Convergence Criteria of a Common Fixed Point Iterative Process with Errors for Quasi-Nonexpansive Mappings in Banach Spaces

Jutharat Boonsanit

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics and Statistics

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Author	Miss Jutharat Boonsanit
Major Program	Mathematics and Statistics

Major Advisor:	**Examining Committee:**
……………….………………………………	…………….…………….……Chairperson
(Assoc. Prof. Dr. Jantana Ayaragarnchanakul)	(Dr. Pisamai Kittipoom)
	……………………………….……….…...
	(Dr. Suwicha Imnang)
	……………………………………….…….
	(Assoc. Prof. Dr. Jantana Ayaragarnchanakul)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Mathematics and Statistics

...……………………..……..

(Prof. Dr. Amornrat Phongdara)

Dean of Graduate School

ชื่อวิทยานิพนธ์	เกณฑ์การลู่เข้าของกระบวนการทำซ้ำที่มีค่าคลาดเคลื่อนกำกับสำหรับการแปลงชนิดควอไซนอนเอกซ์แพนซีฟในปริภูมิบานาค
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บทคัดย่อ

ให้ X เป็นปริภูมิบานาคเชิงจริง และ C เป็นเซตย่อยไม่ว่างของ X ซึ่งเป็นเซตปิด เมื่อ $i = 1, 2$ ให้ $T_i : C \to C$ เป็นการส่งแบบควอไซนอนเอกซ์แพนซีฟ ซึ่ง $F(T_1) \cap F(T_2) \neq \phi$ ใน C และให้ $\{\alpha_n\}, \{\beta_n\}$ เป็นลำดับใน [0,1) และ $\{u_n\}, \{v_n\}$ เป็นลำดับใน C สำหรับ $x_1 \in C$ และ $n \geq 1$ นิยามลำดับ $\{x_n\}$ และ $\{y_n\}$ ดังนี้

$$y_n = \beta_n T_2 x_n + (1 - \beta_n) x_n + v_n$$
$$x_{n+1} = \alpha_n T_1 y_n + (1 - \alpha_n) y_n + u_n$$

เราให้เงื่อนไขที่จำเป็นและเพียงพอที่จะทำให้ได้ว่า ลำดับ $\{x_n\}$ ที่นิยามข้างต้น ลู่เข้าสู่จุดตรึงร่วมบางจุด ของ T_1 และ T_2

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ABSTRACT

Let X be a real Banach space and let C be a nonempty closed subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \varnothing$ in C. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. For $x_1 \in C$ and $n \geq 1$, define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1-\beta_n)x_n + v_n \\ x_{n+1} &= \alpha_n T_1 y_n + (1-\alpha_n)y_n + u_n. \end{aligned}$$

We give sufficient and necessary conditions so that the sequence $\{x_n\}$ defined above converges to some common fixed point of T_1 and T_2.

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Jutharat Boonsanit

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CHAPTER 1

Introduction

Let X be a real Banach space, C a closed subset of X and T a mapping of C into X such that T has a nonempty set of fixed point $F(T) \subset C$ and

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C, p \in F(T)$. We shall refer to T satisfying the above conditions as *quasi-nonexpansive*. It is introduced by Tricomi for real functions and further studied by Diaz and Metcalf and Dotson for mapping in Banach spaces.

In 1972, Petryshyn and Williamson [7] presented two new theorems which provided necessary and sufficient conditions for the convergence of the successive approximation method and of the convex combination iteration method for quasi-nonexpansive mapping defined on suitable subsets of Banach spaces and with nonempty sets of fixed points as follows.

Theorem 1.1. *Let X be a real Banach space, C a closed subset of X, and T a quasi-nonexpansive mapping of C into C with nonempty fixed point set $F(T)$. Suppose there exists a point x_0 in C such that the sequence $\{x_n\}$ of iterates lies in C, where x_n is given by*

$$(S1) \qquad x_n = Tx_{n-1}, \qquad n = 1, 2, ...$$

Then $\{x_n\}$ *converges to a fixed point of* T *in* C *if and only if* $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Theorem 1.2. *Let* X *be a Banach space,* C *a closed convex subset of* X*, and* T *a quasi-nonexpansive map of* C *into* C*. Suppose there exists a point* x_0 *in* C *such that, for some* λ *in* $(0, 1)$*, the sequence* $\{x_n\} = \{T_\lambda^n(x_0)\}$ *given by* $(S2)$ *lies in* C*, where*

$$(S2) \qquad x_n = T_\lambda(x_{n-1}), \quad x_0 \in C, \quad T_\lambda = \lambda T + (1-\lambda)I, \quad \lambda \in (0, 1).$$

Then $\{x_n\}$ *converges to a fixed point of* T *in* C *if and only if*

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

They also indicated briefly how these theorems were used to deduce a number of known, as well as some new, convergence results for various special classes of mappings of nonexpansive, P-compact, and 1-set-contractive type which recently have been extensively studied by a number of authors.

In this thesis, inspired by the previous theorem we construct a new iterative procedure to approximate a common fixed point of two quasi-nonexpansive mappings and prove some convergence theorems as follows.

Theorem 1.3. *Let* X *be a real Banach space and let* C *be a nonempty closed subset of* X*. For* $i = 1, 2$*, let* $T_i : C \to C$ *be a quasi-nonexpansive mapping such that* $F(T_1) \cap F(T_2) \neq \varnothing$ *in* C*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, 1)$ *and* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* C*. Let* $x_1 \in C$ *be such that the iterative sequences* $\{x_n\}$

and $\{y_n\}$ *are in* C*, where define the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1-\beta_n)x_n + v_n \\ x_{n+1} &= \alpha_n T_1 y_n + (1-\alpha_n)y_n + u_n. \end{aligned} \tag{1.1}$$

Assume that

(i) $u_n = u'_n + u''_n$ *for* $n \geq 1$, $\sum_{n=1}^{\infty} \|v_n\| < \infty, \sum_{n=1}^{\infty} \|u'_n\| < \infty$ *and* $\|u''_n\| = o(1-\alpha_n)$*;*

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$.

Then the iterative sequence $\{x_n\}$ *converges strongly to a common fixed point of* T_1 *and* T_2 *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

Theorem 1.4. *Let* $X, C, T_i (i = 1, 2)$ *and the iterative sequence* $\{x_n\}$ *be as in Theorem* 1.3*. Assume further that the mapping* $T_i (i = 1, 2)$ *is asymptotically regular in* x_n *and there exists an increasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(r) > 0$ *for all* $r > 0$ *such that for* $i = 1, 2$*, we have*

$$\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2))) \quad \textit{for all} \quad n \geq 1.$$

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* T_1 *and* T_2*.*

CHAPTER 2

Preliminaries

In this chapter, we first collect, in section 2.1, some basic knowledge from mathematical analysis (Definition 2.2-Definition 2.10) and elementary functional analysis (Definition 2.11-Definition 2.20). Then we give, in section 2.2, detail on classical Banach Contraction theorem and in section 2.3, fixed point theorems for quasi-nonexpansive mapping.

2.1 Basic knowledge without proof

In this section, we give some well-known definitions and theorems without proof. Definitions $2.2 - 2.9$ and Theorems $2.1 - 2.13$ are from [8], Definitions 2.10 is from [2] and Definitions $2.11 - 2.20$ and Theorems $2.14 - 2.15$ are from [6].

Definition 2.1. Let S be a nonempty subset of $\mathbb{R}$.

(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound of* S and the set S is said to be *bounded above*.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound of* S and the set S is said to be *bounded below*.

(c) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

Definition 2.2. (**Supremum and infimum**). Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and S has a least upper bound, then we will call it the *supremum of* S and denote it by $\sup S$.

(b) If S is bounded below and S has a greatest lower bound, then we will call it the *infimum of* S and denote it by $\inf S$.

Definition 2.3. (**Convergent sequence**). A sequence $\{s_n\}$ of real numbers is said to *converge* to the real number s provided that

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$n > N \text{ implies } |s_n - s| < \varepsilon.$$

If $\{s_n\}$ converges to s, then we will write $\lim_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. The number s is call the *limit* of the sequence of $\{s_n\}$. A sequence that dose not converge to some real number is said to *diverge*.

Definition 2.4. (**Bounded sequence**). A sequence $\{s_n\}$ of real numbers is said to be *bounded* if there exists a constant M such that $|s_n| \leq M$ for all n.

Theorem 2.1. *Convergent sequences are bounded.*

Definition 2.5. (**Monotone sequence**). A sequence $\{s_n\}$ of real numbers is call a *nondecreasing sequence* if $s_n \leq s_{n+1}$ for all n and $\{s_n\}$ is called a *nonincreasing sequence* if $s_n \geq s_{n+1}$ for all n. Note that if $\{s_n\}$ is nondecreasing then $s_n \leq s_m$ whenever $n < m$. A sequence that is nondecreasing or nonincreasing will be called a *monotone sequence* or a *monotonic sequence*.

Theorem 2.2. (***Monotone Convergence Theorem***). *All bounded monotone sequences converge.*

Theorem 2.3.

(1) *If* $\{s_n\}$ *is an unbounded nondecreasing sequence, then* $\lim s_n = +\infty$.

(2) *If* $\{s_n\}$ *is an unbounded nonincreasing sequence, then* $\lim s_n = -\infty$.

Corollary 2.4. *If* $\{s_n\}$ *is a monotone sequence, then the sequence either converges, diverges to* $+\infty$*, or diverges to* $-\infty$*. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Definition 2.6. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}.$$

Theorem 2.5. *Let* $\{s_n\}$ *be a sequence in* $\mathbb{R}$.

(1) *If* $\lim_{n\to\infty} s_n$ *is defined* $[$*as a real number*$, +\infty$ *or* $-\infty]$*, then*

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

(2) *If* $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$*, then* $\lim_{n\to\infty} s_n$ *is defined and*

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Definition 2.7. (**Cauchy sequence**). A sequence $\{s_n\}$ of real numbers is called a *Cauchy sequence* if

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$m, n > N \text{ implies } |s_n - s_m| < \varepsilon.$$

Theorem 2.6. (***Cauchy Completeness Theorem***). *A sequence in* $\mathbb{R}$ *is convergent if and only if it is a Cauchy sequence.*

Theorem 2.7. (***Sandwich Theorem***). *Let* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be sequences and* $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$*. If* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$*, then* $\lim_{n\to\infty} b_n = L$.

Definition 2.8. (**Subsequence**). Suppose that $\{s_n\}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $\{t_k\}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{2.1}$$

and

$$t_k = s_{n_k}. \tag{2.2}$$

Thus $\{t_k\}$ is just a selection of some [possibly all] of the s_n's, taken in order.

Theorem 2.8. *If the sequence $\{s_n\}$ converges, then every subsequence converges to the same limit.*

Theorem 2.9. *Every sequence has a monotonic subsequence.*

Corollary 2.10. *Let $\{s_n\}$ be any sequence. There exists a monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$ and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.*

Definition 2.9. (**The Cauchy Criterion for Series**). We say that a series $\sum_{n=1}^{\infty} a_n$ satisfies the *Cauchy criterion* if its sequence $\{s_n\}$ of *partial sum* is a Cauchy sequence:

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$m, n > N \ \text{ implies } \ |s_n - s_m| < \varepsilon. \tag{2.3}$$

Nothing is lost in this definition if we impose the restriction $n > m$. Moreover, it is only a natural matter to work with $m-1$ where $m \leq n$ instead of m where $m < n$. Therefore (2.3) is equivalent to

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$n \geq m > N \ \text{ implies } \ |s_n - s_{m-1}| < \varepsilon. \tag{2.4}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, condition (2.4) can be written for each $\varepsilon > 0$ there exists a number N such that

$$n \geq m > N \text{ implies } \left| \sum_{k=m}^{n} a_k \right| < \varepsilon. \tag{2.5}$$

Theorem 2.11. *A series converges if and only if it satisfies the Cauchy criterion.*

Theorem 2.12. *Let $\{a_n\}$ be a sequence such that $\sum_{n=0}^{\infty} a_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Theorem 2.13. (***Mean Value Theorem***). *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Definition 2.10. (**Little-o notation**). Given two functions f and g, the statement $f = o(g)$ is equivalent to the statement

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

This statement is voiced f *is little* $-$ o *of* g or simply f *is little* $-$ o g.

The following are some basic knowledge about metric spaces and normed spaces.

Definition 2.11. (**Metric space, metric**). Let X be a nonempty set. A function d defined on $X \times X$ is called a *metric* on X (or *distance function on* X) if it satisfies the following properties :

(M1) d is a real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$. (**Symmetry**)

(M4) $d(x, z) \leq d(x, y) + d(y, z)$. (**Triangle inequality**)

In this case, a pair (X, d) is called a *metric space*.

Definition 2.12. (**Ball and Sphere**). Given a point $x_0 \in X$ and real number $r > 0$, we define three types of sets:

(1) $B(x_0;r) = \{x \in X | d(x, x_0) < r\}$. (**Open ball**)

(2) $\widetilde{B}(x_0;r) = \{x \in X | d(x, x_0) \leq r\}$. (**Close ball**)

(3) $S(x_0;r) = \{x \in X | d(x, x_0) = r\}$. (**Sphere**)

In all three cases, x_0 is called the *center* and r is called the *radius*.

Definition 2.13. (**Open set, Closed set**). A subset M of a metric space X is said to be *open* if it contains a ball about each of its points. A subset K of X is said to be *closed* if its complement (in X) is open, that is, $K^c = X - K$ is open.

Definition 2.14. (**Convergence of a sequence, limit**). A sequence $\{x_n\}$ is a metric space $X = (X, d)$ is said to *converge* or to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x$$

or, simply,

$$x_n \to x.$$

We say that $\{x_n\}$ *converges* to x or *has the limit* x. If $\{x_n\}$ is not convergent, it is said to be *divergent*.

Definition 2.15. (**Cauchy sequence, Completeness**). A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* (or *fundamental*) if for every $\varepsilon > 0$ there is an N such that

$$d(x_m, x_n) < \varepsilon \quad \text{for every} \quad m, n > N.$$

The space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Theorem 2.14. (***Closed set***). *Let M be a nonempty subset of a metric space $X = (X, d)$. M is closed if and only if the situation $x_n \in M, x_n \to x$ implies that $x \in M$.*

Definition 2.16. (**Distance**). The *distance* $d(x, A)$ from a point x to a nonempty subset A of a metric space (X, d) is defined to be

$$d(x, A) = \inf_{a \in A} d(x, a).$$

This infimum certainly exists in $\mathbb{R}$ and is nonnegative. If x is already in A, then, of course, $d(x, A) = 0$.

Definition 2.17. (**Normed space, Banach space**). Let X be a vector space. A norm $\|\cdot\|$ defined on X is called a *norm* on X if it satisfies the following properties :

(N1) $\|x\| \geq 0$

(N2) $\|x\| = 0 \Leftrightarrow x = 0$

(N3) $\|\alpha x\| = |\alpha|\|x\|$ (**Absolute homogeneity**)

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (**Triangle inequality**);

here x and y are arbitrary vectors in X and α is any scalar. In this case, a pair $(X, \|\cdot\|)$ is called a *normed space*. Note that a complete normed space is called a *Banach space*.

Theorem 2.15. (***Subspace of a Banach space***). *A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Definition 2.18. (**Linear operator**). A *linear operator* T is an operator such that
(i) the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in the vector space over the same field,
(ii) for all $x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty$$
$$T(\alpha x) = \alpha Tx.$$

Definition 2.19. (**Strong convergence**). A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in the norm*) if there is an

$x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

That is written

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

x is called the *strong limit* of $\{x_n\}$, and we say that $\{x_n\}$ *converges strongly* to x.

Definition 2.20. (**Fixed point**). Let X be a set and $T : X \to X$ be a self mapping. A *fixed point* of T is an $x \in X$ such that $Tx = x$. The set of all fixed points of T is denoted by $F(T)$, that is,

$$F(T) = \{x \in X | x = Tx\}.$$

Example 2.1. Let $X = \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x^2 - 3x + 4$. We show that T has a fixed point. By definition, x is a fixed point of T if and only if $Tx = x$. So

$$\begin{aligned} x^2 - 3x + 4 &= x \\ x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

Therefore T has exactly one fixed point and $F(T) = \{2\}$.

Example 2.2. Let $X = \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x^2 - 2x - 4$. We show that T have two fixed points. By definition, x is a fixed point of T if and only if

$Tx = x$. So

$$\begin{aligned} x^2 - 2x - 4 &= x \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x &= -1, 4 \end{aligned}$$

Therefore -1 and 4 are fixed point of T, i.e., $F(T) = \{-1, 4\}$.

Example 2.3. Let $X = \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = x - 1$. We show that T does not have any fixed point. Suppose x is a fixed point of T. Then $Tx = x$, i.e., $x - 1 = x$ which implies that $-1 = 0$. This a contradiction. Therefore T has no fixed point, i.e., $F(T) = \varnothing$.

2.2 Basic knowledge with proof

In this section, we give some basic knowledge which is known, but the proof cannot be easily found. Some are very old results while the other have proof but we want to give more detail here so that those who are interested in this area may study and understand more easily.

Lemma 2.16. *Let $\{a_n\}$ be a sequence of real numbers. Then*

$$\lim_{m\to\infty} \sup_{n\geq m} a_{n+m} = \lim_{m\to\infty} \sup_{n\geq m} a_n.$$

Proof. Let $L_1 = \lim\limits_{m\to\infty} \sup\limits_{n\geq m} a_{n+m}$ and $L_2 = \lim\limits_{m\to\infty} \sup\limits_{n\geq m} a_n$. We will prove that $L_1 = L_2$. Since $\{a_{n+m} : n \geq m\} \subseteq \{a_n : n \geq m\}$, we see that

$$\sup\{a_{n+m} : n \geq m\} \leq \sup\{a_n : n \geq m\}.$$

That is $\lim\limits_{m\to\infty} \sup\{a_{n+m} : n \geq m\} \leq \lim\limits_{m\to\infty} \sup\{a_n : n \geq m\}$, i.e., $L_1 \leq L_2$. Next, we will show that $L_1 \geq L_2$. We prove this by a contradiction. Suppose that $L_1 < L_2$. Since $\lim\limits_{m\to\infty} \sup\limits_{n\geq m} a_{n+m} = L_1$, there exists N in $\mathbb{N}$ such that $m > N$ implies

$$|\sup\{a_{n+m} : n \geq m\} - L_1| < L_2 - L_1.$$

Thus

$$a_{n+m} < L_2, \quad \forall n \geq m > N,$$

which implies that

$$\begin{aligned} a_r &< L_2, \quad \forall r \geq m > N, \\ \sup\{a_r : r \geq m\} &< L_2, \quad \forall m > N. \end{aligned}$$

Taking $m \to \infty$, we have $\limsup_{r\to\infty} a_r < L_2$, i.e., $L_2 < L_2$, a contradiction. From this, we get $L_1 \geq L_2$, as desired. □

Lemma 2.17. [*1*] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \quad \textit{for all} \quad n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n < \infty$ *exists.*

(2) *If $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \quad \text{for all} \quad n,$$

where $\{b_n\}$ and $\{\delta_n\}$ converges. We will show that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ which implies $\lim_{n\to\infty} a_n$ exists. Since we know that $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$, we need only prove that $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n$. Since $a_{n+1} \leq (1+\delta)a_n + b_n$, we have

$$\begin{aligned} a_{n+m} &\leq (1+\delta_{n+m-1})a_{n+m-1} + b_{n+m-1} \\ &\leq (e^{\delta_{n+m-1}})a_{n+m-1} + b_{n+m-1} \\ &\leq e^{\delta_{n+m-1}}\{(1+\delta_{n+m-2})a_{n+m-2} + b_{n+m-2}\} + b_{n+m-1} \end{aligned}$$

$$\begin{aligned} a_{n+m} &\leq e^{\delta_{n+m-1}}\{e^{\delta_{n+m-2}}a_{n+m-2} + b_{n+m-2}\} + b_{n+m-1} \\ &= (e^{\delta_{n+m-1}+\delta_{n+m-2}})a_{n+m-2} + e^{\delta_{n+m-1}}b_{n+m-2} + b_{n+m-1} \\ &\vdots \\ &\leq a_n e^{\left(\sum_{k=n}^{n+m-1}\delta_k\right)} + \left(\sum_{k=n}^{n+m-1} b_k\right) e^{\left(\sum_{k=n}^{n+m-1}\delta_k\right)}, \forall n,m \in \mathbb{N}. \end{aligned} \tag{2.6}$$

Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty}\delta_k < \infty$ and $\sum_{k=1}^{\infty} b_k < \infty$ converge, there exists N in $\mathbb{N}$ such that

$$\sum_{k=1}^{\infty}\delta_k < \varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} b_k < \varepsilon \quad \text{for all} \quad n > N. \tag{2.7}$$

From (2.6) and (2.7), for all $n, m \geq N$, we have

$$a_{n+m} \leq a_n e^{\varepsilon} + \varepsilon e^{\varepsilon}.$$

Thus

$$\sup_{n\geq m} a_{n+m} \leq e^{\varepsilon} \inf_{n\geq m} a_n + \varepsilon e^{\varepsilon}, \quad \forall m \geq N.$$

We see that

$$\begin{aligned} \lim_{m\to\infty}\sup_{n\geq m} a_{n+m} &\leq e^{\varepsilon}\lim_{m\to\infty}\inf_{n\geq m} a_n + \varepsilon e^{\varepsilon} \\ &= \liminf_{n\to\infty} a_n + \varepsilon e^{\varepsilon}. \end{aligned}$$

By Lemma 2.16, we see that

$$\limsup_{n\to\infty} a_n = \lim_{m\to\infty}\sup_{n\geq m} a_{n+m} \leq e^{\varepsilon}\liminf_{n\to\infty} a_n + \varepsilon e^{\varepsilon}.$$

Taking $\varepsilon \to 0$, we have

$$\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n.$$

Thus $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Hence $\lim_{n\to\infty} a_n$ exists. If $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to zero, by Theorem 2.8 we get $\lim_{n\to\infty} a_n = 0$. □

Lemma 2.18. *Let $\{x_n\}$ be a sequence in a normed space X. Assume that for any $\varepsilon > 0$, there exists an N such that*

$$\|x_{n+N} - x_N\| < \varepsilon \textit{ for all } n.$$

Then $\{x_n\}$ *is a Cauchy sequence in* X.

Proof. Let $\varepsilon > 0$. By definition, there exists N such that

$$\|x_{n+N} - x_N\| < \frac{\varepsilon}{2} \quad \text{for all } n. \tag{2.8}$$

For $m, n > N$, we have

$$\|x_n - x_N\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|x_m - x_N\| < \frac{\varepsilon}{2}. \tag{2.9}$$

Thus

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_N\| + \|x_m - x_N\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X, as desired. □

2.3 Banach Contraction Theorem

Here we study a classical theorem in fixed point theory. It is Banach Contraction Theorem. We first give the definition of a contraction. Finally, we give the proof which is from [6] with more detail.

Definition 2.21. (**Contraction**). Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a *contraction* on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Geometrically this means that any point x and y have images that are closer together than those points x and y; more precisely, the ratio $d(Tx, Ty)/d(x, y)$ does not exceed a constant α which is strictly less than 1.

Example 2.4. Let $X = \mathbb{R}$. Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \dfrac{x}{4}$. We claim that T is contraction mapping. For x and y are in $\mathbb{R}$, we see that

$$|Tx - Ty| = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}\,|x - y|.$$

Hence T is a contraction mapping.

Example 2.5. Let $X = \mathbb{R}$. Define $f : [0,1] \to [0,1]$ by $f(x) = \cos x$. To show that f is a contraction mapping on $[0,1]$, we use the mean value theorem. Since the function cosine is continuous on $[0,1]$ and differentiable on $(0,1)$. By the mean value theorem for x and y are in $[0,1]$, there exists $t \in (x,y)$ such that

$$\begin{aligned} f'(t) &= \frac{f(x) - f(y)}{x - y} \\ f(x) - f(y) &= f'(t)(x - y) \\ |f(x) - f(y)| &= |f'(t)||x - y|. \end{aligned}$$

Since $f(x) = \cos x$, $f'(t) = -\sin t$ and

$$|\cos x - \cos y| = |\sin t||x - y|.$$

Since the function sine is increasing on $[0,1]$, $|\sin t| = \sin t \leq \sin 1 \approx .8415 < 1$. Thus

$$|\cos x - \cos y| \leq (\sin 1)|x - y|.$$

Therefore f is a contraction mapping on $[0,1]$.

The following is the classical iterative process that helps us reach the fixed point of a contraction.

Definition 2.22. (**Picard iteration**). Let $X = (X,d)$ be a metric space and $T : X \to X$. *Picard iteration* of T is a recursive sequence $x_0, x_1, x_2, ...$ from a relation of the form

$$x_{n+1} = Tx_n \qquad\qquad n = 0, 1, 2, ...$$

with arbitrary $x_0 \in X$.

Note that

$$\begin{aligned} x_1 &= Tx_0 \\ x_2 &= Tx_1 = T(Tx_0) = T^2x_0 \\ &\vdots \\ x_n &= T^nx_0. \end{aligned}$$

This shows that $x_n = Tx_{n-1} = T^nx_o$.

We now ready to prove the very first fixed point theorem in the history of functional analysis.

Theorem 2.19. (***Banach Fixed Point Theorem or Contraction Theorem***). *Consider a metric space $X = (X, d)$, where $X \neq \varnothing$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.*

Proof. We will show that a sequence $\{x_n\}$ of Picard iteration of T is Cauchy, so that it converges in the complete space X, and then we prove that its limit x is a fixed point of T and T has no further fixed points. Let $x_0 \in X$ and define the $\{x_n\}$ to be a sequence of Picard iteration that is $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ Let $\varepsilon > 0$ and $m, n \in \mathbb{N} \cup \{0\}$. By definition of Picard iteration and the assumption on contraction of T, we obtain that

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1}) \\ &\leq \alpha d(x_m, x_{m-1}) \\ &= \alpha d(Tx_{m-1}, Tx_{m-2}) \\ &\leq \alpha^2 d(x_{m-1}) \\ &\vdots \\ &\leq \alpha^m d(x_1, x_0). \end{aligned} \tag{2.10}$$

Assume that $n > m$. By the triangle inequality, the formula for the sum of a geometric progression and (2.10), we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{n-1}, x_n) \\ &\leq \alpha^m d(x_1, x_0) + \alpha^{m+1} d(x_1, x_0) + ... + \alpha^{n-1} d(x_1, x_0) \\ &= (\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}) d(x_1, x_0) \\ &\leq \alpha^m (1 + \alpha + \alpha^2 + ...) d(x_1, x_0) \\ &= \frac{\alpha^m}{1-\alpha} d(x_1, x_0). \end{aligned} \tag{2.11}$$

Since $0 < \alpha < 1$, we have $\lim_{n\to\infty} \alpha^m = 0$. Since $\dfrac{\varepsilon(1-\alpha)}{d(x_0, x_1)+1} > 0$, by definition of limit, there exists N_1 such that

$$\alpha^m < \frac{\varepsilon(1-\alpha)}{d(x_0, x_1)+1} \qquad \text{for all } m > N_1.$$

By the inequality (2.11), we get

$$\begin{aligned} d(x_n, x_m) &\leq \alpha^m \frac{1}{1-\alpha} d(x_0, x_1) \\ &< \frac{\varepsilon(1-\alpha)}{d(x_0, x_1)+1} \cdot \frac{1}{1-\alpha} d(x_0, x_1) \\ &= \frac{\varepsilon d(x_0, x_1)}{d(x_0, x_1)+1} < \varepsilon \quad \text{for all} \quad m, n > N_1. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges, say $x_n \to x \in X$. That is, $\lim_{n\to\infty} d(x_n, x) = 0$.

We next show that this limit x is a fixed point of the mapping T. By the triangle inequality and definition of contraction mapping we have

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(x_n, Tx) \\ &= d(x, x_n) + d(Tx_{n-1}, Tx) \\ &\leq d(x, x_n) + \alpha d(x_{n-1}, x). \end{aligned}$$

Taking $n \to \infty$, we have

$$d(x, Tx) \leq 0.$$

By property (M1) of the metric d, we obtain that $d(x, Tx) \geq 0$ and so $d(x, Tx) = 0$. By property (M2) of metric d, we get $Tx = x$. This show that x is a fixed point of T

Finally we show that x is the only fixed point of T. Let x and y be fixed points of T. Thus $Tx = x$ and $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y)$$

and so

$$\begin{aligned} d(x, y) - \alpha d(x, y) &\leq 0 \\ (1 - \alpha) d(x, y) &\leq 0. \end{aligned}$$

Since $1 - \alpha > 0$, we have $d(x, y) = 0$. By property (M2) of the metric d, we get $x = y$ and the theorem is proved. □

2.4 Fixed Point Theorems for Quasi-nonexpansive Mapping

Here three classical theorems about fixed point of a quasi-nonexpansive mapping are studied. We first give the definition of a quasi-nonexpansive mapping which is from [7].

Definition 2.23. (**Quasi-nonexpansive mapping**). Let X be a real Banach space, C a closed subset of X and T a mapping of C into X such that T has a nonempty set of fixed points $F(T)$ in C. T is called a $quasi-nonexpansive$ $mapping$, if

$$\|Tx - p\| \leq \|x - p\|,$$

for all x in C and p in $F(T)$. If the range of T is C, i.e., $T : C \to C$, we called T a self mapping.

Example 2.6. Let $X = \mathbb{R}$ and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = \frac{x}{2} \text{ for all } x \in [0, 1].$$

Since $T0 = 0$, 0 is a fixed point of T. We see that

$$|Tx - 0| = |x - 0| = |\frac{x}{2}| \leq |x| = |x - 0|.$$

Hence T is a quasi-nonexpansive mapping.

Example 2.7. Let $X = \mathbb{R}$ and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = x^2 - x + 1.$$

Find fixed point of T.

$$\begin{aligned} x^2 - x + 1 &= x \\ x^2 - 2x + 1 &= 0 \\ (x - 1)^2 &= 0 \\ x &= 1. \end{aligned}$$

Since $T1 = 1$, 1 is a fixed point of T. We see that

$$|Tx - 1| = |x^2 - x + 1 - 1| = |x^2 - x| = |x(x - 1)| = |x||x - 1| \leq |x - 1|,$$

since $x \in [0, 1]$. Hence T is a quasi-nonexpansive mapping.

Definition 2.24. [5](**Asymptotically regular**). Let X be a real Banach space, C a closed subset of X, and T a quasi-nonexpansive mapping of C into C with nonempty fixed point set $F(T)$. T is said to be
(1) *asymptotically regular* at x_0 if $\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0$,
(2) *asymptotically regular* in x_n if $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

The following are useful lemmas we will use to obtain the theorems. They are well known. Lemma 2.20 is exercise in some text book. For Lemma 2.21 is special for our mappings and sets in the main theorem and Lemma 2.22, the proof are very hard to be found. We give proof in detail here for those who want to study them.

Lemma 2.20. *If C is a nonempty closed subset of a normed space X, $x \in X$ and $d(x, C) = 0$, then $x \in C$.*

Proof. Let C be a nonempty closed subset of a normed space X, $x \in X$ and $d(x, C) = 0$, i.e., $\inf_{y \in C} d(x, y) = 0$. Using Theorem 2.14, we will show that $x \in C$. That is we construct a sequence $y_n \in C$ such that $y_n \to x$ as $n \to \infty$. For $n \in \mathbb{N}$ we get that

$$\inf_{y \in C} d(x, y) < \inf_{y \in C} d(x, y) + \frac{1}{n}.$$

Thus by definition of infimum we obtain that for each $n \in \mathbb{N}$, there exists $y_n \in C$ such that

$$0 = \inf_{y \in C} d(x, y) < d(x, y_n) < \inf_{y \in C} d(x, y) + \frac{1}{n}.$$

By the sandwich theorem we have

$$\lim_{n \to \infty} d(x, y_n) = 0.$$

This means that $y_n \to x$. Since C is closed $y_n \in C$ and $y_n \to x$, by Theorem 2.14 we have $x \in C$. □

Lemma 2.21. *Let X be a real Banach space, C a closed subset of X and T a quasi-nonexpansive mapping of C into C such that $F(T) \neq \varnothing$ in C. Then $F(T)$ is a closed subset of C.*

Proof. We will prove that $F(T)$ is closed. To apply Theorem 2.14, we let $\{x_n\}$ in $F(T)$ and $x_n \to x$, then we will show $x \in F(T)$ i.e., $Tx = x$. By the triangle inequality and since T is a quasi-nonexpansive mapping, we have

$$\begin{aligned} 0 \leq \|Tx - x\| &= \|Tx - x_n + x_n - x\| \\ &\leq \|Tx - x_n\| + \|x_n - x\| \\ &\leq \|x - x_n\| + \|x_n - x\| \\ &= 2\|x_n - x\|. \end{aligned} \tag{2.12}$$

Since $x_n \to x$, $\lim_{n\to\infty} \|x_n - x\| = 0$. From this, (2.12) and by the sandwich theorem we get

$$\|Tx - x\| = 0.$$

By the property (N2) of norm we have $Tx = x$. Hence by Theorem 2.14 we conclude that $F(T)$ is closed. $\square$

Note that for a quasi-nonexpansive mapping $T_i : C \to C$ $(i = 1, 2)$ with the common fixed point set $F(T_1) \cap F(T_2) \neq \varnothing$, we have that $F(T_1) \neq \varnothing$ and $F(T_2) \neq \varnothing$. From Lemma 2.21 we get that $F(T_1)$ and $F(T_2)$ are closed. Thus $F(T_1) \cap F(T_2)$ is closed.

Lemma 2.22. *Let X be a metric space and C a nonempty subset of X. If $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} d(x_n, C) = d(x, C)$.*

Proof. Let $x_n \to x$. We will prove that $\lim_{n\to\infty} d(x_n, C) = d(x, C)$. By the triangle inequality, for each $n \in \mathbb{N}$, we obtain

$$d(x_n, C) \leq d(x, C) + d(x_n, x).$$

From this, for each $n \in \mathbb{N}$, we get

$$d(x_n, C) - d(x, C) \leq d(x_n, x). \tag{2.13}$$

Similarly, for each $n \in \mathbb{N}$, we can obtain that

$$d(x, C) \leq d(x_n, C) + d(x_n, x),$$

so, for each $n \in \mathbb{N}$, we get

$$-d(x_n, x) \leq d(x_n, C) - d(x, C). \tag{2.14}$$

From (2.13) and (2.14), we get

$$|d(x_n, C) - d(x, C)| \leq d(x_n, x). \tag{2.15}$$

Since $x_n \to x$, $\lim_{n\to\infty} d(x_n, x) = 0$. From this, (2.15) and the sandwich theorem we get

$$\lim_{n\to\infty} |d(x_n, C) - d(x, C)| = 0.$$

Hence $\lim_{n\to\infty} d(x_n, C) = d(x, C)$, as desired. □

Note that, for a quasi-nonexpansive mapping $T_i : C \to C$ $(i = 1, 2)$ with the common fixed point set $F(T_1) \cap F(T_2) \neq \varnothing$ and $\{x_n\}$ is a sequence in X such that $x_n \to x$. From Lemma 2.22 we get that

$$\lim_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = d(x, F(T_1) \cap F(T_2)).$$

Theorem 2.23, Proposition 2.24 and Theorem 2.25 are mentioned in [7] with no proof. We give the proof here for the same purpose as before.

Theorem 2.23. *Let X be a real Banach space, C a closed subset of X, and T a quasi-nonexpansive mapping of C into C with nonempty fixed point set $F(T)$. Suppose there exists a point x_0 in C such that the sequence $\{x_n\}$ of iterates lies in C, where x_n is given by*

$$(S1) \qquad x_n = Tx_{n-1}, \qquad n = 1, 2, ...$$

Then $\{x_n\}$ converges to a fixed point of T in C if and only if $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. We first assume that $\{x_n\}$ converges to a fixed point of T in C. Let x be a fixed point such that $x_n \to x$. We know that

$$0 \leq d(x_n, F(T)) = \inf_{y\in F(T)} d(x_n, y) \leq d(x_n, x), \quad \text{for all } n \in \mathbb{N}.$$

Since $x_n \to x$, we have $\lim_{n\to\infty} d(x_n, x) = 0$. From this and by the sandwich theorem, we get

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Conversely, we assume that $\lim_{n\to\infty} d(x_n, F(T)) = 0$, We will show that $\{x_n\}$ is a Cauchy sequence so that it converges to a point in C.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists N such that for all $n > N$ we have

$$d(x_n, F(T)) = \inf_{y\in F(T)} d(x_n, y) = \inf_{y\in F(T)} \|x_n - y\| < \frac{\varepsilon}{2}.$$

From this and definition of infimum we see that for $n > N$, there exists $y_n \in F(T)$ such that

$$\|x_n - y_n\| < \frac{\varepsilon}{2}. \tag{2.16}$$

That is, for $m, n > N$ where $n = m + k$ there exists $y_{m+k} \in F(T)$ such that

$$\|x_{m+k} - y_{m+k}\| < \frac{\varepsilon}{2}. \tag{2.17}$$

Since T is quasi-nonexpansive, $y_m \in F(T)$, $x_n = Tx_{n-1}$ and by (2.16), considering $\|x_{m+k} - y_m\|$ we get

$$\begin{aligned} \|x_{m+k} - y_m\| &= \|Tx_{m+k-1} - y_m\| \\ &\leq \|x_{m+k-1} - y_m\| \\ &= \|Tx_{m+k-2} - y_m\| \\ &\leq \|x_{m+k-2} - y_m\| \\ &\vdots \\ &\leq \|x_m - y_m\| < \frac{\varepsilon}{2}. \end{aligned}$$

Thus

$$\|x_{m+k} - y_m\| < \frac{\varepsilon}{2} \tag{2.18}$$

Hence for $n = m + k > m > N$, by (2.16) and (2.18) we have

$$\begin{aligned} \|x_{m+k} - x_m\| &= \|x_{m+k} - y_m + y_m - x_m\| \\ &\leq \|x_{m+k} - y_m\| + \|x_m - y_m\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We conclude that, for $\varepsilon > 0$ there exists N such that,

$$\|x_n - x_m\| < \varepsilon \quad \text{for all} \quad n > m > N.$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. Since X is complete and $C \subseteq X$ is closed, by Theorem 2.15 C is complete. Thus there exists $x \in C$ such that $x_n \to x$. Next we show that x is a fixed point of T. Since $x_n \to x$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$, by Lemma 2.22 we obtain that

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T)).$$

That is $d(x, F(T)) = 0$. By Lemma 2.21, $F(T)$ is closed and by Lemma 2.20, we conclude that $x \in F(T)$. □

The following is a sufficient condition for a quasi-nonexpansive mapping to have a fixed point.

Proposition 2.24. *Suppose X, C, T and x_0 satisfy the conditions of Theorem* 2.23*. Suppose further that*

(a) T is asymptotically regular at x_0.

(b) If $\{y_n\}$ is any sequence in C such that $\|(I - T)y_n\| \to 0$ as $n \to \infty$, then $\liminf_{n\to\infty} d(y_n, F(T)) = 0$.

Then $\{x_n\}$ determined by the process $(S1)$ of Theorem 2.23 *converges to a fixed point of T in C.*

Proof. Let $p \in F(T)$. Since T is quasi-nonexpansive,

$$\begin{aligned} \|x_{n+1} - p\| &= \|T^{n+1}x_0 - p\| \\ &\leq \|T^n x_0 - p\| \\ &= \|x_n - p\|, \end{aligned}$$

for all n. From this we can say that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) \quad \text{for all } n,$$

which means the sequence $\{d(x_n, F(T))\}$ is nonincreasing. Note that $\{d(x_n, F(T))\}$ is bounded below by 0. By Monotone convergence theorem (Theorem 2.2) we get

$\lim_{n\to\infty} d(x_n, F(T))$ exists. Since T is asymptotically regular at x_0, i.e., $\|T^n x_0 - T^{n+1}x_0\| \to 0$ as $n \to \infty$, by definition of x_n, we obtain that

$$\begin{aligned}\|T^n x_0 - T^{n+1}x_0\| &= \|x_n - Tx_n\| \\ &= \|(I-T)x_n\|.\end{aligned}$$

We get $\|(I-T)x_n\| \to 0$ as $n \to \infty$. By condition (b),

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus by (1) of Theorem 2.5, $\lim_{n\to\infty} d(x_n, F(T)) = \liminf_{n\to\infty} d(x_n, F(T)) = 0$.
Hence Theorem 2.23 tell us that $\{x_n\}$ converges to a fixed point of T in C. $\square$

Let X be a Banach space, C a closed convex subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such that, for some λ in $(0,1)$, the sequence $\{x_n\}$ given by $(S2)$ lies in C.

$(S2)$ $\quad x_n = T_\lambda(x_{n-1}), \quad x_0 \in C, \quad T_\lambda = \lambda T + (1-\lambda)I, \quad \lambda \in (0,1).$

We see that

$$\begin{aligned}x_1 &= T_\lambda(x_0) \\ x_2 &= T_\lambda(x_1) = T_\lambda(T_\lambda x_0) = T_\lambda^2(x_0) \\ &\vdots \\ x_n &= T_\lambda^n(x_0)\end{aligned}$$

Thus we conclude that $x_n = T_\lambda(x_{n-1}) = T_\lambda^n(x_o)$. We also see that, since T is quasi-nonexpansive , by the triangle inequality and absolute homogeneity, we obtain

$$\begin{aligned}\|T_\lambda(x) - p\| &= \|\lambda Tx + (1-\lambda)Ix - p\| \\ &= \|\lambda Tx + (1-\lambda)x - (1-\lambda)p - \lambda p\| \\ &= \|\lambda(Tx-p) + (1-\lambda)(x-p)\| \\ &\leq \lambda\|Tx-p\| + (1-\lambda)\|x-p\| \\ &\leq \lambda\|x-p\| + (1-\lambda)\|x-p\| \\ &= \|x-p\|.\end{aligned}$$

Hence $\|T_\lambda(x) - p\| \leq \|x - p\|$ for $x \in C$ and $p \in F(T)$. Therefore T_λ is quasi-nonexpansive.

Theorem 2.25. *Let X be a Banach space, C a closed convex subset of X, and T a quasi-nonexpansive map of C into C. Suppose there exists a point x_0 in C such that, for some λ in $(0,1)$, the sequence $\{x_n\} = \{T_\lambda^n(x_0)\}$ given by $(S2)$ lies in C. Then $\{x_n\}$ converges to a fixed point of T in C if and only if*

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. We first assume that $\{x_n\} = \{T_\lambda^n(x_0)\}$ converges to a fixed point of T in C. Let x be a fixed point of T such that $x_n \to x$. Since $x_n \to x$, $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We know that

$$0 \leq d(x_n, F(T)) = \inf_{y\in F(T)} d(x_n, y) \leq d(x_n, x).$$

By the sandwich theorem, we get

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Conversely, we assume that $F(T) \neq \varnothing$ and $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence so that it converges to a point in C. Let $\varepsilon > 0$, since $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$, there exists N such that

$$n > N \Longrightarrow d(x_n, F(T)) < \frac{\varepsilon}{2}. \tag{2.19}$$

By definition of $d(x_n, F(T))$ as an infimum, for $n > N$ there exists $y_n \in F(T)$ such that

$$\|x_n - y_n\| < \frac{\varepsilon}{2}. \tag{2.20}$$

For all $n = m + k > m > N$, since T_λ is a quasi-nonexpansive mapping and by (2.20), we have

$$\begin{aligned}\|x_n - y_m\| = \|x_{m+k} - y_m\| &= \|T_\lambda(x_{m+k-1}) - y_m\| \\ &\le \|x_{m+k-1} - y_m\| \\ &= \|T_\lambda(x_{m+k-2}) - y_m\| \\ &\vdots \\ &\le \|x_m - y_m\| < \frac{\varepsilon}{2}.\end{aligned}$$

Hence for $n, m > N$ we have

$$\|x_n - x_m\| \le \|x_n - y_m| + \|x_m - y_m\| < \varepsilon.$$

This shows that $\{x_n\} = \{T_\lambda^n(x_0)\}$ is Cauchy in C. Since X is complete and $C \subseteq X$ is closed, by Theorem 2.15, C is complete. Thus there exists $x \in C$ such that $x_n \to x$. Next we will show that x is a fixed point of T. By Lemma 2.22, we have

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d(x, F(T)),$$

Since $F(T)$ is closed and $d(x, F(T)) = 0$, $x \in F(T)$ by Lemma 2.20. $\square$

CHAPTER 3

Main Results

Let X be a real Banach space and let C be a nonempty closed subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \varnothing$ in C. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. We are interested in sequences in the following process. For $x_1 \in C$ and $n \geq 1$, define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} y_n &= \beta_n T_2 x_n + (1 - \beta_n) x_n + v_n \\ x_{n+1} &= \alpha_n T_1 y_n + (1 - \alpha_n) y_n + u_n. \end{aligned} \tag{3.1}$$

If $T_1 = T_2 = T$, then

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n) x_n + v_n \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) y_n + u_n. \end{aligned} \tag{3.2}$$

3.1 Main Theorems

We have the following which is the main theorem of this thesis.

Theorem 3.1. *Let X be a real Banach space and let C be a nonempty closed subset of X. For $i = 1, 2$, let $T_i : C \to C$ be a quasi-nonexpansive mapping such that $F(T_1) \cap F(T_2) \neq \varnothing$ in C. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. Let $x_1 \in C$ be such that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.1) are in C. Assume that*

(i) $u_n = u'_n + u''_n$ *for* $n \geq 1$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, $\sum_{n=1}^{\infty} \|u'_n\| < \infty$ *and* $\|u''_n\| = o(1 - \alpha_n)$;

(ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) < \infty$.

Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 if and only if

$$\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0. \tag{3.3}$$

Proof. For the necessity, we assume that $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2, i.e., there exists $p \in F(T_1)\cap F(T_2)$ such that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

From this, we have

$$\liminf_{n\to\infty} \|x_n - p\| = 0.$$

We see that

$$d(x_n, F(T_1)\cap F(T_2)) = \inf_{q\in F(T_1)\cap F(T_2)} \|x_n - q\| \le \|x_n - p\|,$$

for all n. Taking limit infimum as $n \to \infty$ and using the sandwich theorem, we obtain that

$$\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0,$$

as desired. For the sufficiency, we let $p \in F(T_1)\cap F(T_2)$. Since $T_i : C \to C$ is a quasi-nonexpansive mapping for $i = 1, 2$ and by the triangle inequality, we get

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n T_2 x_n + (1-\beta_n)x_n + v_n - p\| \\
&= \|\beta_n T_2 x_n + (1-\beta_n)x_n + v_n - p - \beta_n T_2 p + \beta_n p\| \\
&= \|\beta_n (T_2 x_n - T_2 p) + (1-\beta_n)(x_n - p) + v_n\| \\
&= \|\beta_n (T_2 x_n - p) + (1-\beta_n)(x_n - p) + v_n\| \\
&\le \beta_n \|T_2 x_n - p\| + (1-\beta_n)\|x_n - p\| + \|v_n\| \\
&\le \beta_n \|x_n - p\| + \|x_n - p\| - \beta_n \|x_n - p\| + \|v_n\| \\
&= \|x_n - p\| + \|v_n\|,
\end{aligned} \tag{3.4}$$

for all n. By assumption, we have $\|u''_n\| = o(1-\alpha_n)$, so by definition of the little-o notation, we get

$$\lim_{n\to\infty} \frac{\|u''_n\|}{1-\alpha_n} = 0.$$

We let $\varepsilon_n = \dfrac{\|u''_n\|}{1-\alpha_n}$, so $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. From this we get

$$\|u''_n\| = \varepsilon_n(1-\alpha_n). \tag{3.5}$$

Since T_1 is a quasi-nonexpansive mapping, by the triangle inequality, condition (i), (3.4) and (3.5) we obtain that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n T_1 y_n + (1-\alpha_n)y_n + u_n - p\| \\
&= \|\alpha_n T_1 y_n + (1-\alpha_n)y_n + u_n - p - \alpha_n T_1 p + \alpha_n p\| \\
&= \|\alpha_n(T_1 y_n - T_1 p) + (1-\alpha_n)(y_n - p) + u_n\| \\
&\leq \alpha_n\|T_1 y_n - p\| + (1-\alpha_n)\|y_n - p\| + \|u_n\| \\
&\leq \alpha_n\|y_n - p\| + (1-\alpha_n)\|y_n - p\| + \|u_n\| \\
&= \|y_n - p\| + \|u_n\| \\
&\leq \|x_n - p\| + \|v_n\| + \|u'_n\| + \|u''_n\| \\
&= \|x_n - p\| + \|v_n\| + \|u'_n\| + \varepsilon_n(1-\alpha_n). \qquad (3.6)
\end{aligned}$$

Let $b_n = \|v_n\| + \|u'_n\| + \varepsilon_n(1-\alpha_n)$. Therefore

$$\|x_{n+1}-p\| \leq \|x_n - p\| + b_n \tag{3.7}$$

By assumptions that $\sum_{n=1}^{\infty}\|v_n\| < \infty, \sum_{n=1}^{\infty}\|u'_n\| < \infty, \sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and $\{\varepsilon_n\}$ is bounded, thus $\sum_{n=1}^{\infty} b_n < \infty$. Hence by Lemma 2.17 $\lim_{n\to\infty}\|x_n - p\|$ exists. By (3.7) and induction, we obtain, for $m, n \geq 1$ and $p \in F(T_1)\cap F(T_2)$ that

$$\|x_{n+m}-p\| \leq \|x_n - p\| + \sum_{i=n}^{n+m-1} b_i. \tag{3.8}$$

By (3.7) and taking infimum over $p \in F(T_1)\cap F(T_2)$, we obtain

$$d(x_{n+1}, F(T_1)\cap F(T_2)) \leq d(x_n, F(T_1)\cap F(T_2)) + b_n.$$

The assumption $\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0$ implies that there exists a subsequence of $\{d(x_n, F(T_1)\cap F(T_2))\}$ converging to zero, Lemma 2.17 tells us that

$$\lim_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0. \tag{3.9}$$

We will show that $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$. From (3.9) and $\sum_{n=1}^{\infty} b_n < \infty$, there exists k such that, for $n \geq k$, we have

$$d(x_n, F(T_1) \cap F(T_2)) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{i=k}^{\infty} b_i < \frac{\varepsilon}{2}. \tag{3.10}$$

By the first inequality in (3.10) and the definition of infimum, there exists $q \in F(T_1) \cap F(T_2)$ such that

$$\|x_n - q\| < \frac{\varepsilon}{4}, \tag{3.11}$$

for $n \geq k$. We combine (3.8), (3.10), (3.11) and use the triangle inequality to get that, for $n \geq k$,

$$\begin{aligned} \|x_{n+k} - x_k\| &\leq \|x_{n+k} - q\| + \|x_k - q\| \\ &\leq \left(\|x_k - q\| + \sum_{i=n}^{n+k-1} b_i \right) + \|x_k - q\| \\ &= 2\|x_k - q\| + \sum_{i=n}^{n+k-1} b_i \\ &< 2\left(\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which means that $\{x_n\}$ is a Cauchy sequence in X by Lemma 2.18. But X is a Banach space, so there exists $x \in X$ such that $x_n \to x$. Since C is closed and $\{x_n\}$ is a sequence in C converging to x, we have $x \in C$. Since $\varnothing \neq F(T_1) \cap F(T_2) \subseteq C$ and $x_n \to x$, by Lemma 2.22, we have

$$0 = \lim_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = d(x, F(T_1) \cap F(T_2)).$$

Thus $d(x, F(T_1) \cap F(T_2)) = 0$. From this and since $F(T_1) \cap F(T_2)$ is closed, then by Lemma 2.20 and $x \in F(T_1) \cap F(T_2)$. Therefore $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 in C, as desired. □

The following corollary comes directly from the Theorem.

Corollary 3.2. *Let $X, C, T_i (i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem* 3.1. *Suppose that* (*i*) *The mapping* $T_i(i = 1, 2)$ *is asymptotically regular in* x_n *and*

(*ii*) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ *implies that* $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* T_1 *and* T_2*.*

The following theorem gives some other sufficient conditions.

Theorem 3.3. *Let* $X, C, T_i(i = 1, 2)$ *and the iterative sequence* $\{x_n\}$ *be as in Theorem* 3.1. *Assume further that the mapping* $T_i(i = 1, 2)$ *is asymptotically regular in* x_n *and there exists an increasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(r) > 0$ *for all* $r > 0$ *such that for* $i = 1, 2$*, we have*

$$\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2))) \quad \textit{for all} \quad n \geq 1.$$

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* T_1 *and* T_2*.*

Proof. By assumption that $\|x_n - T_i x_n\| \geq f(d(x_n, F(T_1) \cap F(T_2)))$ for all $n \geq 1$ and since $T_i(i = 1, 2)$ is asymptotically regular in $\{x_n\}$, we conclude that

$$0 \geq \liminf_{n\to\infty} f(d(x_n, F(T_1) \cap F(T_2))).$$

Since $f : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$\liminf_{n\to\infty} f(d(x_n, F(T_1) \cap F(T_2))) = 0. \tag{3.12}$$

We claim that $\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = 0$. Suppose not, i.e.,

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) \neq 0.$$

From this and $f : \mathbb{R}^+ \to \mathbb{R}^+$, we get

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = L > 0.$$

From this we see that

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2)) = \lim_{N\to\infty} \inf_{n>N} d(x_n, F(T_1) \cap F(T_2)) = L > 0.$$

Thus for all $\varepsilon = L > 0$, there exists $N_1 \in \mathbb{N}$ such that $N > N_1$ implies

$$\left| \inf_{n \geq N} d(x_n, F(T_1) \cap F(T_2)) - L \right| < \frac{L}{2}.$$

From this we get

$$\frac{L}{2} < \inf_{n \geq N} d(x_n, F(T_1) \cap F(T_2)) < \frac{3L}{2} \quad \text{for all} \quad N > N_1.$$

That is

$$\frac{L}{2} < d(x_n, F(T_1) \cap F(T_2)) \quad \text{for all} \quad n \geq N > N_1.$$

By the assumption, f is increasing, so

$$f(\frac{L}{2}) < f(d(x_n, F(T_1) \cap F(T_2))) \quad \text{for all} \quad n \geq N > N_1.$$

Taking infimum over n, we get

$$\begin{aligned} f(\frac{L}{2}) &\leq \inf\{f(d(x_n, F(T_1) \cap F(T_2))); n \geq N\}, \forall N > n_1 \\ &\leq \lim_{N \to \infty} \inf\{f(d(x_n, F(T_1) \cap F(T_2))); n \geq N\} \\ &= \liminf_{n \to \infty} f(d(x_n, F(T_1) \cap F(T_2))). \end{aligned}$$

Since $f(r) > 0$ if $r > 0$, we get

$$0 < f(\frac{L}{2}) \leq \liminf_{n \to \infty} f(d(x_n, F(T_1) \cap F(T_2))),$$

contradiction with (3.12). We obtain that

$$\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

By Theorem 3.1, we conclude that $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2, as desired. $\square$

If $T_1 = T_2 = T$, we have the following result.

Corollary 3.4. *Let X be a real Banach space, and let C be a nonempty closed subset of X. Let $T : C \to C$ be a quasi-nonexpansive mapping such that the fixed point set $F(T) \neq \varnothing$ in C. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequence in $[0, 1)$ and $\{u_n\}$ and $\{v_n\}$ be sequences in C. Suppose there exists an element $x_1 \in C$ for which the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined in (3.2) are in C. Assume that* *(i)* $u_n = u_n' + u_n''$ *for* $n \geq 1$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, $\sum_{n=1}^{\infty} \|u_n'\| < \infty$ *and* $\|u_n''\| = o(1-\alpha_n)$;

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$.

Then the iterative $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Corollary 3.5. *Let* X, C, T *and the iterative sequence* $\{x_n\}$ *be as in Corollary* 3.4. *Suppose that*

(i) The mapping T *is asymptotically regular in* x_n *and*

(ii) $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ *implies that* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T *in* C.

Corollary 3.6. *Let* X, C, T *and the iterative sequence* $\{x_n\}$ *be as in Corollary* 3.4. *Assume further that the mapping* T *is asymptotically regular in* x_n *and there exists an increasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(r) > 0$ *for all* $r > 0$ *such that*

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))) \quad \textit{for all} \quad n \geq 1.$$

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T *in* C.

3.2 Example

Let $X = \mathbb{R}$ and $C = [0,1]$. Then X is a Banach space with C as a closed subset. For $i = 1, 2$, define $T_i : [0,1] \to [0,1]$ by

$$T_1x = \frac{x}{10} \text{ and } T_2x = \frac{x}{5}.$$

Then

$$T_1x = x \iff x = 0 \text{ and } T_2x = x \iff x = 0.$$

Thus 0 is the only common fixed point of T_1 and T_2. That is $F(T_1) \cap F(T_2) = \{0\}$. Consider, for all $x \in [0,1]$, we get

$$\begin{aligned} |T_1x - 0| &= \left|\frac{x}{10} - 0\right| = \left|\frac{x}{10}\right| \leq |x| = |x-0| \quad \text{and} \\ |T_2x - 0| &= \left|\frac{x}{5} - 0\right| = \left|\frac{x}{5}\right| \leq |x| = |x-0|. \end{aligned}$$

Hence T_1 and T_2 are quasi-nonexpansive mapping on $[0,1]$. Let $u_n = e^{-\frac{(\sqrt{5}+1)}{2}n}$, $v_n = \frac{1}{5^n}$, $\alpha_n = 1 - \frac{1}{3^n}$, $\beta_n = \frac{1}{4^n}$. Consider the condition (i),

$$\begin{aligned} u_n' &= -\left(\frac{\sqrt{5}+1}{2}\right) e^{-\frac{(\sqrt{5}+1)}{2}n} \\ u_n'' &= \left(\frac{\sqrt{5}+1}{2}\right)^2 e^{-\frac{(\sqrt{5}+1)}{2}n} \\ u_n' + u_n'' &= \left[\frac{-\sqrt{5}-1}{2} + \frac{5+2\sqrt{5}+1}{4}\right] e^{-\frac{(\sqrt{5}+1)}{2}n} \\ &= e^{-\frac{(\sqrt{5}+1)}{2}n} \\ &= u_n. \end{aligned}$$

Hence $u_n = u_n' + u_n''$ for $n \geq 1$. Consider $\sum_{n=1}^{\infty} |u_n'|$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |u_n'| &= \sum_{n=1}^{\infty} \left| -\left(\frac{\sqrt{5}+1}{2}\right) e^{-\frac{(\sqrt{5}+1)}{2}n} \right| \\ &= \left|\frac{\sqrt{5}+1}{2}\right| \sum_{n=1}^{\infty} \left|\frac{1}{e^{\frac{(\sqrt{5}+1)}{2}n}}\right|. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} |u_n'| < \infty$. Consider $\sum_{n=1}^{\infty} |v_n|$, we have

$$\sum_{n=1}^{\infty} |v_n| = \sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n.$$

This is a geometric series with $|r| = \frac{1}{5} < 1$. Hence $\sum_{n=1}^{\infty} |v_n| < \infty$. Therefore the condition (i) holds. Next, we consider the condition (ii). Since $\alpha = 1 - \frac{1}{3^n}$,

$$1-\alpha_n = 1-\left(1-\frac{1}{3^n}\right) = \frac{1}{3^n}.$$

From this, we get $\sum_{n=1}^{\infty}(1-\alpha_n) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$. This is a geometric series with $|r| = \frac{1}{3} < 1$. So $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$. Therefore the condition (ii) holds. Choose $x_1 = 0.5$. Then the iteration in (3.1) becomes

$$\begin{aligned} y_n &= \frac{1}{4^n}\left(\frac{1}{5}\right)x_n + \left(1-\frac{1}{4^n}\right)x_n + \frac{1}{5^n} \\ x_{n+1} &= \left(1-\frac{1}{3^n}\right)\left(\frac{1}{10}\right)y_n + \frac{1}{3^n}y_n + e^{-\frac{(\sqrt{5}+1)}{2}n} \end{aligned}$$

We show that with $x_1 = 0.5, \{x_n\}$ and $\{y_n\}$ are sequence in $[0,1]$ by calculation using Microsoft office Excel. See Appendix A.

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APPENDIX

APPENDIX A

Let $u_n = e^{-\left(\frac{\sqrt{5}+1}{2}\right)n}$, $v_n = \frac{1}{5^n}$, $\alpha_n = 1 - \frac{1}{3^n}$, $\beta_n = \frac{1}{4^n}$, $T_1 x = \frac{x}{10}$ and $T_2 x = \frac{x}{5}$. Choose $x_1 = 0.5$, then $\{x_n\}$ and $\{y_n\}$ are as follows.

$$y_n = \frac{1}{4^n}\left(\frac{1}{5}\right)x_n + \left(1 - \frac{1}{4^n}\right)x_n + \frac{1}{5^n}$$

$$x_{n+1} = \left(1 - \frac{1}{3^n}\right)\left(\frac{1}{10}\right)x_n + \frac{1}{3^n}x_n + e^{-\left(\frac{\sqrt{5}+1}{2}\right)n}$$

We use Microsoft office Excel to obtain the following table for the iterative process above.

Table 3.1: Value of α_n, β_n, u_n and v_n

n	α_n	$1-\alpha_n$	u_n	β_n	$1-\beta_n$	v_n
1	0.666666667	0.333333333	0.198288153	0.25	0.75	0.2
2	0.888888889	0.111111111	0.039318192	0.0625	0.9375	0.04
3	0.962962963	0.037037037	0.007796332	0.015625	0.984375	0.008
4	0.987654321	0.012345679	0.00154592	0.00390625	0.99609375	0.0016
5	0.995884774	0.004115226	0.000306538	0.000976563	0.999023438	0.00032
6	0.998628258	0.001371742	6.07828E-05	0.000244141	0.999755859	0.000064
7	0.999542753	0.000457247	1.20525E-05	6.10352E-05	0.999938965	0.0000128
8	0.999847584	0.000152416	2.38987E-06	1.52588E-05	0.999984741	0.00000256
9	0.999949195	5.08053E-05	4.73883E-07	3.8147E-06	0.999996185	0.000000512
10	0.999983065	1.69351E-05	9.39653E-08	9.53674E-07	0.999999046	1.024E-07

Table 3.2: Value of y_n and x_{n+1}

n	y_n	x_{n+1}
1	0.975	0.58828815
2	0.598873745	0.15909294
3	0.165104279	0.02981024
4	0.031317078	0.0050256
5	0.005341669	0.00086049
6	0.00092432	0.00015436
7	0.000167148	2.8836E-05
8	3.13958E-05	5.5338E-06
9	6.04574E-06	1.0787E-06
10	1.18113E-06	2.121E-07

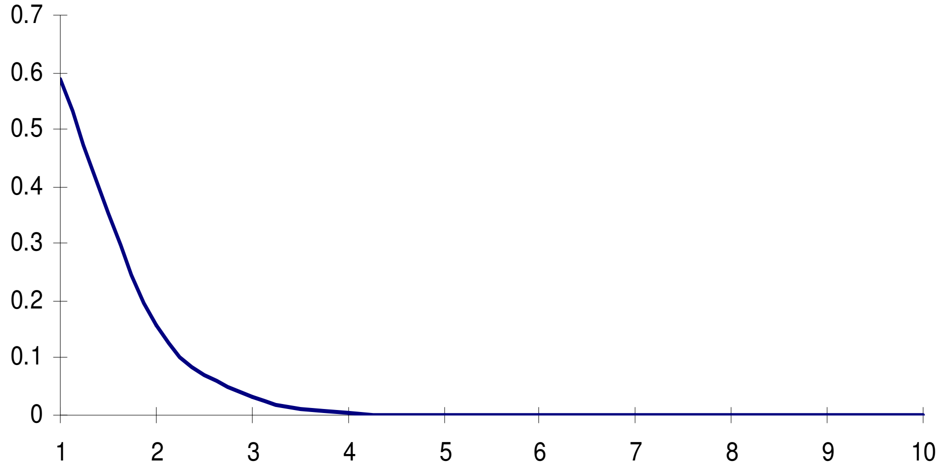

Figure 3.1: Graph of $\{x_n\}$ in our iteration

Next, we will compare sequence $\{x_n\}$ between using Picard iteration and our iteration, where $T_1x = T_2x = Tx = \frac{x}{5}$, $\alpha_n = 1 - \frac{1}{3^n}$, $\beta_n = \frac{1}{4^n}$ and $u_n = e^{-\left(\frac{\sqrt{5}+1}{2}\right)n}$. Choose $x_1 = 0.5$, then sequence $\{x_n\}$ and $\{y_n\}$ of our iteration are as follows

$$y_n = \beta_n Tx_n + (1-\beta_n)x_n + u_n$$

$$x_{n+1} = \alpha_n Ty_n + (1-\alpha_n)y_n + u_n .$$

Picard iteration is

$$x_{n+1} = Tx_n + u_n .$$

Use Microsoft office Excel, we obtain the following table.

Table 3.3: The comparison $\{x_{n+1}\}$ of our iteration and Picard iteration

n	Our iteration		Picard iteration
	y_n	x_{n+1}	x_{n+1}
1	0.973288153	0.652489291	0.298288153
2	0.659183018	0.229748841	0.098975822
3	0.234673312	0.061684277	0.027591496
4	0.063037434	0.014775999	0.007064219
5	0.015070993	0.003370353	0.001719382
6	0.003430477	0.000750643	0.000404659
7	0.000762659	0.000164863	9.29843E-05
8	0.000167251	3.58605E-05	2.09867E-05
9	3.63343E-05	7.74221E-06	4.67123E-06
10	7.83617E-06	1.66131E-06	1.02821E-06

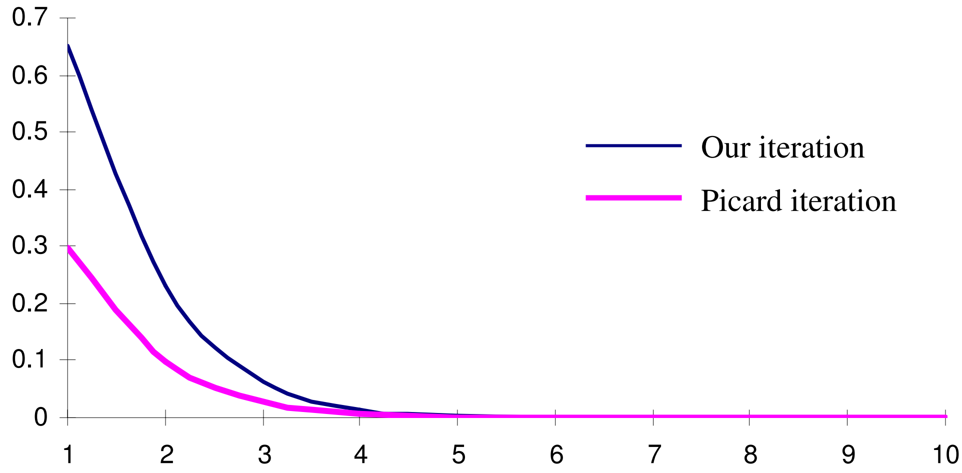

Figure 3.2: The comparison of our iteration and Picard iteration

This shows that our iteration is as good as Picard iteration. Thus, our iteration is an alternative iteration for approximation a fixed point of quasi-nonexpansive mapping.

VITAE

Name Miss Jutharat Boonsanit

Student ID 5110220125

Educational Attainment

Degree	**Name of Institution**	**Year of Graduation**
B.Sc. (Mathematics)	Prince of Songkla University	2003
GRAD.DIP. (Teaching profession)	Chiang Mai University	2004

Scholarship Awards during Enrolment

Teaching Assistant ship from Faculty of Scince, Prince of Songkla University, 2008-2010.

Scholarship from Teacher Professional Dvelopment Project, the Institute for the Promotion of Teaching Science and Technology, 2008-2010.

List of Publication and Proceeding

Jutharat Boonsanit and Jantana Ayaragarnchanakul. 2010. Convergence Criteria of a Common Fixed Point Iterative Process with Errors for Quasi-Nonexpansive Mappings in Banach Spaces. The 20^{th} Thaksin University Annual Conference : Thai Society Development with Creative Research, J.B. Hat-Yai Hotel, Hat-Yai District, Songkhla Province, 16-18 September 2010, page 399-405.