Some Properties of Ternary Semirings

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics and Statistics
Prince of Songkla University
2011

Thesis Title	Some Properties of Ternary Semirings
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Major Program	Mathematics and Statistics
Academic Year	2010

ABSTRACT

Let S be a nonempty set together with a binary operation and a ternary operation called addition $+$ and ternary multiplication, respectively, is said to be a *ternary semiring* if $(S,+)$ is a commutative semigroup satisfying the following conditions: for all $a,b,c,d,e \in S$,

(1) $(abc)de = a(bcd)e = ab(cde)$,

(2) $(a+b)cd = acd + bcd$,

(3) $a(b+c)d = abd + acd$ and

(4) $ab(c+d) = abc + abd$.

In this thesis, we study k-fuzzy ideals and L-fuzzy ideals in ternary semirings and investigate interesting results. Also, we give some properties and examples of such ideals.

ACKNOWLEDGEMENTS

I would like to express my deep appreciation and sincere gratitude to my advisor, Assoc. Prof. Dr. Ronnason Chinram for his guidance and encouragement throughout this work which enabled me to carry out my study.

My special appreciation is expressed to Asst. Dr. Sarachai Kongsiriwong, Dr. Supawadee Prugsapitak and Dr. Samruam Baupradist for many valuable comments and helpful suggestions.

I would like to thank Science Achievement Scholarship of Thailand for financial support all my Bachelor and Master degree education. Moreover, I would like to thank all of my teachers for sharing their knowledge and support so that I can obtain this Master degree. In addition, I am grateful to all my friends for their helpful suggestions and friendship over the course of this study.

Finally, I wish to thank my parents for their love who always encourage and support me in all my life and thank everyone, who support me, but I did not mention above.

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CONTENTS

CHAPTER 1

Introduction and Preliminaries

The notion of ternary algebraic structures was introduced by Lehmer (Lehmer, 1932), but earlier such structure was studied by Kasner (Kasner, 1904) and Prüfer (Prüfer, 1924). Lehmer investigated certain ternary algebraic systems called triplexes. Lister (Lister, 1971) characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar in the year 2003 introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [(Dutta and Kar, 2003), (Dutta and Kar, 2004), (Dutta and Kar, 2005), (Dutta and Kar, 2006) and (Kar, 2005), etc.].

The theory of fuzzy sets was first studied by Zadeh (Zadeh, 1965) . Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Fuzzy ideals of semirings were studied by some authors [(Akram and Dudek, 2008), (Dutta and Biswas, 1995), (Ghosh, 1998), (Hedayati, 2009), (Jun, Neggers and Kim, 1998) and (Kim and Park, 1996), etc.]. The notion of k-fuzzy ideals of semirings was studied by Kim and Park (Kim and Park, 1996). Zhang (Zhang, 1988) studied prime L-fuzzy ideals and primary L-fuzzy ideals in rings where L is a complete distributive lattice. The concepts of L-fuzzy ideals in semirings were studied by Jun, Neggers and Kim in (Jun and Kim, 1996), (Jun and Kim, 1998), (Neggers, Jun and Kim, 1998) and (Neggers, Jun and Kim, 1999). Recently Kavikumar, Khamis and Jun studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings in (Kavikumar and Khamis, 2007) and (Kavikumar, Khamis and Jun, 2009). The fuzzy ideal of ternary semirings is a good tool for us to study the fuzzy algebraic structure.

In this thesis, we study k-fuzzy ideals and L-fuzzy ideals in ternary semirings and investigate interesting results. Also, we give some properties and examples of such ideals.

We refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic structures that are necessary for this thesis.

Definition 1.1. A nonempty set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) is called a *semiring* if $(S,+)$ is a commutative semigroup, $(S,\cdot)$ is a semigroup and multiplicative distributes over addition both from the left and from the right, i.e., $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a,b,c \in S$.

We give some examples of semirings.

Example 1.1. Any ring is also a semiring.

Example 1.2. $(\mathbb{Q}^+ \cup \{0\}, +, \cdot)$ and $(\mathbb{R}^+ \cup \{0\}, +, \cdot)$ are semirings but are not rings.

Definition 1.2. Let S be a semiring. A nonempty subset I of S is called a *left (resp. right) ideal* of S if $x,y \in I$ and $s \in S$ imply that $x+y \in I$ and $sx \in I$ (resp. $xs \in I$). If I is both left and right ideal of S, then I is called an *ideal* of S.

Definition 1.3. Let S and R be semiring. A mapping $\varphi : S \to R$ is called a *homomorphism* if

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

and

$$\varphi(xy) = \varphi(x)\varphi(y)$$

for all $x,y \in S$.

We note that if $\varphi : S \to R$ is an onto homomorphism and I is an ideal of S, then $\varphi(I)$ is an ideal of R.

Dutta and Kar (Dutta and Kar, 2003) have given the definition of a ternary semiring as following:

Definition 1.4. A nonempty set S together with a binary operation and a ternary operation called addition $+$ and ternary multiplication, respectively, is said to be a *ternary semiring* if $(S, +)$ is a commutative semigroup satisfying the following conditions: for all $a, b, c, d, e \in S$,

(1) $(abc)de = a(bcd)e = ab(cde)$,

(2) $(a + b)cd = acd + bcd$,

(3) $a(b + c)d = abd + acd$ and

(4) $ab(c + d) = abc + abd$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example.

Example 1.3. We consider $\mathbb{Z}_0^-$, the set of all non-positive integers under usual addition and multiplication, we see that $\mathbb{Z}_0^-$ is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, $\mathbb{Z}_0^-$ is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 1.5. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the *zero element* or simply the *zero* of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Definition 1.6. An additive subsemigroup T of a ternary semiring S is called a *ternary subsemiring* of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 1.7. An additive subsemigroup I of a ternary semiring S is called a *left [resp. right, lateral] ideal* of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I, s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an *ideal* of S.

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 1.8. Let S and R be ternary semirings. A mapping $\varphi : S \to R$ is said to be a *homomorphism* if

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

and

$$\varphi(xyz) = \varphi(x)\varphi(y)\varphi(z)$$

for all $x, y, z \in S$.

Let $\varphi : S \to R$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S, then $\varphi(I)$ is an ideal of R. If S and R be ternary semirings with zero 0, then $\varphi(0) = 0$.

Definition 1.9. Let S be a nonempty set. A mapping $f : S \to [0, 1]$ is called a *fuzzy subset* of S.

Definition 1.10. Let A be a subset of a nonempty set S. The *characteristic function* χ_A of A is a fuzzy subset of S defined as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 1.11. Let f be a fuzzy subset of a nonempty subset S. For $t \in [0, 1]$, the set

$$f_t = \{x \in S \mid f(x) \geq t\}$$

is called a *level subset* of S with respect to f.

CHAPTER 2

k-Fuzzy ideals of ternary semirings

In 1996, Kim and Park (Kim and Park, 1996) studied the k-fuzzy ideal of a semiring and the properties of the image and the preimage of a k-fuzzy ideal in a semiring under epimorphism. Also they constructed an extension of a fuzzy ideal in a k-semiring and studied the quotient structure of a k-semiring by a fuzzy ideal.

In this chapter, we separate into two sections. In the first section, we study k-fuzzy ideals of ternary semirings analogous to that of semirings considered by Kim and Park (Kim and Park, 1996) and give some of its examples. In the last section, fuzzy k-ideals for ternary semirings are considered.

2.1 k-Fuzzy ideals

First, we give the definition of k-fuzzy ideals in ternary semirings.

Definition 2.1. An ideal I of a ternary semiring S is said to be a *k-ideal* if for $x, y \in S, x + y, y \in I \Rightarrow x \in I$.

Example 2.1. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication, let $I = \{0, -3\} \cup \{-5, -6, -7, \ldots\}$. It is easy to see that I is an ideal of $\mathbb{Z}_0^-$ but not a k-ideal of $\mathbb{Z}_0^-$ because

$$-3, (-2) + (-3) \in I \text{ but } -2 \notin I.$$

Example 2.2. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication, let $I = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$. It is easy to see that I is a k-ideal of $\mathbb{Z}_0^-$.

Definition 2.2. For each ideal I of a ternary semiring S, the *k-closure* $\overline{I}$ of I is defined by

$$\overline{I} = \{x \in S \mid a + x = b \text{ for some } a, b \in I\}.$$

Example 2.3. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication, let $I = \{0, -3\} \cup \{-5, -6, -7, \ldots\}$ is an ideal of $\mathbb{Z}_0^-$. It is easy to see that $\bar{I} = \mathbb{Z}_0^-$.

Example 2.4. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication, let $I = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$ is a k-ideal of $\mathbb{Z}_0^-$. It is easy to see that $\bar{I} = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$.

Theorem 2.1. *Let I be an ideal of a ternary semiring S with zero. Then I is a k-ideal of S if and only if $I = \overline{I}$.*

Proof. Let $x \in I$. Since $0 \in I$ and $0 + x = x$, $x \in \bar{I}$.

Conversely, let $x \in \bar{I}$. Then there exist $a, b \in I$ such that $a + x = b$. So $x + a \in I$ and $a \in I$, this implies $x \in I$. □

Definition 2.3. A fuzzy subset f of a ternary semiring S is called a *fuzzy ideal* of S if for all $x, y, z \in S$,

(1) $f(x + y) \geq \min\{f(x), f(y)\}$ and

(2) $f(xyz) \geq \max\{f(x), f(y), f(z)\}$.

By the definitions of ideals and fuzzy ideals of ternary semirings, the following lemma holds.

Lemma 2.2. *Let I be a nonempty subset of a ternary semiring S. Then I is an ideal of S if and only if the characteristic function χ_I is a fuzzy ideal of S.*

Proof. Assume I is an ideal of S.

(1) Let $x, y \in S$.

case I $x \in I$ and $y \in I$. So $x + y \in I$.

Thus $\chi_I(x + y) = 1 \geq \min\{\chi_I(x), \chi_I(y)\}$.

case II $x \notin I$ or $y \notin I$. Then $\chi_I(x) = 0$ or $\chi_I(y) = 0$.

So $\min\{\chi_I(x), \chi_I(y)\} = 0 \leq \chi_I(x + y)$

(2) Let $x, y, z \in S$.

case I $x \in I$ or $y \in I$ or $z \in I$. So $xyz \in I$.

Thus $\chi_I(xyz) = 1 \geq \max\{\chi_I(x), \chi_I(y), \chi_I(z)\}$.

case II $x \notin I$, $y \notin I$ and $z \notin I$. Then $\chi_I(x) = 0$, $\chi_I(y) = 0$ and $\chi_I(z) = 0$.

So $\max\{\chi_I(x), \chi_I(y), \chi_I(z)\} = 0 \leq \chi_I(xyz)$.

Hence χ_I is a fuzzy ideal of S.

Conversely, assume χ_I is a fuzzy ideal of S.

(1) Let $x, y \in I$. So $\chi_I(x) = \chi_I(y) = 1$. Then $\chi_I(x + y) \geq \min\{\chi_I(x), \chi_I(y)\} = 1$. Thus $x + y \in I$.

(2) Let $x, y \in S$ and $z \in I$. So $\chi_I(xyz) \geq \chi_I(z) = 1$, $\chi_I(zxy) \geq \chi_I(z) = 1$ and $\chi_I(xzy) \geq \chi_I(z) = 1$. Then $xyz, zxy, xzy \in I$.

Hence I is an ideal of S. □

Lemma 2.3. *Let f be a fuzzy ideal of a ternary semiring S with zero 0. Then $f(x) \leq f(0)$ for all $x \in S$.*

Proof. For any $x \in S, f(0) = f(00x) \geq \max\{f(0), f(x)\} \geq f(x)$. □

Definition 2.4. A fuzzy ideal f of a ternary semiring S with zero 0 is said to be a *k-fuzzy ideal* of S if

$$f(x + y) = f(0) \text{ and } f(y) = f(0) \Rightarrow f(x) = f(0)$$

for all $x, y \in S$.

Example 2.5. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

To show f is a fuzzy ideal of $\mathbb{Z}_0^-$, let $x, y, z \in \mathbb{Z}_0^-$.

1. **case I** $x = -1$ or $y = -1$. So $\min\{f(x), f(y)\} = 0$.

   Then $f(x+y) \geq \min\{f(x), f(y)\}$.

   case II $x \neq -1$ and $y \neq -1$. So $x + y \neq -1$.

   Then $f(x+y) = 0.5 \geq \min\{f(x), f(y)\}$.

2. **case I** $x = -1$, $y = -1$ and $z = -1$. So $\max\{f(x), f(y), f(z)\} = 0$.

   Then $f(xyz) \geq \max\{f(x), f(y), f(z)\}$.

   case II $x \neq -1$, $y \neq -1$ or $z \neq -1$. So $xyz \neq -1$.

   Then $f(xyz) = 0.5 \geq \max\{f(x), f(y), f(z)\}$.

However, f is not a k-fuzzy ideal of $\mathbb{Z}_0^-$ because

$$f((-1) + (-2)) = f(-3) = 0.5 = f(0)$$

and $f(-2) = 0.5 = f(0)$ but $f(-1) = 0 \neq 0.5 = f(0)$.

Example 2.6. Let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd,} \\ 0.5 & \text{if } x \text{ is even.} \end{cases}$$

Therefore f is a fuzzy ideal of $\mathbb{Z}_0^-$. Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. So $f(x+y) = 0.5$ and $f(y) = 0.5$. Thus $x+y$ and y are even. Hence x is even, this implies $f(x) = 0.5 = f(0)$. Therefore f is a k-fuzzy ideal of $\mathbb{Z}_0^-$.

From the condition of Definition 2.4 and Lemma 2.2, the following theorem holds.

Theorem 2.4. *Let S be a ternary semiring with zero 0 and I be a nonempty subset of S. Then I is a k-ideal of S if and only if the characteristic function χ_I is a k-fuzzy ideal of S.*

Proof. Assume I is a k-ideal of S. By Lemma 2.2, χ_I is a fuzzy ideal of S. Next, let $x, y \in S$ and assume $\chi_I(x+y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Since I is an ideal of S, $0 \in I$. Thus $\chi_I(0) = 1$, this implies $\chi_I(x+y) = 1$ and $\chi_I(y) = 1$. Then $x+y, y \in I$. Since I is a k-ideal of S, $x \in I$. Hence $\chi_I(x) = 1 = \chi_I(0)$. Therefore χ_I is a k-fuzzy ideal of S.

Conversely, assume characteristic function χ_I is a k-fuzzy ideal of S. By Lemma 2.2, I is an ideal of S. So $0 \in I$, this implies $\chi_I(0) = 1$. Let $x, y \in S$ such that $x+y, y \in I$. So $\chi_I(x+y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Then $\chi_I(x) = \chi_I(0) = 1$. So $x \in I$. Hence I is a k-ideal of S. □

Theorem 2.5. *Let f be a fuzzy subset of a ternary semiring S. Then f is a fuzzy ideal of S if and only if for any $t \in [0,1]$ such that $f_t \neq \emptyset$, f_t is an ideal of S.*

Proof. Let f be a fuzzy ideal of S. Let $t \in [0,1]$ such that $f_t \neq \emptyset$. Let $x, y \in f_t$. Then $f(x), f(y) \geq t$. Then

$$f(x+y) \geq \min\{f(x), f(y)\} \geq t.$$

Thus $x+y \in f_t$. Next, let $x, y \in S$ and $a \in f_t$. We have

$$f(xya) \geq \max\{f(x), f(y), f(a)\} \geq f(a) \geq t.$$

Thus $xya \in f_t$. Similarly, $xay, axy \in f_t$. Therefore f_t is an ideal of S.

Conversely, let $x, y, z \in S$ and $t = \min\{f(x), f(y)\}$. Then $f(x), f(y) \geq t$. Thus $x, y \in f_t$. By assumption, $x+y \in f_t$. So

$$f(x+y) \geq t = \min\{f(x), f(y)\}.$$

Next, let $s = \max\{f(x), f(y), f(z)\}$. Then $f(x) = s$ or $f(y) = s$ or $f(z) = s$. Thus $x \in f_s$ or $y \in f_s$ or $z \in f_s$. By assumption, $xyz \in f_s$. So

$$f(xyz) \geq s = \max\{f(x), f(y), f(z)\}.$$

Therefore f is a fuzzy ideal of S. $\square$

However, it is not true in general that f is a fuzzy ideal of a ternary semiring S with zero 0, then for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S. We can see from the following example.

Example 2.7. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$ but $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k-ideal of $\mathbb{Z}_0^-$ because $(-1) + (-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Theorem 2.6. *Let f be a fuzzy subset of a ternary semiring S with zero 0. If for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S, then f is a k-fuzzy ideal of S.*

Proof. By Theorem 2.5, f is a fuzzy ideal of S. Next, let $x, y \in S$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Then $x+y, y \in f_{f(0)}$. By assumption, $x \in f_{f(0)}$. Hence $f(x) \geq f(0)$. Since f is a fuzzy ideal of S, by Lemma 2.3, $f(x) = f(0)$. Therefore f is a k-fuzzy ideal of S. $\square$

However, the converse of Theorem 2.6 does not hold which can be shown the following example.

Example 2.8. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication. Let f be a fuzzy subset of $\mathbb{Z}_0^-$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$. Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(x) = f(0)$. So $f(x+y) = 1$ and $f(y) = 1$. Thus $x+y$ and y are even. Hence x is even, this implies $f(x) = 1 = f(0)$. Therefore f is a k-fuzzy ideal of $\mathbb{Z}_0^-$. However, $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k-ideal of $\mathbb{Z}_0^-$ because $(-1)+(-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Definition 2.5. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. The *k-fuzzy closure* $\overline{f}$ of f is defined by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \notin \overline{f_{f(0)}}, \\ f(0) & \text{if } x \in \overline{f_{f(0)}}. \end{cases}$$

The next theorem holds.

Theorem 2.7. *Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. Then f is a k-fuzzy ideal of S if and only if $f = \overline{f}$.*

Proof. Assume f is a k-fuzzy ideal of S and let $x \in \overline{f_{f(0)}}$. Then $\overline{f}(x) = f(0)$. Since $x \in \overline{f_{f(0)}}$, there exist $a, b \in f_{f(0)}$ such that $a + x = b$. Thus $f(a) = f(0)$ and $f(x+a) = f(b) = f(0)$. Then $f(x) = f(0)$. Then $f = \overline{f}$.

Conversely, assume $f = \overline{f}$. So $f_{f(0)} = \overline{f_{f(0)}}$, by Theorem 2.1, $f_{f(0)}$ is a k-ideal of S. Let $x, y \in S$ such that $f(x+y) = f(0) = f(y)$. So $x+y, y \in f_{f(0)}$. Then $x \in f_{f(0)}$. So $f(x) = f(0)$. Hence f is a k-fuzzy ideal of S. □

Definition 2.6. Let $\varphi : S \to R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of R. We define a fuzzy subset $\varphi^{-1}(f)$ of S by

$$\varphi^{-1}(f)(x) = f(\varphi(x)) \text{ for all } x \in S.$$

We call $\varphi^{-1}(f)$ the *preimage* of f under φ.

Theorem 2.8. *Let $\varphi : S \to R$ be an homomorphism of ternary semirings. If f is a fuzzy ideal of R, then $\varphi^{-1}(f)$ is a fuzzy ideal of S.*

Proof. Let f be a fuzzy ideal of R. Then for any $x, y, z \in S$,

$$\begin{aligned}\varphi^{-1}(f)(x+y) &= f(\varphi(x+y)) \\ &= f(\varphi(x)+\varphi(y)) \\ &\geq \min\{f(\varphi(x)), f(\varphi(y))\} \\ &= \min\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(f)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \max\{f(\varphi(x)), f(\varphi(y)), f(\varphi(z))\} \\ &= \max\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y), \varphi^{-1}(f)(z)\}.\end{aligned}$$

This shows that $\varphi^{-1}(f)$ is a fuzzy ideal of S. □

Theorem 2.9. *Let S and R be ternary semirings with zero 0 and $\varphi : S \to R$ an onto homomorphism. Let f be a fuzzy ideal of R. Then f is a k-fuzzy ideal of R if and only if $\varphi^{-1}(f)$ is a k-fuzzy ideal of S.*

Proof. Suppose that f is a k-fuzzy ideal of R. By Theorem 2.8, $\varphi^{-1}(f)$ is a fuzzy ideal. Let $x, y \in S$. Assume

$$\varphi^{-1}(f)(x+y) = \varphi^{-1}(f)(0) \text{ and } \varphi^{-1}(f)(y) = \varphi^{-1}(f)(0).$$

Then

$$f(\varphi(x+y)) = f(\varphi(0)) = f(0) \text{ and } f(\varphi(y)) = f(\varphi(0)) = f(0).$$

Since f is a k-fuzzy ideal of R,

$$f(\varphi(x)) = f(0) = f(\varphi(0)).$$

Thus $\varphi^{-1}(f)(x) = \varphi^{-1}(f)(0)$. Hence $\varphi^{-1}(f)$ is a k-fuzzy ideal of S.

Conversely, assume $\varphi^{-1}(f)$ is a k-fuzzy ideal of S. Let $x, y \in R$ such that

$$f(x+y) = f(0) \text{ and } f(y) = f(0).$$

Since φ is onto, there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$. So

$$f(\varphi(a) + \varphi(b)) = f(\varphi(0)) \text{ and } f(\varphi(b)) = f(\varphi(0)).$$

Hence

$$\varphi^{-1}(f)(a+b) = \varphi^{-1}(f)(0) \text{ and } \varphi^{-1}(f)(b) = \varphi^{-1}(f)(0).$$

Since $\varphi^{-1}(f)$ is a k-fuzzy ideal of S, $\varphi^{-1}(f)(a) = \varphi^{-1}(f)(0)$, this implies

$$f(x) = f(\varphi(a)) = f(\varphi(0)) = f(0).$$

Hence f is a k-fuzzy ideal of R. □

Definition 2.7. Let $\varphi : S \to R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of S. We define a fuzzy subset $\varphi(f)$ of R by

$$\varphi(f)(y) = \begin{cases} \sup\limits_{x \in \varphi^{-1}(y)} f(x) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\varphi(f)$ the *image* of f under φ.

The following lemma is case $L = [0, 1]$ of Proposition 8 in (Jun, Neggers and Kim, 1998).

Lemma 2.10. *(Jun, Neggers and Kim, 1998) Let φ be a mapping from a set X to a set Y and f a fuzzy subset of X. Then for every $t \in (0, 1]$,*

$$(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s}).$$

Lemma 2.11. *The intersection of arbitrary set of ideals of a ternary semiring S is either empty or an ideal of S.*

Proof. Let A_α be an ideal of ternary semiring S for all $\alpha \in I$. Assume

$$\bigcap_{\alpha \in I} A_\alpha \neq \emptyset.$$

Let $x, y \in \bigcap_{\alpha \in I} A_\alpha$ and $s, r \in S$. So $x, y \in A_\alpha$ for all $\alpha \in I$. Since A_α is an ideal of S for all $\alpha \in I$, $x + y, srx, xsr, sxr \in A_\alpha$ for all $\alpha \in I$. Then

$$x + y, srx, xsr, sxr \in \bigcap_{\alpha \in I} A_\alpha.$$

Hence $\bigcap_{\alpha \in I} A_\alpha$ is an ideal of S. □

Theorem 2.12. *Let $\varphi : S \to R$ be an onto homomorphism of ternary semirings. If f is a fuzzy ideal of S, then $\varphi(f)$ is a fuzzy ideal of R.*

Proof. By Theorem 2.5, it is sufficient to show that each nonempty level subset of $\varphi(f)$ is an ideal of R. Let $t \in [0, 1]$ such that $(\varphi(f))_t \neq \emptyset$. If $t = 0$, then $(\varphi(f))_t = R$. Assume that $t \neq 0$. By Lemma 2.10,

$$(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s}).$$

Then $\varphi(f_{t-s}) \neq \emptyset$ for all $0 < s < t$, and so $f_{t-s} \neq \emptyset$ for all $0 < s < t$. By Theorem 2.5, f_{t-s} is an ideal of S for all $0 < s < t$. Since φ is an onto homomorphism, $\varphi(f_{t-s})$ is an ideal of R for all $0 < s < t$. By Lemma 2.11, $(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s})$ is an ideal of R. □

Definition 2.8. Let S and R be any two sets and $\varphi : S \to R$ be any function. A fuzzy subset f of S is called *φ-invariant* if $\varphi(x) = \varphi(y)$ implies $f(x) = f(y)$ where $x, y \in S$.

Lemma 2.13. *Let S and R be ternary semirings and $\varphi : S \to R$ a homomorphism. Let f be a φ-invariant fuzzy ideal of S. If $x = \varphi(a)$, then $\varphi(f)(x) = f(a)$.*

Proof. If $t \in \varphi^{-1}(x)$, then $\varphi(t) = x = \varphi(a)$. Since f is φ-invariant, $f(t) = f(a)$. This implies

$$\varphi(f)(x) = \sup_{t \in \varphi^{-1}(x)} f(t) = f(a).$$

Hence $\varphi(f)(x) = f(a)$. □

Theorem 2.14. *Let S and R be ternary semirings with zero* 0 *and $\varphi : S \to R$ an onto homomorphism. Let f be a φ-invariant fuzzy ideal of S. Then f is a k-fuzzy ideal S if and only if $\varphi(f)$ is a k-fuzzy ideal of R.*

Proof. Suppose that f is a k-fuzzy ideal of S and let $x, y \in R$ such that

$$\varphi(f)(x+y) = \varphi(f)(0) \text{ and } \varphi(f)(y) = \varphi(f)(0).$$

Since φ is onto, there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$. By Lemma 2.13, $\varphi(f)(0) = f(0)$,

$$\varphi(f)(x+y) = f(a+b) \text{ and } \varphi(f)(y) = f(b).$$

Thus $f(a+b) = f(0)$ and $f(b) = f(0)$. Since f is a k-fuzzy ideal of S, $f(a) = f(0)$. By Lemma 2.13, $\varphi(f)(x) = f(a) = f(0) = \varphi(f)(0)$. Hence $\varphi(f)$ is a k-fuzzy ideal of R.

Conversely, if $\varphi(f)$ is a k-fuzzy ideal of R, then for any $x \in S$,

$$\varphi^{-1}(\varphi(f))(x) = \varphi(f)(\varphi(x)) = f(x).$$

So $\varphi^{-1}(\varphi(f)) = f$. Since $\varphi(f)$ is a k-fuzzy ideal of R, by Theorem 2.9, $f = \varphi^{-1}(\varphi(f))$ is a k-fuzzy ideal of S. □

2.2 Fuzzy k-ideals

In this section, we will study fuzzy k-ideals of ternary semirings analogous to fuzzy k-ideals of semirings.

Definition 2.9. A fuzzy ideal f of a ternary semiring S is said to be a *fuzzy k-ideal* of S if

$$f(x) \geq \min\{f(x+y), f(y)\}$$

for all $x, y \in S$.

Example 2.9. Let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication defined by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$ but not a fuzzy k-ideal of $\mathbb{Z}_0^-$ because set $x = -1$ and $y = -2$, we have $f(x) = 0 < 0.5 = \min\{f(x+y), f(y)\}$.

Example 2.10. let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd,} \\ 0.5 & \text{if } x \text{ is even.} \end{cases}$$

It is easy to show that f is a fuzzy k-ideal of $\mathbb{Z}_0^-$.

Lemma 2.15. *Let S be a ternary semiring and f a fuzzy ideal of S. Then f is a fuzzy k-ideal of S if and only if for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S.*

Proof. By Theorem 2.5, f_t is an ideal of S. Let $x, y \in f_t$ and assume $x+y, y \in f_t$. Then $f(x+y), f(y) \geq t$. Since f is a fuzzy k-ideal of S,

$$f(x) \geq \min\{f(x+y), f(y)\} \geq t.$$

So $x \in f_t$. Therefore f_t is a k-ideal of S. Conversely, assume for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S. By Theorem 2.5, f is a fuzzy ideal of S. Next, let $x, y \in S$. Set $t = \min\{f(x+y), f(y)\}$ Then $f(x+y), f(y) \geq t$. So $x+y, y \in f_t$. By assumption, f_t is a k-ideal of S, this implies $x \in f_t$. Hence

$$f(x) \geq t = \min\{f(x+y), f(y)\}.$$

Therefore f is a fuzzy k-ideal of S. □

Theorem 2.16. *Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. If f is a fuzzy k-ideal of S, then f is a k-fuzzy ideal of S.*

Proof. Let $x, y \in S$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Set $t = f(0)$. So $x+y, y \in f_t$. By Lemma 2.16, the level subset f_t is a k-ideal of S. So $x \in f_t$. This implies $f(x) \geq t = f(0)$. By Lemma 2.3, $f(x) = f(0)$. □

However, the converse of Theorem 2.17 does not hold. We can see this example.

Example 2.11. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

By Example 2.8, we known that f is a k-fuzzy ideal of $\mathbb{Z}_0^-$. However, f is not a fuzzy k-ideal of $\mathbb{Z}_0^-$ because $f((-1)+(-2)) = f(-3) = 0.5$ and $f(-2) = 1$ but $f(-1) = 0 < 0.5 = \min\{f((-1)+(-2)), f(-2)\}$.

Definition 2.10. Let f and g be fuzzy subset of a nonempty subset S. A fuzzy subset $f \cap g$ of S is defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$ for all $x \in S$.

Lemma 2.17. *Let f and g be fuzzy subset of a ternary semiring S. If f and g are fuzzy ideals of S, then $f \cap g$ is a fuzzy ideal of S.*

Proof. Let $x, y, z \in S$. We have

$$\begin{aligned}(f \cap g)(x+y) &= \min\{f(x+y), g(x+y)\} \\ &\geq \min\{f(x), f(y), g(x), g(y)\} \\ &= \min\{(f \cap g)(x), (f \cap g)(y)\}\end{aligned}$$

and

$$\begin{aligned}(f \cap g)(xyz) &= \min\{f(xyz), g(xyz)\} \\ &\geq \min\{\max\{f(x), f(y), f(z)\}, \max\{g(x), g(y), g(z)\}\} \\ &\geq \max\{(f \cap g)(x), (f \cap g)(y), (f \cap g)(z)\}.\end{aligned}$$

Hence $f \cap g$ is a fuzzy ideal of S. □

Theorem 2.18. *Let f and g be fuzzy subset of a ternary semiring S. If f and g are fuzzy k-ideals of S, then $f \cap g$ is a fuzzy k-ideal of S.*

Proof. By Lemma 2.18, $f \cap g$ is a fuzzy ideal of S. Let $x, y \in S$. We have

$$\begin{aligned}(f \cap g)(x) &= \min\{f(x), g(x)\} \\ &\geq \min\{f(x+y), f(y), g(x+y), g(y)\} \\ &= \min\{(f \cap g)(x+y), (f \cap g)(y)\}.\end{aligned}$$

Hence $f \cap g$ is a fuzzy k-ideal of S. □

Let f and g be k-fuzzy ideals of a ternary semiring S. In general, a fuzzy ideal $f \cap g$ need not be a k-fuzzy ideal of S. See this example.

Example 2.12. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and the ternary multiplication. Let f and g be fuzzy subsets on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0.3 & \text{if } x = 0, \\ 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise} \end{cases}$$

and $g(x) = 0.2$ for all $x \in \mathbb{Z}_0^-$. It is easy to verify that f and g are k-fuzzy ideal of $\mathbb{Z}_0^-$. We have

$$(f \cap g)(x) = \begin{cases} 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise.} \end{cases}$$

Set $x = -1$ and $y = -2$. We have $(f \cap g)(x + y) = 0.2 = (f \cap g)(0)$ and $(f \cap g)(y) = 0.2 = (f \cap g)(0)$ but $(f \cap g)(x) = 0.1 \neq 0.2 = (f \cap g)(0)$. Thus $f \cap g$ is not k-fuzzy ideal of $\mathbb{Z}_0^-$.

CHAPTER 3

L-Fuzzy ideals of ternary semirings

In this chapter, we study L-fuzzy ternary subsemirings and L-fuzzy ideals in ternary semirings. We demonstrate this chapter in two sections. In the first section, we study the notion of L-fuzzy ideals in ternary semirings. In the last section, we give some properties of normal L-fuzzy ideals in ternary semirings. And now, we give the definition of completely distributive lattices as follows

Let $L = (L, \leq, \wedge, \vee)$ be a completely distributive lattice which has the least and the greatest elements, say 0 and 1, respectively.

Definition 3.1. Let X be a nonempty set. An *L-fuzzy set* of X is a map $\mu : X \to L$, and $\mathcal{F}(X)$ denote the set of all L-fuzzy subsets of X.

Definition 3.2. If $\mu, \nu \in \mathcal{F}(X)$, then $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$ and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$.

It is easy to see that $\mathcal{F}(X) = (\mathcal{F}(X), \subseteq, \wedge, \vee)$ is a completely distributive lattice, which has the least and the greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively, where $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$.

Proposition 3.1. *(Jun, Neggers and Kim, 1998). Let f be a mapping from a set X to a set Y and $\mu \in F(X)$. Then for every $t \in L, t \neq 0$,*

$$(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s}).$$

Given any two sets X and Y, let $\mu \in \mathcal{F}(X)$ and let $f : X \to Y$ be a function. Define $\nu \in \mathcal{F}(Y)$ by for $y \in Y$,

$$\nu(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We call ν the *image* of μ under f which is denoted by $f(\mu)$. Conversely, for $\nu \in \mathcal{F}(f(x))$, define $\mu \in \mathcal{F}(X)$ by $\mu(x) = \nu(f(x))$ for all $x \in X$, and we call μ the *preimage* of ν under f which is denoted by $f^{-1}(\nu)$.

3.1 L-Fuzzy ideals

Definition 3.3. An L-fuzzy subset μ of a ternary semiring S is called an *L-fuzzy ternary subsemiring* of S if for all $x, y, z \in S$,

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and

(2) $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$

Definition 3.4. An L-fuzzy ternary subsemiring μ of a ternary semiring S is called an *L-fuzzy left [resp. right, lateral] ideal* of S if $\mu(xyz) \geq \mu(z)$ [resp. $\mu(xyz) \geq \mu(x), \mu(xyz) \geq \mu(y)$] for all $x, y, z \in S$. If μ is an L-fuzzy left, right and lateral ideal of S, then μ is called an *L-fuzzy ideal* of S.

Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. Let

$$\mu_t = \{x \in S \mid \mu(x) \geq t\},$$

which is called a t-level subset of μ. Note that

(1) for $s, t \in L, s < t$ implies $\mu_t \subseteq \mu_s$ and

(2) for $t \in L$, $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s \in L$ such that $s > t$.

Theorem 3.2. *Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. The following statements are true.*

(1) μ is an L-fuzzy ternary subsemiring of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is a ternary subsemiring of S.

(2) μ is an L-fuzzy left [resp. right, lateral] ideal of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is a left [resp. right, lateral] ideal of S.

(3) μ is an L-fuzzy ideal of S if and only if for any $t \in L$ such that $\mu_t \neq \emptyset, \mu_t$ is an ideal of S.

Proof. (1) Let μ be an L-fuzzy ternary subsemiring of S. Let $t \in L$ such that $\mu_t \neq \emptyset$. Let $x, y, z \in \mu_t$. Then $\mu(x), \mu(y), \mu(z) \geq t$. Then

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \geq t$$

and

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq t.$$

So $xyz \in \mu_t$. Hence μ_t is a ternary subsemiring of S.

Conversely, let $x, y, z \in S$ and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x), \mu(y) \geq t$. Thus $x, y \in \mu_t$. By assumption, $x + y \in \mu_t$. So

$$\mu(x+y) \geq t = \min\{\mu(x), \mu(y)\}.$$

Next, let $s = \min\{\mu(x), \mu(y), \mu(z)\}$. Then $\mu(x), \mu(y), \mu(z) \geq s$. Thus $x, y, z \in \mu_s$. By assumption, $xyz \in \mu_s$. So

$$\mu(xyz) \geq s = \min\{\mu(x), \mu(y), \mu(z)\}.$$

Therefore μ is an L-fuzzy ternary subsemiring of S.

(2) Let μ be an L-fuzzy left ideal of S. Let $t \in L$ such that $\mu_t \neq \emptyset$. Let $x, y \in \mu_t$. Then $\mu(x), \mu(y) \geq t$. Then

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \geq t.$$

Next, let $x, y \in S$ and $z \in \mu_t$. We have $\mu(xyz) \geq \mu(z) \geq t$. Thus $xyz \in \mu_t$. Therefore μ_t is a left ideal of S.

Conversely, let $x, y, z \in S$ and $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x), \mu(y) \geq t$. Thus $x, y \in \mu_t$. By assumption, $x + y \in \mu_t$. So

$$\mu(x+y) \geq t = \min\{\mu(x), \mu(y)\}.$$

Next, let $s = \mu(z)$. Then $\mu(z) \geq s$. Thus $z \in \mu_s$. By assumption, $xyz \in \mu_s$. So

$$\mu(xyz) \geq s = \mu(z).$$

Therefore μ is an L-fuzzy left ideal of S.

The proofs of other cases are similar.

(3) follows from (2). □

Theorem 3.3. *Let S be a ternary semiring. The following statements are true.*

(1) If A is a ternary subsemiring of S, then there exists an L-fuzzy ternary subsemiring μ of S such that $\mu_t = A$ for some $t \in L$.

(2) If A is a left [resp. right, lateral] ideal of S, then there exists an L-fuzzy left [resp. right, lateral] ideal μ of S such that $\mu_t = A$ for some $t \in L$.

(3) If A is an ideal of S, then there exists an L-fuzzy ideal μ of S such that $\mu_t = A$ for some $t \in L$.

Proof. (1) Let $t \in L$ and define an L-fuzzy set of S by

$$\mu(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\mu_t = A$. For $s \in L$ we have

$$\mu_s = \begin{cases} S & \text{if } s = 0, \\ A & \text{if } 0 < s \leq t, \\ \emptyset & \text{otherwise .} \end{cases}$$

Since A and S are ternary subsemirings of S, it follows that every nonempty level subset μ_s of μ is a ternary subsemiring of S. By Theorem 3.2 (1), μ is an L-fuzzy ternary subsemiring of S.

The proofs of (2) and (3) are similar to the proof of (1). □

Theorem 3.4. *Let S be a ternary semiring with zero and*

$$S_\mu = \{x \in S \mid \mu(x) \geq \mu(0)\}.$$

The following statements are true.

(1) *If μ is an L-fuzzy ternary subsemiring of S, then S_μ is a ternary subsemiring of S.*

(2) *If μ is an L-fuzzy left [resp. right, lateral] ideal of S, then S_μ is a left [resp. right, lateral] ideal of S.*

(3) *If μ is an L-fuzzy ideal of S, then S_μ is an ideal of S.*

Proof. (1) Let μ be an L-fuzzy ternary subsemiring of S and $x, y, z \in S_\mu$. So $\mu(x), \mu(y), \mu(z) \geq \mu(0)$. Thus

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \geq \mu(0)$$

and

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\} \geq \mu(0).$$

So $x+y, xyz \in S_\mu$. Thus S_μ is a ternary subsemiring of S.

The proof of (2) is similar to the proof of (1).

(3) follows from (2). □

Let S be a ternary semiring. If $\mu \in \mathcal{F}(S)$ is an L-fuzzy ternary subsemiring of S, we call $\mu_t (\neq \emptyset)$ a level ternary subsemiring of μ. The level left ideals [resp. right ideals, lateral ideals, ideals] of μ are defined analogously.

Lemma 3.5. *Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. If μ is an L-fuzzy subset of S, then two level set μ_s and μ_t (with $s < t$ in L) of μ are equal if and only if there is no $x \in S$ such that $s \leq \mu(x) < t$.*

Proof. Let $s, t \in L$ such that $s < t$ and $\mu_s = \mu_t$. If there exist $x \in S$ such that $s \leq \mu(x) < t$. Then $x \in \mu_s$ but $x \notin \mu_t$, a contradiction.

Conversely, assume that there is no $x \in S$ such that $s \leq \mu(x) < t$. Let $y \in \mu_s$. Then $\mu(y) \geq s$. By assumption, $\mu(y) \geq t$. Hence $y \in \mu_t$. This implies that $\mu_s = \mu_t$. □

Let S be a ternary semiring. For any $\mu \in \mathcal{F}(S)$, we denote by $Im(\mu)$ the image set of μ.

Theorem 3.6. *Let S be a ternary semiring. The following statements are true.*

(1) Let μ be an L-fuzzy subset of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of set μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level subset of μ.

(2) Let μ be an L-fuzzy ternary subsemiring of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of ternary subsemirings μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level ternary subsemirings of μ.

(3) Let μ be an L-fuzzy left [resp. right, lateral] ideal of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of left [resp. right, lateral] ideals μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level left [resp. right, lateral] ideals of μ.

(4) Let μ be an L-fuzzy ideal of S. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, then the family of ideals μ_{t_i} (where $i = 1, 2, \ldots, n$) constitutes the collection of all level ideals of μ.

Proof. (1) If $t \in L$ with $t < t_1$, we have that $\mu_t = \mu_{t_1}$. If $t \in L$ with $t > t_n$, we have that $\mu_t = \emptyset$. If $t \in L$ with $t_i < t < t_{i+1}$ for some $i = 1, 2, \ldots, n-1$. By assumption, there is no $x \in R$ such that $t \leq \mu(x) < t_{i+1}$. By Lemma 3.5, $\mu_t = \mu_{t_{i+1}}$.

The proofs of (2), (3) and (4) are similar to the proof of (1). □

Theorem 3.7. *Let S and R be ternary semirings and $\varphi : S \to R$ be a homomorphism. The following statements are true.*

(1) Let μ be an L-fuzzy ternary subsemiring of R. Then the preimage of μ under φ is an L-fuzzy ternary subsemiring of S.

(2) Let μ be an L-fuzzy left [resp. right, lateral] ideal of R. Then the preimage of μ under φ is an L-fuzzy left [resp. right, lateral] ideal of S.

(3) *Let μ be an L-fuzzy ideal of R. Then the preimage of μ under φ is an L-fuzzy ideal of S.*

Proof. (1) Let $\mu \in \mathcal{F}(R)$ be an L-fuzzy ternary subsemiring and ν be the preimage of μ under φ. Then for any $x, y, z \in S$,

$$\begin{aligned}
\nu(x+y) &= \mu(\varphi(x+y)) \\
&= \mu(\varphi(x)+\varphi(y)) \\
&\geq \min\{\mu(\varphi(x)), \mu(\varphi(y))\} \\
&= \min\{\nu(x), \nu(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu(xyz) &= \mu(\varphi(xyz)) \\
&= \mu(\varphi(x)\varphi(y)\varphi(z)) \\
&\geq \min\{\mu(\varphi(x)), \mu(\varphi(y)), \mu(\varphi(z))\} \\
&= \min\{\nu(x), \nu(y), \nu(z)\}
\end{aligned}$$

This shows that ν is an L-fuzzy ternary subsemiring of S.

The proofs of (2) and (3) are similar to the proof of (1). □

Theorem 3.8. *Let S and R be ternary semirings and $\varphi : S \to R$ be an onto homomorphism. The following statements are true.*

(1) *Let μ be an L-fuzzy ternary subsemiring of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy ternary subsemiring of R.*

(2) *Let μ be an L-fuzzy left [resp. right, lateral] ideal of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy left [resp. right, lateral] ideal of R.*

(3) *Let μ be an L-fuzzy ideal of S. Then the homomorphic image $\varphi(\mu)$ of μ under φ is an L-fuzzy ideal of R.*

Proof. (1) By Theorem 3.2 (1), it is sufficient to show that each nonempty level subset of $\varphi(\mu)$ is a ternary subsemiring of R. Let $t \in L$ such that $(\varphi(\mu))_t \neq \emptyset$. If $t = 0$, then $(\varphi(\mu))_t = R$. Assume that $t \neq 0$. By Proposition 3.1,

$$(\varphi(\mu))_t = \bigcap_{0<s<t} \varphi(\mu_{t-s}).$$

Then $\varphi(\mu_{t-s}) \neq \emptyset$ for all $0 < s < t$, and so $\mu_{t-s} \neq \emptyset$ for all $0 < s < t$. By Theorem 3.2 (1), μ_{t-s} is a ternary subsemiring of S for all $0 < s < t$. Since φ is an onto homomorphism, $\varphi(\mu_{t-s})$ is a ternary subsemiring of R for all $0 < s < t$. Then $(\varphi(\mu))_t = \bigcap_{0<s<t} \varphi(\mu_{t-s})$ is a ternary subsemiring of R.

The proofs of (2) and (3) are similar to the proof of (1). □

Definition 3.5. A ternary subsemiring A of a ternary semiring S is said to be *characteristic* if $\varphi(A) = A$ for all $\varphi \in Aut(S)$ where $Aut(S)$ is the set of all automorphisms of S. The characteristic left ideals [resp. right ideals, lateral ideals, ideals] are defined analogously.

Definition 3.6. An L-fuzzy ternary subsemiring μ of S is said to be *L-fuzzy characteristic ternary subsemiring* if $\mu(\varphi(x)) = \mu(x)$ for all $x \in S$ and $\varphi \in Aut(S)$. The L-fuzzy characteristic left ideals [resp. right ideals, lateral ideals, ideals] are defined analogously.

Theorem 3.9. *Let S be a ternary semiring, $\varphi : S \to S$ an homomorphism and $\mu \in \mathcal{F}(S)$. Define $\mu^{\varphi} \in \mathcal{F}(S)$ by $\mu^{\varphi}(x) = \mu(\varphi(x))$ for all $x \in S$. The following statements are true.*

(1) If μ is an L-fuzzy ternary subsemiring of S, then μ^{φ} is an L-fuzzy ternary subsemiring of S.

(2) If μ is an L-fuzzy left [resp. right, lateral] ideal of S, then μ^{φ} is an L-fuzzy left [resp. right, lateral] ideal of S.

(3) If μ is an L-fuzzy ideal of S, then μ^{φ} is an L-fuzzy ideal of S.

Proof. (1) Let $x, y, z \in S$. We have

$$\begin{aligned}\mu^{\varphi}(x+y) &= \mu(\varphi(x+y)) \\ &= \mu(\varphi(x)+\varphi(y)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y))\} \\ &= \min\{\mu^{\varphi}(x), \mu^{\varphi}(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^{\varphi}(xyz) &= \mu(\varphi(xyz)) \\ &= \mu(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y)), \mu(\varphi(z))\} \\ &= \min\{\mu^{\varphi}(x), \mu^{\varphi}(y), \mu^{\varphi}(z)\}\end{aligned}$$

Therefore μ^{φ} is an L-fuzzy ternary subsemiring of S.

The proofs of (2) and (3) are similar to the proof of (1). □

Theorem 3.10. *Let S be a ternary semiring. The following statements are true.*

(1) If μ is an L-fuzzy characteristic ternary subsemiring of S, then each level ternary subsemiring of μ is characteristic.

(2) If μ is an L-fuzzy characteristic left [resp. right, lateral] ideal of S, then each level left [resp. right, lateral] ideal of μ is characteristic.

(3) If μ is an L-fuzzy characteristic ideal of S, then each level ideal of μ is characteristic.

Proof. (1) Let μ be an L-fuzzy characteristic ternary subsemiring of $S, \varphi \in Aut(S)$ and $t \in L$. If $y \in \varphi(\mu_t)$. Then there exists $x \in S$ such that $\varphi(x) = y$ and so $\mu(y) = \mu(\varphi(x)) = \mu(x) \geq t$. Thus $y \in \mu_t$.

Conversely, if $y \in \mu_t$, then $\mu(x) = \mu(\varphi(x)) = \mu(y) \geq t$ where $\varphi(x) = y$. It follows that $y \in \varphi(\mu_t)$. Thus $\varphi(\mu_t) = \mu_t$. Then μ_t is characteristic.

The proofs of (2) and (3) are similar to the proof of (1). □

The following theorem is the converse of Theorem 3.10.

Theorem 3.11. *Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. The following statements are true.*

(1) If each level ternary subsemiring of μ is characteristic, then μ is an L-fuzzy characteristic ternary subsemiring of S.

(2) If each level left [resp. right, lateral] ideal is characteristic, then μ is an L-fuzzy characteristic left [resp. right, lateral] ideal of S.

(3) If each level ideal is characteristic of μ, then μ is an L-fuzzy characteristic ideal of S.

Proof. (1) Let $x \in S, \varphi \in Aut(S)$ and $t = \mu(x)$. Then $x \in \mu_t$ and $x \notin \mu_s$ for all $s \in L$ with $s > t$. Since each level ternary subsemiring of μ is characteristic, $\varphi(x) \in \varphi(\mu_t) = \mu_t$. Thus $\mu(\varphi(x)) \geq t$. Suppose $\mu(\varphi(x)) = r > t$. Then

$$\varphi(x) \in \mu_r = \varphi(\mu_r).$$

This implies that $x \in \mu_r$, a contradiction. Hence $\mu(\varphi(x)) = \mu(x)$. Therefore μ is an L-fuzzy characteristic ternary subsemiring of S.

The proofs of (2) and (3) are similar to the proof of (1). □

3.2 Normal L-fuzzy ideals

Definition 3.7. An L-fuzzy subset μ of a ternary semiring with zero S is said to be *normal* if $\mu(0) = 1$.

Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. Define an L-fuzzy subset μ^+ of S by

$$\mu^+(x) = \mu(x) + 1 - \mu(0) \text{ for all } x \in S.$$

Proposition 3.12. *Let S be a ternary semiring and $\mu \in \mathcal{F}(S)$. The following statements are true.*

(1) μ^+ *is a normal L-fuzzy subset of* S *containing* μ.

(2) $(\mu^+)^+ = \mu^+$.

(3) μ *is normal if and only if* $\mu = \mu^+$.

Proof. (1) We can see that $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1$ and for $x \in S$,

$$\mu(x) \leq \mu^+(x),$$

completing the proof.

(2) By (1), we have

$$(\mu^+)^+(x) = \mu^+(x) + 1 - \mu^+(0) = \mu^+(x) + 1 - 1 = \mu^+(x),$$

completing the proof.

(3) Assume that μ is normal. Then

$$\mu^+(x) = \mu(x) + 1 - \mu(0) = \mu(x) + 1 - 1 = \mu(x).$$

The converse is obvious by (i). □

Corollary 3.13. *Let* S *be a ternary semiring,* $\mu \in \mathcal{F}(S)$ *and* $x \in S$. *If* $\mu^+(x) = 0$, *then* $\mu(x) = 0$.

Proof. By Proposition 3.12 (1), we have $\mu(x) \leq \mu^+(x)$, this implies that

$$\mu(x) = 0.$$

□

Theorem 3.14. *Let* S *be a ternary semiring and* $\mu \in \mathcal{F}(S)$. *The following statements are true.*

(1) If μ *is an L-fuzzy ternary subsemiring of* S, *then* μ^+ *is a normal L-fuzzy ternary subsemiring of* S *containing* μ.

(2) If μ *is an L-fuzzy left [resp. right, lateral] ideal of* S, *then* μ^+ *is a normal L-fuzzy left [resp. right, lateral] ideal of* S *containing* μ.

(3) If μ *is an* L*-fuzzy ideal of* S*, then* μ^+ *is a normal* L*-fuzzy ideal of* S *containing* μ*.*

Proof. (1) Let $x, y, z \in S$. Then

$$\begin{aligned}\mu^+(x+y) &= \mu(x+y) + 1 - \mu(0) \\ &\geq \min\{\mu(x), \mu(y)\} + 1 - \mu(0) \\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\ &= \min\{\mu^+(x), \mu^+(y)\}\end{aligned}$$

and

$$\begin{aligned}\mu^+(xyz) &= \mu(xyz) + 1 - \mu(0) \\ &\geq \min\{\mu(x), \mu(y), \mu(z)\} + 1 - \mu(0) \\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0), \\ &\quad \mu(z) + 1 - \mu(0)\} \\ &= \min\{\mu^+(x), \mu^+(y), \mu^+(z)\}.\end{aligned}$$

Hence μ^+ is an L-fuzzy ternary subsemiring of S. By Proposition 3.12 (1), μ^+ is a normal L-fuzzy ternary subsemiring of S containing μ.

The proofs of (2) and (3) are similar to the proof of (1) □

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