# On Riemann Approach to Stieltjes Integral

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## ABSTRACT

The Riemann-Stieltjes integral is defined in some text books on analysis. Kenneth A. Ross proved in his Elementary Analysis: The Theory of Calculus, that if the integrand is a continuous function and integrator is an increasing function then the Riemann-Stieltjes integral exists.

In this thesis, we shall weaken the condition on the Riemann-Stieltjes integral. More precisely, we prove that if $f \in RF[a,b]$ and $g \in BV[a,b]$, then $(\mathcal{R}S)\int_a^b f dg$ exists.

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Weerachai Thadee

# CONTENTS

# CHAPTER 1

## Introduction

The Riemann-Stieltjes integral was introduced in some text books on analysis. This work is a generalization of the Riemann integral. It is important for probability and statistics areas. Moreover, this integral will be used in other area in mathematics.

In 1980, Kenneth A. Ross presented Darboux integral, Riemann integral, Darboux-Stieltjes integral and Riemann-Stieltjes integral in his book Elementary Analysis: The Theory of Calculus. In this study, our concern is the Riemann-Stieltjes integral.

He shown in his book that if the integrand is a continuous function and integrator is an increasing function then the Riemann-Stieltjes integral exists.

From the condition of existence of integral above, we see that this condition is too strong and the spaces of the integrands and the integrators small. Thus, in this study, we shall weaken the condition on the Riemann-Stieltjes integral. More precisely, we prove that if a function $f$ is a regulated function on $[a, b]$ and a function $g$ is of bounded variation on $[a, b]$ then the Riemann-Stieltjes integral exists.

# CHAPTER 2

## Preliminaries

In this chapter, We first collect some basic knowledge used in this thesis.

**Definition 2.1.** Let $S$ be a non empty subset of $\mathbb{R}$.

**(a)** If a real number $M$ satisfies $s \leq M$ for all $s \in S$, then $M$ is called an *upper bound* of $S$ and the set $S$ is said to be bounded above.

**(b)** If a real number $m$ satisfies $m \leq s$ for all $s \in S$, then $m$ is called an *lower bound* of $S$ and the set $S$ is said to be bounded below.

**(c)** The set $S$ is said to be *bounded* if it is bounded above and bounded below. Thus $S$ is bounded if there exist real number $m$ and $M$ such that $S \subseteq [m, M]$.

**Definition 2.2.** Let $S$ be a non empty subset of $\mathbb{R}$.

**(a)** If $S$ is bounded above and $S$ has least upper bound, then we will call it the *supremum of* $S$ and denote it by $\sup S$.

**(b)** If $S$ is bounded below and $S$ has greatest lower bound, then we will call it the *infimum of* $S$ and denote it by $\inf S$.

**Definition 2.3.** A real-valued function $f$ defined on $A$ is said to be bounded if there exists a real number $M$ such that $|f(x)| \leq M$ for all $x \in \text{dom}(f)$.

**Theorem 2.1. (Triangle Inequality)** *Let $a, b \in \mathbb{R}$. we have*

$$|a + b| \leq |a| + |b|.$$

**Definition 2.4.** A sequence $\{s_n\}$ of real number is called a *Cauchy sequence* if for each $\epsilon > 0$ there exists a number $N$ such that $m, n > N$ implies $|s_m - s_n| < \epsilon$.

**Definition 2.5.** Let $f$ be a real-valued function defined on $A$ and $x_0 \in A$. We say that $f$ is *continuous at* $x_0$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in A$ with $|x - x_0| < \delta$ imply that

$$|f(x) - f(x_0)| < \epsilon.$$

**Definition 2.6.** Let $f$ be a real-valued function defined on $A$ and $B \subseteq A$. We say that $f$ is *continuous on* $B$ if $f$ is continuous at every point in $B$.

# CHAPTER 3

# Riemann Integral

In this chapter we will present the definitions of the Darboux integral, the Riemann integral and give their properties. Some of the results and proofs are known, see [5]. We give proof here for easy reference.

## 3.1 Darboux integral

Let $P = \{[u_i, v_i]\}_{i=1}^{n}$ be a finite collection of non-overlapping subintervals of $[a, b]$, then $P$ is said to be a *partition* of $[a, b]$. If, in addition, $\bigcup_{i=1}^{n}[u_i, v_i] = [a, b]$, then $P$ is said to be a partition of $[a, b]$.

If $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ and $Q = \{[y_{j-1}, y_j]\}_{j=1}^{m}$ are partitions of $[a, b]$, we say that $Q$ is a *refinement* of $P$ if each one of the interval $[x_{k-1}, x_k]$ from $P$ can be written as the union of intervals from $Q$ that is

$$[x_{k-1}, x_k] = [y_{r-1}, y_r] \cup [y_r, y_{r+1}] \cup \ldots \cup [y_{s-1}, y_s].$$

We now will construct a new partition from partition $P$ and $Q$ by rearrange the end points of the intervals in $P$ and $Q$ such that $a = z_0 < z_1 < \ldots < z_r = b$. We see that this new partition $\{[z_{t-1}, z_t]\}_{t=1}^{r}$ is a refinement of $P$ and $Q$ and we denote this partition by $P \nabla Q = \{[z_{t-1}, z_t]\}_{t=1}^{r}$.

**Definition 3.1.** Let $f$ be a bounded function on $[a, b]$. For $S \subseteq [a, b]$, we use the notation

$$M(f, S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f, S) = \inf\{f(x) : x \in S\}.$$

The *upper Darboux sum* is defined by

$$U(f, P) = \sum_{k=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

and the *lower Darboux sum* is

$$L(f,P)=\sum_{k=1}^{n} m(f,[x_{i-1},x_i])\cdot(x_i-x_{i-1}).$$

Note that for any partition $P$ of $[a,b]$, we have

$$U(f,P)\leq\sum_{i=1}^{n} M(f,[a,b])\cdot(x_i-x_{i-1})=M(f,[a,b])(b-a).$$

Similarly, $L(f,P)\geq m(f,[a,b])(b-a)$. Thus

$$m(f,[a,b])(b-a)\leq L(f,P)\leq U(f,P)\leq M(f,[a,b])(b-a).$$

The *upper Darboux integral* is defined by

$$U(f)=\inf\{U(f,P):P \text{ is a partition of } [a,b]\}$$

and the *lower Darboux integral* is

$$L(f)=\sup\{L(f,P):P \text{ is a partition of } [a,b]\}.$$

We say that $f$ is *Darboux integrable* or $\mathcal{D}$-integrable on $[a,b]$ if $L(f)=U(f)$. In this case, we write $(\mathcal{D})\displaystyle\int_a^b f dx=L(f)=U(f)$.

**Lemma 3.1.** [5] *Let $f$ be a bounded function on $[a,b]$. If $P$ is a partition of $[a,b]$ and $Q$ is a refinement of $P$, then*

$$L(f,P)\leq L(f,Q)\leq U(f,Q)\leq U(f,P). \tag{3.1}$$

*Proof.* First, we see that the second inequality in (3.1) is obvious. It is sufficient to show that

$$L(f,P)\leq L(f,Q) \quad\text{and}\quad U(f,Q)\leq U(f,P).$$

We may assume that

$$Q=\{[x_0,x_1],\ldots,[x_{k-1},u],[u,x_k],\ldots,[x_{m-1},x_m]\},$$

for some $k=1,2,\ldots,m$. The lower Darboux sum for $P$ and $Q$ are the same except for the terms involving $x_{k-1}$ or $x_k$. Note that

$$m(f,[x_{k-1},x_k])\leq m(f,[x_{k-1},u]) \quad\text{and}\quad m(f,[x_{k-1},x_k])\leq m(f,[u,x_k]).$$

Hence, we have

$$\begin{aligned} m(f,[x_{k-1},x_k])(x_k-x_{k-1}) &= m(f,[x_{k-1},x_k])[(x_k-u)+(u-x_{k-1})] \\ &\le m(f,[u,x_k])(x_k-u)+m(f,[x_{k-1},u])(u-x_{k-1}). \end{aligned}$$

Therefore $L(f,P) \le L(f,Q)$. Similarly $U(f,Q) \le U(f,P)$. □

**Lemma 3.2.** [5] *If $f$ is bounded function on $[a,b]$, and if $P$ and $Q$ are partitions of $[a,b]$, then $L(f,P) \le U(f,Q)$.*

*Proof.* Let $P$ and $Q$ are partitions of $[a,b]$, we have that $P\nabla Q$ is also a partition of $[a,b]$. We have that $P\nabla Q$ is a refinement of $P$ and $Q$, we can apply Lemma 3.1 to have

$$L(f,P) \le L(f,P\nabla Q) \le U(f,P\nabla Q) \le U(f,Q).$$

□

**Theorem 3.3.** [5] *If $f$ is bounded function on $[a,b]$, then $L(f) \le U(f)$.*

*Proof.* Fix a partition $P$ of $[a,b]$. By Lemma 3.2, $L(f,P)$ is a lower bound of the set

$$\{U(f,Q) : Q \text{ is a partition of } [a,b]\}.$$

Hence $L(f,P)$ is less than or equal to the infimum of the above set. That is

$$L(f,P) \le U(f).$$

Hence $U(f)$ is an upper bound for the set of $L(f,P)$ and we get $U(f) \ge L(f)$. □

**Theorem 3.4.** [5] *A bounded function $f$ on $[a,b]$ is $\mathcal{D}$-integrable if and only if for each $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \epsilon.$$

*Proof.* Suppose that $f$ is $\mathcal{D}$-integrable on $[a,b]$. Let $\epsilon > 0$ be given. There exists partition $P_1$ and $P_2$ of $[a,b]$ such that

$$L(f,P_1) > L(f) - \frac{\epsilon}{2} \quad \text{and} \quad U(f) + \frac{\epsilon}{2} > U(f,P_2).$$

Let $P = P_1 \nabla P_2$, thus $P$ is a refinement of $P_1$ and $P_2$. By Lemma 3.1 we have

$$U(f,P) \leq U(f,P_2) \quad \text{and} \quad L(f,P) \geq L(f,P_1).$$

Hence,

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\epsilon}{2} - (L(f) - \frac{\epsilon}{2}) \\ &= U(f) - L(f) + \epsilon \\ &= \epsilon. \end{aligned}$$

The last equation above holds by integrability of $f$ on $[a,b]$.

Conversely, suppose that for each $\epsilon > 0$, there exists a partition $P$ of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Then we have

$$\begin{aligned} U(f) &< U(f,P) \\ &= U(f,P) - L(f,P) + L(f,P) \\ &< \epsilon + L(f,P) \\ &\leq \epsilon + L(f). \end{aligned}$$

That is $U(f) - L(f) \leq \epsilon$. By Theorem 3.3, since $\epsilon$ was arbitrary, we can conclude that $U(f) \leq L(f)$. Therefore, we have that $f$ is $\mathcal{D}-$integrable on $[a,b]$. □

Let $\delta$ be a positive constant, $[u,v] \subseteq [a,b]$. Then an interval $[u,v]$ is said to be $\delta$-fine if $|v-u| < \delta$. Let $P = \{[u_i,v_i]\}_{i=1}^n$ be a finite collection of intervals. Then $P$ is said to be a $\delta$-fine partial partition of $[a,b]$ if $P$ is a partial partition of $[a,b]$ and each $[u_i,v_i]$ is $\delta$-fine. In addition, if $P$ is a partition of $[a,b]$, then $P$ is said to be a $\delta$-fine partition of $[a,b]$.

**Theorem 3.5.** [5]**(Cauchy's criterion for Darboux intgral)**
*A bounded function $f$ is $\mathcal{D}$-integrable on $[a,b]$ if and only if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a,b]$ we have*

$$U(f,P) - L(f,P) < \epsilon.$$

*Proof.* First, assume that for every $\epsilon > 0$, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a, b]$, we have

$$U(f, P) - L(f, P) < \epsilon.$$

By Theorem 3.4, $f$ is $\mathcal{D}$-integrable on $[a, b]$.

Conversely, assume that $f$ is $\mathcal{D}$-integrable on $[a, b]$. Let $\epsilon > 0$ be given and there is a partition $P_0 = \{[t_{i-1}, t_i]\}_{i=1}^{m}$ of $[a, b]$ such that

$$U(f, P_0) - L(f, P_0) < \frac{\epsilon}{2}. \tag{3.2}$$

Since $f$ is bounded function, there exists a positive real number $B$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

Let $\delta = \dfrac{\epsilon}{8mB}$, where $m$ is a number of interval in $P_0$. Let $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a $\delta$-fine partition of $[a, b]$. Since $P$ is a $\delta$-fine partition of $[a, b]$, thus for any two points in $\delta$-fine partition $P$, we have $|x' - x''| < \delta$. Let $Q = P \nabla P_0$. Since $Q$ is a refinement of $P$, we have

$$\begin{aligned} L(f, Q) - L(f, P) &\leq 2mB|x' - x''| \\ &< 2mB\delta \\ &= \frac{\epsilon}{4}. \end{aligned}$$

By Lemma 3.1, we have $L(f, P_0) \leq L(f, Q)$ and so

$$L(f, P_0) - L(f, P) < \frac{\epsilon}{4}.$$

Similarly we have $U(f, P) - L(f, P_0) < \dfrac{\epsilon}{4}$ and so

$$U(f, P) - L(f, P) < U(f, P_0) - L(f, P_0) + \frac{\epsilon}{2}.$$

Now, by an inequality (3.2), we have

$$U(f, P) - L(f, P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, this proof is complete. □

A finite collection $\{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ of *interval point pairs* is said to be a *division* of $[a, b]$ if $\{[x_{i-1}, x_i]\}_{i=1}^n$ is a partition of $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $i$. The point $\xi_i$ is called a *tag* or an *associate point* of $[x_{i-1}, x_i]$.

Let $\delta$ be a positive constant. An interval point pair $([u, v], \xi)$ is said to be $\delta$-*fine* if $|v - u| < \delta$, $\xi$ is any point in $[u, v]$.

A division $\{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is said to be $\delta$-fine if each $([x_{i-1}, x_i], \xi_i)$ is $\delta$-fine.

## 3.2 Riemann integral

**Definition 3.2.** A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* or $\mathcal{R}$-integrable to $A$ on [a,b] if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that whenever $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is a $\delta$-fine division of $[a, b]$, we have

$$|S(f, \delta, D) - A| \leq \epsilon,$$

where $S(f, \delta, D) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$. We denoted a constant $A$ by $(R)\int_a^b f dx$.

**Theorem 3.6. (Cauchy's criterion for Riemann integral)**
*Let $f : [a, b] \to \mathbb{R}$. Then $f$ is $\mathcal{R}$-integrable on $[a, b]$ if and only if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that whenever $D_1, D_2$ are two $\delta$-fine divisions of $[a, b]$, we have*

$$|S(f, \delta, D_1) - S(f, \delta, D_2)| \leq \epsilon.$$

*Proof.* First, we assume that $f$ is $\mathcal{R}$-integrable on $[a, b]$. Let $\epsilon > 0$ be given. There exists a positive constant $\delta$ on $[a, b]$ such that for every $D_1, D_2$ are two $\delta$-fine divisions of $[a, b]$, we get

$$|S(f, \delta, D_1) - A| \leq \frac{\epsilon}{2}$$

and

$$|S(f, \delta, D_2) - A| \leq \frac{\epsilon}{2}.$$

By triangle inequality we have that

$$\begin{aligned}|S(f,\delta,D_1)-S(f,\delta,D_2)| &= |(S(f,\delta,D_1)-A)-(S(f,\delta,D_2)-A)| \\ &\leq |S(f,\delta,D_1)-A|+|S(f,\delta,D_2)-A| \\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Conversely, we assume that for each $\epsilon > 0$, there exists a positive constant $\delta$ on $[a, b]$ such that for every $\delta$-fine division $D_1, D_2$ of $[a, b]$ we have that

$$|S(f,\delta,D_1)-S(f,\delta,D_2)| \leq \frac{\epsilon}{2}.$$

Let $\epsilon_n = \dfrac{2}{n}$ for any $n \in \mathbb{N}$ and $\delta_n$ is a positive constant on $[a, b]$. We may assume that if $m \geq n$ then $\delta_m \leq \delta_n$ for any $m, n \in \mathbb{N}$. Thus for every $\delta_m$-fine division on $[a, b]$ is also a $\delta_n$-fine division on $[a, b]$. Hence we get

$$\left|S(f,\delta_{m_1},D_{m_1})-S(f,\delta_{m_2},D_{m_2})\right| \leq \frac{1}{n},$$

where $m_1, m_2 \geq n$.

We can conclude that $\{S(f,\delta_n,D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. We have that $\{S(f,\delta_n,D_n)\}_{n\in\mathbb{N}}$ is also a convergent sequence in $\mathbb{R}$, there exists a constant $A$ such that $\lim\limits_{n\to\infty} |S(f,\delta_n,D_n)-A| = 0$.

Let $\epsilon > 0$ be given, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have that

$$|S(f,\delta_n,D_n)-A| \leq \frac{\epsilon}{2}.$$

Let $\dfrac{1}{N_2} \leq \dfrac{\epsilon}{2}$, $N = \max\{N_1, N_2\}$ and $\delta = \delta_N$, we get

$$\begin{aligned}|S(f,\delta,D)-A| &= |S(f,\delta,D)-S(f,\delta_N,D_N)+S(f,\delta_N,D_N)-A| \\ &\leq |S(f,\delta,D)-S(f,\delta_N,D_N)|+|S(f,\delta_N,D_N)-A| \\ &\leq \frac{1}{N}+\frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence $f$ is $\mathcal{R}$-integrable on $[a, b]$. □

**Theorem 3.7.** [3] *If $f$ is $\mathcal{R}$-integrable on $[a,b]$, then $f$ is bounded on $[a,b]$.*

*Proof.* Let $\epsilon = 1$ be given, by Cauchy's criterion for Riemann integral, there exists positive constant $\delta$ such that whenever $D_1, D_2$ are $\delta$-fine division of $[a,b]$, we have

$$|S(f,\delta,D_1) - S(f,\delta,D_2)| \leq 1.$$

Let $D_1 = \{([x_{i-1},x_i],\xi_i)\}_{i=1}^n$ be a fixed $\delta$-fine division of $[a,b]$ and $x$ be a point in $[a,b]$, so $x \in [x_{j-1},x_j]$ for some $j$. We define a new division $D_2$ form division $D_1$ and replacing a point $\xi_j$ by $x$, we have $D_2$ is a $\delta$-fine division of $[a,b]$. Hence

$$\begin{aligned} |(f(\xi_j) - f(x))(x_j - x_{j-1})| &= |S(f,\delta,D_1) - S(f,\delta,D_2)| \\ &\leq 1. \end{aligned}$$

Hence we get

$$\begin{aligned} |f(x)| &\leq \frac{(1 + |f(\xi_j)(x_j - x_{j-1})|)}{(x_j - x_{j-1})} \\ &\leq \frac{(1 + \sum_{i=1}^{n} |f(\xi_j)(x_j - x_{j-1})|)}{\beta}, \end{aligned}$$

where $\beta = \min\{|x_j - x_{j-1}| : j = 1,2,\ldots,n\}$. Therefore, $f$ is bounded on $[a,b]$. □

**Theorem 3.8.** [5] *A bounded function $f$ on $[a,b]$ is $\mathcal{R}$-integrable on $[a,b]$ if and only if it is $\mathcal{D}$-integrable on $[a,b]$.*

*Proof.* Assume that $f$ is $\mathcal{D}$-integrable on $[a,b]$. Let $\epsilon > 0$ be given. By Theorem 3.5, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a,b]$, we have

$$U(f,P) - L(f,P) < \epsilon.$$

Let $D$ be a $\delta$-fine division of $[a,b]$. Let $P$ be an associated partition of $D$. Clear that $P$ form a $\delta$-fine partition of $[a,b]$. Thus

$$S(f,\delta,D) \leq U(f,P) < L(f,P) + \epsilon \leq L(f) + \epsilon = (\mathcal{D})\int_a^b f dx + \epsilon,$$

and

$$S(f,\delta,D) \geq L(f,P) > U(f,P) - \epsilon \geq U(f) - \epsilon = (\mathcal{D})\int_a^b f dx - \epsilon.$$

Hence, we have $\left|S(f,\delta,D) - \int_a^b f dx\right| < \epsilon$. We can conclude that $f$ is $\mathcal{R}$-integrable on $[a,b]$ and

$$(\mathcal{R})\int_a^b f dx = (\mathcal{D})\int_a^b f dx.$$

Conversely, assume that $f$ is $\mathcal{R}$-integrable on $[a,b]$. Thus for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that whenever $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is a $\delta$-fine division of $[a,b]$, we have

$$\left|S(f,\delta,D) - (\mathcal{R})\int_a^b f dx\right| < \epsilon.$$

Let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a $\delta$-fine partition of $[a,b]$. For each $k$, $k = 1, 2, \ldots, n$, choose $\gamma_k \in [x_{k-1}, x_k]$ such that

$$f(\gamma_k) < m(f, [x_{k-1}, x_k]) + \epsilon.$$

Then $D$ form a $\delta$-fine division of $[a,b]$ such that

$$S(f,\delta,D) \leq L(f,P) + \epsilon(b-a).$$

Thus we have

$$L(f) \geq L(f,P) \geq S(f,\delta,D) - \epsilon(b-a) > (\mathcal{R})\int_a^b f dx - \epsilon - \epsilon(b-a).$$

We see that $L(f) \geq (\mathcal{R})\int_a^b f dx$. Similarly, $U(f) \leq (\mathcal{R})\int_a^b f dx$. Since we have $L(f) \leq U(f)$, we get

$$L(f) = U(f) = (\mathcal{R})\int_a^b f dx.$$

Therefore, $f$ is $\mathcal{D}$-integrable on $[a,b]$ and

$$(\mathcal{D})\int_a^b f dx = (\mathcal{R})\int_a^b f dx.$$

□

# CHAPTER 4

## The Riemann-Stieltjes Integral

In this chapter, we shall $(i)$ present the definitions of usual Darboux-Stieltjes integral, usual Riemann-Stieltjes integral and give their properties; $(ii)$ give the definitions of Darboux-Stieltjes integral, Riemann-Stieltjes integral and give their properties; $(iii)$ present the definition of regulated function, bounded variation and prove their properties; $(iv)$ present the definition of null set and give its properties; $(v)$ prove the main result.

## 4.1 Usual Darboux-Stieltjes and usual Riemann-Stieltjes integral

In this section,we give the definitions of usual Darboux-Stieltjes integral and usual Riemann-Stieltjes integral that can be found in [5]. We also give the useful lemmas and theorems that Ross stated in [5]. However, he gave them without proof. We provide proof with detail in this section.

**Definition 4.1.** Let $f$ be a bounded function and $g$ an increasing functions on $[a,b]$. For any subset $[x_{i-1},x_i]$ of $[a,b]$, we use the notation the *upper usual Darboux-Stieltjes sum* of $f$ with respect to $g$ is defined by

$$\tilde{U}(f,g,P)=\sum_{i=1}^{n} M(f,[x_{i-1},x_i])(g(x_i)-g(x_{i-1})),$$

and the *lower usual Darboux-Stieltjes sum* of $f$ with respect to $g$ is defined by

$$\tilde{L}(f,g,P)=\sum_{i=1}^{n} m(f,[x_{i-1},x_i])(g(x_i)-g(x_{i-1})).$$

We define the *upper usual Darboux-Stieltjes integral* by

$$\tilde{U}(f,g)=\inf_{P}\{\tilde{U}(f,g,P)\}$$

and the *lower usual Darboux-Stieltjes integral* is

$$\tilde{L}(f,g) = \sup_{P}\{\tilde{L}(f,g,P)\}.$$

We say that $f$ is *usual Darboux-Stieltjes integrable* or $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$ if $\tilde{L}(f,g) = \tilde{U}(f,g)$. In this case, we write $(\mathcal{UDS})\displaystyle\int_a^b f dg = \tilde{U}(f,g) = \tilde{L}(f,g)$.

**Lemma 4.1.** *Let $f$ be a bounded fucntion and $g$ an increasing function on $[a,b]$. If $P$ is a partition of $[a,b]$, and $Q$ is a refinement of $P$, then*

$$\tilde{L}(f,g,P) \leq \tilde{L}(f,g,Q) \leq \tilde{U}(f,g,Q) \leq \tilde{U}(f,g,P). \tag{4.1}$$

*Proof.* First, we see that the second inequality in (4.1) is obvious. It is sufficient to show that

$$\tilde{L}(f,g,P) \leq \tilde{L}(f,g,Q) \quad \text{and} \quad \tilde{U}(f,g,Q) \leq \tilde{U}(f,g,P).$$

We may assume that

$$Q = \{[x_0,x_1],\ldots,[x_{k-1},u],[u,x_k],\ldots,[x_{m-1},x_m]\}$$

for some $k = 1,2,\ldots,m$. The lower usual Darboux-Stieltjes sum for $P$ and $Q$ are the same except for the terms involving $x_{k-1}$ or $x_k$. Note that

$$m(f,[x_{k-1},x_k]) \leq m(f,[x_{k-1},u]) \quad \text{and} \quad m(f,[x_{k-1},x_k]) \leq m(f,[u,x_k])$$

Hence, we have

$$\begin{aligned} m(f,[x_{k-1},x_k])(g(x_k)-g(x_{k-1})) &= m(f,[x_{k-1},x_k])\cdot[(g(x_k)-g(u))+(g(u)-g(x_{k-1}))] \\ &\leq m(f,[u,x_k])(g(x_k)-g(u)) \\ &\quad +m(f,[x_{k-1},u])(g(u)-g(x_{k-1})). \end{aligned}$$

Therefore $\tilde{L}(f,g,P) \leq \tilde{L}(f,g,Q)$. Similarly $\tilde{U}(f,g,Q) \leq \tilde{U}(f,g,P)$. $\square$

**Lemma 4.2.** *If $f$ is a bounded function and $g$ is an increasing function on $[a,b]$, and if $P$ and $Q$ are partitions of $[a,b]$, then $\tilde{L}(f,g,P) \leq \tilde{U}(f,g,Q)$.*

*Proof.* Let $P$ and $Q$ be partition of $[a,b]$, we have that $P\nabla Q$ is also a partition of $[a,b]$. We have that $P\nabla Q$ is a refinement of $P$ and $Q$. We can apply Lemma 4.1 to have

$$\tilde{L}(f,g,P) \leq \tilde{L}(f,g,P\nabla Q) \leq \tilde{U}(f,g,P\nabla Q) \leq \tilde{U}(f,g,Q).$$

□

**Theorem 4.3.** *If $f$ be a bounded function and $g$ is an increasing function on $[a,b]$, then $\tilde{L}(f,g) \leq \tilde{U}(f,g)$.*

*Proof.* Fix a partition $P$ of $[a,b]$. By Lemma 4.2, $\tilde{L}(f,g,P)$ is a lower bound of the set

$$\{\tilde{U}(f,g,Q) : Q \text{ is a partition of } [a,b]\}.$$

Hence $\tilde{L}(f,g,P)$ is less than or equal to the infimum of the above set. That is

$$\tilde{L}(f,g,P) \leq \tilde{U}(f,g).$$

Hence $\tilde{U}(f,g)$ is an upper bound for the set of $\tilde{L}(f,g,P)$ and we get $\tilde{U}(f,g) \geq \tilde{L}(f,g)$. □

**Theorem 4.4.** *Let $f$ be a bounded function and $g$ an increasing function on $[a,b]$. $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$ if and only if for each $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that*

$$\tilde{U}(f,g,P) - \tilde{L}(f,g,P) < \epsilon.$$

*Proof.* Suppose that $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$. Let $\epsilon > 0$ be given. There exists partition $P_1$ and $P_2$ of $[a,b]$ such that

$$\tilde{L}(f,g,P_1) > \tilde{L}(f,g) - \frac{\epsilon}{2} \quad \text{and} \quad \tilde{U}(f,g) + \frac{\epsilon}{2} > \tilde{U}(f,g,P_2).$$

Let $P = P_1 \nabla P_2$, thus $P$ is a refinement of $P_1$ and $P_2$. By Lemma 4.1 we have

$$\tilde{U}(f,g,P) \leq \tilde{U}(f,g,P_2) \quad \text{and} \quad \tilde{L}(f,g,P) \geq \tilde{L}(f,g,P_1).$$

Hence,

$$\begin{aligned}\tilde{U}(f,g,P)-\tilde{L}(f,g,P) &\leq \tilde{U}(f,g,P_2)-\tilde{L}(f,g,P_1)\\ &< \tilde{U}(f,g)+\frac{\epsilon}{2}-(\tilde{L}(f,g)-\frac{\epsilon}{2})\\ &= \tilde{U}(f,g)-\tilde{L}(f,g)+\epsilon\\ &= \epsilon.\end{aligned}$$

The last equation above holds by integrability of $f$ on $[a,b]$.

Conversely, suppose that for each $\epsilon>0$, there exists a partition $P$ of $[a,b]$ such that $\tilde{U}(f,g,P)-\tilde{L}(f,g,P)<\epsilon$. Thus we have

$$\begin{aligned}\tilde{U}(f,g) &< \tilde{U}(f,g,P)\\ &= \tilde{U}(f,g,P)-\tilde{L}(f,g,P)+\tilde{L}(f,g,P)\\ &< \epsilon+\tilde{L}(f,g,P)\\ &\leq \epsilon+\tilde{L}(f,g).\end{aligned}$$

That is $\tilde{U}(f,g)-\tilde{L}(f,g)\leq\epsilon$. By Theorem 4.3, since $\epsilon$ was arbitrary, we can conclude that $\tilde{U}(f,g)\leq\tilde{L}(f,g)$. Therefore, we have that $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$. □

**Theorem 4.5. (Cauchy's criterion for usual Darboux-Stieltjes integral)** *Let $f$ be a bounded function and $g$ an increasing function on $[a,b]$. $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$ if and only if for each $\epsilon>0$, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a,b]$ with respect to $g$, we have $\tilde{U}(f,g,P)-\tilde{L}(f,g,P)<\epsilon$.*

*Proof.* Assume that there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a,b]$ with respect to $g$, we have $\tilde{U}(f,g,P)-\tilde{L}(f,g,P)<\epsilon$. By Theorem 4.4, $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$.

Conversely, assume that $f$ is $\mathcal{UDS}$-integrable on $[a,b]$ with respect to $g$. Let $\epsilon>0$ be given. There is a partition $P_0=\{[t_{i-1},t_i]\}_{i=1}^{m}$ of $[a,b]$ such that

$$\tilde{U}(f,g,P_0)-\tilde{L}(f,g,P_0)<\frac{\epsilon}{2}. \tag{4.2}$$

Since $f$ is bounded function, there exists a positive real number $B$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

Let $\delta = \dfrac{\epsilon}{8mB}$, where $m$ is a number of interval in $P_0$. Let $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be $\delta$-fine partition of $[a, b]$ with respect to $g$. Thus for any two points in $\delta$-fine partition $P$, we have $|x' - x''| < \delta$.

Let $Q = P \nabla P_0$. Since $Q$ is a refinement of $P$, we get

$$\begin{aligned} \tilde{L}(f, g, Q) - \tilde{L}(f, g, P) &\leq 2mB|x' - x''| \\ &< 2mB\delta \\ &= \frac{\epsilon}{4}. \end{aligned}$$

By Lemma 4.1, we have $\tilde{L}(f, g, P_0) \leq \tilde{L}(f, g, Q)$ and so

$$\tilde{L}(f, g, P_0) - \tilde{L}(f, g, P) < \frac{\epsilon}{4}.$$

Similarly, we have $\tilde{U}(f, g, P) - \tilde{L}(f, g, P_0) < \dfrac{\epsilon}{4}$ and so

$$\tilde{U}(f, g, P) - \tilde{L}(f, g, P) < \tilde{U}(f, g, P_0) - \tilde{L}(f, g, P_0) + \frac{\epsilon}{2}.$$

Now, by inequality (4.2) we have that

$$\tilde{U}(f, g, P) - \tilde{L}(f, g, P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, this proof is complete. □

**Definition 4.2.** A function $f : [a, b] \to \mathbb{R}$ is said to be *usual Riemann-Stieltjes integrable* or $\mathcal{URS}$-integrable to $A$ on $[a, b]$ with respect to $g$ if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that whenever $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^{n}$ is a $\delta$-fine division of $[a, b]$ with respect to $g$, we have

$$|\tilde{S}(f, g, D) - A| \leq \epsilon,$$

where $\tilde{S}(f, g, D) = \displaystyle\sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1}))$. We denoted a constant $A$ by $(\mathcal{URS}) \displaystyle\int_a^b f dg$.

**Theorem 4.6. (Cauchy's criterion for usual Riemann-Stieltjes integral)** *Let $f : [a,b] \to \mathbb{R}$. Then $f$ is $\mathcal{URS}$-integrable on $[a,b]$ with respect to $g$ if and only if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that whenever $D_1, D_2$ are two $\delta$-fine divisions of $[a,b]$, we have*

$$|\tilde{S}(f,g,D_1) - \tilde{S}(f,g,D_2)| \leq \epsilon.$$

*Proof.* First, we assume that $f$ is $\mathcal{URS}$-integrable on $[a,b]$ with respect to $g$ and let $\epsilon > 0$ be given, there exists a positive constant $\delta$ on $[a,b]$ such that for every $D_1, D_2$ are two $\delta$-fine divisions of $[a,b]$ with respect to $g$, we get

$$|\tilde{S}(f,g,D_1) - A| \leq \frac{\epsilon}{2}$$

and

$$|\tilde{S}(f,g,D_2) - A| \leq \frac{\epsilon}{2}.$$

By triangle inequality we have that

$$\begin{aligned} |\tilde{S}(f,g,D_1) - \tilde{S}(f,g,D_2)| &= |(\tilde{S}(f,g,D_1) - A) - (\tilde{S}(f,g,D_2) - A)| \\ &\leq |\tilde{S}(f,g,D_1) - A| + |\tilde{S}(f,g,D_2) - A| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Conversely, we assume that for each $\epsilon > 0$, there exists a positive constant $\delta$ on $[a,b]$ such that for every $\delta$-fine division $D_1, D_2$ of $[a,b]$ with respect to $g$, we have that

$$|\tilde{S}(f,g,D_1) - \tilde{S}(f,g,D_2)| \leq \frac{\epsilon}{2}.$$

Let $\epsilon_n = \dfrac{2}{n}$ for any $n \in \mathbb{N}$ and $\delta_n$ is a positive constant on $[a,b]$. We may assume that if $m \geq n$ then $\delta_m \leq \delta_n$ for any $m, n \in \mathbb{N}$. Thus for every $\delta_m$-fine division on $[a,b]$ with respect to $g$ is also a $\delta_n$-fine division on $[a,b]$ with respect to $g$. Hence we get

$$\left|\tilde{S}(f,g,D_{m_1}) - \tilde{S}(f,g,D_{m_2})\right| \leq \frac{1}{n},$$

where $m_1, m_2 \geq n$.

We can conclude that $\{\tilde{S}(f, g, D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. we have that $\{\tilde{S}(f, g, D_n)\}_{n\in\mathbb{N}}$ is also a convergent sequence in $\mathbb{R}$, there exists a constant $A$ such that $\lim\limits_{n\to\infty} |\tilde{S}(f, g, D_n) - A| = 0$.

Let $\epsilon > 0$ be given, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have that

$$|\tilde{S}(f, g, D_n) - A| \leq \frac{\epsilon}{2}.$$

Let $\frac{1}{N_2} \leq \frac{\epsilon}{2}$, $N = \max\{N_1, N_2\}$ and $\delta = \delta_N$, we get

$$\begin{aligned}
|\tilde{S}(f, g, D) - A| &= |\tilde{S}(f, g, D) - \tilde{S}(f, g, D_N) + \tilde{S}(f, g, D_N) - A| \\
&\leq |\tilde{S}(f, g, D) - \tilde{S}(f, g, D_N)| + |\tilde{S}(f, g, D_N) - A| \\
&\leq \frac{1}{N} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $f$ is $\mathcal{URS}$-integrable on $[a, b]$ with respect to $g$. □

It is known that the $\mathcal{URS}$-integrability criterion implies the $\mathcal{UDS}$-integrability criterion.

**Example 4.1.** Let $s : [0, 1] \to \mathbb{R}$ be a step function defined by

$$s(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We shall show that a step function $s$ above neither $\mathcal{UDS}$-integrable nor $\mathcal{URS}$-integrable with respect to $s$ on $[a, b]$.

Consider

$$L(s, s, P) = \sum_{i=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (g(t_k) - g(t_{k-1})) = 0 \cdot 1 = 0$$

and

$$U(s, s, P) = \sum_{i=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (g(t_k) - g(t_{k-1})) = 1 \cdot 1 = 1.$$

Hence, we have

$$U(s,s) \neq L(s,s).$$

Thus, a step function $s$ neither $\mathcal{UDS}$-integrable nor $\mathcal{URS}$-integrable with respect to $s$ on $[0,1]$.

## 4.2 Darboux-Stieltjes integral

In this section, let $f$ be a bounded function and $g$ an increasing function on $[a,b]$.

**Definition 4.3.** For a bounded function $f$, increasing function $g$ on $[a,b]$ and a partial partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$, we write

$$g(x^+) = \lim_{t \to x^+} g(t) \text{ and } g(x^-) = \lim_{t \to x^-} g(t),$$

and let

$$J(f,g,P) = \sum_{i=1}^{n} f(x_i)(g(x_i^+) - g(x_i^-)).$$

The *upper Darboux-Stieltjes sum* is defined by

$$U(f,g,P) = J(f,g,P) + \sum_{i=1}^{n} M(f,(x_{i-1},x_i)) \cdot (g(x_i^-) - g(x_{i-1}^+))$$

and the *lower Darboux-Stieltjes sum* is defined by

$$L(f,g,P) = J(f,g,P) + \sum_{i=1}^{n} m(f,(x_{i-1},x_i)) \cdot (g(x_i^-) - g(x_{i-1}^+)).$$

We define *the upper Darboux-Stieltjes integral* by

$$U(f,g) = \inf\{U(f,g,P) : P \text{ is a partition of } [a,b]\},$$

and the *lower Darboux-Stieltjes integral* is defined by

$$L(f,g) = \sup\{L(f,g,P) : P \text{ is a partition of } [a,b]\}.$$

We say that $f$ is *Darboux-Stieltjes integrable* or $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ if $L(f,g) = U(f,g)$. In this case, we write $(\mathcal{DS})\int_a^b f dg = L(f,g) = U(f,g)$.

**Lemma 4.7.** *Let $f$ be a bounded function and $g$ an increasing function on $[a, b]$. Let $P$ and $Q$ be partitions of $[a, b]$ such that $Q$ is a refinement of $P$. Then*

$$L(f, g, P) \leq L(f, g, Q) \leq U(f, g, Q) \leq U(f, g, P) \tag{4.3}$$

*Proof.* First, We see that the second inequality in (4.3) is obvious. Hence we will show that

$$L(f, g, P) \leq L(f, g, Q) \text{ and } U(f, g, Q) \leq U(f, g, P).$$

We may assume that

$$Q = \{[x_0, x_1], \ldots, [x_{k-1}, u], [u, x_k], \ldots, [x_{m-1}, x_m]\}$$

for some $k = 1, 2, \ldots, n$. The lower Darboux-Stieltjes sum for $P$ and $Q$ are the same except for the terms involving $x_{k-1}$ or $x_k$. Note that

$$m(f, (x_{k-1}, x_k)) \leq m(f, (x_{k-1}, u)) \text{ and } m(f, (x_{k-1}, x_k)) \leq m(f, (u, x_k)).$$

Hence, we have

$$\begin{aligned} m(f, (x_{k-1}, x_k))(g(x_x^-) - g(x_{k-1}^+)) &\leq m(f, (x_{k-1}, x_k))(g(x_x^-) - g(u^+)) \\ &\quad + f(u)(g(u^+) - g(u^-)) \\ &\quad + m(f(x_{k-1}, x_k))(g(u^-) - g(x_{k-1}^+)) \\ &\leq m(f, (x_{k-1}, u))(g(x_x^-) - g(u^+)) \\ &\quad + f(u)(g(u^+) - g(u^-)) \\ &\quad + m(f(u, x_k))(g(u^-) - g(x_{k-1}^+)) \end{aligned}$$

Therefore, $L(f, g, P) \leq L(f, g, Q)$. Similarly $U(f, g, Q) \leq U(f, g, P)$. □

**Lemma 4.8.** *If $f$ is a bounded function and $g$ an increasing function on $[a, b]$. If $P$ and $Q$ are partition of $[a, b]$, then $L(f, g, P) \leq U(f, g, Q)$.*

*Proof.* Let $P$ and $Q$ be partition of $[a, b]$. Thus $P \nabla Q$ is also a partition of $[a, b]$. We have that $P \nabla Q$ is a refinement of $P$ and $Q$. We can apply Lemma (4.7) to have that $L(f, g, P) \leq L(f, g, P \nabla Q) \leq U(f, g, P \nabla Q) \leq U(f, g, Q)$. □

**Theorem 4.9.** *For every bounded function $f$ and increasing function $g$ on $[a,b]$, we have $L(f,g) \leq U(f,g)$*

*Proof.* Fix a partition $P$ of $[a,b]$. By Lemma 4.8, $L(f,g,P)$ is a lower bound of the set

$$\{U(f,g,Q) : Q \text{ is a partition of } [a,b]\}.$$

Hence $L(f,g,P)$ is less than or equal to the infimum of the above set. That is

$$L(f,g,P) \leq U(f,g).$$

Hence $U(f,g)$ is an upper bound for the set of $L(f,g,P)$ and we have $L(f,g) \leq U(f,g)$. □

## 4.3 Basic properties for Darboux-Stieltjes integral

**Theorem 4.10.** *Let $f$ be a bounded function and $g$ an increasing function on $[a,b]$. $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respest to $g$ if and only if for each $\epsilon > 0$ there exists a partition $P$ of $[a,b]$ such that*

$$U(f,g,P) - L(f,g,P) < \epsilon.$$

*Proof.* Suppose that $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$. Let $\epsilon > 0$ be given. There exists partition $P_1$ and $P_2$ of $[a,b]$ such that

$$L(f,g,P_1) > L(f,g) - \frac{\epsilon}{2} \quad \text{and} \quad U(f,g) + \frac{\epsilon}{2} > U(f,g,P_2).$$

Let $P = P_1 \nabla P_2$, thus $P$ is a refinement of $P_1$ and $P_2$. By Lemma 4.7 we have

$$U(f,g,P) \leq U(f,g,P_2) \quad \text{and} \quad L(f,g,P) \geq L(f,g,P_1).$$

Hence,

$$\begin{aligned} U(f,g,P) - L(f,g,P) &\leq U(f,g,P_2) - L(f,g,P_1) \\ &< U(f,g) + \frac{\epsilon}{2} - (L(f,g) - \frac{\epsilon}{2}) \\ &= U(f,g) - L(f,g) + \epsilon \\ &= \epsilon. \end{aligned}$$

The last equation above holds by integrability of $f$ on $[a, b]$.

Conversely, suppose that for each $\epsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $U(f, g, P) - L(f, g, P) < \epsilon$. Then we have

$$\begin{aligned} U(f, g) &< U(f, g, P) \\ &= U(f, g, P) - L(f, g, P) + L(f, g, P) \\ &< \epsilon + L(f, g, P) \\ &\leq \epsilon + L(f, g). \end{aligned}$$

That is $U(f, g) - L(f, g) \leq \epsilon$. By Theorem 4.9, since $\epsilon$ was arbitrary, we can conclude that $U(f, g) \leq L(f, g)$. Therefore, we have that $f$ is $\mathcal{DS}$-integrable on $[a, b]$ with respect to $g$. □

Let $\delta$ be a positive constant, $[u, v] \subseteq [a, b]$ then an interval $[u, v]$ is said to be $\delta$-fine with respect to $g$ if

$$|g(v^-) - g(u^+)| < \delta.$$

Let $P = \{[u_i, v_i]\}_{i=1}^n$ be a finite collection of intervals. Then $P$ is said to be a $\delta$-fine partial partition with respect to $g$ of $[a, b]$ if $P$ is a partial partition of $[a, b]$ and each $[u_i, v_i]$ is $\delta$-fine with respect to $g$. In addition, if $P$ is a partition of $[a, b]$, then $P$ is said to be a $\delta$-fine partition with respect to $g$ of $[a, b]$.

Note that for any increasing function $g$ on $[a, b]$, given a positive constant $\delta$, there exists a $\delta$-fine partition on $[a, b]$ with respect to $g$.

**Theorem 4.11. (Cauchy's criterion for Darboux-Stieltjes intgral)** *Let $f$ be a bounded function and $g$ an increasing function on $[a, b]$. $f$ is $\mathcal{DS}$-integrable on $[a, b]$ with respect to $g$ if and only if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a, b]$ with respect to $g$, we have $U(f, g, P) - L(f, g, P) < \epsilon$.*

*Proof.* Assume that there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a, b]$ with respect to $g$, we have $U(f, g, P) - L(f, g, P) < \epsilon$. By Theorem 4.10, $f$ is $\mathcal{DS}$-integrable on $[a, b]$ with respect to $g$.

Conversely, assume that $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$. Let $\epsilon > 0$ be given. There is a partition $P_0 = \{[t_{i-1}, t_i]\}_{i=1}^m$ of $[a,b]$ such that

$$U(f,g,P_0) - L(f,g,P_0) < \frac{\epsilon}{2}. \tag{4.4}$$

Since $f$ is bounded function, there exists a positive real number $B$ such that $|f(x)| \leq B$ for all $x \in [a,b]$.

Let $\delta = \dfrac{\epsilon}{8mB}$, where $m$ is a number of interval in $P_0$. Let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be $\delta$-fine partition of $[a,b]$ with respect to $g$. Since $P$ is a $\delta$-fine partition of $[a,b]$ with respect to $g$, for any two points in $\delta$-fine partition $P$, we have $|x' - x''| < \delta$. Let $Q = P \nabla P_0$. Since $Q$ is a refinement of $P$, we get

$$\begin{aligned} L(f,g,Q) - L(f,g,P) &\leq 2mB|x' - x''| \\ &< 2mB\delta \\ &= \frac{\epsilon}{4}. \end{aligned}$$

By Lemma 4.7 we have $L(f,g,P_0) \leq L(f,g,Q)$ and so

$$L(f,g,P_0) - L(f,g,P) < \frac{\epsilon}{4}.$$

Similarly we have $U(f,g,P) - L(f,g,P_0) < \dfrac{\epsilon}{4}$ and so

$$U(f,g,P) - L(f,g,P) < U(f,g,P_0) - L(f,g,P_0) + \frac{\epsilon}{2}.$$

Now, by inequality (4.4) we have that

$$U(f,g,P) - L(f,g,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, this proof is complete. □

We shall use the short hand notation in following prove,

$$U'(f,g,P) = \sum_{i=1}^{n} M(f,(x_{i-1},x_i))(g(x_i^-) - g(x_{i-1}^+))$$

and

$$L'(f,g,P) = \sum_{i=1}^{n} m(f,(x_{i-1},x_i))(g(x_i^-) - g(x_{i-1}^+)).$$

**Theorem 4.12.** *Let $f$ and $h$ be bounded functions and $g$ an increasing function on $[a,b]$. If $f$ and $h$ are $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and if $\alpha \geq 0$, then*

**(i)** *$f+h$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and*

$$(\mathcal{DS})\int_a^b (f+h)dg = (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b hdg.$$

**(ii)** *$\alpha f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and*

$$(\mathcal{DS})\int_a^b \alpha fdg = \alpha(\mathcal{DS})\int_a^b fdg.$$

*Proof.* (i) Let $f$ and $h$ be bounded functions on $[a,b]$. Assume that $f$ and $h$ are $\mathcal{D}S$-integrable on $[a,b]$ with respect to $g$, we have

$$\begin{aligned} J(f+h,g,P) &= \sum_{i=1}^{n}(f+h)(x_i)(g(x_i^+)-g(x_i^-)) \\ &= \sum_{i=1}^{n}(f(x_i)+h(x_i))(g(x_i^+)-g(x_i^-)) \\ &= \sum_{i=1}^{n} f(x_i)(g(x_i^+)-g(x_i^-)) + \sum_{i=1}^{n} h(x_i)(g(x_i^+)-g(x_i^-)) \\ &= J(f,g,P)+J(h,g,P) \end{aligned}$$

and

$$\begin{aligned} U'(f+h,g,P) &= \sum_{i=1}^{n} M((f+h),(x_{i-1},x_i))(g(x_i^-)-g(x_{i-1}^+)) \\ &\leq \sum_{i=1}^{n} M(f,(x_{i-1},x_i))(g(x_i^-)-g(x_{i-1}^+)) \\ &\quad + \sum_{i=1}^{n} M(h,(x_{i-1},x_i))(g(x_i^-)-g(x_{i-1}^+)) \\ &= U'(f,g,P)+U'(h,g,P). \end{aligned}$$

Similarly, $L'(f+h,g,P) \geq L'(f,h,P)+L'(h,g,P)$.

For any partition $P$ of $[a,b]$, we have

$$\begin{aligned} U(f+h,g,P) &= J(f+h,g,P)+U'(f+h,g,P) \\ &\leq J(f,g,P)+J(h,g,P)+U'(f,g,P)+U'(h,g,P) \\ &= J(f,g,P)+U'(f,g,P)+J(h,g,P)+U'(h,g,P) \\ &= U(f,g,P)+U(h,g,P). \end{aligned}$$

Similarly, we have $L(f+h,g,P) \geq L(f,g,P)+L(h,g,P)$.

Let $\epsilon > 0$ be given, Since $f$ and $h$ are $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$, by Theorem 4.11, there exists a $\delta > 0$ such that for any $\delta$-fine partition $P$ of $[a,b]$ we have

$$U(f,g,P)-L(f,g,P) < \frac{\epsilon}{2} \text{ and } U(h,g,P)-L(h,g,P) < \frac{\epsilon}{2}.$$

Hence we have

$$\begin{aligned} U(f+h,g,P)-L(f+h,g,P) &\leq U(f,g,P)-L(f,g,P)+U(h,g,P)-L(h,g,P) \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Theorem 4.11 implies that $f+h$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$, and we have

$$\begin{aligned} (\mathcal{DS})\int_a^b (f+h)dg &\leq U(f+h,g,P) \\ &\leq U(f,g,P)+U(h,g,P) \\ &\leq L(f,g,P)+L(h,g,P)+\epsilon \\ &\leq (\mathcal{DS})\int_a^b fdg+(\mathcal{DS})\int_a^b hdg+\epsilon \end{aligned}$$

and

$$\begin{aligned} (\mathcal{DS})\int_a^b (f+h)dg &\geq L(f+h,g,P) \\ &\geq L(f,g,P)+L(h,g,P) \\ &\geq U(f,g,P)+U(h,g,P)-\epsilon \\ &= (\mathcal{DS})\int_a^b fdg+(\mathcal{DS})\int_a^b hdg-\epsilon. \end{aligned}$$

Thus we have

$$(\mathcal{DS})\int_a^b (f+h)dg = (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b hdg$$

(ii) In case of $\alpha = 0$, it clearly that

$$(\mathcal{DS})\int_a^b 0\cdot fdg = 0 = 0\cdot(\mathcal{DS})\int_a^b fdg.$$

Next, we let $\alpha > 0$ be given. We consider

$$\begin{aligned} J(\alpha f, g, P) &= \sum_{i=1}^{n} \alpha f(x_i)(g(x_i^+) - g(x_i^-)) \\ &= \alpha \sum_{i=1}^{n} f(x_i)(g(x_i^+) - g(x_i^-)) \\ &= \alpha J(f, g, P) \end{aligned}$$

and

$$\begin{aligned} U'(f, g, P) &= \sum_{i=1}^{n} M(\alpha f, (x_{i-1}, x_i))(g(x_i^-) - g(x_{i-1}^+)) \\ &= \alpha \sum_{i=1}^{n} M(f, (x_{i-1}, x_i))(g(x_i^-) - g(x_{i-1}^+)) \\ &= \alpha U'(f, g, P). \end{aligned}$$

Similarly, we have $L'(\alpha f, g, P) = \alpha L'(f, g, P)$.

Since $f$ is $\mathcal{DS}$-integrable on $[a, b]$ with respect to $g$, by Theorem 4.11, there exists $\delta > 0$ such that for any $\delta$-fine partition $P$ of $[a, b]$ with respect to $g$ we have

$$U(f, g, P) - L(f, g, P) < \frac{\epsilon}{\alpha}.$$

Hence we have

$$\begin{aligned} U(\alpha f, g, P) - L(\alpha f, g, P) &= \alpha U(f, g, P) - \alpha L(f, g, P) \\ &= \alpha(U(f, g, P) - L(f, g, P)) \\ &< \alpha\frac{\epsilon}{\alpha} \\ &= \epsilon. \end{aligned}$$

Theorem 4.11 implies that $\alpha f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and we have

$$\begin{aligned}(\mathcal{DS})\int_a^b \alpha f dg &\leq U(\alpha f,g,P) < L(\alpha f,g,P)+\epsilon \\ &= \alpha L(f,g,P)+\epsilon \\ &\leq \alpha(\mathcal{DS})\int_a^b fdg+\epsilon\end{aligned}$$

and $(\mathcal{DS})\int_a^b \alpha f dg \geq \alpha(\mathcal{DS})\int_a^b fdg-\epsilon$.
Thus we have

$$(\mathcal{DS})\int_a^b \alpha f dg = \alpha(\mathcal{DS})\int_a^b fdg.$$

□

**Theorem 4.13.** *Let $f$ be a bounded function and $g,h$ be increasing functions on $[a,b]$. If $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and $h$ and $\alpha \geq 0$, then*

**(i)** *$f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g+h$ and*

$$(\mathcal{DS})\int_a^b fd(g+h) = (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b fdh.$$

**(ii)** *$f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $\alpha g$ and*

$$(\mathcal{DS})\int_a^b fd(\alpha g) = \alpha(\mathcal{DS})\int_a^b fdg.$$

*Proof.* (i) Let $g$ and $h$ be increasing functions on $[a,b]$. If $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and $h$, we have

$$\begin{aligned}(g+h)(x^+) &= \lim_{t\to x^+}(g(t)+h(t)) \\ &= \lim_{t\to x^+} g(t) + \lim_{t\to x^+} h(t) \\ &= g(x^+)+h(x^+).\end{aligned}$$

Similarly, we have $(g+h)(x^-) = g(x^-)+h(x^-)$.

We consider

$$\begin{aligned}
J(f,g+h,P) &= \sum_{i=1}^{n} f(x_i)((g+h)(x_i^+)-(g+h)(x_i^-)) \\
&= \sum_{i=1}^{n} f(x_i)(g(x_i^+)+h(x_i^+)-g(x_i^-)-h(x_i^-)) \\
&= \sum_{i=1}^{n} f(x_i)((g(x_i^+)-g(x_i^-))+(h(x_i^+)-h(x_i^-))) \\
&= \sum_{i=1}^{n} f(x_i)((g(x_i^+)-g(x_i^-))+\sum_{i=1}^{n} f(x_i)(h(x_i^+)-h(x_i^-)) \\
&= J(f,g,P)+J(f,h,P)
\end{aligned}$$

and

$$\begin{aligned}
U'(f,g+h,P) &= \sum_{i=1}^{n} M(f,(x_{i-1},x_i))((g+h)(x_i^-)-(g+h)(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} M(f,(x_{i-1},x_i))(g(x_i^-)+h(x_i^-)-(g(x_{i-1}^+)+h(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} M(f,(x_{i-1},x_i))((g(x_i^-)-g(x_{i-1}^+))+(h(x_i^-)-h(x_{i-1}^+))) \\
&= \sum_{i=1}^{n} M(f,(x_{i-1},x_i))(g(x_i^-)-g(x_{i-1}^+)) \\
&\quad +\sum_{i=1}^{n} M(f,(x_{i-1},x_i))(h(x_i^-)-h(x_{i-1}^+)) \\
&= U'(f,g,P)+U'(f,h,P).
\end{aligned}$$

Similarly, $L'(f,g+h,P)=L'(f,g,P)+L'(f,h,P)$.

For any partition $P$ of $[a,b]$, we have

$$\begin{aligned}
U(f,g+h,P) &= J(f,g+h,P)+U'(f,g+h,P) \\
&= J(f,g,P)+J(f,h,P)+U'(f,g,P)+U'(f,h,P) \\
&= J(f,g,P)+U'(f,g,P)+J(f,h,P)+U'(f,h,P) \\
&= U(f,g,P)+U(f,h,P).
\end{aligned}$$

Similarly, we have $L(f,g+h,P)=L(f,g,P)+L(f,h,P)$.

Let $\epsilon > 0$ be given, Since $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$ and $h$, by Theorem 4.11, there exists a $\delta > 0$ such that for any $\delta$-fine partition $P$ of $[a,b]$ with respect to $g$, we have

$$U(f,g,P) - L(f,g,P) < \frac{\epsilon}{2} \text{ and } U(f,h,P) - L(f,h,P) < \frac{\epsilon}{2}.$$

Hence we have

$$\begin{aligned} U(f,g+h,P) - L(f,g+h,P) &= U(f,g,P) - L(f,g,P) + U(f,h,P) - L(f,h,P) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Theorem 4.11 implies that $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g+h$, and we have

$$\begin{aligned} (\mathcal{DS})\int_a^b fd(g+h) &\leq U(f,g+h,P) \\ &< L(f,g+h,P) + \epsilon \\ &= L(f,g,P) + L(f,h,P) + \epsilon. \\ &\leq (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b fdh + \epsilon \end{aligned}$$

and $(\mathcal{DS})\int_a^b fd(g+h) \geq (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b fdh - \epsilon.$
Thus we have

$$(\mathcal{DS})\int_a^b fd(g+h) = (\mathcal{DS})\int_a^b fdg + (\mathcal{DS})\int_a^b fdh.$$

(ii) In the case of $\alpha = 0$, it clearly that

$$(\mathcal{DS})\int_a^b fd(0 \cdot g) = 0 = 0 \cdot (\mathcal{DS})\int_a^b fdg.$$

We now let $\alpha > 0$, we have $(\alpha g)(x^+) = \alpha g(x^+)$, $(\alpha g)(x^-) = \alpha g(x^-)$, $U(f,\alpha g,P) = \alpha U(f,g,P)$ and $L(f,\alpha g,P) = \alpha L(f,g,P)$. Since $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$, by Theorem 4.11, there exists a $\delta > 0$ such that for any $\delta$-fine partition $P$ of $[a,b]$ with respect to $g$, we have

$$U(f,g,P) - L(f,g,P) < \frac{\epsilon}{\alpha}.$$

Hence we have

$$\begin{aligned} U(f,\alpha g,P)-L(f,\alpha g,P) &= \alpha U(f,g,P)-\alpha L(f,g,P) \\ &= \alpha(U(f,g,P)-L(f,g,P)) \\ &< \alpha\frac{\epsilon}{\alpha} \\ &= \epsilon. \end{aligned}$$

Theorem 4.11 implies that $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $\alpha g$ and we have

$$\begin{aligned} (\mathcal{DS})\int_a^b f d(\alpha g) &\leq U(f,\alpha g,P) < L(f,\alpha g,P)+\epsilon \\ &= \alpha L(f,g,P)+\epsilon \\ &\leq \alpha(\mathcal{DS})\int_a^b f dg+\epsilon \end{aligned}$$

and $(\mathcal{DS})\int_a^b f d(\alpha g) > \alpha(\mathcal{DS})\int_a^b f dg-\epsilon$.
Thus we have

$$(\mathcal{DS})\int_a^b f d(\alpha g) = \alpha(\mathcal{DS})\int_a^b f dg.$$

□

## 4.4 Regulated functions

**Definition 4.4.** Let $f:[a,b]\to\mathbb{R}$. Then $f$ is said to be *regulated* if $f$ has one side limits at every point of $[a,b]$, i.e., $\lim\limits_{x\to a^+} f(x)$, $\lim\limits_{x\to b^-} f(x)$, $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ exist, for each $c\in(a,b)$. The set of all regulated function defined on $[a,b]$ is denoted by $RF[a,b]$.

**Lemma 4.14.** *If $f:[a,b]\to\mathbb{R}$ is regulated, then for every $\epsilon>0$ there exists a partition $P=\{[x_{i-1},x_i]\}_{i=1}^n$ such that for each $i=1,2,\ldots,n$ whenever $\xi,\eta\in(x_{i-1},x_i)$, we have*

$$|f(\xi)-f(\eta)|<\epsilon. \tag{4.5}$$

*Proof.* Let $\epsilon > 0$ be given and let $\mathcal{B}$ be the set of all $\zeta \in (a, b]$ such that there is a finite sequence $a = x_1 < x_2 < \ldots x_{k+1} = \zeta$ satisfying (4.5) for $i = 1, 2, \ldots, k+1$.

Since $f(a^+) = \lim_{x \to a^+} f(x)$ exists, there is $\zeta > a$ such that for any $x \in (a, \zeta)$

$$|f(x) - f(a^+)| < \frac{\epsilon}{2}.$$

Then for all $x^{'}, x^{''} \in (a, \zeta)$ we have that

$$\begin{aligned}
|f(x^{'}) - f(x^{''})| &= |f(x^{'}) - f(a^+) - f(x^{''}) + f(a^+)| \\
&= |f(x^{'}) - f(a^+) + (f(a^+) - f(x^{''}))| \\
&\leq |f(x^{'}) - f(a^+)| + |f(a^+) - f(x^{''})| \\
&= |f(x^{'}) - f(a^+)| + |f(x^{''}) - f(a^+)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence we get

$$|f(x^{'}) - f(x^{''})| < \epsilon.$$

Hence, $\zeta \in \mathcal{B}$ thus $\mathcal{B}$ is nonempty. Let $d = \sup\{\mathcal{B}\}$. We will show that $d \in \mathcal{B}$. Since $f(d^-) = \lim_{x \to d^-} f(x)$ exists, there is $\delta > 0$ such that for every $x \in (d - \delta, d)$, $|f(x) - f(d^-)| \leq \frac{\epsilon}{2}$. Let $\zeta \in \mathcal{B} \cup (d - \delta, d)$. Since $\zeta \in \mathcal{B}$, there exists a finite sequence $a = x_1 < x_2 < \ldots x_{k+1} = \zeta$ such that (4.5) holds for $i = 1, 2, \ldots, k+1$. We denote $x_{k+2} = d$, thus for any $x^{'}, x^{''} \in (\zeta, d) = (\zeta, x_{k+2})$. Similar above we have that

$$|f(x^{'}) - f(x^{''})| \leq |f(x^{'}) - f(d^-)| + |f(x^{''}) - f(x^-)| \leq \epsilon.$$

Thus $d \in \mathcal{B}$. We now suppose that $d \neq b$, i.e., $d < b$ . Since $f(d^+) = \lim_{x \to d^+} f(x)$ exists, there is $x^* > d$ with $x^* < b$ such that for any $x \in (d, x^*)$ we have

$$|f(x) - f(d^+)| < \frac{\epsilon}{2}.$$

Hence for any $x^{'}, x^{''} \in (d, x^*)$

$$|f(x^{'}) - f(x^{''})| \leq |f(x^{'}) - f(d^+)| + |f(x^{''}) - f(d^+)| \leq \epsilon.$$

Hence $x^* \in \mathcal{B}$, we get contradiction with $d = \sup\{\mathcal{B}\}$.

Therefore, we can conclude that $d = b$. □

## 4.5 Bounded $p$-variation

**Definition 4.5.** Let $f$ be a real-valued function defined on $[a,b]$ and let $0 < p < \infty$. Given a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a,b]$, let

$$V_p(f, P, [a,b]) = \big[\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p\big]^{\frac{1}{p}}.$$

The $p$-variation of $f$ defined by

$$V_p(f, [a,b]) = \sup_P V_p(f, P, [a,b]),$$

where supremum is taken over all partition $P$. We say that $f \in BV_p[a,b]$ if $V_p(f, [a,b]) < \infty$.

In this study, we consider as $p = 1$ and the set of all function of bounded variation defined on $[a,b]$ is denoted by $BV[a,b]$.

**Lemma 4.15.** *Let $f : [a,b] \to \mathbb{R}$. If $f$ is monotonically increasing function. Then for any partition $P = \{[x_i, x_{i-1}]\}_{i=1}^n$ of $[a,b]$, we have that $f \in BV[a,b]$.*

*Proof.* Let $f$ is monotonically increasing function. Thus we have

$$\begin{aligned} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= f(x_n) - f(x_0) \\ &= f(b) - f(a). \end{aligned}$$

Hence $f \in BV[a,b]$. □

**Theorem 4.16.** *If $f, g \in BV[a,b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in BV[a,b]$ and*

$$V(\alpha f + \beta g, [a,b]) \le |\alpha| V(f, [a,b]) + |\beta| V(g, [a,b]).$$

*Proof.* Let $f, g \in BV[a,b]$ and a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be given, we have that

$$\begin{aligned}\sum_{i=1}^{n} |(\alpha f + \beta g)(x_i) \quad - \quad & (\alpha f + \beta g)(x_{i-1})| \\ &= \sum_{i=1}^{n} |\alpha f(x_i) + \beta g(x_i) - \alpha f(x_{i-1}) - \beta g(x_{i-1})| \\ &= \sum_{i=1}^{n} |\alpha f(x_i) - \alpha f(x_{i-1}) + (\beta g(x_i) - \beta g(x_{i-1}))| \\ &\leq \sum_{i=1}^{n} \{|\alpha||f(x_i) - f(x_{i-1})| + |\beta||g(x_i) - g(x_{i-1})|\} \\ &= |\alpha| \sum_{i=n}^{n} |f(x_i) - f(x_{i-1})| + |\beta| \sum_{i=n}^{n} |g(x_i) - g(x_{i-1})| \\ &\leq |\alpha| V(f, [a,b]) + |\beta| V(g, [a,b]).\end{aligned}$$

Therefore, this prove is complete. □

**Lemma 4.17.** *If $f \in BV_p[a,b]$, $p > 0$, then $f \in RF[a,b]$.*

*Proof.* Assume that $f \notin RF[a,b]$, without lost of generality, there exists $x_0 \in [a,b]$ such that right limit of $f$ at $x_0$ does not exist. Hence there exists $\epsilon > 0$, such that for every $\delta_i$, there exists $[u_i, v_i] \in (x_0, x_0 + \delta_i)$ such that

$$|f(v_i) - f(u_i)| > \epsilon,$$

for each $i$. We may assume that $[u_i, v_i], i = 1, 2, \ldots,$ are pairwise disjoint. Thus we have

$$V_p(f, [a,b]) \geq \Big[\sum_{i=1}^{n} |f(v_i) - f(u_i)|^p\Big]^{\frac{1}{p}} > n^{\frac{1}{p}}\epsilon.$$

for every integer $n$. Hence, $f \notin BV_p[a,b]$, it leads to a contradiction. Therefore, $f \in RF[a,b]$. □

## 4.6 Riemann-Stieltjes integral

**Definition 4.6.** Let $g : [a,b] \to \mathbb{R}$. $g$ is said to satisty $\gamma$-condition if for every $\delta > 0$, there exists a $\delta$-fine partition with respect to $g$ of $[a,b]$.

Let $\delta < 0$ be a positive constant. An interval point pair $([u, v], \xi)$ is said to be $\delta$-fine with respect to $g$ if $|g(v^-) - g(u^+)| < \delta$ and $\xi$ is any point in $[u, v]$.

A division $\{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is said to be $\delta$-fine of $[a, b]$ with respect to $g$ if each $([x_{i-1}, x_i], \xi_i)$ is $\delta$-fine with respect to $g$.

**Definition 4.7.** Let $g : [a, b] \to \mathbb{R}$ satisfy $\gamma$-condition. A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann-Stieltjes integrable* or $\mathcal{RS}$-integrable to $A$ on $[a, b]$ with respect to $g$, if for each $\epsilon > 0$, there exists a positive constant $\delta$ such that whenever $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is a $\delta$-fine division of $[a, b]$ with respect to $g$, we get

$$|S(f, g, D) - A| \leq \epsilon,$$

where $S(f, g, D) = J(f, g, D) + \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) - g(x_{i-1}^+))$. We denoted a constant $A$ by $(\mathcal{R}S) \int_a^b f dg$.

**Lemma 4.18.** *If $g \in RF$, then $g$ satisfy $\gamma$-condition.*

*Proof.* Let $\delta > 0$ be given. Since $g \in RF[a, b]$, by Lemma 4.14, there exists a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a, b]$ such that for each $i = 1, 2, \ldots, n$, whenever $\xi, \eta \in (x_{i-1}, x_i)$, we have,

$$|g(\xi) - g(\eta)| < \delta.$$

Hence $g$ satisfy $\gamma$-condition. □

**Theorem 4.19.** *Let $f, g \in RF[a, b]$. If $f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$, then $f$ is bounded on $[a, b]$.*

*Proof.* Let $\epsilon = 1$ be given, by Cauchy's criterion for $\mathcal{RS}$-integrable, there exists positive constant $\delta$ such that whenever $D_1, D_2$ are two $\delta$-fine divisions of $[a, b]$ with respect to $g$, we have

$$|S(f, g, D_1) - S(f, g, D_2)| \leq 1.$$

Let $D_1 = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ be a fixed $\delta$-fine division of $[a, b]$ with respect to $g$ and $x$ be a point in $[a, b]$, so $x \in (x_{j-1}, x_j)$ for some $j$. We define a new division

$D_2$ form division $D_1$ and replacing a point $\xi_j$ by $x$, we have $D_2$ is a $\delta$-fine division of $[a, b]$ with respect to $g$. Hence

$$\begin{aligned} |(f(\xi_j) - f(x))(g(x_j^-) - g(x_{j-1}^+))| &= |S(f, g, D_1) - S(f, g, D_2)| \\ &\leq 1 \end{aligned}$$

Hence we get

$$\begin{aligned} |f(x)| &\leq \frac{(1 + |f(\xi_j)(g(x_j^-) - g(x_{j-1}^+))|)}{(g(x_j^-) - g(x_{j-1}^+))} \\ &\leq \frac{(1 + \sum_{i=1}^{n} |f(\xi_j)(g(x_j^-) - g(x_{j-1}^+))|)}{\beta}, \end{aligned}$$

where $\beta = \min\{|g(x_j^-) - g(x_{j-1}^+)| : j = 1, 2, \ldots, n\}$. Therefore, $f$ is bounded on $[a, b]$. □

## 4.7 Basic properties for Riemann-Stieltjes integral

For the proof of Theorems 4.20 - 4.21, we shall use the shorthand notation

$$S'(f, g, D) = \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) - g(x_{i-1}^+))$$

**Theorem 4.20.** *Let $\alpha \in \mathbb{R}$. If $f, h : [a, b] \to \mathbb{R}$ are $\mathcal{RS}$-integrable on [a,b] with respect to $g$, then*

**(i)** *f+h is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and*

$$(\mathcal{RS})\int_a^b (f + h)dg = (\mathcal{RS})\int_a^b fdg + (\mathcal{RS})\int_a^b hdg.$$

**(ii)** *$\alpha f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and*

$$(\mathcal{RS})\int_a^b \alpha fdg = \alpha(\mathcal{RS})\int_a^b fdg.$$

*Proof.* (i) Let $\epsilon > 0$ and assume that $f, h$ are $\mathcal{RS}$-integrable function on $[a, b]$ with respect to $g$ such that

$$(\mathcal{RS})\int_a^b fdg = A$$

and

$$(\mathcal{RS})\int_a^b hdg = B.$$

Then there exist a positive constant $\delta_1$ and $\delta_2$ such that

$$|S(f,g,D_1) - A| \leq \frac{\epsilon}{2},$$

and

$$|S(h,g,D_2) - B| \leq \frac{\epsilon}{2}.$$

for every $\delta_1$, $\delta_2$-fine division $D_1$, $D_2$ of $[a,b]$ with respect to $g$, respectively.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for every $\delta$-fine division $D$ of $[a,b]$ with respect to $g$, we have $|S(f,g,D) - A| \leq \frac{\epsilon}{2}$ and $|S(h,g,D) - B| \leq \frac{\epsilon}{2}$. We have that

$$\begin{aligned}
S'(f+h,g,D) &= \sum_{i=1}^{n}(f+h)(\xi_i)(g(x_i^-) - g(x_{i-1}^+)) \\
&= \sum_{i=1}^{n}(f(\xi_i) + h(\xi_i))(g(x_i^-) - g(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) - g(x_{i-1}^+)) + \sum_{i=1}^{n} h(\xi_i)(g(x_i^-) - g(x_{i-1}^+)) \\
&= S'(f,g,D) + S'(h,g,D)
\end{aligned}$$

and

$$\begin{aligned}
J(f+h,g,D) &= \sum_{i=1}^{n}(f+h)(x_i)(g(x_i^+) - g(x_i^-)) \\
&= \sum_{i=1}^{n} f(x_i)(g(x_i^+) - g(x_i^-)) + \sum_{i=1}^{n} h(x_i)(g(x_i^+) - g(x_i^-)) \\
&= J(f,g,D) + J(h,g,D).
\end{aligned}$$

Hence

$$\begin{aligned}
S(f+h,g,D) &= S'(f+h,g,D) + J(f+h,g,D) \\
&= S'(f,g,D) + S'(h,g,D) + J(f,g,D) + J(h,g,D) \\
&= S(f,g,D) + S(h,g,D).
\end{aligned}$$

Thus

$$\begin{aligned}|S(f+h,g,D)-(A+B)| &= |S(f,g,D)+S(h,g,D)-(A+B)| \\ &\leq |S(f,g,D)-A|+|S(h,g,D)-B| \\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Hence $f+h$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$ and

$$(\mathcal{RS})\int_a^b (f+h)dg = (\mathcal{RS})\int_a^b fdg + (\mathcal{RS})\int_a^b hdg.$$

(ii) It is clearly that

$$(\mathcal{RS})\int_a^b 0\cdot fdg = 0 = 0\cdot(\mathcal{RS})\int_a^b fdg.$$

Assume that $\alpha \neq 0$ and let $\epsilon > 0$ be given, then there exists a positive constant $\delta$ such that for every $\delta$-fine division $D$ of $[a,b]$, we have

$$|S(f,g,D)-A| \leq \frac{\epsilon}{|\alpha|}, \text{ where } A = (\mathcal{RS})\int_a^b fdg.$$

We know that

$$S(\alpha f,g,D) = \alpha\cdot S(f,g,D).$$

Thus

$$\begin{aligned}|S(\alpha f,g,D)-\alpha A| &= |\alpha S(f,g,D)-\alpha A| \\ &= |\alpha|\,|S(f,g,D)-A| \\ &\leq |\alpha|(\frac{\epsilon}{|\alpha|}) \\ &= \epsilon.\end{aligned}$$

Hence $\alpha f$ is $\mathcal{RS}$-integrable function on $[a,b]$ with respect to $g$ and

$$(\mathcal{RS})\int_a^b \alpha fdg = \alpha\cdot(\mathcal{RS})\int_a^b fdg.$$

□

**Theorem 4.21.** *Let $\alpha \in \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$ is $\mathcal{RS}$-integrable on [a,b] with respect to $g, h$, then*

**(i)** *$f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g + h$, and*

$$(\mathcal{RS})\int_a^b fd(g+h) = (\mathcal{RS})\int_a^b fdg + (\mathcal{RS})\int_a^b fdh.$$

**(ii)** *$f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $\alpha g$, and*

$$(\mathcal{RS})\int_a^b fd(\alpha g) = \alpha(\mathcal{RS})\int_a^b fdg.$$

*Proof.* (i) Let $\epsilon > 0$ and assume that $f, h$ are $\mathcal{RS}$-integrable function on $[a, b]$ with respect to $g$ such that

$$(\mathcal{RS})\int_a^b fdg = A$$

and

$$(\mathcal{RS})\int_a^b fdh = B.$$

Then there exist a positive constant $\delta_1$ and $\delta_2$ such that

$$|S(f, g, D_1) - A| \leq \frac{\epsilon}{2},$$

and

$$|S(f, h, D_2) - B| \leq \frac{\epsilon}{2}.$$

for every $\delta_1$, $\delta_2$-fine division $D_1$, $D_2$ of $[a, b]$ with respect to $g$, respectively.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for every $\delta$-fine division $D$ of $[a, b]$ with respect to $g$, we have $|S(f, g, D) - A| \leq \frac{\epsilon}{2}$ and $|S(f, h, D) - B| \leq \frac{\epsilon}{2}$. We know that

$$\begin{aligned}
S'(f, g+h, D) &= \sum_{i=1}^{n} f(\xi_i)((g+h)(x_i^-) - (g+h)(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) + h(x_i^-) - g(x_{i-1}^+) - h(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) - g(x_{i-1}^+) + h(x_i^-) - h(x_{i-1}^+)) \\
&= \sum_{i=1}^{n} f(\xi_i)(g(x_i^-) - g(x_{i-1}^+)) + \sum_{i=1}^{n} f(\xi_i)(h(x_i^-) - h(x_{i-1}^+)) \\
&= S'(f, g, D) + S'(f, h, D)
\end{aligned}$$

and

$$\begin{aligned}
J(f, g+h, D) &= \sum_{i=1}^{n}(f(\xi_i))((g+h)(x_i^+) - (g+h)(x_i^-)) \\
&= \sum_{i=1}^{n}(f(\xi_i))(g(x_i^+) + h(x_i^+) - g(x_i^-) - h(x_i^-)) \\
&= \sum_{i=1}^{n}(f(\xi_i))(g(x_i^+) - g(x_i^-) + h(x_i^+) - h(x_i^-)) \\
&= \sum_{i=1}^{n}(f(\xi_i))(g(x_i^+) - g(x_i^-)) + \sum_{i=1}^{n}(f(\xi_i))(h(x_i^+) - h(x_i^-)) \\
&= J(f, g, D) + J(f, h, D).
\end{aligned}$$

Thus

$$\begin{aligned}
S(f, g+h, D) &= S'(f, g+h, D) + J(f, g+h, D) \\
&= S'(f, g, D) + S'(f, h, D) + J(f, g, D) + J(f, h, D) \\
&= S(f, g, D) + S(f, h, D).
\end{aligned}$$

So we have that

$$\begin{aligned}
|S(f, g+h, D) - (A+B)| &= |S(f, g, D) + S(f, h, D) - (A+B)| \\
&\leq |S(f, g, D) - A| + |S(f, h, D) - B| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g + h$ and

$$(\mathcal{RS})\int_a^b f d(g+h) = (\mathcal{RS})\int_a^b f dg + (\mathcal{RS})\int_a^b f dh.$$

(ii) It easy to see that

$$(\mathcal{RS})\int_a^b f d0 \cdot g = 0 = 0 \cdot (\mathcal{RS})\int_b^b f dg.$$

Assume that $\alpha \neq 0$ and let $\epsilon > 0$ be given, then there exists a positive constant $\delta$ such that for every $\delta$-fine division $D$ of $[a, b]$, we have

$$|S(f, g, D) - A| \leq \frac{\epsilon}{|\alpha|}, \text{ where } A = (\mathcal{RS})\int_a^b f dg.$$

We know that

$$\begin{aligned} S(f,\alpha g,D) &= S'(f,\alpha g,D)+J(f,\alpha g,D) \\ &= \alpha S'(f,g,D)+\alpha J(f,g,D) \\ &= \alpha[S'(f,g,D)+J(f,g,D)] \\ &= \alpha S(f,g,D). \end{aligned}$$

Then

$$\begin{aligned} \big|S(f,\alpha g,D)-\alpha A\big| &= \big|\alpha S(f,g,D)-\alpha A\big| \\ &= |\alpha|\big|S(f,g,D)-A\big| \\ &\leq |\alpha|\frac{\epsilon}{|\alpha|} \\ &= \epsilon. \end{aligned}$$

Therefore, $f$ is $\mathcal{RS}$-integrable function on $[a,b]$ with respect to $\alpha g$ and

$$(\mathcal{RS})\int_a^b f d\alpha g = \alpha\cdot(\mathcal{RS})\int_a^b f dg.$$

□

**Theorem 4.22. (Cauchy's criterion for Riemann-Stieltjes integral)** *Let $f:[a,b]\to\mathbb{R}$. Then $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$ if and only if for each $\epsilon>0$ there exists a positive constant $\delta$ such that whenever $D_1, D_2$ are two $\delta$-fine divisions of $[a,b]$ with respect to $g$, we have*

$$|S(f,g,D_1)-S(f,g,D_2)|\leq\epsilon.$$

*Proof.* First, we assume that $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$ and let $\epsilon>0$ be given, there exists a positive constant $\delta$ on $[a,b]$ such that for every $D_1, D_2$ are two $\delta$-fine divisions of $[a,b]$ with respect to $g$, we get

$$|S(f,g,D_1)-A|\leq\frac{\epsilon}{2}$$

and

$$|S(f,g,D_2)-A|\leq\frac{\epsilon}{2},$$

By triangle inequality we have that

$$\begin{aligned} |S(f,g,D_1) - S(f,g,D_2)| &= |(S(f,g,D_1) - A) - (S(f,g,D_2) - A)| \\ &\leq |S(f,g,D_1) - A| + |S(f,g,D_2) - A| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Conversely, we assume that for each $\epsilon > 0$, there exists a positive constant $\delta$ on $[a,b]$ such that for every $\delta$-fine division $D_1, D_2$ of $[a,b]$ with respect to $g$, we have that

$$|S(f,g,D_1) - S(f,g,D_2)| \leq \frac{\epsilon}{2}.$$

Let $\epsilon_n = \dfrac{2}{n}$ for any $n \in \mathbb{N}$ and $\delta_n$ is a positive constant on $[a,b]$. We may assume that if $m \geq n$ then $\delta_m \leq \delta_n$ for any $m,n \in \mathbb{N}$. Thus for every $\delta_m$-fine division on $[a,b]$ with respect to $g$ is also a $\delta_n$-fine division on $[a,b]$ with respect to $g$. Hence we get

$$\left|S(f,g,D_{m_1}) - S(f,g,D_{m_2})\right| \leq \frac{1}{n},$$

where $m_1, m_2 \geq n$.

We can conclude that $\{S(f,g,D_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. we have that $\{S(f,g,D_n)\}_{n\in\mathbb{N}}$ is also a convergent sequence in $\mathbb{R}$, there exists a constant $A$ such that $\lim\limits_{n\to\infty} |S(f,g,D_n) - A| = 0$.

Let $\epsilon > 0$ be given, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have that

$$|S(f,g,D_n) - A| \leq \frac{\epsilon}{2}.$$

Let $\dfrac{1}{N_2} \leq \dfrac{\epsilon}{2}$, $N = \max\{N_1, N_2\}$ and $\delta = \delta_N$, we get

$$\begin{aligned} |S(f,g,D) - A| &= |S(f,g,D) - S(f,g,D_N) + (S(f,g,D_N) - A)| \\ &\leq |S(f,g,D) - S(f,g,D_N)| + |S(f,g,D_N) - A| \\ &\leq \frac{1}{N} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $f$ is $\mathcal{RS}$-integrable on $[a,b]$. $\square$

**Theorem 4.23.** *Let $f$ be a bounded function and $g$ an increasing function. $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$ if and only if it is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$.*

*Proof.* Assume that $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$. By Theorem 4.11, there exists a positive constant $\delta$ such that for any $\delta$-fine partition $P$ of $[a,b]$ with respect to $g$, we have

$$U(f,g,P) - L(f,g,P) < \epsilon.$$

Let $D$ be a $\delta$-fine division of $[a,b]$ and $P$ an associated partition of $D$. We have that $P$ form $\delta$-fine partition of $[a,b]$ with respect to $g$. Thus

$$\begin{aligned} S(f,g,D) &\leq U(f,g,P) \\ &< L(f,g,P) + \epsilon \\ &\leq L(f,g) + \epsilon \\ &= (\mathcal{DS})\int_a^b f dg + \epsilon \end{aligned}$$

and

$$\begin{aligned} S(f,g,D) &\geq L(f,g,P) \\ &> U(f,g,P) - \epsilon \\ &\geq U(f,g) - \epsilon \\ &= (\mathcal{DS})\int_a^b f dg - \epsilon. \end{aligned}$$

Hence we have

$$\left| S(f,g,D) - (\mathcal{DS})\int_a^b f dg \right| < \epsilon.$$

We can conclude that $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$ and

$$(\mathcal{RS})\int_a^b f dg = (\mathcal{DS})\int_a^b f dg.$$

Conversely, assume that $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$. Thus $g$ is satisfy $\gamma$-condition and for any $\epsilon > 0$, there exists a constant $\delta > 0$

such that whenever $D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$ is a $\delta$-fine division of $[a,b]$ with respect to $g$, we have

$$\left| S(f,g,D) - (\mathcal{RS})\int_a^b f dg \right| < \epsilon.$$

Let $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a $\delta$-fine partition of $[a,b]$ with respect to $g$. For each $k$, $k = 1, 2, \ldots, n$. We choose $\gamma_k \in (x_{k-1}, x_k)$ such that

$$f(\gamma_k) < m(f, (x_{k-1}, x_k)) + \epsilon.$$

Then $D$ form a $\delta$-fine division of $[a,b]$ with respect to $g$ such that

$$S(f,g,D) \leq L(f,g,P) + \epsilon(b-a).$$

Thus we have

$$\begin{aligned} L(f,g) \geq L(f,g,P) &\geq S(f,g,D) - \epsilon(b-a) \\ &> (\mathcal{RS})\int_a^b f dg - \epsilon - \epsilon(b-a). \end{aligned}$$

We have that $L(f,g) \geq (\mathcal{RS})\int_a^b f dg$. Similarly, $U(f,g) \leq (\mathcal{RS})\int_a^b f dg$. Since we have $L(f,g) \leq U(f,g)$, we get

$$L(f,g) = U(f,g) = (\mathcal{R}S)\int_a^b f dg.$$

Therefore, $f$ is $\mathcal{D}S$-integrable on $[a,b]$ with respect to $g$ and

$$(\mathcal{DS})\int_a^b f dg = (\mathcal{RS})\int_a^b f dg.$$

□

## 4.8 Null set

In this section, we follow ideas of Chew, see [3], to prove the results.

**Definition 4.8.** Let $E \subset [a,b]$. Denote the characteristic function of $E$ by

$$I_E(x) \begin{cases} 1, & x \in E; \\ 0, & x \in [a,b] \setminus E. \end{cases}$$

Then $E$ is said to be *null set* if $I_E$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$. and $(\mathcal{RS})\displaystyle\int_a^b I_E dg = 0$.

It is clear that any subset of null set is a null set.

**Lemma 4.24.** *Let $E$ be a null subset of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a bounded function. Then $fI_E$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and $(\mathcal{RS})\displaystyle\int_a^b fI_E dg = 0$.*

*Proof.* Let $E$ be a null set and $f : [a, b] \to \mathbb{R}$ a bounded function. We have that

$$\begin{aligned} |S(fI_E, g, D) - 0| &= |S(fI_E, g, D)| \\ &\leq M|S(I_E, g, D)| \\ &= M|S(I_E, g, D) - 0| \\ &\leq M\frac{\epsilon}{M} \\ &= \epsilon, \end{aligned}$$

where $M = \sup|f|$. Thus $fI_E$ is is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and $(\mathcal{RS})\displaystyle\int_a^b fI_E dg = 0$. □

**Corollary 4.25.** *Let $f$ be $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and $E$ a null subset of $[a, b]$. Then $fI_{[a,b]\setminus E}$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and*

$$(\mathcal{RS})\int_a^b fI_{[a,b]\setminus E} dg = (\mathcal{RS})\int_a^b f dg.$$

*Proof.* We know that

$$f = fI_{[a,b]\setminus E} + fI_E.$$

By assumption and Lemma 4.24, we have that $fI_{[a,b]\setminus E}$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$ and

$$(\mathcal{RS})\int_a^b fI_{[a,b]\setminus E} dg = (\mathcal{RS})\int_a^b f dg.$$

□

**Theorem 4.26.** *Let $f, h : [a, b] \to \mathbb{R}$ and $f = h$ except on a null set $E$. Suppose $f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$. Then $h$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$.*

*Proof.* Assume that $f$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$. Let $f = h$ except on a null set $E$. We have that

$$\begin{aligned}(\mathcal{RS})\int_a^b f dg &= (\mathcal{RS})\int_a^b f I_{[a,b]\setminus E} dg \\ &= (\mathcal{RS})\int_a^b h I_{[a,b]\setminus E} dg \\ &= (\mathcal{RS})\int_a^b h dg.\end{aligned}$$

Hence $h$ is $\mathcal{RS}$-integrable on $[a, b]$ with respect to $g$. □

**Definition 4.9.** A property is said to hold almost everywhere (abbreviated a.e.) if the set of points where it fails to hold is a null set.

## 4.9 Integrable functions

**Lemma 4.27.** *If $f \in RF[a, b]$ and $g$ is an increasing function, then $f$ is $\mathcal{DS}$-integrable on $[a, b]$ with respect to $g$.*

*Proof.* Let $f \in RF[a, b]$, we may assume that $f$ is bounded. Let $g$ be an increasing function. by Lemma 4.14, we have that for every $\epsilon > 0$, there exists a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a, b]$ such that for each $i = 1, 2, \ldots, n$ whenever $\xi, \eta \in (x_{i-1}, x_i)$ we have

$$|f(\xi) - f(\eta)| < \frac{\epsilon}{g(b) - g(a) + 1}.$$

Then we have

$$M(f, (x_{i-1}, x_i)) - m(f, (x_{i-1}, x_i)) < \frac{\epsilon}{g(b) - g(a) + 1}.$$

Hence

$$\begin{aligned}&\sum_{i=1}^n [M(f, (x_{i-1}, x_i)) - m(f, (x_{i-1}, x_i))](g(x_i^-) - g(x_{i-1}^+)) \\ &< \sum_{i=1}^n \frac{\epsilon}{g(b) - g(a) + 1}(g(x_i^-) - g(x_{i-1}^+)) \\ &= \frac{\epsilon}{g(b) - g(a) + 1}\sum_{i=1}^n (g(x_i^-) - g(x_{i-1}^+)).\end{aligned}$$

Since $g$ is an increasing function,

$$\sum_{i=1}^{n}(g(x_i^-) - g(x_{i-1}^+)) \leq g(b) - g(a).$$

Thus

$$\begin{aligned}
&[J(f,g,P) + \sum_{i=1}^{n} M(f,(x_{i-1},x_i))(g(x_i^-) - g(x_{i-1}^+))] \\
&-[J(f,g,P) + \sum_{i=1}^{n} m(f,(x_{i-1},x_i))(g(x_i^-) - g(x_{i-1}^+))] \\
&< \frac{\epsilon}{g(b)-g(a)+1} g(b) - g(a) \\
&< \epsilon.
\end{aligned}$$

Hence, we have

$$U(f,g,P) - L(f,g,P) < \epsilon.$$

Therefore, by Cauchy's criterion for Darboux-Stieltjes integral, $f$ is $\mathcal{DS}$-integrable on $[a,b]$ with respect to $g$. □

**Corollary 4.28.** *If $f \in RF[a,b]$ and $g$ an increasing function, then $f$ is $\mathcal{RS}$-integrable on $[a,b]$ with respect to $g$.*

*Proof.* By Lemma 4.27 and Theorem 4.23, we have this prove. □

**Theorem 4.29.** *If $f \in BV[a,b]$, then the function $V(f,[a,x])$ and $V(f,[a,x]) - f(x)$ are both increasing function on $[a,b]$.*

*Proof.* Let $a \leq x_1 \leq x_2 \leq b$ and assume that $V(x) = V(f,[a,x])$. Since $[a,x_1] \subset [a,x_2]$, we have

$$V(x_1) = V(f,[a,x_1]) \leq V(f,[a,x_2]) = V(x_2).$$

Thus

$$\begin{aligned}
[V(x_2) - f(x_2)] \quad - \quad & [V(x_1) - f(x_1)] \\
&= V(f,[a,x_2]) - f(x_2) - V(f,[a,x_1]) + f(x_1) \\
&= V(f,[a,x_2]) - V(f,[a,x_1]) - [f(x_2) - f(x_1)] \\
&= V(f,[x_1,x_2]) - [f(x_2) - f(x_1)] \\
&\geq V(f,[x_1,x_2]) - |f(x_2) - f(x_1)|.
\end{aligned}$$

So the inequality $[V(x_2) - f(x_2)] \geq [V(x_1) - f(x_1)]$ follows from

$$|f(x_2) - f(x_1)| = \sum_{\{x_1, x_2\}} |f(x_i) - f(x_{i-1})| \leq V(f, [x_1, x_2]).$$

□

**Theorem 4.30.** *The function $f \in BV[a,b]$ if and only if it is the difference of two increasing functions.*

*Proof.* First, assume that $f \in BV[a,b]$ then

$$f(x) = V(f, [a,x]) - [V(f, [a,x]) - f(x)].$$

By Theorem 4.29 we have that $f$ is the difference of two increasing functions.

Conversely, assume that $f$ is the difference of two increasing function, thus we have

$$f = g - h,$$

where $g, h$ are increasing functions.
By Lemma 4.15 and Theorem 4.16, we have that $g, h \in BV[a,b]$ and $g - h \in BV[a,b]$. □

**Theorem 4.31.** *If $f \in RF[a,b]$ and $g \in BV[a,b]$, then $f$ is $\mathcal{RS}$-integrable on [a,b] with respect to $g$ .*

*Proof.* Let $f \in RF[a,b]$ and $g \in BV[a,b]$. Since $g \in BV[a,b]$, by Theorem 4.30, we have that $g$ is the difference of two increasing functions. By Theorem 4.21 and Corollary 4.28, we have that $f$ is $\mathcal{RS}$-integrable with respect to $g$ on [a,b]. □

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