

Conversely, assume f' is a prime fuzzy left ideal of T. Let $x, y \in T$ such that $x \leq y$. Since f' is a fuzzy left ideal of $T, f'(x) \geq f'(y)$. Therefore $f(x) \leq f(y)$. Next, let $x, y, z \in T$. Since f' is prime, $f'(xyz) \leq \max\{f'(x), f'(y), f'(z)\}$. By Lemma 3.9, we have $f(xyz) \geq \min\{f(x), f(y), f(z)\}$. Finally, let $x, y, z \in T$. Since f' is a fuzzy left ideal of $T, f'(xyz) \geq f'(z)$. Hence $f(xyz) \leq f(z)$.

The other parts can be proved in a similar way. □

Corollary 34. *Let F be a nonempty subset of an ordered ternary semigroup T. Then F is a left filter (right filter, lateral filter, filter) of T if and only if the complement f'_F of f_F is a prime fuzzy left ideal (fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of T.*

Proof. It follows by Theorem 3.8 and Theorem 3.10. □

CHAPTER 4

Rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups

The notion of rough sets was introduced by Pawlak (Pawlak, 1982). The notion of a rough set has often been compared to the notion of a fuzzy set, sometimes with a view to prove that one is more general, or, more useful than the other. Several researchs were conducted on the generalizations of the notion of fuzzy sets and rough sets. Rough ideals in semigroups were studied by Kuroki (Kuroki, 1997). Xiao and Zhang (Xiao and Zhang, 2006) studied rough prime ideals and rough fuzzy prime ideals. Later, Petchkhaew and Chinram (Petchkhaew and Chinram, 2009) studied rough ideals and fuzzy rough ideals of ternary semigroups analogous to that of semigroups considered by Kuroki (Kuroki, 1997) and Xiao and Zhang (Xiao and Zhang, 2006). Moreover, they studied fuzzy ideals of ternary semigroups analogous to that of semigroups.

In this chapter, we study rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.

4.1 Rough bi-ideals of ternary semigroups

Kar and Maity (Kar and Maity, 2007) studied congruences on ternary semigroups. First, we recall the definition of congruence on ternary semigroups.

Definition 45. Let T be a ternary semigroup. A *congruence* ρ on T is an equivalence relation on T such that for all $a, b, x, y \in T$,

$$(a, b) \in \rho \text{ implies } (xya, xyb), (xay, xby), (axy, bxy) \in \rho.$$

Definition 46. Let T be a ternary semigroup. A congruence ρ of T is called *complete* if $(a\rho)(b\rho)(c\rho) = (abc)\rho$ for all $a, b, c \in T$.

Definition 47. Let T be a ternary semigroup. Let ρ be a congruence on T and A a nonempty subset of T. The sets

$$\rho_-(A) = \{x \in T \mid x\rho \subseteq A\} \text{ and}$$

$$\rho^-(A) = \{x \in T \mid x\rho \cap A \neq \emptyset\}$$

are called the *ρ-lower* and *ρ-upper approximations* of A, respectively.

Example 4.1. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 3 \mid x - y \text{ for all } x, y \in \mathbb{Z}^-.$$

Then ρ is a congruence on $\mathbb{Z}^-$ and $\mathbb{Z}^-/\rho = \{(-1)\rho, (-2)\rho, (-3)\rho\}$. Let $A = \{-3, -6\}$. We have that $\rho_-(A) = \emptyset$ and $\rho^-(A) = (-3)\rho$.

This proposition is similar to the proof of Theorem 2.1 in Kuroki (Kuroki, 1997)

Proposition 35. *(Petchkeaw and Chinram, 2009) Let ρ and λ be congruences on a ternary semigroup T and A and B nonempty subsets of T. The following statements are true.*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$.

(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$.

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$.

(4) $A \subseteq B$ *implies* $\rho_-(A) \subseteq \rho_-(B)$.

(5) $A \subseteq B$ *implies* $\rho^-(A) \subseteq \rho^-(B)$.

(6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$.

(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$.

(8) $\rho \subseteq \lambda$ *implies* $\lambda_-(A) \subseteq \rho_-(A)$.

(9) $\rho \subseteq \lambda$ *implies* $\rho^{-}(A) \subseteq \lambda^{-}(A)$.

Definition 48. A nonempty subset A of a ternary semigroup T is called

(1) a *ρ-upper rough bi-ideal* of T if $\rho^{-}(A)$ is a bi-ideal of T,

(2) a *ρ-lower rough bi-ideal* of T if $\rho_{-}(A)$ is a bi-ideal of T.

Theorem 36. *(Petchkeaw and Chinram, 2009) Let ρ be a complete congruence on a ternary semigroup T and A, B and C nonempty subsets of T. Then*

(1) $\rho^{-}(A)\rho^{-}(B)\rho^{-}(C) \subseteq \rho^{-}(ABC)$ *and*

(2) $\rho_{-}(A)\rho_{-}(B)\rho_{-}(C) \subseteq \rho_{-}(ABC)$.

Lemma 37. *(Petchkeaw and Chinram, 2009) Let ρ be a congruence on a ternary semigroup T and A a nonempty subset of T. If A is a ternary subsemigroup of T, then A is a ρ-upper rough ternary subsemigroup of T.*

Theorem 38. *Let ρ be a congruence on a ternary semigroup T and A a nonempty subset of T. If A is a bi-ideal of T, then A is a ρ-upper rough bi-ideal of T.*

Proof. Assume A is a bi-ideal of T. Then $\rho^{-}(A) \neq \emptyset$. By Lemma 4.3, we have $\rho^{-}(A)\rho^{-}(A)\rho^{-}(A) \subseteq \rho^{-}(A)$. By Theorem 4.2 and Proposition 4.1(5), we have $\rho^{-}(A)T\rho^{-}(A)T\rho^{-}(A) = \rho^{-}(A)\rho^{-}(T)\rho^{-}(A)\rho^{-}(T)\rho^{-}(A) \subseteq \rho^{-}(ATATA) \subseteq \rho^{-}(A)$. □

However, the converse of this theorem is not true in general. For example, we can see in Example 4.1, $\rho^{-}(A)$ is a bi-ideal of $\mathbb{Z}^{-}$ but A is not.

Theorem 39. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_{-}(A) \neq \emptyset$. If A is a bi-ideal of T, then A is a ρ-lower rough bi-ideal of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.4 □

Definition 49. Let ρ be a congruence on a ternary semigroup T. The ρ-lower and ρ-upper approximations can be presented in an equivalent form as shown below:

$$\rho_{-}(A)/\rho = \{x\rho \in T/\rho \mid x\rho \subseteq A\} \text{ and}$$

$$\rho^{-}(A)/\rho = \{x\rho \in T/\rho \mid x\rho \cap A \neq \emptyset\},$$

respectively.

Now we discuss these sets as subsets of a quotient ternary semigroup T/ρ.

Lemma 40. *(Petchkeaw and Chinram, 2009) Let ρ be a complete congruence on a ternary semigroup T. If A is a ternary subsemigroup of T, then $\rho^{-}(A)/\rho$ is a ternary subsemigroup of T/ρ.*

Theorem 41. *Let ρ be a complete congruence on a ternary semigroup T. If A is a bi-ideal of T, then $\rho^{-}(A)/\rho$ is a bi-ideal of T/ρ.*

Proof. Assume A is a bi-ideal of T. By Lemma 4.6, we have $\rho^{-}(A)/\rho$ is a subsemigroup of T/ρ. Let $x\rho, y\rho, z\rho \in \rho^{-}(A)/\rho$ and $a\rho, b\rho \in T/\rho$. Then there exist $x' \in x\rho \cap A, y' \in y\rho \cap A, z' \in z\rho \cap A$ and $a' \in a\rho, b' \in b\rho$. Since ρ is complete, $x'a'y'b'z' \in (x\rho)(a\rho)(y\rho)(b\rho)(x\rho) = (xaybz)\rho$. Since A is a bi-ideal of T, $x'a'y'b'z' \in A$. Then $(xaybz)\rho \cap A \neq \emptyset$. Hence $(x\rho)(a\rho)(y\rho)(b\rho)(z\rho) \in \rho^{-}(A)/\rho$. □

Lemma 42. *(Petchkeaw and Chinram, 2009) Let ρ be a complete congruence on a ternary semigroup T. If A is a ternary subsemigroup of T, then $\rho_{-}(A)/\rho$ is a ternary subsemigroup of T/ρ.*

Theorem 43. *Let ρ be a complete congruence on a ternary semigroup T and A a nonempty subset of T such that $\rho_{-}(A)/\rho \neq \emptyset$. If A is a bi-ideal of T, then $\rho_{-}(A)/\rho$ is a bi-ideal of T/ρ.*

Proof. Assume A is a bi-ideal of T. By Lemma 4.8, we have $\rho_{-}(A)/\rho$ is a subsemigroup of T/ρ. Let $x\rho, y\rho, z\rho \in \rho_{-}(A)/\rho$ and $a\rho, b\rho \in T/\rho$. Then $x\rho \subseteq A, y\rho \subseteq A$ and $z\rho \subseteq A$. Since A is a bi-ideal of T, $(x\rho)(a\rho)(y\rho)(b\rho)(z\rho) \subseteq A$. Therefore $(x\rho)(a\rho)(y\rho)(b\rho)(z\rho) \subseteq \rho_{-}(A)/\rho$. □

4.2 Fuzzy bi-ideals of ternary semigroups

In this section, we study fuzzy bi-ideals of ternary semigroups.

Definition 50. Let T be a ternary semigroup. A fuzzy subset f of T is called a *fuzzy bi-ideal* of T if $f(abc) \geq \min\{f(a), f(b), f(c)\}$ and $f(abcde) \geq \min\{f(a), f(c), f(e)\}$ for all $a, b, c, d, e \in T$.

Theorem 44. *Let T be a ternary semigroup and A a nonempty subset of T. Then A is a bi-ideal of T if and only if f_A is a fuzzy bi-ideal of T.*

Proof. Assume A is a bi-ideal of T. Let $a, b, x, y, z \in T$.
Case 1 : $x, y, z \in A$. Since A is a bi-ideal of T, $xyz, xaybz \in A$. Therefore $f_A(xyz) = 1 \geq$ min$\{f_A(x), f_A(y), f_A(z)\}$ and $f_A(xaybz) = 1 \geq$ min$\{f_A(x), f_A(y), f_A(z)\}$.
Case 2 : $x \notin A$ or $y \notin A$ or $z \notin A$. Thus $f_A(x) = 0$ or $f_A(y) = 0$ or $f_A(z) = 0$. Hence min$\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xyz)$ and min$\{f_A(x), f_A(y), f_A(z)\} = 0 \leq f_A(xaybz)$.

Conversely, assume that f_A is a fuzzy bi-ideal of T. Let $x, y, z \in A$ and $a, b \in T$. By assumption, $f_A(xaybz) \geq min\{f_A(x), f_A(y), f_A(z)\}$. Since $f_A(x) = 1, f_A(y) = 1$ and $f_A(z) = 1$, $min\{f_A(x), f_A(y), f_A(z)\} = 1$. Then $f_A(xaybz) = 1$. Hence $xaybz \in A$. This implies that A is a bi-ideal of T. □

Definition 51. Let T be a ternary semigroup. A nonempty subset S of T is called a *prime subset* of T if for all $x, y, z \in T, xyz \in S$ implies $x \in S$ or $y \in S$ or $z \in S$.

Definition 52. Let T be a ternary semigroup. A bi-ideal S of T is called a *prime bi-ideal* of T if S is a prime subset of T.

Definition 53. Let T be a ternary semigroup. A fuzzy subset f of T is called a *prime fuzzy subset* of T if $f(xyz) \leq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Definition 54. Let T be a ternary semigroup. A fuzzy bi-ideal f of T is called a *prime fuzzy bi-ideal* of T if f is a prime fuzzy subset of T.

Lemma 45. *(Petchkeaw and Chinram, 2009) Let T be a ternary semigroup and A a nonempty subset of T. Then A is a prime subset of T if and only if f_A is a prime fuzzy subset of T.*

Theorem 46. *Let T be a ternary semigroup and A a nonempty subset of T. Then A is a prime bi-ideal of T if and only if f_A is a prime fuzzy bi-ideal of T.*

Proof. It follows from Lemma 4.11 and Theorem 4.10. □

Definition 55. Let f be a fuzzy subset of a ternary semigroup T. For any $t \in [0, 1]$, the set

$$f_t = \{x \in T \mid f(x) \geq t\} \text{ and } f_t^s = \{x \in T \mid f(x) > t\}$$

are called a *t-levelset* and a *t-strong levelset* of f, respectively.

Theorem 47. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a fuzzy bi-ideal of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a bi-ideal of T.*

Proof. Assume f is a fuzzy bi-ideal of T. Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x, y, z \in f_t$. Then $f(x) \geq t$, $f(y) \geq t$ and $f(z) \geq t$. Since f is a fuzzy bi-ideal of T, $f(xyz) \geq \min\{f(x), f(y), f(z)\} \geq t$ and $f(xaybz) \geq \min\{f(x), f(y), f(z)\} \geq t$ for all $a, b \in T$. Therefore $xyz, xaybz \in f_t$. Hence f_t is a bi-ideal of T.

Conversely, assume for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a bi-ideal of T. Let $a, b, x, y, z \in T$. Choose $t = \min\{f(x), f(y), f(z)\}$. Then $x, y, z \in f_t$. This implies that $f_t \neq \emptyset$. By assumption, we have f_t is a bi-ideal of T. So $xyz, xaybx \in f_t$. Therefore $f(xyz) \geq t$ and $f(xaybz) \geq t$. Hence $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ and $f(xaybz) \geq \min\{f(x), f(y), f(z)\}$. □

Lemma 48. *(Petchkeaw and Chinram, 2009) Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime fuzzy subset of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime subset of T.*

Theorem 49. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime fuzzy bi-ideal of T if and only if for all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime bi-ideal of T.*

Proof. It follows from Lemma 4.14 and Theorem 4.13. □

Theorem 50. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a fuzzy bi-ideal of T if and only if for all $t \in [0, 1]$, if $f_t^s \neq \emptyset$, then f_t^s is a bi-ideal of T.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.13 □

Theorem 51. *Let f be a fuzzy subset of a ternary semigroup T. Then f is a prime bi-ideal of T if and only if for all $t \in [0, 1]$, if $f_t^s \neq \emptyset$, then f_t^s is a prime bi-deal of T.*

Proof. It follows from Lemma 4.14 and Theorem 4.16. □

4.3 Rough fuzzy bi-ideals of ternary semigroups

In this section, we study rough fuzzy bi-ideals of ternary semigroups.

Definition 56. Let f be a fuzzy subset of a ternary semigroup. Then the sets

$$\rho^-(f)(x) = \sup_{a \in x\rho} f(a) \text{ and } \rho_-(f)(x) = \inf_{a \in x\rho} f(a)$$

are called the *ρ-upper* and *ρ-lower approximations* of a fuzzy set f, respectively.

Example 4.2. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \leftrightarrow 2 \mid x - y \text{ for all } a, b \in \mathbb{Z}^-.$$

Let $f(x) = \dfrac{1}{-2x}$ for all $x \in \mathbb{Z}^-$. Then

$$\rho_-(f)(x) = 0 \text{ for all } x \in \mathbb{Z}^-$$

and

$$\rho^-(f)(x) = \begin{cases} \dfrac{1}{2} & \text{if } x \in \mathbb{Z}^- \text{ is odd,} \\ \dfrac{1}{4} & \text{if } x \in \mathbb{Z}^- \text{ is even.} \end{cases}$$

Lemma 52. *(Petchkeaw and Chinram, 2009) Let ρ be a congruence on a ternary semigroup T, f a fuzzy subset of T and $t \in [0,1]$, then*

(1) $(\rho_-(f))_t = \rho_-(f_t)$ *and*

(2) $(\rho^-(f))_t^s = \rho^-(f_t^s)$.

Theorem 53. *Let ρ be a complete congruence on a ternary semigroup T. If f is a fuzzy bi-ideal of T, then $\rho^-(f)$ and $\rho_-(f)$ are fuzzy bi-ideals of T.*

Proof. It follows from Theorem 4.13, Theorem 4.16, Theorem 4.4, Theorem 4.5 and Lemma 4.18. $\square$

The converse of this theorem is not true.

4.4 Problems of homomorphisms

In this section, we study problems of homomorphisms.

Lemma 54. *(Petchkeaw and Chinram, 2009) Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) ρ_1 is a congruence on T_1.

(2) If ρ_2 is complete and φ is 1-1, then ρ_1 is complete.

(3) $\varphi(\rho_1^-(A)) = \rho_2^-(\varphi(A))$.

(4) $\varphi(\rho_{1-}(A)) \subseteq \rho_{2-}(\varphi(A))$.

(5) If φ is 1-1, then $\varphi(\rho_{1-}(A)) = \rho_{2-}(\varphi(A))$.

Lemma 55. *(Petchkeaw and Chinram, 2009) Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2.*

Theorem 56. *Let φ be an onto homomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a bi-ideal of T_1 if and only if $\rho_2^-(\varphi(A))$ is a bi-ideal of T_2.*

Proof. Assume $\rho_1^-(A)$ is a bi-ideal of T_1. By Lemma 4.21, we have $\rho_2^-(\varphi(A))$ is a ternary subsemigroup of T_2. $\varphi(\rho_1^-(A))\varphi(T_1)\varphi(\rho_1^-(A))\varphi(T_1)\varphi(\rho_1^-(A)) =$ $\varphi(\rho_1^-(A)T_1\rho_1^-(A)T_1\rho_1^-(A)) \subseteq \varphi(\rho_1^-(A))$. By Lemma 4.20(3), $\rho_2^-(\varphi(A))\rho_2^-(T_2)$ $\rho_2^-(\varphi(A))\rho_2^-(T_2)\rho_2^-(\varphi(A)) \subseteq \rho_2^-(\varphi(A))$. So $\rho_2^-(\varphi(A))$ is a bi-ideal of T_2.

Conversely, the proof is similar. □

Lemma 57. *(Petchkeaw and Chinram, 2009) Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a prime subset of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime subset of T_2.*

Theorem 58. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_1^-(A)$ is a prime bi-ideal of T_1 if and only if $\rho_2^-(\varphi(A))$ is a prime bi-ideal of T_2.*

Proof. It follows from Lemma 4.23 and Theorem 4.22. □

Lemma 59. *(Petchkeaw and Chinram, 2009) Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a ternary subsemigroup of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a ternary subsemigroup of T_2.*

Theorem 60. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x,y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a bi-ideal of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a bi-ideal of T_2.*

Proof. By Lemma 4.20(5), we have $\rho_{1-}(A) = \rho_{2-}(\varphi(A))$.
The proof of this theorem is similar to the proof of Theorem 4.22. $\square$

Lemma 61. *(Petchkeaw and Chinram, 2009) Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a prime subset of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime subset of T_2.*

Theorem 62. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. Then $\rho_{1-}(A)$ is a prime bi-ideal of T_1 if and only if $\rho_{2-}(\varphi(A))$ is a prime bi-ideal of T_2.*

Proof. It follows from Lemma 4.27 and Theorem 4.26. $\square$

We can obtain the following conclusion easily in a quotient ternary semigroup.

Corollary 63. *Let φ be an isomorphism from a ternary semigroup T_1 to a ternary semigroup T_2, ρ_2 a complete congruence on T_2, $\rho_1 = \{(x, y) \in T_1 \times T_1 \mid (\varphi(x), \varphi(y)) \in \rho_2\}$ and A a nonempty subset of T_1. The following statements are true.*

(1) $\rho_{1-}(A)/\rho_1$ is a bi-ideal of T_1/ρ_1 if and only if $\rho_{2-}(\varphi(A))/\rho_2$ is a bi-ideal of T_2/ρ_2.

(2) $\rho_1^-(A)/\rho_1$ is a bi-ideal of T_1/ρ_1 if and only if $\rho_2^-(\varphi(A))/\rho_2$ is a bi-ideal of T_2/ρ_2.

(3) $\rho_{1-}(A)/\rho_1$ is a prime bi-ideal of T_1/ρ_1 if and only if $\rho_{2-}(\varphi(A))/\rho_2$ is a prime bi-ideal of T_2/ρ_2.

(4) $\rho_1^-(A)/\rho_1$ is a prime bi-ideal of T_1/ρ_1 if and only if $\rho_2^-(\varphi(A))/\rho_2$ is a prime bi-ideal of T_2/ρ_2.

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