Another Aspect of a Generalized Bergman Space

Marisa Senmoh

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Author Miss Marisa Senmoh

Major Program Mathematics and Statistics

Advisor :

....

(Dr. Kamthorn Chailuek)

Examination Committee :

....Chairperson

(Dr. Orawan Tripak)

....

(Dr. Kamthorn Chailuek)

....

(Dr. Suwicha Imnang)

The Graduate School, Prince of Songkla University, has approved this thesis as partial fulfillment of the requirements for the Master of Science Degree in Mathematics and Statistics

....

(Prof. Dr. Amornrat Phongdara)

Dean of Graduate School

ชื่อวิทยานิพนธ์	มุมมองอื่นของปริภูมิเบิร์กแมนเชิงทั่วไป
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บทคัดย่อ

ปริภูมิเบิร์กแมนคือปริภูมิของฟังก์ชันฮอลอมอร์ฟิกบนบอลหนึ่งหน่วย $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ ซึ่งค่าสัมบูรณ์ยกกำลังสองสามารถหาปริพันธ์ได้เมื่อเทียบกับเมเชอร์ dv_α โดยที่ $dv_\alpha = c_\alpha(1-|z|^2)^\alpha$ และ $c_\alpha = \dfrac{\Gamma(\alpha+2)}{\pi\Gamma(\alpha+1)}$ นั่นคือ

$$\mathcal{H}L^2(\mathbb{B}, dv_\alpha) = \{f : f \text{ เป็นฟังก์ชันฮอลอมอร์ฟิกบน } \mathbb{B} \text{ และ } \int_{\mathbb{B}} |f(z)|^2 dv_\alpha(z) < \infty\}$$

ปริภูมิเบิร์กแมนไม่เป็นปริภูมิศูนย์ก็ต่อเมื่อ $\alpha > -1$ อย่างไรก็ตามค่ารีโปรดิวซิ่งเคอร์เนล $K(w,z) = \dfrac{1}{\pi(1-\langle z, w\rangle)^{\alpha+2}}$ ยังคงถูกนิยามอย่างบวก เมื่อ $-2 < \alpha \leq -1$ ปริภูมิเบิร์กแมนสามารถขยายไปยังกรณีที่ $-2 < \alpha \leq -1$ โดยพิจารณาให้เป็นปริภูมิย่อยของปริภูมิเบิร์กแมนที่มีอยู่แล้วดังนี้

$$HL^2(\mathbb{B}, \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) : z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) \right\}$$

ซึ่งปริภูมิ $HL^2(\mathbb{B}, \alpha) = \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ เมื่อ $\alpha > -1$ อย่างไรก็ตาม $HL^2(\mathbb{B}, \alpha)$ ไม่เป็นปริภูมิศูนย์เมื่อ $-2 < \alpha \leq -1$ เราเรียกปริภูมิใหม่นี้ว่า ปริภูมิเบิร์กแมนเชิงทั่วไป

ในงานวิจัยครั้งนี้ เราสนใจมุมมองอื่นของปริภูมิเบิร์กแมนเชิงทั่วไปนอกเหนือจากการมองให้เป็นปริภูมิย่อยของปริภูมิเบิร์กแมนดังกล่าวข้างต้น

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ABSTRACT

A Bergman space $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ is a space consisting of all holomorphic functions on the unit ball $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ which are square-integrable with respect to dv_α where $dv_\alpha = c_\alpha(1 - |z|^2)^\alpha$ and $c_\alpha = \dfrac{\Gamma(\alpha+2)}{\pi\Gamma(\alpha+1)}$. That is,

$$\mathcal{H}L^2(\mathbb{B}, dv_\alpha) = \{f : f \text{ is holomorphic on } \mathbb{B} \text{ and} \int_{\mathbb{B}} |f(z)|^2 \, dv_\alpha(z) < \infty\}.$$

The Bergman spaces are non-zero if and only if $\alpha > -1$. However the reproducing kernel of each Bergman space, $K(w, z) = \dfrac{1}{\pi(1 - \langle z, w\rangle)^{\alpha+2}}$, is still positive-definite when $-2 < \alpha \leq -1$. Bergman spaces can be extended to the case $-2 < \alpha \leq -1$ by considering them as subspaces of existing Bergman spaces as follows,

$$HL^2(\mathbb{B}, \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) : z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) \right\}.$$

The space $HL^2(\mathbb{B}, \alpha) = \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ when $\alpha > -1$. However $HL^2(\mathbb{B}, \alpha)$ do exist and non-zero when $-2 < \alpha \leq -1$ and they are called generalized Bergman spaces.

In this research, we are interested in viewing a generalized Bergman space in another aspect besides a subspace of an existing Bergman space.

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Marisa Senmoh

CONTENTS

CHAPTER 1

Introduction

Let $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ and $dv_\alpha = c_\alpha(1-|z|^2)^\alpha$, where $c_\alpha = \dfrac{\Gamma(\alpha+2)}{\pi\Gamma(\alpha+1)}$. A Bergman space $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ is a space consisting of all holomorphic functions on $\mathbb{B}$ which are square-integrable with respect to dv_α. That is, $\displaystyle\int_{\mathbb{B}} |f(z)|^2 c_\alpha(1-|z|^2)^\alpha\, dz < \infty$. This space is a Hilbert space with the inner product and the norm on $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ is defined by $\langle f, g\rangle_\alpha = \displaystyle\int_{\mathbb{B}} f(z)\overline{g(z)}\, dv_\alpha(z)$ and $\|f\|_\alpha = \left\{\displaystyle\int_{\mathbb{B}} |f(z)|^2 dv_\alpha(z)\right\}^{\frac{1}{2}}$, respectively. The Bergman spaces are non-zero if and only if $\alpha > -1$ with reproducing kernel $K(w,z) = \dfrac{1}{\pi(1-\langle z, w\rangle)^{\alpha+2}}$. The formula for the reproducing kernel is still positive-definite when $-2 < \alpha \leq -1$ telling us that we have a chance in extending the space to the range $-2 < \alpha \leq -1$. In [2], the authers extended Bergman spaces to the case $-2 < \alpha$ by defining

$$HL^2(\mathbb{B}, \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) : z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) \right\}$$

and called them **generalized Bergman spaces**.

As inner product spaces, $HL^2(\mathbb{B}, \alpha) = \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ when $\alpha > -1$, see [1]. However $HL^2(\mathbb{B}, \alpha)$ is non-zero for each $-2 < \alpha \leq -1$. The norm of $f \in HL^2(\mathbb{B}, \alpha)$ is defined via the norm and inner product on $\mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$ by

$$\begin{aligned} \|f\|^2_{HL^2_\alpha} \;=\; & \frac{1}{(\alpha+2)(\alpha+3)}\|z\frac{d}{dz}f\|^2_{\alpha+2} + \|f\|^2_{\alpha+2} + \frac{1}{\alpha+2}\langle f, z\frac{d}{dz}f\rangle_{\alpha+2} \\ & + \frac{1}{\alpha+3}\langle z\frac{d}{dz}f, f\rangle_{\alpha+2} \end{aligned}$$

and

$$\langle f, g\rangle_{HL^2_\alpha} = \langle \mathcal{P}_{\alpha+1} f, \mathcal{P}_\alpha g\rangle_{\alpha+2}$$

where $\mathcal{P}_\alpha f = \dfrac{z\dfrac{d}{dz} + \alpha + 2}{\alpha + 2} f$.

The space $HL^2(\mathbb{B}, \alpha)$ has not been extended by using the measure dv_α because it is an infinite measure when $\alpha \leq -1$, especially $\Gamma(\alpha)$ is undefined when $\alpha = -1, -2, -3, \ldots$.

By the definition of a generalized Bergman space, the conditions for a function f to be in $HL^2(\mathbb{B}, \alpha)$ involves in being in another Bergman space with higher parameter α, $\mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$. We note that $\alpha + 2 > -1$ when $\alpha > -2$ which makes the measure $dv_{\alpha+2}$ finite.

In this research, we will present another version of a generalized Bergman space in Theorem 3.1. As well as, some equivalent conditions for a function being in a generalized Bergman space will be given in Theorem 3.2 and Theorem 3.3. The conditions also involve in square-integrability with respect to the measure $dv_{\alpha+2}$. However they do not involve in the integrability of f itself but the function f under the number operator, the gradient and the invariant gradient of f.

CHAPTER 2

Preliminaries

In this chapter, we collect some basic knowledge and notations of operators used in this paper.

Definition 2.1. A **metric space** (X, d) is a nonempty set X of elements together with a distance function d defined on $X \times X$ such that for all x, y and z in X:
(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0$ if and only if $x = y$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.
The function d is called a **metric on** X.

Definition 2.2. Let (X, d) be a metric space and $A \subset X$. We say that A is **open** if, for each point $x \in A$, there is an $\varepsilon > 0$ such that $B_x(\varepsilon) \subset A$, where $B_x(\varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Definition 2.3. Let Ω be a non-empty open set in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is said to be **holomorphic** on Ω if for every point $z \in \Omega$, the limit

$$\lim_{\lambda \to 0} \frac{f(z + \lambda) - f(z)}{\lambda}$$

exists, where $\lambda \in \mathbb{C}$. We denote the set of all holomorphic functions on Ω by $\mathcal{H}(\Omega)$.

Instead of checking the limit $\lim\limits_{\lambda \to 0} \dfrac{f(z + \lambda) - f(z)}{\lambda}$ at every point z, we can verify the holomorphicity of f by using partial derivatives with respect to its components $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.
We define **the differential operators** as follows :

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Theorem 2.1. *A function f is holomorphic on Ω if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$ on Ω.*

Example. Consider $f(z) = z^m$ where $m \in \mathbb{N} \cup \{0\}$, we compute

$$\begin{aligned}
\frac{\partial}{\partial z} z^m &= \frac{\partial (x+yi)^m}{\partial z} \\
&= \frac{1}{2}\left(\frac{\partial (x+yi)^m}{\partial x} - i\frac{\partial (x+yi)^m}{\partial y}\right) \\
&= \frac{1}{2}\left(m(x+yi)^{m-1} + m(x+yi)^{m-1}\right) \\
&= m(x+yi)^{m-1} \\
&= mz^{m-1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}} z^m &= \frac{\partial (x+yi)^m}{\partial \overline{z}} \\
&= \frac{1}{2}\left(\frac{\partial (x+yi)^m}{\partial x} + i\frac{\partial (x+yi)^m}{\partial y}\right) \\
&= \frac{1}{2}\left(m(x+yi)^{m-1} - m(x+yi)^{m-1}\right) \\
&= 0.
\end{aligned}$$

Therefore z^m is holomorphic and $\dfrac{\partial}{\partial z} z^m = mz^{m-1}$.

Theorem 2.2. ***(The Maximum Principle)**. If Ω is connected, $f \in \mathcal{H}(\Omega)$, and $|f(a)| = \sup_{z \in \Omega} |f(z)|$ for some $a \in \Omega$, then f is a constant function.*

This theorem implies that a holomorphic function defined on $\mathbb{B}$ does not process a maximum or a minimum inside the ball $\mathbb{B}$ otherwise it is a constant function.

Definition 2.4. The **Gamma function** is defined by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)...(x+n)} \qquad \text{for each } \ x \in \mathbb{R} - (\mathbb{Z}^- \cup \{0\}).$$

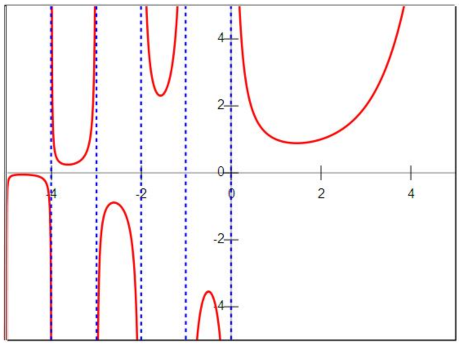
Theorem 2.3. *If $x > 0$, then*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

Corollary 2.4. $\Gamma(n) = (n-1)!$ *for all positive integers* n.



The graph of the Gamma function

Definition 2.5. Let X be a vector space over a field $\mathbb{F}$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a **norm** on X if

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|cx\| = |c|\|x\|$ for any $x \in X$ and $c \in \mathbb{F}$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

A vector space equipped with a norm is called a **normed linear space**, or simply a **normed space**. Property (iii) is referred to as the **triangle inequality**.

Definition 2.6. A sequence $\{x_n\}$ in a metric space (X, d) is a **Cauchy sequence** if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon, \qquad \text{for all } m, n \geq N.$$

Definition 2.7. The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to an element in X. That is if $d(x_n, x_m) \to 0$ as $m, n \to \infty$ then $\{x_n\}$ must converge also in X.

Definition 2.8. A **Banach space** is a normed linear space which is complete in the metric defined by its norm. That is $d(x, y) = \|x - y\|$.

Definition 2.9. An **inner product** on a vector space V is a function that associates a complex number $\langle u, v\rangle$ with each pair of vectors u and v in V which the following axioms are satisfied for all vectors u, v and w in V and all scalars k.

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$

(ii) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(iii) $\langle ku, v\rangle = k\langle u, v\rangle$

(iv) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$

A vector space equipped with an inner product is called an **inner product space**.

So if we define $\|v\| = \sqrt{\langle v, v\rangle}$ then $\|\cdot\|$ is a norm on V.

Definition 2.10. A **Hilbert space** is an inner product space which is complete with respect to the norm given by the inner product.

Definition 2.11. For $1 \leq p < \infty$, the $\mathcal{L}^p(X,\mu)$-space is the collection of all functions $f : X \to \mathbb{C}$ such that

$$\int_X |f(z)|^p \, d\mu(z) < \infty.$$

We define $L^p(X,\mu)$ to be the space of all equivalence classes of functions in $\mathcal{L}^p(X,\mu)$ under the relation $f \sim g$ if and only if $f = g$ almost everywhere with respect to the measure μ.

Definition 2.12. For $\alpha > -1$, the weighted Lebesgue measure dv_α is defined by

$$dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dz$$

where

$$c_\alpha = \frac{\Gamma(\alpha+2)}{\pi\Gamma(\alpha+1)}$$

is the normalizing constant so that dv_α is a probability measure on $\mathbb{B}$.

For $\alpha > -1$, the **Bergman space** (also called weighted Bergman space) $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ consists of all holomorphic functions f in $L^2(\mathbb{B}, dv_\alpha)$, that is

$$\mathcal{H}L^2(\mathbb{B}, dv_\alpha) = L^2(\mathbb{B}, dv_\alpha) \cap \mathcal{H}(\mathbb{B}).$$

The norm and inner product on $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ are defined by

$$\|f\|_\alpha = \left\{\int_{\mathbb{B}} |f(z)|^2 dv_\alpha(z)\right\}^{\frac{1}{2}}$$

and

$$\langle f, g\rangle_\alpha = \int_{\mathbb{B}} f(z)\overline{g(z)} dv_\alpha(z)$$

for $f, g \in \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$.

Proposition 2.5. A Bergman space is not empty.

พิสูจน์. See [1]. □

Actually we can compute directly that $z^m \in \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ as follows. Since z^m is holomorphic function, we have

$$\begin{aligned}
\int_{\mathbb{B}} |z^m|^2 \, dv_\alpha(z) &= c_\alpha \int_{\mathbb{B}} |r^m e^{i\theta m}|^2 (1-r^2)^\alpha \, dz \\
&= c_\alpha \int_{\mathbb{B}} r^{2m} (1-r^2)^\alpha \, dz \\
&= c_\alpha \int_0^{2\pi} \int_0^1 r^{2m} (1-r^2)^\alpha \, r dr d\theta \\
&= 2\pi c_\alpha \int_0^1 r^{2m+1} (1-r^2)^\alpha \, dr \\
&= 2\pi c_\alpha \frac{\Gamma(m+1)\Gamma(\alpha+1)}{\pi\Gamma(m+\alpha+2)} \\
&= 2\pi \frac{\Gamma(\alpha+2)}{\pi\Gamma(\alpha+1)} \frac{\Gamma(m+1)\Gamma(\alpha+1)}{\pi\Gamma(m+\alpha+2)} \\
&= \frac{2\Gamma(\alpha+2)\Gamma(m+1)}{\pi\Gamma(m+\alpha+2)} \\
&= \frac{2m!\Gamma(\alpha+2)}{\pi\Gamma(m+\alpha+2)} \\
&< \infty.
\end{aligned}$$

Therefore $z^m \in \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$.

Definition 2.13. Let X be an inner product space. The vectors $x, y \in X$ are said to be **orthogonal** if $\langle x, y \rangle$=0.

Definition 2.14. Let X be an inner product space. The set $\{e_1, ..., e_k\} \subset X$ is said to be **orthonormal** if $\|e_n\| = 1$ for $1 \leq n \leq k$, and $\langle e_m, e_n \rangle = 0$ for all $1 \leq m, n \leq k$ with $m \neq n$.

Definition 2.15. A maximal orthonormal set in an inner product space V is called an **orthonormal basis** for V or a complete orthonormal set of V.

Proposition 2.6. *In $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$, if $m \neq n$, then $\langle z^m, z^n \rangle_\alpha = 0$.*

พิสูจน์. Since z^m=$r^m e^{i\theta m}$, $\overline{z^n}$=$r^n e^{-i\theta n}$, and $m \neq n$, we obtain

$$\begin{aligned}
\int_{\mathbb{B}} z^m \overline{z^n}\, dz &= \int_0^{2\pi}\int_0^1 r^{m+n} e^{i\theta(m-n)}(1-r^2)^\alpha r\, dr d\theta \\
&= \int_0^{2\pi} e^{i\theta(m-n)} d\theta \int_0^1 r^{m+n}(1-r^2)^\alpha r\, dr \\
&= \int_0^{2\pi} \cos\theta(m-n) + i\sin\theta(m-n)\, d\theta \int_0^1 r^{m+n}(1-r^2)^\alpha r\, dr \\
&= \left[\frac{\sin\theta(m-n)}{m-n} - \frac{i\cos\theta(m-n)}{m-n}\right]_0^{2\pi} \int_0^1 r^{m+n+1}(1-r^2)^\alpha\, dr \\
&= 0.
\end{aligned}$$

□

Proposition 2.7. The set $\{z^m\}$ is an orthonormal basis for $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$.

พิสูจน์. See [1]. □

Theorem 2.8. ***(Riesz Representation)*** *If L is a bounded linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that*

$$L(x) = \langle y, x\rangle \qquad \textit{for each } x \in H.$$

Moreover $\|L\| = \|y\|$ *where* $\|L\| = \sup_{x\in H-\{0\}} \frac{|L(x)|}{\|x\|}$.

Consider the pointwise evaluation in $\mathbb{B}$ which is defined as follows. For a fixed $w \in \mathbb{B}$, we define $T_w : \mathcal{H}L^2(\mathbb{B}, dv_\alpha) \to \mathbb{C}$ by

$$T_w(f) = f(w).$$

Then we have that T_w is a bounded linear functional on $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$. The Riesz representation shows that for each $w \in \mathbb{B}$, there exists a unique function K_w^α in $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ such that

$$f(w) = \langle f, K_w^\alpha\rangle = \int_{\mathbb{B}} f(z) K_w^\alpha(z)\, dv_\alpha(z). \tag{2.1}$$

This will be called the **reproducing formula** for f in $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$. The function $K^\alpha(z,w) = K_w^\alpha(z)$ where $z, w \in \mathbb{B}$ is called the **reproducing kernel** of $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$. When $\alpha = 0$, the reproducing kernel $K(z,w) = K_w^0(z)$ is also called the **Bergman kernel**.

Proposition 2.9. *For each $\alpha > -1$, the reproducing kernel of $\mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ is given by*

$$K^\alpha(z,w) = \frac{1}{\pi(1-\langle z, w\rangle)^{\alpha+2}}.$$

พิสูจน์. See [4]. □

Theorem 2.10. ***(Hölder Inequality)*** *If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}.$$

Theorem 2.11. *Let* $\mathcal{H}L^2(\mathbb{B}, \alpha)$ *be as above. Then there exists a function* $K(z, w)$, $z, w \in \mathbb{B}$, *with the following properties :*

1. $K(z, w)$ *is holomorphic in* z *and anti-holomorphic in* w, *and satisfies*

$$K(w, z) = \overline{K(z, w)}.$$

2. For each fixed $z \in \mathbb{B}, K(z, w)$ *is square-integrable* $d\alpha(w)$. *For all* $F \in \mathcal{H}L^2(\mathbb{B}, \alpha)$

$$F(z) = \int_{\mathbb{B}} K(z, w) F(w) \alpha(w)\, dw.$$

3. If $F \in L^2(\mathbb{B}, \alpha)$, *let* PF *denote the orthogonal projection of* F *onto the closed subspace* $\mathcal{H}L^2(\mathbb{B}, \alpha)$. *Then*

$$PF(z) = \int_{\mathbb{B}} K(z, w) F(w) \alpha(w)\, dw.$$

4. For all $z, u \in \mathbb{B}$,

$$\int_{\mathbb{B}} K(z, w) K(w, u) \alpha(w) = K(z, u).$$

5. For all $z \in \mathbb{B}$,

$$|F(z)|^2 \leq K(z, z) \|F\|^2$$

and the constant $K(z, z)$ *is optimal in the sense that for each* $z \in \mathbb{B}$ *there exists a non-zero* $F(z) \in \mathcal{H}L^2(\mathbb{B}, \alpha)$ *for which equality holds.*

6. Given any $z \in \mathbb{B}$, *if* $\phi_z(\cdot) \in \mathcal{H}L^2(\mathbb{B}, \alpha)$ *satisfies*

$$F(z) = \int_{\mathbb{B}} \overline{\phi_z(w)} F(w) \alpha(w)\, dw$$

for all $F \in \mathcal{H}L^2(\mathbb{B}, \alpha)$ *then* $\overline{\phi_z(w)} = K(z, w)$.

พิสูจน์. See [4]. □

In $\mathbb{C}$, we introduce the "**number operator**" N which is defined by

$$N = z\frac{d}{dz}.$$

It is clear that for any monomial z^m, $N(z^m) = mz^m$.

For a holomorphic function f in $\mathbb{B}$, we write

$$\nabla f(z) = \frac{df}{dz}(z)$$

and call $|\nabla f(z)|$ the holomorphic gradient of f at z. Similarly, we define

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

where φ_z is the biholomorphic mapping of $\mathbb{B}$ that interchanges 0 and z. Precisely, for any point $z \in \mathbb{B} - \{0\}$ and $a \in \mathbb{B}$,

$$\varphi_z(a) = \frac{z - P_z(a) - S_z Q_z(a)}{1 - \langle a, z\rangle}$$

where $S_z = \sqrt{1-|z|^2}$, $P_z(a) = \dfrac{\langle a, z\rangle}{|z|^2} z$ and $Q_z(a) = a - \dfrac{\langle a, z\rangle}{|z|^2} z$. We call $|\widetilde{\nabla} f(z)|$ the invariant gradient of f at z.

Definition 2.16. Let X be a normed linear space. Denote by X^* the set of all bounded linear functionals on X. We call X^* the **dual space** of X

Definition 2.17. For $\alpha > -2$, we define

$$HL^2(\mathbb{B}, \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) : Nf = z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) \right\}$$

with the inner product and norm

$$\langle f, g\rangle_{HL^2_\alpha} = \langle \mathcal{P}_{\alpha+1} f, \mathcal{P}_\alpha g\rangle_{\alpha+2}$$

$$\begin{aligned} \|f\|_{HL^2_\alpha} &= \frac{1}{(\alpha+2)(\alpha+3)}\|z\frac{d}{df}f\|^2_{\alpha+2} + \|f\|^2_{\alpha+2} \\ &\quad + \frac{1}{\alpha+2}\langle f, z\frac{d}{dz}f\rangle_{\alpha+2} + \frac{1}{\alpha+3}\langle z\frac{d}{dz}f, f\rangle_{\alpha+2} \end{aligned}$$

where $\quad \mathcal{P}_\alpha f(z) = \left(\dfrac{z\dfrac{d}{dz} + \alpha + 2}{\alpha+2} \right) f(z).$

Theorem 2.12. *For $\alpha > -1$, $HL^2(\mathbb{B}, \alpha) = \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ as inner product spaces.*

พิสูจน์. See [1]. □

Theorem 2.13. *For $\alpha > -2$,*

$$\left\{ z^m \sqrt{\frac{\Gamma(m+\alpha+2)}{m!\Gamma(\alpha+2)}} \right\}$$

is an orthonormal basis for $HL^2(\mathbb{B}, \alpha)$.

พิสูจน์. See [1]. □

Lemma 2.14. *If f is holomorphic in $\mathbb{B}$, then*

$$|\widetilde{\nabla} f(z)| = (1-|z|^2)|\nabla f(z)|$$

for all $z \in \mathbb{B}$.

พิสูจน์. See [9]. □

Lemma 2.15. *If $f$is holomorphic in $\mathbb{B}$, then*

$$(1-|z|^2)|Nf(z)| \leq (1-|z|^2)|\nabla f(z)| = |\widetilde{\nabla} f(z)|$$

for all $z \in \mathbb{B}$.

พิสูจน์. See [9]. □

The next proposition will show that $|\nabla f(z)|$ is bounded on any compact subset of $\mathbb{B}$.

Proposition 2.16. *Suppose $\alpha > -1$, $0 < r < 1$. Then there exists a positive constant C such that*

$$|\nabla f(z)| = \left| \frac{df}{dz}(z) \right| \leq C\|f\|_\alpha$$

for all $f \in \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ and all $z \in \mathbb{B}_n$ with $|z| \leq r$.

พิสูจน์. See [9]. □

Proposition 2.17. *Suppose α is real and $\int_{\mathbb{B}} |f(z)|(1-|z|^2)^\alpha \, dz < \infty$. Then*

$$\int_{\mathbb{B}} f \circ \varphi(z) \, dv_\alpha(z) = \int_{\mathbb{B}} f(z) \frac{(1-|a|^2)^{2+\alpha}}{|1-\langle z, a\rangle|^{2(2+\alpha)}} \, dv_\alpha(z)$$

where φ is any automorphism of $\mathbb{B}$ and $a = \varphi(0)$.

พิสูจน์. See [9]. □

Theorem 2.18. *Suppose c is real and $t > -1$. Then the integral*

$$J_{c,t}(z) = \int_{\mathbb{B}} \frac{(1-|w|^2)}{|1-\langle z,w\rangle|^{t+c+2}}\, dw \qquad\qquad z \in \mathbb{B}$$

has the following properties.

If $c > 0$, then

$$J_{c,t}(z) \sim (1-|z|^2)^{-c}$$

as $|z| \to 1^-$.

พิสูจน์. See [9]. □

Theorem 2.19. *If $\alpha > -1$ and $\int_{\mathbb{B}} |f(z)|(1-|z|^2)^{\alpha}\, dz < \infty$, then*

$$f(z) = \int_{\mathbb{B}} \frac{f(w)}{(1-\langle z,w\rangle)^{\alpha+2}}\, dv_\alpha(w)$$

for all $z \in \mathbb{B}$.

พิสูจน์. See [9]. □

Theorem 2.20. *The kernel*

$$L(z,w) = \int_0^1 \left[\frac{1}{(1-t\langle z,w\rangle)^{\beta+2}} - 1\right]\frac{dt}{t}$$

satisfies

$$|L(z,w)| \le \frac{C}{|1-\langle z,w\rangle|^{\beta+1}}$$

for any $z, w \in \mathbb{B}$ and C is some positive constant independent of z and w.

พิสูจน์. Consider the kernel

$$\begin{aligned}
|L(z,w)| &\le \int_0^1 \left|\frac{1}{(1-t\langle z,w\rangle)^{\beta+2}} - 1\right| \frac{1}{t}dt \\
&= \int_0^1 \left|\frac{1-(1-t\langle z,w\rangle)^{\beta+2}}{(1-t\langle z,w\rangle)^{\beta+2}}\right| \frac{1}{t}dt \\
&= \frac{1}{|1-\langle z,w\rangle|^{\beta+1}} \int_0^1 \left|\frac{(1-\langle z,w\rangle)^{\beta+1}}{(1-t\langle z,w\rangle)^{\beta+2}} \cdot \frac{1-(1-t\langle z,w\rangle)^{\beta+2}}{t}\right| dt.
\end{aligned}$$

Since $f(t) = \left|\dfrac{(1-\langle z,w\rangle)^{\beta+1}}{(1-t\langle z,w\rangle)^{\beta+2}}\right|$ is continuous on $[0,1]$, we can choose

$A = \max_{0\le t\le 1}\left|\dfrac{(1-\langle z,w\rangle)^{\beta+1}}{(1-t\langle z,w\rangle)^{\beta+2}}\right|$. Thus

$$|L(z,w)| \le \frac{A}{|1-\langle z,w\rangle|^{\beta+1}}\int_0^1 \left|\frac{1-(1-t\langle z,w\rangle)^{\beta+2}}{t}\right| dt.$$

Since $g(t) = \left|\dfrac{1-(1-t\langle z,w\rangle)^{\beta+2}}{t}\right|$ is continuous on $(0,1]$ and

$$\lim_{t\to 0}\frac{1-(1-t\langle z,w\rangle)^{\beta+2}}{t} = (\beta+2)\langle z,w\rangle,$$

we can let $C = A\displaystyle\int_0^1 \max_{0\le t\le 1}\left|\frac{1-(1-t\langle z,w\rangle)^{\alpha+2}}{t}\right| dt$. Therefore we obtain

$$|L(z,w)| \le \frac{C}{|(1-\langle z,w\rangle)|^{\beta+1}}. \tag{2.2}$$

□

Theorem 2.21. *Fix two real parameters a and b and define two integral operators T and S by*

$$Tf(z) = (1-|z|^2)^a\int_{\mathbb{B}}\frac{(1-|w|^2)^b}{(1-\langle z,w\rangle)^{a+b+2}}f(w)\,dw$$

and

$$Sf(z) = (1-|z|^2)^a\int_{\mathbb{B}}\frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^{a+b+2}}f(w)\,dw.$$

Then for $-\infty < t < \infty$ and $1 < p < \infty$ the following conditions are equivalent:

a). T is bounded on $L^p(\mathbb{B}, dv_t)$.

b). S is bounded on $L^p(\mathbb{B}, dv_t)$.

c). $-pa < t+1 < p(b+1)$.

พิสูจน์. See [9]. □

CHAPTER 3

Main Results

We recall that for $\alpha > -2$, a generalized Bergman space is defined by

$$HL^2(\mathbb{B}, \alpha) = \left\{ f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) : z\frac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2}) \right\}.$$

The space $HL^2(\mathbb{B}, \alpha) = \mathcal{H}L^2(\mathbb{B}, dv_\alpha)$ when $\alpha > -1$. However $HL^2(\mathbb{B}, \alpha)$ does exist and non-zero when $-2 < \alpha \leq -1$.

In this chapter, we will present another aspect of a generalized Bergman space. We start with viewing a generalized Bergman space as a dual space of other spaces. Then we will give necessary and sufficient conditions for a function being in a generalized Bergman space.

Theorem 3.1. *Suppose* $\alpha, \beta > -2$ *then*

$$\left(HL^2(\mathbb{B}, \alpha)\right)^* = HL^2(\mathbb{B}, \beta)$$

(with equivalent norms) under the integral pairing

$$\langle f, g \rangle_{HL^2_\gamma} = \int_{\mathbb{B}} \mathcal{P}_{\gamma+1} f(z) \overline{\mathcal{P}_\gamma g(z)} dv_{\gamma+2}(z)$$

where $\qquad \gamma = \dfrac{\alpha + \beta}{2}.$

พิสูจน์. For each $g \in HL^2(\mathbb{B}, \beta)$, we define $T_g : HL^2(\mathbb{B}, \alpha) \to \mathbb{C}$ by

$$T_g(f) = \langle f, g \rangle_{HL^2_\gamma}.$$

Next, we will prove that $T_g \in (HL^2(\mathbb{B}, \alpha))^*$.

$$
\begin{aligned}
\text{Consider } |T_g(f)| &= \left|\langle f, g\rangle_{HL^2_\gamma}\right| \\
&= |\langle \mathcal{P}_{\gamma+1} f, \mathcal{P}_\gamma g\rangle_{\gamma+2}| \\
&= c_{\gamma+2}\left|\int_{\mathbb{B}} \mathcal{P}_{\gamma+1} f(z)\overline{\mathcal{P}_\gamma g(z)}(1-|z|^2)^{\gamma+2}dz\right| \\
&\le c_{\gamma+2}\int_{\mathbb{B}}(1-|z|^2)^{\frac{\alpha+2}{2}}|\mathcal{P}_{\gamma+1} f(z)|(1-|z|^2)^{\frac{\beta+2}{2}}\left|\overline{\mathcal{P}_\gamma g(z)}\right| dz.
\end{aligned}
$$

By Theorem 2.10,

$$
\begin{aligned}
|T_g(f)| &\le c_{\gamma+2}\left(\int_{\mathbb{B}}((1-|z|^2)^{\frac{\alpha+2}{2}}|\mathcal{P}_{\gamma+1} f(z)|)^2 dz\right)^{\frac{1}{2}} \\
&\quad \cdot\left(\int_{\mathbb{B}}((1-|z|^2)^{\frac{\beta+2}{2}}\left|\overline{\mathcal{P}_\gamma g(z)}\right|)^2 dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2}\left(\int_{\mathbb{B}}|\mathcal{P}_{\gamma+1} f(z)|^2(1-|z|^2)^{\alpha+2}dz\right)^{\frac{1}{2}} \\
&\quad \cdot\left(\int_{\mathbb{B}}\left|\overline{\mathcal{P}_\gamma g(z)}\right|^2(1-|z|^2)^{\beta+2}dz\right)^{\frac{1}{2}} \\
&= c_{\gamma+2}\|\mathcal{P}_{\gamma+1} f(z)\|_{\alpha+2}\|\mathcal{P}_\gamma g(z)\|_{\beta+2} \\
&= c_{\gamma+2}\left(\langle P_{\gamma+1} f(z), \mathcal{P}_{\gamma+1} f(z)\rangle_{\alpha+2}\langle \mathcal{P}_\gamma g(z), \mathcal{P}_\gamma g(z)\rangle_{\beta+2}\right)^{\frac{1}{2}} \\
&= c_{\gamma+2}\left(\left\langle \frac{Nf(z)}{\gamma+3}+f(z), \frac{Nf(z)}{\gamma+3}+f(z)\right\rangle_{\alpha+2}\right)^{\frac{1}{2}} \\
&\quad \cdot\left(\left\langle \frac{Ng(z)}{\gamma+2}+g(z), \frac{Ng(z)}{\gamma+2}+g(z)\right\rangle_{\beta+2}\right)^{\frac{1}{2}}.
\end{aligned}
$$

We compute

$$
\begin{aligned}
\left\langle \frac{Nf(z)}{\gamma+3}+f(z), \frac{Nf(z)}{\gamma+3}+f(z)\right\rangle_{\alpha+2} &= \left\langle \frac{Nf(z)}{\gamma+3}, \frac{Nf(z)}{\gamma+3}\right\rangle_{\alpha+2}+\left\langle \frac{Nf(z)}{\gamma+3}, f(z)\right\rangle_{\alpha+2} \\
&\quad +\left\langle f(z), \frac{Nf(z)}{\gamma+3}\right\rangle_{\alpha+2}+\langle f(z), f(z)\rangle_{\alpha+2}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle \frac{Ng(z)}{\gamma+2}+g(z), \frac{Ng(z)}{\gamma+2}+g(z)\right\rangle_{\beta+2} &= \left\langle \frac{Ng(z)}{\gamma+2}, \frac{Ng(z)}{\gamma+2}\right\rangle_{\beta+2}+\left\langle \frac{Ng(z)}{\gamma+2}, g(z)\right\rangle_{\beta+2} \\
&\quad +\left\langle g(z), \frac{Ng(z)}{\gamma+2}\right\rangle_{\beta+2}+\langle g(z), g(z)\rangle_{\beta+2}.
\end{aligned}
$$

Then we multiply $\left\langle \frac{Nf(z)}{\gamma+3} + f(z), \frac{Nf(z)}{\gamma+3} + f(z) \right\rangle_{\alpha+2}$ with $\left\langle \frac{Ng(z)}{\gamma+2} + g(z), \frac{Ng(z)}{\gamma+2} + g(z) \right\rangle_{\beta+2}$. The quantity is equal to the 16 terms showed below.

$$
\begin{aligned}
K \quad := \quad & A_1 \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
& + A_2 \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
& + A_3 \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
& + A_4 \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
& + A_5 \langle Nf(z), f(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
& + A_6 \langle Nf(z), f(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
& + A_7 \langle Nf(z), f(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
& + A_8 \langle Nf(z), f(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
& + A_9 \langle f(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
& + A_{10} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
& + A_{11} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
& + A_{12} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
& + A_{13} \langle f(z), f(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
& + A_{14} \langle f(z), f(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
& + A_{15} \langle f(z), f(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
& + A_{16} \langle f(z), f(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \, ,
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad A_1 &= \frac{1}{(\gamma+3)(\gamma+3)(\gamma+2)(\gamma+2)} \ , \\
A_2 &= A_3 = \frac{1}{(\gamma+3)(\gamma+3)(\gamma+2)} \ , \\
A_4 &= \frac{1}{(\gamma+3)^2} \ , \\
A_5 &= A_9 = \frac{1}{(\gamma+3)(\gamma+2)^2} \ , \\
A_6 &= A_7 = A_{10} = A_{11} = \frac{1}{(\gamma+3)(\gamma+2)} \ , \\
A_8 &= A_{12} = \frac{1}{(\gamma+3)} \ , \\
A_{13} &= \frac{1}{(\gamma+2)^2} \ , \\
A_{14} &= A_{15} = \frac{1}{(\gamma+2)} \quad \text{and} \\
A_{16} &= 1.
\end{aligned}
$$

By the definition, we have

$$
\begin{aligned}
\|f\|^2_{HL^2_\alpha} &= \left\langle \frac{Nf(z)}{\alpha+3}, \frac{Nf(z)}{\alpha+2} \right\rangle_{\alpha+2} + \left\langle \frac{Nf(z)}{\alpha+3}, f(z) \right\rangle_{\alpha+2} \\
&\quad + \left\langle f(z), \frac{Nf(z)}{\alpha+2} \right\rangle_{\alpha+2} + \langle f(z), f(z) \rangle_{\alpha+2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|g\|^2_{HL^2_\beta} &= \left\langle \frac{Ng(z)}{\beta+3}, \frac{Ng(z)}{\beta+2} \right\rangle_{\beta+2} + \left\langle \frac{Ng(z)}{\beta+3}, g(z) \right\rangle_{\beta+2} \\
&\quad + \left\langle g(z), \frac{Ng(z)}{\beta+2} \right\rangle_{\beta+2} + \langle g(z), g(z) \rangle_{\beta+2} \, .
\end{aligned}
$$

We compute

$$\begin{aligned}
\|f\|^2_{HL^2_\alpha}\|g\|^2_{HL^2_\beta} &= B_1\left\langle Nf(z),Nf(z)\right\rangle_{\alpha+2}\left\langle Ng(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_2\left\langle Nf(z),Nf(z)\right\rangle_{\alpha+2}\left\langle Ng(z),g(z)\right\rangle_{\beta+2}\\
&+B_3\left\langle Nf(z),Nf(z)\right\rangle_{\alpha+2}\left\langle g(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_4\left\langle Nf(z),Nf(z)\right\rangle_{\alpha+2}\left\langle g(z),g(z)\right\rangle_{\beta+2}\\
&+B_5\left\langle Nf(z),f(z)\right\rangle_{\alpha+2}\left\langle Ng(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_6\left\langle Nf(z),f(z)\right\rangle_{\alpha+2}\left\langle Ng(z),g(z)\right\rangle_{\beta+2}\\
&+B_7\left\langle Nf(z),f(z)\right\rangle_{\alpha+2}\left\langle g(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_8\left\langle Nf(z),f(z)\right\rangle_{\alpha+2}\left\langle g(z),g(z)\right\rangle_{\beta+2}\\
&+B_9\left\langle f(z),Nf(z)\right\rangle_{\alpha+2}\left\langle Ng(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_{10}\left\langle f(z),Nf(z)\right\rangle_{\alpha+2}\left\langle Ng(z),g(z)\right\rangle_{\beta+2}\\
&+B_{11}\left\langle f(z),Nf(z)\right\rangle_{\alpha+2}\left\langle g(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_{12}\left\langle f(z),Nf(z)\right\rangle_{\alpha+2}\left\langle g(z),g(z)\right\rangle_{\beta+2}\\
&+B_{13}\left\langle f(z),f(z)\right\rangle_{\alpha+2}\left\langle Ng(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_{14}\left\langle f(z),f(z)\right\rangle_{\alpha+2}\left\langle Ng(z),g(z)\right\rangle_{\beta+2}\\
&+B_{15}\left\langle f(z),f(z)\right\rangle_{\alpha+2}\left\langle g(z),Ng(z)\right\rangle_{\beta+2}\\
&+B_{16}\left\langle f(z),f(z)\right\rangle_{\alpha+2}\left\langle g(z),g(z)\right\rangle_{\beta+2}\,, \qquad (3.1)
\end{aligned}$$

where

$$
\begin{aligned}
B_1 &= \frac{1}{(\alpha+3)(\alpha+2)(\beta+3)(\beta+2)} \ , \\
B_2 &= \frac{1}{(\alpha+2)(\alpha+3)(\beta+3)} \ , \\
B_3 &= \frac{1}{(\alpha+2)(\alpha+3)(\beta+2)} \ , \\
B_4 &= \frac{1}{(\alpha+3)(\alpha+2)} \ , \\
B_5 &= \frac{1}{(\alpha+3)(\beta+2)(\beta+3)} \ , \\
B_6 &= \frac{1}{(\alpha+3)(\beta+3)} \ , \\
B_7 &= \frac{1}{(\alpha+3)(\beta+2)} \ , \\
B_8 &= \frac{1}{(\alpha+3)} \ , \\
B_9 &= \frac{1}{(\alpha+2)(\beta+2)(\beta+3)} \ , \\
B_{10} &= \frac{1}{(\alpha+2)(\beta+3)} \ , \\
B_{11} &= \frac{1}{(\alpha+2)(\beta+2)} \ , \\
B_{12} &= \frac{1}{(\alpha+2)} \ , \\
B_{13} &= \frac{1}{(\beta+2)(\beta+3)} \ , \\
B_{14} &= \frac{1}{(\beta+3)} \ , \\
B_{15} &= \frac{1}{(\beta+2)} \quad \text{and} \\
B_{16} &= 1.
\end{aligned}
$$

Let $\mathcal{M} = \min\ B_i$, $\mathcal{N} = \max\ A_i$. Then multiply the quantity K with $\frac{\mathcal{M}}{\mathcal{N}}$, we obtain

$$\begin{aligned}
\left(\frac{\mathcal{M}}{\mathcal{N}}\right) K &= \frac{\mathcal{M}A_1}{\mathcal{N}} \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_2}{\mathcal{N}} \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_3}{\mathcal{N}} \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_4}{\mathcal{N}} \langle Nf(z), Nf(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_5}{\mathcal{N}} \langle Nf(z), f(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_6}{\mathcal{N}} \langle Nf(z), f(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_7}{\mathcal{N}} \langle Nf(z), f(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_8}{\mathcal{N}} \langle Nf(z), f(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_9}{\mathcal{N}} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{10}}{\mathcal{N}} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{11}}{\mathcal{N}} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{12}}{\mathcal{N}} \langle f(z), Nf(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{13}}{\mathcal{N}} \langle f(z), f(z)\rangle_{\alpha+2} \langle Ng(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{14}}{\mathcal{N}} \langle f(z), f(z)\rangle_{\alpha+2} \langle Ng(z), g(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{15}}{\mathcal{N}} \langle f(z), f(z)\rangle_{\alpha+2} \langle g(z), Ng(z)\rangle_{\beta+2} \\
&+\frac{\mathcal{M}A_{16}}{\mathcal{N}} \langle f(z), f(z)\rangle_{\alpha+2} \langle g(z), g(z)\rangle_{\beta+2} \, . \qquad (3.2)
\end{aligned}$$

Compare (3.1) and (3.2) term-by-term, we have that $\frac{\mathcal{M}A_i}{\mathcal{N}} \leq \mathcal{M} \leq B_i$

for all i. That is,

$$\begin{aligned}|T_g(f)| &\leq c_{\gamma+2}(\left\langle \frac{Nf(z)}{\gamma+3}+f(z), \frac{Nf(z)}{\gamma+3}+f(z)\right\rangle_{\alpha+2} \\ &\quad \cdot \left\langle \frac{Ng(z)}{\gamma+2}+g(z), \frac{Ng(z)}{\gamma+2}+g(z)\right\rangle_{\beta+2})^{\frac{1}{2}} \\ &\leq \frac{\mathcal{N}}{\mathcal{M}}(\|g\|^2_{HL^2_\beta}\|f\|^2_{HL^2_\alpha})^{\frac{1}{2}} \\ &= \frac{\mathcal{N}}{\mathcal{M}}\|g\|_{HL^2_\beta}\|f\|_{HL^2_\alpha}.\end{aligned}$$

Therefore T_g is a bounded functional on $HL^2(\mathbb{B},\alpha)$ with $\|T_g\| \leq K_{\gamma,\alpha,\beta}\|g\|_{HL^2_\beta}$ where $K_{\gamma,\alpha,\beta}$ is a positive constant depending on c_α, c_β and c_γ. The map T_g is linear because of the linearity of the inner product. Thus we can define $L : HL^2(\mathbb{B},\beta) \to (HL^2(\mathbb{B},\alpha))^*$ by $L : g \mapsto T_g$. Then we have L is an injective map.

Conversly, let $F \in (HL^2(\mathbb{B},\alpha))^*$. By Riesz representation, there exists some $h \in HL^2(\mathbb{B},\alpha)$ such that $F(f) = \langle f, h\rangle_{HL^2_\alpha}$ for all $f \in HL^2(\mathbb{B},\alpha)$. We will show that for each $F \in (HL^2(\mathbb{B},\alpha))^*$ there exists $g \in HL^2(\mathbb{B},\beta)$ such that

$$F(f) = \langle f, g\rangle_{HL^2_\gamma} \qquad \text{for all } f \in HL^2(\mathbb{B},\alpha).$$

Consider

$$\begin{aligned}F(f) &= \langle f, h\rangle_{HL^2_\alpha} \\ &= \langle \mathcal{P}_{\alpha+1}f, \mathcal{P}_\alpha h\rangle_{\alpha+2} \\ &= \left\langle \frac{N}{\alpha+3}f+f, \frac{N}{\alpha+2}h+h\right\rangle_{\alpha+2} \\ &= \left\langle \frac{N}{\alpha+3}f, \frac{N}{\alpha+2}h\right\rangle_{\alpha+2} + \left\langle \frac{N}{\alpha+3}f, h\right\rangle_{\alpha+2} + \left\langle f, \frac{N}{\alpha+2}h\right\rangle_{\alpha+2} \\ &\quad + \langle f, h\rangle_{\alpha+2} \\ &= \left\langle \frac{N}{\gamma+3}f, \frac{N}{\gamma+2}Ah\right\rangle_{\gamma+2} + \left\langle \frac{N}{\gamma+3}f, Ah\right\rangle_{\gamma+2} + \left\langle f, \frac{N}{\gamma+2}Ah\right\rangle_{\gamma+2} \\ &\quad + \langle f, Ah\rangle_{\gamma+2} \\ &= \left\langle \frac{N}{\gamma+3}f+f, \frac{N}{\gamma+2}Ah+Ah\right\rangle_{\gamma+2} \\ &= \langle \mathcal{P}_{\gamma+1}f, \mathcal{P}_\gamma Ah\rangle_{\gamma+2} \\ &= \langle f, Ah\rangle_{HL^2_\gamma}\end{aligned}$$

where $A = \dfrac{(\gamma+m+3)(\gamma+m+2)(\gamma+m+1)...(\gamma+4)}{(\alpha+m+3)(\alpha+m+2)(\alpha+m+1)...(\alpha+4)}$. Let $g = Ah$. Then, we also $g \in HL^2(\mathbb{B},\alpha) \subset HL^2(\mathbb{B},\beta)$ if $\beta > \alpha$. Therefore there exists $g \in HL^2(\mathbb{B},\beta)$ such that $F(f) = \langle f, g\rangle_{HL^2_\gamma}$, for all $f \in HL^2(\mathbb{B},\alpha)$. Therefore the map $L : HL^2(\mathbb{B},\beta) \to (HL^2(\mathbb{B},\alpha))^*$ is one-to-one and onto. By the Riesz representation, we also have

$$\|L(g)\| = k\|g\|_{HL^2_\beta}.$$

Without eliminating of the constant k, we may say that this is an "essentially "isomorphism. Alternatively, we can say that this is an isomorphism under equivalent norm.

The condition $\beta > \alpha$ restricts us to say that this theorem is valid only for $\beta > \alpha > -2$. However the transitivity of "="(which we mean an isomorphism) allows us to manipulate the numbers α, β which make the theorem to be valid for all $\alpha, \beta > -2$. □

In the next theorem, we will show some equivalent conditions for a function being in a generalized Bergman space.

Theorem 3.2. *Suppose $\alpha > -2$, and f is holomorphic in $\mathbb{B}$. Then the following conditions are equivalent:*

(a). $f \in HL^2(\mathbb{B},\alpha)$

(b). $|\widetilde{\nabla} f(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$ *and* $|\widetilde{S} f(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$

(c). $(1-|z|^2)|\nabla f(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$ *and* $(1-|z|^2)|Sf(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$

(d). $(1-|z|^2)|Nf(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$ *and* $(1-|z|^2)|N^2 f(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$.

Where $Sf(z) = \nabla z \dfrac{d}{dz} f(z)$ *and* $\widetilde{S} f(z) = \widetilde{\nabla} z \dfrac{d}{dz} f(z)$

พิสูจน์. The proof can be adjusted from that of Theorem 2.16 in [9] as follows. To show *(a)* implies *(b)*, we assume that $f \in HL^2(\mathbb{B},\ \alpha)$. That is, $f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$ and $z\dfrac{df}{dz} \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$. For $\beta > \alpha+2$, $g \in \mathcal{H}L^2(\mathbb{B}, dv_\beta)$, and Proposition 2.16, there exists a constant $c > 0$ such that

$$\begin{aligned} |\nabla g(0)|^2 &= \left|\frac{d}{dz} g(0)\right|^2 \\ &\le (c\|g\|_\beta)^2 \\ &= c^2 \int_{\mathbb{B}} |g(w)|^2 dv_\beta(w) \\ &= c^2 \int_{\mathbb{B}} |g(w)|^2\, dv_\beta(w). \end{aligned}$$

Let $g = f \circ \varphi_z$ and by Proposition 2.17, we obtain

$$\begin{aligned} |\nabla g(0)|^2 = |\nabla f \circ \varphi_z(0)|^2 &\leq c^2 \int_{\mathbb{B}} |\nabla f \circ \varphi_z(w)|^2 \, dv_\beta(w) \\ &= c^2 \int_{\mathbb{B}} \frac{|f(w)|^2 (1-|z|^2)^{\beta+2}}{|1-\langle z, w\rangle|^{2(\beta+2)}} \, dv_\beta(w). \end{aligned}$$

Since $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, we obtain

$$|\widetilde{\nabla} f(z)|^2 \leq c^2 \int_{\mathbb{B}} \frac{|f(w)|^2 (1-|z|^2)^{\beta+2}}{|1-\langle z, w\rangle|^{2(\beta+2)}} \, dv_\beta(w).$$

An application of Fubini's theorem and Theorem 2.18 give

$$\begin{aligned} \int_{\mathbb{B}} |\widetilde{\nabla} f(z)|^2 \, dv_{\alpha+2}(z) &\leq \int_{\mathbb{B}} c^2 (1-|z|^2)^{\beta+2} \int_{\mathbb{B}} \frac{|f(w)|^2}{|1-\langle z, w\rangle|^{2(\beta+2)}} dv_\beta(w) \, dv_{\alpha+2}(z) \\ &= c^2 \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+2} |f(w)|^2}{|1-\langle z, w\rangle|^{2(\beta+2)}} dv_\beta(w) \, dv_{\alpha+2}(z) \\ &= c^2 \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+2} |f(w)|^2}{|1-\langle z, w\rangle|^{2(\beta+2)}} dv_{\alpha+2}(z) \, dv_\beta(w) \\ &= c^2 \int_{\mathbb{B}} |f(w)|^2 \int_{\mathbb{B}} \frac{(1-|z|^2)^{\alpha+\beta+4}}{|1-\langle z, w\rangle|^{2(\beta+2)}} dz \, dv_\beta(w) \\ &\leq c_\beta c^2 \int_{\mathbb{B}} |f(w)|^2 (1-|w|^2)^{\alpha-\beta+2} \, dv_\beta(w) \\ &= c_\beta c^2 \int_{\mathbb{B}} |f(w)|^2 (1-|w|^2)^{\alpha+2} \, dw \\ &= c_\beta c^2 \|f\|^2_{\alpha+2} < \infty. \end{aligned}$$

Thus $|\widetilde{\nabla} f(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$.

Let $g = z\dfrac{df}{dz}(z) \circ \varphi_z$ and by Proposition 2.17, we obtain

$$\begin{aligned} |\nabla g(0)|^2 = \left|\nabla z \frac{df}{dz}(z) \circ \varphi_z(0)\right|^2 &\leq c^2 \int_{\mathbb{B}} \left|\nabla z \frac{df}{dz}(z) \circ \varphi_z(w)\right|^2 dv_\beta(w) \\ &= c^2 \int_{\mathbb{B}} \frac{|w\frac{df}{dw}(w)|^2 (1-|z|^2)^{\beta+2}}{|1-\langle z, w\rangle|^{2(\beta+2)}} \, dv_\beta(w). \end{aligned}$$

Since $\widetilde{\nabla} z \dfrac{df}{dz}(z) = \nabla(z \dfrac{df}{dz} \circ \varphi_z)(0)$, we have that

$$\left|\widetilde{\nabla} z \frac{df}{dz}(z)\right|^2 \leq c^2 \int_{\mathbb{B}} \frac{|w\frac{df}{dw}(w)|^2 (1-|z|^2)^{\beta+2}}{|1-\langle z, w\rangle|^{2(\beta+2)}} dv_\beta(w).$$

Again, an application of Fubini's theorem and Theorem 2.18 imply

$$\begin{aligned}
\int_{\mathbb{B}} \left|\tilde{\nabla} z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z) &\leq \int c^2(1-|z|^2)^{\beta+2}\int_{\mathbb{B}} \frac{|w\dfrac{df}{dw}(w)|^2}{|1-\langle z,w\rangle|^{2(\beta+2)}} dv_\beta(w)\, dv_{\alpha+2}(z) \\
&= c^2\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+2}|w\dfrac{df}{dw}(w)|^2}{|1-\langle z,w\rangle|^{2(\beta+2)}} dv_\beta(w)\, dv_{\alpha+2}(z) \\
&= c^2\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{(1-|z|^2)^{\beta+2}|w\dfrac{df}{dw}(w)|^2}{|1-\langle z,w\rangle|^{2(\beta+2)}} dv_{\alpha+2}(z)\, dv_\beta(w) \\
&= c^2\int_{\mathbb{B}} |w\frac{df}{dw}(w)|^2 \int_{\mathbb{B}} \frac{(1-|z|^2)^{\alpha+\beta+4}}{|1-\langle z,w\rangle|^{2(\beta+2)}} dz\, dv_\beta(w) \\
&\leq c_\beta c^2 \int_{\mathbb{B}} |w\frac{df}{dw}(w)|^2 (1-|w|^2)^{\alpha-\beta+2}\, dv_\beta(w) \\
&= c_\beta c^2 \int_{\mathbb{B}} |w\frac{df}{dw}(w)|^2 (1-|w|^2)^{\alpha+2}\, dw \\
&= c_\beta c^2 \|z\frac{df}{dz}\|^2_{\alpha+2} < \infty.
\end{aligned}$$

Thus $|\tilde{S}f(z)| = |\tilde{\nabla} z\frac{df}{dz}(z)| \in L^2(\mathbb{B}, dv_{\alpha+2})$. This proves that *(a)* implies *(b)*. Lemma 2.14 and Lemma 2.15 show that *(b)* implies *(c)*, and *(c)* implies *(d)*.

To prove *(d)* implies *(a)*, we assume that f is a holomorphic function in $\mathbb{B}$ such that the function $(1-|z|^2)|Nf(z)|$ belongs to $L^2(\mathbb{B}, dv_{\alpha+2})$. Let β be a sufficiently large positive constant that $\beta > \alpha$. Then by Theorem 2.19, we obtain

$$Nf(z) = \int_{\mathbb{B}} \frac{Nf(w)}{(1-\langle z,w\rangle)^{\beta+2}} dv_\beta(w), \quad \text{for any } z \in \mathbb{B}.$$

Since $Nf(0) = 0$, we have

$$\begin{aligned}
Nf(z) &= Nf(z) - Nf(0) \\
&= Nf(z) - \int_{\mathbb{B}} Nf(w)\, dv_\beta(w) \\
&= \int_{\mathbb{B}} \frac{Nf(w)}{(1-\langle z,w\rangle)^{\beta+2}}\, dv_\beta(w) - \int_{\mathbb{B}} Nf(w)\, dv_\beta(w) \\
&= \int_{\mathbb{B}} \left(\frac{Nf(w)}{(1-\langle z,w\rangle)^{\beta+2}} - Nf(w) \right) dv_\beta(w) \\
&= \int_{\mathbb{B}} Nf(w) \left(\frac{1}{(1-\langle z,w\rangle)^{\beta+2}} - 1 \right) dv_\beta(w)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{Nf(tz)}{t} dt &= \sum_{n=1}^{\infty} \frac{n a_n t^n z^n}{n} \Big|_0^1 \\
&= \sum_{n=1}^{\infty} a_n t^n z^n \Big|_0^1 \\
&= \sum_{n=1}^{\infty} a_n z^n \\
&= \sum_{n=0}^{\infty} a_n z^n - a_0 \\
&= f(z) - f(0).
\end{aligned}$$

It follows that

$$\begin{aligned}
f(z) - f(0) = \int_0^1 \frac{Nf(tz)}{t}\, dt &= \int_0^1 \int_{\mathbb{B}} Nf(w) \left(\frac{1}{(1-\langle tz,w\rangle)^{\beta+2}} - 1 \right) \frac{dv_\beta(w)}{t}\, dt \\
&= \int_{\mathbb{B}} Nf(w) \int_0^1 \left(\frac{1}{(1-t\langle z,w\rangle)^{\beta+2}} - 1 \right) \frac{dt}{t}\, dv_\beta(w) \\
&= \int_{\mathbb{B}} Nf(w) L(z,w) dv_\beta(w).
\end{aligned}$$

From equation (2.2) of Theorem 2.20, we get

$$|f(z) - f(0)| \le C \int_{\mathbb{B}} \frac{(1-|w|)^2 |Nf(w)|}{|(1-\langle z,w\rangle)|^{\beta+1}}\, dv_{\beta-1}(w).$$

If β is so large that

$$2 < \alpha + 3 < 2(\beta + 1),$$

then Theorem 2.21 implies

$$\begin{aligned}
& \int_{\mathbb{B}} |f(z)-f(0)|^2 dv_{\alpha+2}(z) \\
\leq & \int_{\mathbb{B}} C^2 \left[\int_{\mathbb{B}} \frac{(1-|w|^2)|Nf(w)|dv_{\beta-1}(w)}{|(1-\langle z,w\rangle)|^{\beta+1}}\right]^2 dv_{\alpha+2}(z) \\
= & \int_{\mathbb{B}} C^2 \left[\frac{(1-|z|^2)}{(1-|z|^2)}\int_{\mathbb{B}} \frac{(1-|w|^2)|Nf(w)|dv_{\beta-1}(w)}{|(1-\langle z,w\rangle)|^{\beta+1}}\right]^2 dv_{\alpha+2}(z) \\
= & \int_{\mathbb{B}} C^2 \left[(1-|z|^2)^{-1}\int_{\mathbb{B}} (1-|z|^2)^{\frac{1}{2}} \frac{(1-|w|^2)^{\beta}|Nf(w)|dv(w)}{|1-\langle z,w\rangle|^{\beta+1}}\right]^2 dv_{\alpha+2}(z).
\end{aligned}$$

It follows that

$$\int_{\mathbb{B}} |f(z)-f(0)|^2 dv_{\alpha+2}(z) \leq K\int_{\mathbb{B}} (1-|z|^2)|Nf(z)|^2 dv_{\alpha+2}(z),$$

where $K = \left(\dfrac{Cc_\beta}{c_{\beta-1}}\right)^2$.

Thus if $(1-|z|^2)|Nf(z)|^2 \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2})$ then $\int_{\mathbb{B}} |f(z)-f(0)|^2 dv_{\alpha+2}(z) \leq k < \infty$ which implies $f(z)-f(0) \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2})$, and hence $f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$. Similarly if $(1-|z|^2)|N^2 f(z)|^2 \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2})$ that is $(1-|z|^2)|N(Nf(z))| \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2})$, then $Nf \in \mathcal{H}L^2(\mathbb{B},\ dv_{\alpha+2})$. This completes the proof of the theorem. □

Theorem 3.3. *Suppose $\alpha > -2$, N is a positive integer, and f is holomorphic in $\mathbb{B}$. Then $f \in HL^2(\mathbb{B},\alpha)$ if and only if the functions*

$$(1-|z|^2)^N \frac{\partial^N f}{\partial z^N}(z) \text{ and } (1-|z|^2)^N \frac{\partial^N}{\partial z^N} z\frac{df}{dz}(z)$$

all belong to $L^2(\mathbb{B}, dv_{\alpha+2})$.

พิสูจน์. (⇒) The case $K = 1$ follows from the equivalence of *(a)* and *(c)* in Theorem 3.2. We prove the case $K = 2$, by assuming $f \in HL^2(\mathbb{B},\alpha)$ that is $f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$ and $z\frac{df}{dz}(z) \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$. By using Theorem 3.2, the condition *(a)* implies *(c)* gives $(1-|z|^2)\left|\frac{\partial f}{\partial z}(z)\right|$ and $(1-|z|^2)\left|\frac{\partial}{\partial z}\left(z\frac{df}{dz}(z)\right)\right|$ are in $L^2(\mathbb{B}, dv_{\alpha+2})$ and hence

$$\int_{\mathbb{B}} (1-|z|^2)^2 \left|\frac{\partial f}{\partial z}(z)\right|^2 dv_{\alpha+2}(z) = \int_{\mathbb{B}} \left|\frac{\partial f}{\partial z}(z)\right|^2 dv_{\alpha+4}(z) < \infty$$

and

$$\int_{\mathbb{B}} (1-|z|^2)^2 \left|\frac{\partial}{\partial z} z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z) = \int_{\mathbb{B}} \left|\frac{\partial}{\partial z} z\frac{df}{dz}(z)\right|^2 dv_{\alpha+4}(z) < \infty.$$

So each function $\frac{\partial f}{\partial z}(z)$ and $\frac{\partial}{\partial z}\left(z\frac{df}{dz}(z)\right)$ are in $\mathcal{H}L^2(\mathbb{B}, dv_{\alpha+4})$. It follows from the condition *(a)* implies *(c)* in Theorem 3.2 that $(1-|z|^2)\left|\frac{\partial^2 f}{\partial z}(z)\right|$ and $(1-|z|^2)\left|\frac{\partial^2}{\partial z^2}\left(z\frac{df}{dz}(z)\right)\right|$ are in $L^2(\mathbb{B}, dv_{\alpha+4})$ and hence

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^2 f}{\partial z}(z)\right|^2 dv_{\alpha+4}(z) = \int_{\mathbb{B}}(1-|z|^2)^4\left|\frac{\partial^2 f}{\partial z}(z)\right|^2 dv_{\alpha+2}(z) < \infty$$

and

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^2}{\partial z}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+4}(z) = \int_{\mathbb{B}}(1-|z|^2)^4\left|\frac{\partial^2}{\partial z}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z) < \infty.$$

Therefore each the functions $(1-|z|^2)^2\frac{\partial^2 f}{\partial z^2}(z)$ and $(1-|z|^2)^2\frac{\partial^2}{\partial z}z\frac{df}{dz}(z)$ are in $L^2(\mathbb{B}, dv_{\alpha+2})$. We use the same technique to the case $k = 2$ as follows.

The case $K = 3$, assume $f \in HL^2(\mathbb{B}, \alpha)$ that is, $f \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$ and $z\frac{df}{dz}(z) \in \mathcal{H}L^2(\mathbb{B}, dv_{\alpha+2})$. By using Theorem 3.2, the condition *(a)* implies *(c)* gives $(1-|z|^2)\left|\frac{\partial f}{\partial z}(z)\right|$ are in $L^2(\mathbb{B}, dv_{\alpha+2})$ and hence

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial f}{\partial z}(z)\right|^2 dv_{\alpha+2}(z) = \int_{\mathbb{B}}\left|\frac{\partial f}{\partial z}(z)\right|^2 dv_{\alpha+4}(z) < \infty$$

and

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial}{\partial z}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z) = \int_{\mathbb{B}}\left|\frac{\partial}{\partial z}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+4}(z) < \infty.$$

So each function $\frac{\partial^2 f}{\partial z^2}(z)$ and $\frac{\partial}{\partial z}z\frac{df}{dz}(z)$ are in $\mathcal{H}L^2(\mathbb{B}, dv_{\alpha+4})$. It follows from the condition *(a)* $\rightarrow$ *(c)* in Theorem 3.2 that $(1-|z|^2)\left|\frac{\partial^2 f}{\partial z^2}(z)\right|$ and $(1-|z|^2)\left|\frac{\partial^2}{\partial z}z\frac{df}{dz}(z)\right|$ are in $L^2(\mathbb{B}, dv_{\alpha+4})$ and hence

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^2 f}{\partial z^2}(z)\right|^2 dv_{\alpha+4}(z) = \int_{\mathbb{B}}\left|\frac{\partial^2 f}{\partial z^2}(z)\right|^2 dv_{\alpha+6}(z) < \infty$$

and

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^2}{\partial z^2}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+4}(z) = \int_{\mathbb{B}}\left|\frac{\partial^2}{\partial z^2}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+6}(z) < \infty.$$

So each function $\frac{\partial^2 f}{\partial z^2}(z)$ and $\frac{\partial^2}{\partial z^2}z\frac{df}{dz}(z)$ are in $\mathcal{H}L^2(\mathbb{B}, dv_{\alpha+6})$. Again the condition *(a)* $\rightarrow$ *(c)* in Theorem 3.2 gives $(1-|z|^2)\left|\frac{\partial^3 f}{\partial z^3}(z)\right|$ and $(1-|z|^2)\left|\frac{\partial^3}{\partial z^3}z\frac{df}{dz}(z)\right|$ are in $L^2(\mathbb{B}, dv_{\alpha+6})$ and hence

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^3 f}{\partial z^3}(z)\right|^2 dv_{\alpha+6}(z) = \int_{\mathbb{B}}(1-|z|^2)^6\left|\frac{\partial^3 f}{\partial z^3}(z)\right|^2 dv_{\alpha+2} < \infty$$

and

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^3}{\partial z^3}z\frac{df}{dz}(z)\right|dv_{\alpha+6}(z)=\int_{\mathbb{B}}(1-|z|^2)^6\left|\frac{\partial^3}{\partial z^3}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z)<\infty.$$

Therefore each function $(1-|z|^2)^3\frac{\partial^3 f}{\partial z^3}(z)$ and $(1-|z|^2)^3\frac{\partial^3}{\partial z^3}z\frac{df}{dz}(z)$ are in $L^2(\mathbb{B},dv_{\alpha+2})$. There if we assume $f\in HL^2(\mathbb{B},\alpha)$. We have $f\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+2})$ and $z\frac{df}{dz}(z)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+2})$. By using the condition *(a)* implies *(c)* in Theorem 3.2 repeatedly, we have the following conclusions

$$\frac{\partial f}{\partial z}(z)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+4})\quad\text{and}\quad\frac{\partial}{\partial z}\left(z\frac{df}{dz}(z)\right)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+4})$$

$$\frac{\partial^2 f}{\partial z^2}(z)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+6})\quad\text{and}\quad\frac{\partial^2}{\partial z^2}\left(z\frac{df}{dz}(z)\right)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+6})$$

$$\frac{\partial^3 f}{\partial z^3}(z)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+8})\quad\text{and}\quad\frac{\partial^3}{\partial z^3}\left(z\frac{df}{dz}(z)\right)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+8})$$

$$\vdots$$

$$\frac{\partial^{k-1} f}{\partial z^{k-1}}(z)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+2k})\quad\text{and}\quad\frac{\partial^{k-1}}{\partial z^{k-1}}\left(z\frac{df}{dz}(z)\right)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+2k})$$

$$\frac{\partial^2 f}{\partial z^k}(z)\in\mathcal{H}L^k(\mathbb{B},dv_{\alpha+2(k+1)})\quad\text{and}\quad\frac{\partial^k}{\partial z^k}\left(z\frac{df}{dz}(z)\right)\in\mathcal{H}L^2(\mathbb{B},dv_{\alpha+2(k+1)}).$$

Hence $(1-|z|^2)\left|\frac{\partial^{k+1} f}{\partial z^{k+1}}(z)\right|$ and $(1-|z|^2)\left|\frac{\partial^{k+1}}{\partial z^{k+1}}z\frac{df}{dz}(z)\right|$ are in $L^2(\mathbb{B},dv_{\alpha+2(k+1)})$ such that

$$\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^{k+1} f}{\partial z^{k+1}}(z)\right|^2 dv_{\alpha+2(k+1)}(z)=\int_{\mathbb{B}}(1-|z|^2)^{2k+2}\left|\frac{\partial^{k+1} f}{\partial z^{k+1}}(z)\right|^2 dv_{\alpha+2}(z)<\infty.$$

and

$$\begin{aligned}&\int_{\mathbb{B}}(1-|z|^2)^2\left|\frac{\partial^{k+1}}{\partial z^{k+1}}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2(k+1)}(z)\\ =\ &\int_{\mathbb{B}}(1-|z|^2)^{2k+2}\left|\frac{\partial^{k+1}}{\partial z^{k+1}}z\frac{df}{dz}(z)\right|^2 dv_{\alpha+2}(z)<\infty.\end{aligned}$$

Therefore each function $(1-|z|^2)^{k+1}\left|\frac{\partial^{k+1} f}{\partial z^{k+1}}(z)\right|$ and $(1-|z|^2)^{k+1}\left|\frac{\partial^{k+1}}{\partial z^{k+1}}z\frac{df}{dz}(z)\right|$ are in $L^2(\mathbb{B},dv_{\alpha+2})$.

($\Leftarrow$) The converse is also true because of implication *(c)* $\rightarrow$ *(a)* in Theorem 3.2. We can follow the previous parts conversely, then we prove this theorem. $\square$

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VITAE

Name Miss Marisa Senmoh

Student ID 5210220064

Educational Attainment

Degree	Name of Institution	Year of Graduation
B.Sc. (Mathematics)	Prince of Songkla University	2006

Scholarship Awards during Enrolment

Teaching Assistant from Faculty of Science, Prince of Songkla University, 2010-2011.

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