On Some Fixed Point Theorems for Asymptotically Quasi-Nonexpansive Nonself Mappings on Banach Spaces

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ชื่อวิทยานิพนธ์ ทฤษฎีบทจุดตรึงสำหรับการส่งแบบนอนเซลฟ์กึ่งไม่ขยายเชิงเส้นกำกับบนปริภูมิบานาค

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บทคัดย่อ

ในวิทยานิพนธ์เล่มนี้ เราได้ศึกษาการมีจริงของการส่งแบบหดตัวของเซตย่อยปิดของปริภูมิบานาค จากนั้นเราสร้างและศึกษาระเบียบวิธีการทำซ้ำ 3 ขั้นตอนที่มีความหนืดเพื่อประมาณค่าจุดตรึงร่วมของการส่งแบบนอนเซลฟ์กึ่งไม่ขยายเชิงเส้นกำกับบนปริภูมิบานาค เราได้ศึกษาเกณฑ์ต่าง ๆ สำหรับการลู่เข้าแบบเข้มสำหรับระเบียบวิธีการทำซ้ำที่แนะนำข้างต้น นอกจากนี้เรายังได้ศึกษาระเบียบวิธีการทำซ้ำหลายขั้นตอนที่มีความหนืดสำหรับการประมาณค่าจุดตรึงร่วมของวงศ์จำกัดของการส่งแบบนอนเซลฟ์กึ่งไม่ขยายเชิงเส้นกำกับบนปริภูมิบานาค ในลำดับสุดท้ายนั้นเราได้ให้เงื่อนไขที่เพียงพอสำหรับการลู่เข้าแบบอ่อนและแบบเข้มของระเบียบวิธีการทำซ้ำดังกล่าวในปริภูมิบานาคคอนเวกซ์แบบเอกรูป

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ABSTRACT

In this thesis we study the existence of a retraction of a closed subset of a Banach space. Then we introduce and study a three-step iterative process with viscosity to approximate common fixed points for asymptotically quasi-nonexpansive nonself mappings in Banach spaces. Criteria for strong convergence of such iteration is given. We also introduce and study a multi-step iterative schemes with viscosity to approximate of common fixed points of finite family for asymptotically quasi-nonexpansive nonself mappings in Banach spaces. Finally, weak and strong convergence theorems for such iteration in uniformly convex Banach spaces are established under some sufficient conditions.

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Supamit Wiriyakulopast

CONTENTS

CHAPTER 1

Introduction

The concept of asymptotically nonexpansive self mappings which is a generalization of the class of nonexpansive self mappings was first introduced in 1972 by Goebel and Kirk [5]. They proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space and T is an asymptotically nonexpansive self mapping of C, then T has a fixed point. Since then, the weak and strong convergence problem of iterative sequences (with errors) for asymptotically nonexpansive self mappings have been studied by many authors. In 2003, Chidume et al [2] introduced the concept of asymptotically nonexpansive nonself mappings, which is a generalization of asymptotically nonexpansive mappings. Similarly, the concept of asymptotically quasi-nonexpansive nonself mappings can also be defined as a generalization of asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive nonself mappings. These mappings are defined as follows. Let X be a real Banach space and C be a nonempty subset of X.

(i) A mapping P from X onto C is said to be a *retraction*, if $P^2 = P$;

(ii) If there exists a continuous retraction $P : X \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *retract* of X.

(iii) In particular, if there exists a nonexpansive retraction $P : X \to C$ such that

$Px = x$ for all $x \in C$, then the set C is said to be a *nonexpansive retract* of X.

Let $T : C \to X$ be a nonself mapping.

(i) T is said to be an *asymptotically nonexpansive nonself mapping*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|x - y\| \ ,$$

for all $x,\ y \in C$ and $n \ge 1$.

(ii) T is said to be an *asymptotically quasi-nonexpansive nonself mapping*, if the set of fixed points of mapping T is denoted by $F(T)$ which is nonempty and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - p\| \le k_n\|x - p\|$$

for all $x \in C,\ p \in F(T)$ and $n \ge 1$.

Recall that a self mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$.

In 2004, Xu [15] defined the following one viscosity iteration for nonexpansive mappings in uniformly smooth Banach space. The Banach space X is said to be uniformly smooth if

$$\rho_x^{'}(0) = \lim_{t \to 0} \frac{\rho_x(t)}{t} = 0,$$

where the function $\rho_x : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\begin{aligned}\rho_x(t) &= \sup\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\},\\ &= \sup\{\frac{\|x+ty\|+\|x-ty\|}{2} - 1 : \|x\| = \|y\| = 1\}, t \geq 0.\end{aligned}$$

Theorem 1.1. *Let X be a uniformly smooth Banach space, C be a nonempty closed convex subset of $X, T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \Pi_C$ denotes the set of all contractions on C. Then $\{x_t\}$ defined by the following:*

$$x_t = tf(x_t) + (1-t)Tx_t, x_t \in C$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t \to 0} x_t, f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, f \in \Pi_C, p \in F(T).$$

In 2005, Song and Chen [12] extended Theorem 1.1 to nonexpansive nonself mapping in a reflexive Banach space : for $t \in (0,1)$,

$$x_t = P(tf(x_t) + (1-t)Tx_t)$$

where P is nonexpansive retraction and proved that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 0$.

Recently in 2011, Ayaragarnchanakul [1] constructed an iterative procedure to approximate common fixed points with viscosity of two asymptotically nonexpansive nonself mappings:

$$
\begin{aligned}
y_n &= \alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_2(PT_2)^{n-1}x_n) \\
x_{n+1} &= \gamma_n f(y_n) + (1-\gamma_n)(\delta_n y_n + (1-\delta_n)T_1(PT_1)^{n-1}y_n)
\end{aligned}
$$

and proved some strong convergence theorems for such mappings in arbitrary real Banach spaces and Tripak and Kongsiriwong [13] proved weak and strong convergence theorems of a finite family of generalized asymptotically nonexpansive nonself mappings in uniformly Banach space.

The purpose of this thesis is to extend and to improve some results announced by Ayaragarnchanakul [1] , define a new iteration scheme for approximating common fixed points of a finite family of asymptotically quasi-nonexpansive nonself mapping in Banach space, and prove weak and strong convergence of new iteration scheme in a uniformly convex Banach space.

CHAPTER 2

Preliminaries

The purpose of this chapter is to explain certain notations, terminologies and elementary results used throughout the thesis. Although details are included in some cases, many of the fundamental principles of real and functional analysis are merely stated without proof.

We first collect some basic knowledge from mathematical analysis. Definition 2.1 - Theorem 2.14 are from [9].

Definition 2.1. Let S be a nonempty subset of $\mathbb{R}$.

(i) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound of* S and the set S is said to be *bounded above*.

(ii) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound of* S and the set S is said to be *bounded below*.

(iii) The set S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subseteq [m, M]$.

Definition 2.2. (**Supremum and infimum**) Let S be a nonempty subset of $\mathbb{R}$.

(i) If S is bounded above and S has the least upper bound, then we will call it the *supremum of* S and denote it by $\sup S$.

(ii) If S is bounded below and S has the greatest lower bound, then we will call it the *infimum of* S and denote it by $\inf S$.

Axiom 2.1. ***(Completeness Axiom)*** *Every subset S of $\mathbb{R}$ that is bounded above has the least upper bound. In other words, $\sup S$ exists and is a real number.*

Definition 2.3. (**Convergent sequence**) A sequence $\{s_n\}$ of real numbers is said to *converge* to the real number s provided that

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$n > N \text{ implies } |s_n - s| < \varepsilon.$$

If $\{s_n\}$ converges to s, then we will write $\lim_{n\to\infty} s_n = s$, $\lim s_n = s$, or $s_n \to s$. The number s is called the *limit* of the sequence $\{s_n\}$. A sequence that dose not converge to some real number is said to be divergent.

Definition 2.4. (**Bounded sequence**) A sequence $\{s_n\}$ of real numbers is said to be *bounded* if there exists a constant M such that $|s_n| \leq M$ for all n.

Theorem 2.2. *Convergent sequences are bounded.*

Definition 2.5. (**Monotone sequence**) A sequence $\{s_n\}$ of real numbers is called a *nondecreasing sequence* if $s_n \leq s_{n+1}$ for all n and $\{s_n\}$ is called a *nonincreasing sequence* if $s_n \geq s_{n+1}$ for all n. We note that if $\{s_n\}$ is nondecreasing then $s_n \leq s_m$ whenever $n < m$. A sequence that is nondecreasing or nonincreasing will be called a *monotone sequence* or a *monotonic sequence*.

Theorem 2.3. *(**Monotone Convergence Theorem**) All bounded monotone sequences converge.*

Theorem 2.4.

(i) *If $\{s_n\}$ is an unbounded nondecreasing sequence, then* $\lim s_n = +\infty$.

(ii) *If $\{s_n\}$ is an unbounded nonincreasing sequence, then* $\lim s_n = -\infty$.

Corollary 2.5. *If $\{s_n\}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Definition 2.6. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

and

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}.$$

Theorem 2.6. *Let $\{s_n\}$ be a sequence in $\mathbb{R}$.*

(*i*) *If $\lim_{n\to\infty} s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then*

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

(*ii*) *If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$, then $\lim_{n\to\infty} s_n$ is defined and*

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Definition 2.7. (**Cauchy sequence**) A sequence $\{s_n\}$ of real numbers is called a *Cauchy sequence* if

for each $\varepsilon > 0$ there exists a number N such that

$m, n > N$ implies $|s_n - s_m| < \varepsilon$.

Theorem 2.7. ***(Cauchy Completeness Theorem)*** *A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

Theorem 2.8. ***(Sandwich Theorem)*** *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.*

Definition 2.8. (**Subsequence**) Suppose that $\{s_n\}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $\{t_k\}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{2.1}$$

and

$$t_k = s_{n_k}. \tag{2.2}$$

Thus $\{t_k\}$ is just a selection of some [possibly all] of the s_n's, taken in order.

Theorem 2.9. *If the sequence $\{s_n\}$ converges, then every subsequence converges to the same limit.*

Theorem 2.10. *Every sequence has a monotonic subsequence.*

Corollary 2.11. *Let $\{s_n\}$ be any sequence. There exists a monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$ and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.*

Theorem 2.12. ***(Bolzano-Weierstrass Theorem)*** *Every bounded sequence has a convergent subsequence.*

Definition 2.9. (**The Cauchy Criterion for Series**) We say that a series $\sum_{n=1}^{\infty} a_n$ satisfies the *Cauchy criterion* if its sequence $\{s_n\}$ of the *partial sum* is a Cauchy sequence :

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < \varepsilon. \tag{2.3}$$

Nothing is lost in this definition if we impose the restriction $n > m$. Moreover, it is only a natural matter to work with $m - 1$ where $m \leq n$ instead of m where $m < n$. Therefore (2.3) is equivalent to

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$n \geq m > N \quad \text{implies} \quad |s_n - s_{m-1}| < \varepsilon. \tag{2.4}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, condition (2.4) can be written

$$\text{for each } \varepsilon > 0 \text{ there exists a number } N \text{ such that}$$

$$n \geq m > N \quad \text{implies} \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon. \tag{2.5}$$

Theorem 2.13. *A series converges if and only if it satisfies the Cauchy criterion.*

Theorem 2.14. *Let $\{a_n\}$ be a sequence such that $\sum_{n=0}^{\infty} a_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Then we collect some basic knowledge from elementary functional analysis. Definition 2.10 - Definition 2.20 are from [4].

The following are some basic knowledge about metric spaces and normed spaces.

Definition 2.10. (**Metric space, metric**) Let X be a nonempty set. A function d defined on $X \times X$ is called a *metric* on X (or *distance function on* X) if it satisfies the following properties :

(M1) d is a real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$. (**Symmetry**)

(M4) $d(x, z) \leq d(x, y) + d(y, z)$. (**Triangle inequality**)

In this case, a pair (X, d) is called a *metric space*.

Definition 2.11. (**Convergence of a sequence, limit**) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to *converge* or to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0,$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x$$

or, simply,

$$x_n \to x.$$

We say that $\{x_n\}$ *converges* to x or *has the limit* x. If $\{x_n\}$ is not convergent, it is said to be *divergent*.

Definition 2.12. (**Distance**) The *distance* $d(x, A)$ from a point x to a nonempty subset A of a metric space (X, d) is defined to be

$$d(x, A) = \inf_{a\in A} d(x, a).$$

This infimum certainly exists in $\mathbb{R}$ and is nonnegative. If x is already in A, then, of course, $d(x, A) = 0$.

Definition 2.13. (**Ball and Sphere**) Given a point $x_0 \in X$ and real number $r > 0$, we define three types of sets:

(i) $B(x_0; r) = \{x \in X | d(x, x_0) < r\}$. (**Open ball**)

(ii) $\widetilde{B}(x_0; r) = \{x \in X | d(x, x_0) \leq r\}$. (**Close ball**)

(iii) $S(x_0; r) = \{x \in X | d(x, x_0) = r\}$. (**Sphere**)

In all three cases, x_0 is called the *center* and r is called the *radius*.

Definition 2.14. (**Open set, Closed set**) A subset M of a metric space X is said to be *open* if it contains an open ball about each of its points. A subset K of X is said to be *closed* if it complement(in X) is open, that is, $K^c = X - K$ is open.

Definition 2.15. (**Cauchy sequence, Completeness**) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* (or *fundamental*) if for every $\varepsilon > 0$ there is an N such that

$$d(x_m, x_n) < \varepsilon \quad \text{for every} \quad m, n > N.$$

The space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Theorem 2.15. *Let M be a nonempty subset of a metric space $X = (X, d)$. M is closed if and only if the situation $x_n \in M, x_n \to x$ implies that $x \in M$.*

Definition 2.16. (**Normed space, Banach space**) Let X be a vector space. A norm $\|\cdot\|$ defined on X is called a *norm* on X if it satisfies the following properties:

(N1) $\|x\| \geq 0$

(N2) $\|x\| = 0 \Leftrightarrow x = 0$

(N3) $\|\alpha x\| = |\alpha| \|x\|$ (**Absolute homogeneity**)

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (**Triangle inequality**);

here x and y are arbitrary vectors in X and α is any scalar. In this case, a pair $(X, \|\cdot\|)$ is called a *normed space*. Note that a complete normed space is called a *Banach space*.

Theorem 2.16. *A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Definition 2.17. (**Linear operator**) Let X and Y be two linear spaces over the same field $\mathbb{F}$ and $T : X \to Y$ an operator with domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$. Then T is said to be a *linear operator* if

(i) T is additive : $T(x + y) = Tx + Ty$ for all $x, y \in X$;

(ii) T is homogeneous : $T(\alpha x) = \alpha Tx$ for all $x \in X, \alpha \in \mathbb{F}$.

Otherwise, the operator is called *nonlinear*. The linear operator is called a *linear functional* if $Y = \mathbb{R}$.

Definition 2.18. (**Bounded linear operator**) Let X and Y be normed space and $T : \mathcal{D}(T) \to Y$ a linear operator, where $\mathcal{D}(T) \subseteq X$. The operator T is said to be *bounded* if there is a real number c such that for all $x \in \mathcal{D}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Definition 2.19. (**Convex set**) A subset C of a vector space X is said to be *convex* if $x, y \in C$ implies $M = \{z \in X | z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subset C$.

M is called a closed segment with boundary points x and y; any other $z \in M$ is called an interior point of M.

Definition 2.20. (**Fixed point**) A *fixed point* of a mapping $T : C \to X$ of a set C into X is an $x \in C$ which is mapped onto C, that is, $Tx = x$, the image Tx coincides with x. The set of all fixed points of T is denoted by $F(T)$, that is,

$$F(T) = \{x \in C | x = Tx\}.$$

Example 2.1. Let $X = [1, 5]$ and $C = [1, 2]$. Define $T : [1, 2] \to [1, 5]$ by $Tx = x^2 + x - 1$. We show that T has a fixed point. By definition, x is a fixed point of T if and only if $Tx = x$. Therefore T has only one fixed point and $F(T) = \{1\}$.

A fixed point theorem for asymptotically quasi-nonexpansive nonself mapping

Here a classical theorem about fixed point of asymptotically nonexpansive nonself mapping are from [1]. We first give the definition of retraction and then we give the definition of asymptotically nonexpansive nonself mapping.

Definition 2.21. (**Retraction**) Let X be a real Banach space and C be a nonempty subset of X.

(i) A mapping P from X onto C is said to be a *retraction*, if $P^2 = P$;

(ii) If there exists a continuous retraction $P : X \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *retract* of X.

(iii) In particular, if there exists a nonexpansive retraction $P : X \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *nonexpansive retract* of X.

Definition 2.22. (**Asymptotically Nonexpansive Nonself Mapping**) Let C be a nonempty subset of Banach space X. A mapping $T : C \to X$ is a said to be an asymptotically nonexpansive nonself mapping , P is nonexpansive retraction, if there exists a sequence $k_n \subset [0, 1)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + k_n)\|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$.

Definition 2.23. (**Asymptotically Quasi-Nonexpansive Nonself Mapping**) Let C be a nonempty subset of Banach space X. A mapping $T : C \to X$ is a said to be *asymptotically quasi-nonexpansive nonself mapping*, P is nonexpansive retraction, if $F(T) \neq \emptyset$ and there exists a sequence $k_n \subset [0, 1)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - p\| \leq (1 + k_n)\|x - p\|$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$. $F(T)$ is the set of fixed points of mapping T.

Now, we give definitions and theorems about reflexivity, weak convergence, weak compactness and lower semicontinuous. Definition 2.24 - Theorem 2.23 are from [7].

Reflexivity

Let $X_1, X_2, \cdots, X_n$ be n linear space over the same field $\mathbb{F}$. Then a functional $f : X_1 \times X_2 \times \cdots \times X_n \to \mathbb{R}$ is said to be an $n-linear(multilinear)functional$ on $X = X_1 \times X_2 \times \cdots \times X_n$ if it is linear with respect to each of the variables separately.

Definition 2.24. (**Dual space**) The space of all bounded linear functionals on a normed space X is called *the dual* of X and is denoted by $X^*.X^*$ is a normed space with norm denoted and defined by

$$\|f\|_* = \sup\{|f(x)| : x \in S_X\},$$

where $S_X = \{x \in X : \|x\| = 1\}$.

Definition 2.25. (**Duality pairing**) Given a normed space X and its dual X^*, we define the duality pairing as the functional $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{F}$ such that

$$\langle x, j \rangle = j(x) \text{ for all } x \in X \quad \text{and} \quad j \in X^*.$$

Theorem 2.17. *Let X^* be the dual of normed space X. Then we have the following :*

(i) The duality pairing is a bilinear functional on $X \times X^*$:

(a) $\langle ax + by, j \rangle = a\langle x, j \rangle + b\langle y, j \rangle$ for all $x, y \in X; j \in X^*$ and $a, b \in \mathbb{F}$;

(b) $\langle x, \alpha j_1 + \beta j_2 \rangle = \alpha\langle x, j_1 \rangle + \beta\langle y, j_2 \rangle$ for all $x \in X; j_1, j_2 \in X^*$ and $\alpha, \beta \in \mathbb{F}$.

(ii) $\langle x, j \rangle = 0$ for all $x \in X$ implies $j = 0$.

(iii) $\langle x, j \rangle = 0$ for all $j \in X^*$ implies $x = 0$.

Definition 2.26. (**Natural embedding mapping**) Let $(X, \|\cdot\|)$ be a normed space. Then $(X^*, \|\cdot\|_*)$ is a Banach space. Let $j \in X^*$. Hence for given $x \in X$, the equation

$$f_x(j) = \langle x, j \rangle$$

defines a functional f_x on the dual space X^*.

Define a mapping $\varphi : X \to X^{**}$ by $\varphi(x) = f_x, x \in X$. Then φ is called *the natural embedding mapping* from X into X^{**}. It has the following properties :

(i) φ is linear : $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ for all $x, y \in X, \alpha, \beta \in \mathbb{F}$;

(ii) $\varphi(x)$ is isometry : $\|\varphi(x)\| = \|x\|$ for all $x \in X$.

In general, the natural embedding mapping φ from X into X^{**} is not onto. It means that there may be elements in X^{**} that can not be represented by elements in X.

In the case when φ is onto, we have an important class of normed space.

Definition 2.27. A normed space X is said to be reflexive if the natural embedding mapping $\varphi : X \to X^{**}$ is onto.

Theorem 2.18. *(**Jame theorem**) A Banach space X is reflexive if and only if for each $j \in S_{X^*}$, there exists $x \in S_X$ such that $j(x) = 1$.*
Note that $S_{X^*} = \{j \in X^* : \|j\|_* = 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

Theorem 2.19. *A normed space X is reflexive if and only if every bounded sequence has a weakly convergent subsequence.*

Theorem 2.20. *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X. Then for $x \in X$, there exists a unique point $z_x \in C$ such that $\|x - z_x\| = d(x, C)$.*

Convergence of sequences of elements in a metric space that defined in Definition 2.11 will be called strong convergence, to distinguish it from weak convergence.

Definition 2.28. (**Strong convergence**) A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in the norm*) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

That is written

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

x is called the *strong limit* of $\{x_n\}$, and we say that $\{x_n\}$ *converges strongly* to x.

Weak Convergence and Weak compactness

We are now in a position to define weakly convergence and weakly compact.

Definition 2.29. (**Weak convergence**) A sequence $\{x_n\}$ in a normed space X is said to converge weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, we write $x_n \rightharpoonup x$ or weak-$\lim\limits_{n\to\infty} x_n = x$.

Theorem 2.21. *Let $\{x_n\}$ be a sequence in a Banach space X. Then we have the following :*

(i) $x_n \rightharpoonup x$ (in X) implies $\{x_n\}$ is bounded and $\|x\| \leq \liminf\limits_{n\to\infty} \|x_n\|$.

(ii) $x_n \rightharpoonup x$ in X and $f_n \to f$ in X^* imply $f_n(x_n) \to f(x)$ in $\mathbb{R}$.

Definition 2.30. (**Weak topology**) The weak topology on X is the topology with the fewest open sets.

Definition 2.31. (**Compact in the weak topology**) A subset C of a normed space X is said to be compact in the weak topology. For every sequence $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ converges weakly in C.

Definition 2.32. (**Weak compactness**) A subset C of a normed space X is said to be *weakly compact* if C is compact in the weak topology.

Theorem 2.22. *If X is a Banach space. Then X is reflexive if and only if every closed convex bounded subset of X is weakly compact .*

Definition 2.33. (**Lower semicontinuous**) Let X be a topological space and $f : X \to (-\infty, \infty]$ a proper function. Then f is said to be *lower semicontinuous (l.s.c.)* at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x) = \sup_{V \in U_{x_0}} \inf_{x \in V} f(x),$$

where U_{x_0} is a base of neighborhoods of the point $x_0 \in X$. f is said to be *lower semicontinuous on* X if it is lower semicontinuous on each point of X, i.e., for each $x \in X$

$$x_n \to x \Rightarrow f(x) \leq \liminf_{n \to \infty} f(x_n).$$

Note that f is said to be proper if there exists $x \in X$ such that $f(x) < \infty$.

Theorem 2.23. *Let C be a weakly compact convex subset of Banach space and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function. Then there exists x_0 in domain of f such that $f(x_0) = \inf\{f(x) : x \in C\}$.*

Finally, we give other definitions, theorems and lemmas which are used throughout the proof of this thesis (Definition 2.32 - Lemma 2.26).

Definition 2.34. [14](**Completely continuous**) Let X be Banach spaces and C be a nonempty subset of X. A mapping $T : C \to X$ is said to be *completely continuous* if, for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$, we have $\|Tx_n - Tx\| \to 0$.

Definition 2.35. [14](**Demiclose**) Let X be a Banach space. A mappings T with domain D and Range R in X is said be *demiclosed* at 0 if, for any sequence $\{x_n\}$ in D such that $\{x_n\}$ converges weakly to $x \in D$ and Tx_n converges strongly to 0 imply $Tx = 0$.

Definition 2.36. [14](**Demicompactness**) Let X be Banach spaces and C be a nonempty subset of X. A mapping $T : C \to X$ is said to be *demicompact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x \in C$ such that $\|x_{n_j} - x\| \to 0$.

Definition 2.37. [14](**Opail's property**) A Banach space X is said to satisfy *Opail's property* if for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Definition 2.38. [13] Let X be a Banach space and let C be a subset of X. For $i = 1, 2, 3, \cdots, k$, let $\{T_i\}$ be a family of nonself mappings from C to X with a nonempty set F of common fixed points. We say that $\{T_i\}$ satisfies condition $(\overline{A})$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{k}\sum_{i=1}^{k} \|x - T_i x\| \geq f(d(x, F)),$$

for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 2.24. [1] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad \textit{for all} \quad n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n < \infty$ exists.

(ii) If $\{a_n\}$ *has a subsequence converging to zero, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.25. [10] *Let* X *be a Banach space and let* C *be a nonempty closed convex subset of* X *which satisfies* Opial's condition *and let* $\{x_n\}$ *be a sequence in* X*. Let* $u, v \in X$ *be such that* $\lim_{n\to\infty} \|x_n - u\|$ *and* $\lim_{n\to\infty} \|x_n - v\|$ *exist. If* $\{x_{n_j}\}$ *and* $\{x_{n_k}\}$ *are subsequence of* $\{x_n\}$ *which converge weakly to* u *and* v*, respectively, then* $u = v$.

Lemma 2.26. *If* C *is a nonempty closed subset of a real Banach space* X, $x \in X$ *and* $d(x, C) = 0$*, then* $x \in C$.

Proof. Let C be a nonempty closed subset of a normed space X, $x \in X$ and $d(x, C) = 0$, that is, $\inf_{y\in C} d(x, y) = 0$. Using Theorem 2.15, we will show that $x \in C$. That is we construct a sequence $\{y_n\} \in C$ such that $y_n \to x$ as $n \to \infty$. For $n \in \mathbb{N}$ we get that

$$\inf_{y\in C} d(x, y) < \inf_{y\in C} d(x, y) + \frac{1}{n}.$$

Thus by definition of infimum, we obtain that for each $n \in \mathbb{N}$, there exists $y_n \in C$ such that

$$0 = \inf_{y\in C} d(x, y) < d(x, y_n) < \inf_{y\in C} d(x, y) + \frac{1}{n}.$$

By the sandwich theorem we have

$$\lim_{n\to\infty} d(x, y_n) = 0.$$

This means that $y_n \to x$. Since C is closed, $y_n \in C$ and $y_n \to x$, by Theorem 2.15 we have $x \in C$. □

Lemma 2.27. *Let* C *be a nonempty closed subset of a Banach space* X *and* $T : C \to X$ *be an asymptotically quasi-nonexpansive nonself mapping with the fixed point set* $F(T) \neq \emptyset$*. Then* $F(T)$ *is a closed subset in* C.

Proof. Assume that $T : C \to X$ is an asymptotically quasi-nonexpansive nonself mapping with respect to $\{k_n\}$. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$

as $n \to \infty$. Since C is closed and $\{p_n\}$ is a sequence in C, we must have $p \in C$. Since $T : C \to X$ is asymptotically quasi-nonexpansive, we obtain

$$\|Tp - p_n\| = \|Tp - Tp_n\| \leq (1 + k_1)\|p - p_n\|.$$

Taking limit as $n \to \infty$ and using the continuity of the norm, we obtain $\|Tp-p\| \leq 0$, which implies that $Tp = p$. The proof is complete. □

Lemma 2.28. *Let C be a nonempty closed subset of a Banach space X and $T : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping with the fixed point set $F(T) \neq \emptyset$. If $x_n \to x$, then $d(x_n, F(T)) \to d(x, F(T))$.*

Proof. Let $x_n \to x$. We will prove that $\lim_{n\to\infty} d(x_n, C) = d(x, C)$. By the triangle inequality, for each $n \in \mathbb{N}$, we obtain

$$d(x_n, C) \leq d(x, C) + d(x_n, x).$$

From this, for each $n \in \mathbb{N}$, we get

$$d(x_n, C) - d(x, C) \leq d(x_n, x). \tag{2.6}$$

Similarly, for each $n \in \mathbb{N}$, we can obtain that

$$d(x, C) \leq d(x_n, C) + d(x_n, x),$$

so, for each $n \in \mathbb{N}$, we get

$$-d(x_n, x) \leq d(x_n, C) - d(x, C). \tag{2.7}$$

From (2.6) and (2.7), we get

$$|d(x_n, C) - d(x, C)| \leq d(x_n, x). \tag{2.8}$$

Since $x_n \to x$, $\lim_{n\to\infty} d(x_n, x) = 0$. From this, (2.8) and the sandwich theorem we get

$$\lim_{n\to\infty} |d(x_n, C) - d(x, C)| = 0.$$

Hence $\lim_{n\to\infty} d(x_n, C) = d(x, C)$, as desired. □

CHAPTER 3

Banach Retraction

In this chapter, the existence of the Banach retraction of mapping is studied. At first of this chapter, some preliminary definitions and theorems which are used throughout the proof that when the mapping has a retraction are presented. Then we prove the theorem that confirms the existence of a retraction of a closed subset of a Banach space.

Uniform convexity

The strict convexity of a normed space X says that the midpoint $\frac{x+y}{2}$ of the segment joining two distinct points $x, y \in S_X$ with $\|x-y\| \geq \epsilon > 0$ does no lie on S_X ,that is,

$$\|\frac{x+y}{2}\| < 1.$$

In such spaces, we have no information about $1 - \|\frac{x+y}{2}\|$, the distance of midpoints from the unit sphere S_X. A stronger property than the strict convexity that provides information about the distance $1 - \|\frac{x+y}{2}\|$ is uniform convexity.

Definition 3.1. (**Uniform convexity**). A Banach space X is said to be uniformly convex if for any $\epsilon \in (0, 2]$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$ imply there exists a $\delta = \delta(\epsilon) > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$.

This says that if x and y are in the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ with $\|x - y\| \geq \epsilon > 0$, the midpoint of x and y lies inside the unit ball B_X at a distance of at least δ from the unit sphere S_X.

Example 3.1. Every Hilbert space H is uniformly convex space.

Proof. By the parallelogram law, we have

$$\|x+y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 \text{ for all } x, y \in H$$

Assume $x, y \in B_H$ with $x \neq y$, and $\|x-y\| > \epsilon$ for $\epsilon \in (0,2]$, we get

$$\begin{aligned}\|x+y\|^2 &= 2(\|x\|^2+\|y\|^2) - \|x-y\|^2 \\ &\leq 2(1+1) - \|x-y\|^2 \\ &\leq 4-\epsilon^2,\end{aligned}$$

Thus $\|\frac{x+y}{2}\|^2 \leq 1-\frac{\epsilon^2}{4}$, so it follows that

$$\|\frac{x+y}{2}\| \leq 1-\delta,$$

where $\delta = 1-\sqrt{1-\frac{\epsilon^2}{4}}$. Therefore, H is uniformly convex. □

Example 3.2. The space l_1 and l_∞ are not uniformly convex.

Proof. Let $x = (1,0,0,0,...), y = (0,-1,0,0,...) \in l_1$ and $\epsilon = 1$. Then

$$\|x\|_1 = 1, \|y\|_1 = 1, \|x-y\|_1 = 2 > 1 = \epsilon.$$

However, $\|\frac{x+y}{2}\|_1 = 1$ and there is no $\delta > 0$ such that $\|\frac{x+y}{2}\|_1 \leq 1-\delta$. Thus l_1 is not uniformly convex.

Similarly, if we let $x = (1,1,1,0,0,...), y = (1,1,-1,0,0,...) \in l_\infty$ and $\epsilon = 1$, then

$$\|x\|_\infty = 1, \|y\|_\infty = 1, \|x-y\|_\infty = 2 > 1 = \varepsilon.$$

Because $\|\frac{x+y}{2}\|_\infty = 1, l_\infty$ is not uniformly convex. □

From the definition of uniform convexity, we can derive some theorems as follows :

Theorem 3.1. *Every uniformly convex Banach space is strictly convex.*

Proof. Let X be a uniformly convex Banach space with $x \neq y$ and $x, y \in S_x$ where S_x is a unit sphere of Banach space. For $\epsilon \in (0,2]$, it follows from Definition 3.1 that X is strictly convex. If $\epsilon > 2$, it does not satisfy the condition of strictly convex because $1 = \frac{\|x\|+\|y\|}{2} \geq \|\frac{x+y}{2}\|$. Therefore uniformly convex Banach space is strictly convex. □

Theorem 3.2. *Let X be a Banach space. Then the following are equivalent:*

(i) X is uniformly convex;

(ii) For two sequences $\{x_n\}$ and $\{y_n\}$ in X, if $\|x_n\| \leq 1, \|y_n\| \leq 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. $(i) \Rightarrow (ii)$. Suppose X is uniformly convex. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\|x_n\| \leq 1, \|y_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Suppose to the contrary that $\lim_{n\to\infty} \|x_n - y_n\| \neq 0$ that is there exists $\epsilon > 0$ such that for all N there exists $n_N > N$ such that

$$\|x_{n_N} - y_{n_N}\| \geq \epsilon.$$

Since X is uniformly convex, there exists $\delta > 0$ such that

$$\|x_{n_N} + y_{n_N}\| \leq 2(1 - \delta). \tag{3.1}$$

By assumption, we know that $\lim_{n\to\infty} \|x_n + y_n\| = 2$, and from (3.1) we obtain

$$2 \leq 2(1 - \delta),$$

which is a contradiction. Therefore $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
$(ii) \Rightarrow (i)$. Suppose (ii) holds. If X is not uniformly convex that is there exists $\epsilon \in (0, 2]$ such that for all $\delta > 0$ such that

$$\|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \quad \text{but} \quad \|x + y\| > 2(1 - \delta),$$

and then we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $\|x_n\| \leq 1, \|y_n\| \leq 1$;

(ii) $\|x_n + y_n\| > 2(1 - \frac{1}{n})$;

(iii) $\|x_n - y_n\| \geq \epsilon$.

Clearly $\|x_n - y_n\| \geq \epsilon$, which contradicts to the hypothesis that $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Thus, X must be uniformly convex. □

Next we show the important result for the class of uniformly convex Banach spaces.

Theorem 3.3. *Every uniformly convex Banach space is reflexive.*

Proof. Let X be uniformly convex Banach space. Let $S_{X^*} = \{j \in X^* : \|j\| = 1\}$ be a unit sphere in X^* and $f \in S_{X^*}$. Assume that $\{x_n\}$ is a sequence in S_X such that $\lim_{n\to\infty} f(x_n) = 1$. We claim that $\{x_n\}$ is a Cauchy sequence. Assume $\{x_n\}$ is not a Cauchy sequence, that is, there exists $\epsilon > 0$ such that for all N there exists $n_j, n_k > N$ such that $\|x_{n_j} - x_{n_k}\| \geq \epsilon$. Since X is a uniformly convex Banach space, we have there exists $\delta > 0$ such that $\|\frac{x_{n_j} + x_{n_k}}{2}\| < 1 - \delta$. We see that

$$|f(\frac{x_{n_j} + x_{n_k}}{2})| \leq \|f\|_* \|\frac{x_{n_j} + x_{n_k}}{2}\| < 1 - \delta,$$

since $\lim_{n\to\infty} f(x_n) = 1$, which is a contradiction. Hence $\{x_n\}$ is Cauchy. Thus there exists a point x in X such that $\lim_{n\to\infty} x_n = x$ because X is a Banach space. Now, by continuity of $\|\cdot\|$, we see that

$$\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty} \|x_n\| = 1.$$

So $x \in S_x$. By Theorem 2.18, we conclude that X is reflexive. □

We now introduce a useful property.

Definition 3.2. (**Kadec - Klee property**). A Banach space X is said to have the Kadec - Klee property for every sequence $\{x_n\}$ in X that converges weakly to x where also $\|x_n\| \to \|x\|$, then $\{x_n\}$ converges strongly to x.

The following result has a very useful property:

Theorem 3.4. *Every uniformly convex Banach space has the Kadec - Klee property.*

Proof. Let X be a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$. We claim that $x_n \to x$. If $x = 0$, then

$\lim_{n\to\infty} \|x_n\| = 0$, that is , for all $\epsilon > 0$, there exists N such that $n > N$ implies $|\|x_n\| - 0| < \epsilon$, that is $\|x_n\| < \epsilon$ which yields that $\lim_{n\to\infty} x_n = 0$.

Assume that $x \neq 0$. We are going to show that $\lim_{n\to\infty} x_n = x$. We prove this by contradiction, suppose that $\lim_{n\to\infty} x_n \neq x$ and $\|x_n\| \neq 0$. We can show that $\lim_{n\to\infty} \frac{x_n}{\|x_n\|} \neq \frac{x}{\|x\|}$,where $\|x_n\| \neq 0$ and $\|x\| \neq 0$. Then there exists $\epsilon > 0$, for all N such that there exists $n_i > N$ such that

$$\|\frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|}\| \geq \epsilon.$$

Since X is uniformly convex, there exists $\delta > 0$ such that

$$\frac{1}{2}\|\frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|}\| \leq 1 - \delta. \tag{3.2}$$

Taking limit infimum as $i \to \infty$ both sides, we have

$$\liminf_{i\to\infty} \frac{1}{2}\|\frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|}\| \leq 1 - \delta. \tag{3.3}$$

Since $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we claim that $\frac{x_{n_i}}{\|x_{n_i}\|} \rightharpoonup \frac{x}{\|x\|}$. Let $f \in X^*$ and $\epsilon > 0$, there exists N such that

$$|f(x_n) - f(x)| < \frac{\epsilon\|x\|}{2}$$

and

$$|\|x_n\| - \|x\|| < \frac{\epsilon\|x\|}{2\|f\|_*},$$

for all $n > N$.

Now we consider

$$\begin{aligned}|f(\frac{x_n}{\|x_n\|})-f(\frac{x}{\|x\|})| &= |\frac{1}{\|x_n\|}f(x_n)+\frac{1}{\|x\|}f(x)| \\ &= \frac{|\|x\|f(x_n)-\|x_n\|f(x)|}{\|x\|\|x_n\|} \\ &= \frac{|\|x\|f(x_n)-\|x_n\|f(x_n)+\|x_n\|f(x_n)-\|x_n\|f(x)|}{\|x\|\|x_n\|} \\ &= \frac{|(\|x\|-\|x_n\|)f(x_n)+\|x_n\|(f(x_n)-f(x))|}{\|x\|\|x_n\|} \\ &\leq \frac{|\|x\|-\|x_n\|\|\|f\|_*\|x_n\|}{\|x\|\|x_n\|}+\frac{\|x_n\||f(x_n)-f(x)|}{\|x\|\|x_n\|} \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon, \end{aligned} \tag{3.4}$$

for all $n > N$. Thus we can conclude that $f(\frac{x_n}{\|x_n\|}) \to f(\frac{x}{\|x\|})$. Since $f \in X^*$ was arbitrary, we have $\frac{x_{n_i}}{\|x_{n_i}\|} \rightharpoonup \frac{x}{\|x\|}$. It follows that $\frac{1}{2}(\frac{x_{n_i}}{\|x_{n_i}\|}+\frac{x}{\|x\|}) \rightharpoonup \frac{x}{\|x\|}$, By Theorem 2.21, we have

$$\|\frac{x}{\|x\|}\| \leq \liminf_{n\to\infty}\frac{1}{2}\|\frac{x_{n_i}}{\|x_{n_i}\|}+\frac{x}{\|x\|}\| \leq 1-\delta,$$

which is a contradiction. Therefore $\{x_n\}$ converges strongly to $x \in X$. □

Metric projection

Let X be a normed space and C be a nonempty subset of X. Let $x \in X$ and $y_0 \in C$, we say that y_0 is a *best approximation* to x if

$$\|x-y_0\| = d(x, C).$$

Let $P_C(x) = \{y \in C : \|x-y\| = d(x, C)\}$ denote the (possibly empty) set of all best approximations from x to C which is called the metric projection onto C such that we define a mapping P_C from X into the power set of C. We can call metric projection mapping which are the nearest point projection mapping, proximity mapping and best approximation operator.

Lemma 3.5. *The set of best approximation is convex if C is convex.*

Proof. Let C be a convex set and $P_C(x) = \{y \in C : \|x-y\| = d(x,C)\}$ is the set of all best approximation from X to C. Let $a, b \in P_C(x)$, we have $a, b \in C$, $\|x-a\| = d(x,C)$ and $\|x-b\| = d(x,C)$. We claim that $\lambda a+(1-\lambda)b \in P_C(x)$ for all $\lambda \in [0,1]$. Since $a, b \in C$ and C is convex,that is for $\lambda \in [0,1]$, we get $\lambda a + (1-\lambda)b \in C$. To complete the proof, we show that $\|x - (\lambda a + (1-\lambda)b)\| = d(x,C)$. Clearly, $\|x-(\lambda a+(1-\lambda)b)\| \geq d(x,C)$. Then we claim that $\|x-(\lambda a+(1-\lambda)b)\| \leq d(x,C)$.

$$\begin{aligned}
\|x - (\lambda a + (1-\lambda)b)\| &= \|\lambda x + (1-\lambda)x - (\lambda a + (1+\lambda)b)\| \\
&= \|\lambda(x-a) + (1-\lambda)(x-b)\| \\
&\leq \lambda\|x-a\| + (1-\lambda)\|x-b\| \\
&= \lambda d(x,C) + (1-\lambda)d(x,C) \\
&= d(x,C).
\end{aligned}$$

Thus $\lambda a + (1-\lambda)b \in P_C(x)$. That is the set of best approximation is convex. □

We say C is the proximal set if each $x \in X$ has at least one best approximation in C.

Some results on proximal sets as follow :

Theorem 3.6 (**The existence of best approximation**). *Let C be a nonempty weakly compact convex subset of a Banach space X and $x \in X$. Then x has a best approximation in C, that is, $P_C(x) \neq \emptyset$.*

Proof. We define the function $f : C \to \mathbb{R}^+$ by

$$f(y) = \|x-y\|, \qquad y \in C$$

Let $\{a_n\}$ be a sequence in C such that $a_n \to a$.

$$f(a) = \|x-a\| \leq \liminf_{n\to\infty} \|x-a_n\| = \liminf_{n\to\infty} f(a_n).$$

Thus f is lower semicontinuous. Since C is weakly compact, by Theorem 2.21, there exists $a_0 \in C$ such that $\|x-a_0\| = \inf_{y\in C} \|x-y\|$. □

Theorem 3.7 (The uniqueness of best approximation). *Let C be a nonempty convex subset of a strictly convex Banach space X. Then for each $x \in X$, C has at most one best approximation.*

Proof. We prove this by contradiction. Let y_1, y_2 be elements in C which are best approximations to x in X. Since C is convex, by Lemma 3.5, set of best approximations is convex. Therefore $\frac{y_1 + y_2}{2}$ is also a best approximation to x. Let $r = d(x, C)$, then

$$r = \|x - y_1\| = \|x - y_2\| = \|x - \frac{y_1 + y_2}{2}\|.$$

Since

$$\begin{aligned} r &= \|x - \frac{y_1 + y_2}{2}\| \\ &= \|(\frac{x}{2} - \frac{y_1}{2}) + (\frac{x}{2} - \frac{y_2}{2})\|, \\ 2r &= \|(x - y_1) + (x - y_2)\|, \end{aligned} \tag{3.5}$$

and

$$\|x - y_1\| + \|x - y_2\| = r + r = 2r. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\|x - y_1\| + \|x - y_2\| = \|(x - y_1) + (x - y_2)\|.$$

By the strict convexity of X, we obtain

$$(x - y_2) = a(x - y_1); \quad a \geq 0.$$

Taking the norm in both sides, we have

$$\|x - y_2\| = a\|x - y_1\|$$

$$r = ar$$

Thus $a = 1$. From this, we can conclude that $y_1 = y_2$. □

Banach Ratraction.

Let C be a nonempty subset of a topological space X and D a nonempty subset of C. Then a continuous mapping $P : C \to D$ is said to be a retraction if $Px = x$ for all $x \in D$, that is, $P^2 = P$. If there exists a continuous retraction $P : X \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *retract* of X.

Theorem 3.8. *Every nonempty closed convex bounded subset C of a uniformly convex Banach space X is a retract of X.*

Proof. Let X is a uniformly convex Banach space and $x \in X$. By Theorem 2.22, Theorem 3.3 and Theorem 3.5, x has a best approximation in C, that is, $P_C(x) \neq \emptyset$. From this, Theorem 2.20, Theorem 3.1 and Theorem 3.7, we get, C has the unique best approximation. That is, $P_C(\cdot)$ is a single-valued metric projection mapping from X onto C. It remains to show that P_C is continuous. We prove this by contradiction. Let P_C is not continuous. There exists sequence $\{x_n\}$ in X with $\lim_{n\to\infty} x_n = x \in X$ such that $\lim_{n\to\infty} P_C(x_n) \neq P_C(x)$ that is there exists $\epsilon > 0$, for all N such that there exists $n > N$ and

$$\|P_C(x_n) - P_C(x)\| \geq \epsilon.$$

Since

$$|d(x_n, C) - d(x, C)| = |\inf_{y\in C} \|x_n - y\| - \inf_{y\in C} \|x - y\|| \leq \|x_n - x\|,$$

we have, by Theorem 2.20,

$$|\|x_n - P_C(x_n)\| - \|x - P_C(x)\|| \leq \|x_n - x\|.$$

This implies that

$$\lim_{n\to\infty} \|x_n - P_C(x_n)\| = \|x - P_C(x)\|. \tag{3.7}$$

Since $\{P_C(x_n)\}$ is bounded in C by (3.7), there exists a subsequence $\{P_C(x_{n_i})\}$ of $\{P_C(x_n)\}$ such that weak $-\lim_{i\to\infty} P_C(x_{n_i}) = z \in C$. Note

$$\text{weak} - \lim_{i\to\infty} (x_{n_i} - P_C(x_{n_i})) = x - z. \tag{3.8}$$

By Theorem 2.21, we have

$$\|x - z\| \leq \liminf_{i\to\infty} \|x_{n_i} - P_C(x_{n_i})\| = \|x - P_C(x)\|.$$

This implies $z = P_C(x)$ by definition of the function P_C. From (3.7) and (3.8)

$$\text{weak}-\lim_{i\to\infty}(x_{n_i} - P_C(x_{n_i})) = x - P_C(x) \quad \text{and} \quad \lim_{i\to\infty} \|x_{n_i} - P_C(x_{n_i})\| = \|x - P_C(x)\|.$$

Since X is uniformly convex, X has the Kadec-Klee property. So

$$\lim_{i\to\infty}(x_{n_i} - P_C(x_{n_i})) = x - P_C(x),$$

which implies that $\lim_{i\to\infty} P_C(x_{n_i}) = P_C(x)$ which is a contradiction. Therefore P_C is continuous. $\square$

CHAPTER 4

Main Results

The propose of this chapter is to introduce and to study iterative schemes for a viscosity approximation common fixed points for three-steps and a finite family of asymptotically quasi-nonexpansive nonself mappings in Banach spaces. The convergence theorems in Banach spaces are proved in Section 4.1 and weak and strong convergence theorems of the iterative schemes in a uniformly convex Banach space are also proved in Section 4.2.

Let X be a real Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. A mapping $f : C \to C$ is called a *contractive mapping* if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|,$$

for all $x, y \in C$. For $i = 1, 2, 3$, let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping such that the fixed point set $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping. We are interested in sequences in the following process. For $x_1 \in C$ and $n \geq 1$, define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} z_n &= P(a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)T_3(PT_3)^{n-1}x_n)) \\ y_n &= P(c_n f(z_n) + (1 - c_n)(d_n z_n + (1 - d_n)T_2(PT_2)^{n-1}z_n)) \\ x_{n+1} &= P(e_n f(y_n) + (1 - e_n)(g_n y_n + (1 - g_n)T_1(PT_1)^{n-1}y_n)) \end{aligned} \tag{4.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{g_n\}$ are appropriate sequences in $[0, 1]$.

4.1 Convergence Theorems in Banach Spaces

In this section, we established strong convergence theorems in Banach spaces of the iterative sequence $\{x_n\}$ defined in (4.1) converges to a common fixed point of $T_i (i = 1, 2, 3)$. At the end of this section, we proved some strong convergence theorems of finite family of $\{T_i : C \to X, i = 1, 2, 3, \dots, k\}$ where each T_i is an asymptotically quasi-nonexpansive nonself mapping.

Theorem 4.1. *Let X be a real Banach space, and let C be a nonempty closed convex nonexpansive retract of X with a nonexpansive retraction P. For $i = 1, 2, 3,$ let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = max\{h_1^{(n)}, h_2^{(n)}, h_3^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{g_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined in (4.1) converges strongly to a common fixed point of T_1, T_2 and T_3 if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = 0.$$

Proof. We first prove the necessity. Assume that $\{x_n\}$ converges strongly to a common fixed point of T_1 , T_2 and T_3, that is, there exists $x \in F(T_1) \cap F(T_2) \cap F(T_3)$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

From this, we have

$$\liminf_{n\to\infty} \|x_n - x\| = 0.$$

We see that

$$d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = \inf_{x^* \in F(T_1)\cap F(T_2)\cap F(T_3)} d(x_n, x^*) \le \|x_n - x\|$$

for all n. Taking limit infimum as $n \to \infty$ and using the sandwich theorem, we obtain that

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = 0,$$

as desired. Now we prove the sufficiency. Assume that $T_i : C \to X$ is an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ for $i = 1, 2, 3$. Let $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Note that $T_i(PT_i)^{n-1}p = p$. By assumption, we have

$$
\begin{aligned}
\|z_n - p\| &= \|P(a_n f(x_n) + (1-a_n)(b_n x_n + (1-b_n)T_3(PT_3)^{n-1}x_n)) - Pp\| \\
&\leq \|a_n f(x_n) + (1-a_n)(b_n x_n + (1-b_n)T_3(PT_3)^{n-1}x_n) - p\| \\
&= \|a_n f(x_n) - a_n p + (1-a_n)(b_n x_n + (1-b_n)T_3(PT_3)^{n-1}x_n - p)\| \\
&= \|a_n(f(x_n) - p) + (1-a_n)(b_n(x_n - p) \\
&\quad +(1-b_n)(T_3(PT_3)^{n-1}x_n - p))\| \\
&\leq a_n\|f(x_n) - p\| + (1-a_n)b_n\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\|T_3(PT_3)^{n-1}x_n - p\| \\
&\leq a_n\|f(x_n) - f(p)\| + a_n\|f(p) - p\| \\
&\quad +(1-a_n)b_n\|x_n - p\| + (1-a_n)(1-b_n)(1+h_3^{(n)})\|x_n - p\| \\
&\leq a_n a\|x_n - p\| + a_n\|f(p) - p\| + (1-a_n)b_n\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\|x_n - p\| + (1-a_n)(1-b_n)h_3^{(n)}\|x_n - p\| \\
&\leq (1-(1-a)a_n + h_3^{(n)})\|x_n - p\| + a_n\|f(p) - p\| \\
&\leq (1+h_n)\|x_n - p\| + a_n\|f(p) - p\| \qquad (4.2)
\end{aligned}
$$

$$
\begin{aligned}
\|y_n - p\| &= \|P(c_n f(z_n) + (1-c_n)(d_n z_n + (1-d_n)T_2(PT_2)^{n-1}z_n)) - Pp\| \\
&\leq \|c_n f(z_n) + (1-c_n)(d_n z_n + (1-d_n)T_2(PT_2)^{n-1}z_n) - p\| \\
&= \|c_n f(z_n) - c_n p + (1-c_n)(d_n z_n + (1-d_n)T_2(PT_2)^{n-1}z_n - p)\| \\
&= \|c_n(f(z_n) - p) + (1-c_n)(d_n(z_n - p) \\
&\quad +(1-d_n)(T_2(PT_2)^{n-1}z_n - p))\| \\
&\leq c_n\|f(z_n) - p\| + (1-c_n)d_n\|z_n - p\| \\
&\quad +(1-c_n)(1-d_n)\|T_2(PT_2)^{n-1}z_n - p\|
\end{aligned}
$$

$$\begin{aligned}
&\leq c_n\|f(z_n)-f(p)\|+c_n\|f(p)-p\|+(1-c_n)d_n\|z_n-p\| \\
&\quad +(1-c_n)(1-d_n)(1+h_2^{(n)})\|z_n-p\| \\
&\leq c_n a\|z_n-p\|+c_n\|f(p)-p\|+(1-c_n)d_n\|z_n-p\| \\
&\quad +(1-c_n)(1-d_n)\|z_n-p\|+(1-c_n)(1-d_n)h_2^{(n)}\|z_n-p\| \\
&\leq (1-(1-a)c_n+h_2^{(n)})\|z_n-p\|+c_n\|f(p)-p\| \\
&\leq (1+h_n)\|z_n-p\|+c_n\|f(p)-p\| \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-p\| &= \|P(e_nf(y_n)+(1-e_n)(g_ny_n+(1-g_n)T_1(PT_1)^{n-1}y_n))-Pp\| \\
&\leq \|e_nf(y_n)+(1-e_n)(g_ny_n+(1-g_n)T_1(PT_1)^{n-1}y_n)-p\| \\
&= \|e_nf(y_n)-e_np+(1-e_n)(g_ny_n+(1-g_n)T_1(PT_1)^{n-1}y_n-p)\| \\
&= \|e_n(f(y_n)-p)+(1-e_n)(g_n(y_n-p) \\
&\quad +(1-g_n)(T_1(PT_1)^{n-1}y_n-p))\| \\
&\leq e_n\|f(y_n)-p\|+(1-e_n)g_n\|y_n-p\| \\
&\quad +(1-e_n)(1-g_n)\|T_1(PT_1)^{n-1}y_n-p\| \\
&\leq e_n\|f(y_n)-f(p)\|+e_n\|f(p)-p\| \\
&\quad +(1-e_n)g_n\|y_n-p\|+(1-e_n)(1-g_n)(1+h_1^{(n)})\|y_n-p\| \\
&\leq e_na\|y_n-p\|+e_n\|f(p)-p\|+(1-e_n)g_n\|y_n-p\| \\
&\quad +(1-e_n)(1-g_n)\|y_n-p\|+(1-e_n)(1-g_n)h_1^{(n)}\|y_n-p\| \\
&\leq (1-(1-a)e_n+h_1^{(n)})\|y_n-p\|+e_n\|f(p)-p\| \\
&\leq (1+h_n)\|y_n-p\|+e_n\|f(p)-p\|. \qquad (4.4)
\end{aligned}$$

Substituting (4.2) into (4.3), we obtain

$$\begin{aligned}
\|y_n-p\| &\leq (1+h_n)((1+h_n)\|x_n-p\|+a_n\|f(p)-p\|)+c_n\|f(p)-p\| \\
&= (1+h_n)(1+h_n)\|x_n-p\|+(1+h_n)a_n\|f(p)-p\|+c_n\|f(p)-p\| \\
&= (1+h_n)^2\|x_n-p\|+(a_n+a_nh_n+c_n)\|f(p)-p\| \\
&= (1+h_n(2+h_n))\|x_n-p\|+(a_n+a_nh_n+c_n)\|f(p)-p\| \\
&= (1+m_n)\|x_n-p\|+s_n \qquad (4.5)
\end{aligned}$$

where $m_n = h_n(2+h_n)$ and $s_n = (a_n + a_n h_n + c_n)\|f(p) - p\|$. Since $\sum_{n=1}^{\infty} h_n < \infty$, we have that $\{2+h_n\}$ and $\{1+h_n\}$ are bounded. Thus $\sum_{n=1}^{\infty} m_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ because $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Substituting (4.5) into (4.4), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1+h_n)((1+h_n)^2\|x_n - p\| + s_n) + e_n\|f(p) - p\| \\
&= (1+h_n)^3\|x_n - p\| + (1+h_n)s_n + e_n\|f(p) - p\| \\
&= (1+t_n)\|x_n - p\| + u_n \qquad (4.6)
\end{aligned}$$

where $t_n = (1+h_n)^3 - 1$ and $u_n = (1+h_n)s_n + e_n\|f(p) - p\|$. Since $\sum_{n=1}^{\infty} h_n < \infty, \sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, then $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Hence Lemma 2.24 implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus $\|x_n - p\|$ is bounded. Let $L = \sup_n \|x_n - p\|$. We can rewrite (4.6) as

$$\|x_{n+1} - p\| \leq \|x_n - p\| + Lt_n + u_n \quad \text{for} \quad n \geq 1 \qquad (4.7)$$

Now, for any positive integers $m,\ n \geq 1$, $p \in F(T_1) \cap F(T_2) \cap F(T_3)$ and induction, we have

$$\|x_{n+m} - p\| \leq \|x_n - p\| + L\sum_{i=n_0}^{n+m-1} t_i + \sum_{i=n_0}^{n+m-1} u_i. \qquad (4.8)$$

By (4.7) and taking infimum over $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, we obtain

$$d(x_{n+1},\ F(T_1) \cap F(T_2) \cap F(T_3)) \leq d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)) + Lt_n + u_n.$$

The assumption $\liminf_{n\to\infty} d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)) = 0$ implies that there exists a subsequence of $\{d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))\}$ converging to zero. This result together with the fact $\sum_{n=1}^{\infty}(Lt_n + u_n) < \infty$ and Lemma 2.24,we have

$$\lim_{n\to\infty} d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = 0. \qquad (4.9)$$

We now show that $\{x_n\}$ is a Cauchy sequence in X. Let $\epsilon > 0$. By (4.9) and two facts that $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$, there exists n_0 such that, for $n \geq n_0$, we have

$$d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) < \frac{\epsilon}{6}, \sum_{i=n_0}^{\infty} t_i < \frac{\epsilon}{3(L+1)}, \sum_{i=n_0}^{\infty} u_i < \frac{\epsilon}{3}. \tag{4.10}$$

By the first inequality of (4.10) and the definition of infimum, there exists $p_0 \in F(T_1) \cap F(T_2) \cap F(T_3)$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{6}. \tag{4.11}$$

By combining (4.7), (4.10) and (4.11), we have

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + L \sum_{i=n_0}^{n_0+m-1} t_i + \sum_{i=n_0}^{n_0+m-1} u_i \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. But X is a Banach space, so there must be some $q \in X$ such that $x_n \to q$. Since C is closed and $\{x_n\}$ is a sequence in C, we have that $q \in C$. Since $\emptyset \neq F(T_1) \cap F(T_2) \cap F(T_3) \subseteq C$ and $x_n \to q$ by Lemma 2.28, we have

$$0 = \lim_{n \to \infty} d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = d(q, F(T_1) \cap F(T_2) \cap F(T_3)).$$

Form this and since $F(T_1) \cap F(T_2) \cap F(T_3)$ is closed, so $q \in F(T_1) \cap F(T_2) \cap F(T_3)$ by Lemma 2.26. Therefore $\{x_n\}$ converges strongly to a common fixed point of T_1 , T_2 and T_3 as desired. □

If $T_1 = T_2 = T_3 = T$, then the iterative sequences in (4.1) become

$$\begin{aligned} z_n &= P(a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)T(PT)^{n-1} x_n)) \\ y_n &= P(c_n f(z_n) + (1 - c_n)(d_n z_n + (1 - d_n)T(PT)^{n-1} z_n)) \\ x_{n+1} &= P(e_n f(y_n) + (1 - e_n)(g_n y_n + (1 - g_n)T(PT)^{n-1} y_n)),\ n \geq 1. \end{aligned} \tag{4.12}$$

We then have the following result for fixed point of a single asymptotically qausi-nonexpansive nonself mapping.

Corollary 4.2. *Let X be a real Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_n\}$ such that $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$. Let $f : C \to C$ be a contractive mapping and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{g_n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Then the iterative sequence $\{x_n\}$ defined in (4.12) converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Corollary 4.3. *Let $X, C, T_i (i = 1, 2, 3)$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.1. Suppose that conditions in Theorem 4.1 hold and*

(i) the mapping $T_i (i = 1, 2, 3)$ is asymptotically regular in x_n, that is,

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \ i = 1, 2, 3;$$

(ii) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ implies that $\liminf_{n\to\infty} d(x_n, \ F(T_1) \cap F(T_2) \cap F(T_3)) = 0.$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Since T is asymptotically regular in x_n

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0; \ i = 1, \ 2, \ 3.$$

From (ii), $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$. By Theorem 4.1, we see that the sequence $\{x_n\}$ converges to a common fixed point p of $T_1, \ T_2$ and T_3. □

Theorem 4.4. *Let $X,\ C,\ T_i(i=1,2,3)$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.1. Suppose that conditions in Theorem 4.1 hold. Assume further that the mapping $T_i(i=1,2,3)$ is asymptotically regular in x_n and satisfies condition $(\overline{A})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. To apply Theorem 4.1, we prove that $\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)\cap F(T_3)) = 0$. Since $\{T_i, i = 1,2,3\}$ satisfies condition $(\overline{A})$, there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$\frac{1}{3}\sum_{i=1}^{3} \|x_n - T_i x_n\| \geq f(d(x_n, F(T_1)\cap F(T_2)\cap F(T_3))),$$

for all $n \geq 1$. Since each T_i is asymptotically regular in x_n for $i = 1,2,3$,

$$\liminf_{n\to\infty} f(d(x_n, F(T_1)\cap F(T_2)\cap F(T_3))) \leq 0.$$

Since $f : [0,\infty) \to [0,\infty)$, we have that

$$\liminf_{n\to\infty} f(d(x_n, F(T_1)\cap F(T_2)\cap F(T_3))) = 0 \tag{4.13}$$

We claim that $\liminf_{n\to\infty} d(x_n,\ F(T_1)\cap F(T_2)\cap F(T_3)) = 0$. Suppose not, that is

$$\liminf_{n\to\infty} d(x_n,\ F(T_1)\cap F(T_2)\cap F(T_3)) \neq 0.$$

From this and $f : [0,\infty) \to [0,\infty)$, we get

$$\liminf_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)\cap F(T_3)) = L > 0.$$

Since $\liminf_{n\to\infty} d(x_n,\ F(T_1)\cap F(T_2)\cap F(T_3)) = L > 0$, thus for all $\epsilon = L > 0$, there exists $N_1 \in \mathbb{N}$ such that $N > N_1$ implies

$$|\inf_{n\geq N} d(x_n,\ F(T_1)\cap F(T_2)\cap F(T_3)) - L| < \frac{L}{3}$$

From this we get

$$\frac{2L}{3} < \inf_{n\geq N} d(x_n,\ F(T_1)\cap F(T_2)\cap F(T_3)) < \frac{4L}{3},\quad \text{for all}\quad N > N_1,$$

That is

$$\frac{2L}{3} < d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)), \quad \text{for all} \quad n \geq N > N_1.$$

Since f is nondecreasing,

$$f(\frac{2L}{3}) \leq f(d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))), \quad \text{for all} \quad n \geq N > N_1.$$

We get

$$\begin{aligned} f(\frac{2L}{3}) &\leq \inf_{n \geq N} f(d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))), \quad \text{for all} \quad N > N_1 \\ &\leq \lim_{N \to \infty} \inf\{f(d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))) \ \ ; n \geq N\} \\ &= \liminf_{n \to \infty} \{f(d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))). \end{aligned}$$

Since $f(t) > 0$ if $t > 0$, we have

$$0 < f(\frac{2L}{3}) \leq \liminf_{n \to \infty} f(d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3))),$$

which contradicts (4.13). Hence $\liminf_{n \to \infty} d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)) = 0$. We see that $\{x_n\}$ converges strongly to a common fixed point p of $T_1,\ T_2$ and T_3, by Theorem 4.1, as desired. □

If for $i = 1, 2, 3$, T_i is a self mapping, then the iterative sequences (4.1) become

$$\begin{aligned} z_n &= a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n) T_3 x_n) \\ y_n &= c_n f(z_n) + (1 - c_n)(d_n z_n + (1 - d_n) T_2 z_n) \qquad (4.14) \\ x_{n+1} &= e_n f(y_n) + (1 - e_n)(g_n y_n + (1 - g_n) T_1 y_n), \quad n \geq 1. \end{aligned}$$

We have the following theorem for common fixed points of three asymptotically quasi-nonexpansive self mappings.

Corollary 4.5. *Let X be a real Banach space and let C be a nonempty closed convex subset of X. For $i = 1, 2, 3$, let $T_i : C \to C$ be an asymptotically quasi-nonexpansive self mapping with respect to $\{h_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = max\{h_1^{(n)}, h_2^{(n)}, h_3^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{g_n\}$ be real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Then the iterative sequence $\{x_n\}$ defined in (4.14) converges strongly to a common fixed point of T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)) = 0$.*

Now, we introduce a new iteration process for a finite family $\{T_i : C \to X, i = 1, 2, 3, ..., k\}$ of asymptotically quasi - nonexpansive nonself mapping as follows :

Let X be a real arbitrary Banach space and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. For $i = 1, 2, 3, ..., k$, let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping such that $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. We are interested in sequences in the following process. For $x_1 \in C$, fixed $k \in \mathbb{N}$ and $n \geq 1$, The iteration scheme is defined as follows :

$$\begin{aligned}
x_{n+1} &= P[\alpha_k^{(n)} f(y_{(k-1)n}) + (1-\alpha_k^{(n)})(\beta_k^{(n)} y_{(k-1)}^{(n)} + (1-\beta_k^{(n)}) T_k (PT_k)^{n-1} y_{(k-1)}^{(n)})] \\
y_{(k-1)}^{(n)} &= P[\alpha_{(k-1)}^{(n)} f(y_{(k-2)}^{(n)}) + (1-\alpha_{(k-2)}^{(n)})(\beta_{(k-1)}^{(n)} y_{(k-2)}^{(n)} + \\
&\quad (1-\beta_{(k-1)}^{(n)}) T_{(k-1)} (PT_{(k-1)})^{n-1} y_{(k-2)}^{(n)})] \\
y_{(k-2)}^{(n)} &= P[\alpha_{(k-2)}^{(n)} f(y_{(k-3)}^{(n)}) + (1-\alpha_{(k-3)}^{(n)})(\beta_{(k-2)}^{(n)} y_{(k-3)}^{(n)} + \\
&\quad (1-\beta_{(k-2)}^{(n)}) T_{(k-2)} (PT_{(k-2)})^{n-1} y_{(k-3)}^{(n)})] \\
&\ \vdots \\
y_2^{(n)} &= P[\alpha_2^{(n)} f(y_1^{(n)}) + (1-\alpha_2^{(n)})(\beta_2^{(n)} y_1^{(n)} + (1-\beta_2^{(n)}) T_2 (PT_2)^{n-1} y_1^{(n)})] \\
y_1^{(n)} &= P[\alpha_1^{(n)} f(y_0^{(n)}) + (1-\alpha_1^{(n)})(\beta_1^{(n)} y_0^{(n)} + (1-\beta_1^{(n)}) T_1 (PT_1)^{n-1} y_0^{(n)})]
\end{aligned} \tag{4.15}$$

where $y_0^{(n)} = x_n$, for all n, $\{\alpha_i^{(n)}\}$ and $\{\beta_i^n\}$, $n = 1, 2, 3, ...$ and $i = 1, 2, 3, ..., k$ are appropriate sequences in $[0, 1]$.

Theorem 4.6. *Let X be a real arbitrary Banach space and let C be a nonempty closed convex nonexpansive retract of X with a nonexpansive retraction P. For $i = 1, 2, 3, ..., k$, let $T_i : C \to X$ be an asymptotically quasi nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ such that $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = \max_{1 \le i \le k}\{h_i^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_i^{(n)}\}$ and $\{\beta_i^{(n)}\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_i^{(n)} < \infty$ for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, 3, ..., k$. Then the iterative sequence $\{x_n\}$ defined in (4.15) converges strongly to a common fixed point of $\{T_i, i = 1, 2, 3, ..., k\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. For the necessity, we assume that $\{x_n\}$ converges to a common fixed point of $\{T_i, i = 1, 2, 3, ..., k\}$, that is, there exists $p \in F$ such that $\lim_{n\to\infty} \|x_n - p\| = 0$, so $\liminf_{n\to\infty} \|x_n - p\| = 0$. We have, by definition of distance function,

$$d(x_n, F) = \inf_{p^* \in F} \|x_n - p^*\| \le \|x_n - p\|.$$

By taking limit infimum as $n \to \infty$ and using the sandwich theorem , we have $\liminf_{n\to\infty} d(x_n, F) = 0$, as desired. Now, we prove the sufficiency. Assume that $T_i : C \to X$ is an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ for $i = 1, 2, 3, ..., k$. Let $p \in F$ and $\alpha_n = \max_{1\le i\le k}\{\alpha_i^{(n)}\}$. Note that $T_i(PT_i)^{n-1}p = p$. By assumption, we have

$$\begin{aligned}
\|y_1^{(n)} - p\| &= \|P[\alpha_1^{(n)} f(x_n) + (1 - \alpha_1^{(n)})(\beta_1^{(n)} x_n + (1 - \beta_1^{(n)})T_1(PT_1)^{n-1}x_n)] - P_p\| \\
&\le \|\alpha_1^{(n)} f(x_n) + (1 - \alpha_1^{(n)})(\beta_1^{(n)} x_n + (1 - \beta_1^{(n)})T_1(PT_1)^{n-1}x_n) - p\| \\
&\le \alpha_1^{(n)}\|f(x_n) - p\| + (1 - \alpha_1^{(n)})\beta_1^{(n)}\|x_n - p\| + (1 - \alpha_1^{(n)})(1 - \beta_1^{(n)}) \\
&\quad \|T_1(PT_1)^{n-1}x_n - p\| \\
&\le \alpha_1^{(n)}\alpha\|x_n - p\| + (1 - \alpha_1^{(n)})\beta_1^{(n)}\|x_n - p\| + (1 - \alpha_1^{(n)})(1 - \beta_1^{(n)})\|x_n - p\| \\
&\quad + (1 - \alpha_1^{(n)})(1 - \beta_1^{(n)})h_1^{(n)}\|x_n - p\| + \alpha_1^{(n)}\|f(p) - p\| \\
&\le (1 - (1 - \alpha)\alpha_1^{(n)} + h_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\le (1 + h_n)\|x_n - p\| + \alpha_n\|f(p) - p\|
\end{aligned}$$

Assume that $\|y_l^{(n)} - p\| \leq (1+h_n)^l \|x_n - p\| + \sum_{i=0}^{l-1} (1+h_n)^i \alpha_n \|f(p) - p\|$ holds for some $1 \leq l \leq k-2$. Then

$$\begin{aligned}
\|y_{(l+1)}^{(n)} - p\| &= \|P[\alpha_{(l+1)}^{(n)} f(y_l^{(n)}) + (1-\alpha_{(l+1)}^{(n)})(\beta_{(l+1)}^{(n)} y_l^{(n)} \\
&\quad +(1-\beta_{(l+1)}^{(n)}) T_{l+1}(PT_{l+1})^{n-1} y_l^{(n)})] - P_p\| \\
&\leq \|\alpha_{(l+1)}^{(n)} f(y_l^{(n)}) + (1-\alpha_{(l+1)}^{(n)})(\beta_{(l+1)}^{(n)} y_l^{(n)} \\
&\quad +(1-\beta_{(l+1)}^{(n)}) T_{(l+1)}(PT_{(l+1)})^{n-1} y_l^{(n)}) - p\| \\
&\leq \alpha_{(l+1)}^{(n)} \|f(y_l^{(n)}) - p\| + (1-\alpha_{(l+1)}^{(n)}) \beta_{(l+1)}^{(n)} \|y_l^{(n)} - p\| \\
&\quad +(1-\alpha_{(l+1)}^{(n)})(1-\beta_{(l+1)}^{(n)}) \|T_{(l+1)}(PT_{(l+1)})^{n-1} y_l^{(n)} - p\| \\
&\leq \alpha_{(l+1)}^{(n)} \alpha \|y_l^{(n)} - p\| + (1-\alpha_{(l+1)}^{(n)}) \beta_{(l+1)}^{(n)} \|y_l^{(n)} - p\| \\
&\quad +(1-\alpha_{(l+1)}^{(n)})(1-\beta_{(l+1)}^{(n)}) \|y_l^{(n)} - p\| + (1-\alpha_{(l+1)}^{(n)})(1-\beta_{(l+1)}^{(n)}) h_{(l+1)}^{(n)} \\
&\quad \|y_l^{(n)} - p\| + \alpha_{(l+1)}^{(n)} \|f(p) - p\| \\
&\leq (1-(1-\alpha)\alpha_{(l+1)}^{(n)} + h_n) \|y_l^{(n)} - p\| + \alpha_n \|f(p) - p\| \\
&\leq (1+h_n)(1+h_n)^l \|x_n - p\| + (1+h_n) \sum_{i=0}^{l-1} (1+h_n)^i \alpha_n \|f(p) - p\| \\
&\quad +\alpha_n \|f(p) - p\| \\
&= (1+h_n)^{l+1} \|x_n - p\| + \sum_{i=0}^{l} (1+h_n)^i \alpha_n \|f(p) - p\|
\end{aligned}$$

Thus, by induction, we have

$$\|y_i^{(n)} - p\| \leq (1+h_n)^i \|x_n - p\| + \sum_{j=0}^{i-1} (1+h_n)^j \alpha_n \|f(p) - p\|, \tag{4.16}$$

for all $i = 1, 2, 3, ..., k-1$. Now, by (4.16), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|P[\alpha_k^{(n)} f(y_{(k-1)}^{(n)}) + (1-\alpha_k^{(n)})(\beta_k^{(n)} y_{(k-1)}^{(n)} \\
&\quad +(1-\beta_k^{(n)}) T_k(PT_k)^{n-1} y_{(k-1)}^{(n)})] - P_p\| \\
&\leq \|\alpha_k^{(n)} f(y_{(k-1)}^{(n)}) + (1-\alpha_k^{(n)})(\beta_k^{(n)} y_{(k-1)}^{(n)} \\
&\quad +(1-\beta_k^{(n)}) T_k(PT_k)^{n-1} y_{(k-1)}^{(n)}) - p\| \\
&\leq \alpha_k^{(n)} \|f(y_{(k-1)}^{(n)}) - p\| + (1-\alpha_k^{(n)})(\beta_k^{(n)} \|y_{(k-1)}^{(n)} - p\|
\end{aligned}$$

$$
\begin{aligned}
& +(1-\alpha_k^{(n)})(1-\beta_k^{(n)})\|T_k(PT_k)^{n-1}y_{(k-1)}^{(n)})-p\| \\
\le\ & \alpha_k^{(n)}\alpha\|f(y_{(k-1)}^{(n)})-p\|+(1-\alpha_k^{(n)})(\beta_k^{(n)}\|y_{(k-1)}^{(n)}-p\| \\
& +(1-\alpha_k^{(n)})(1-\beta_k^{(n)}\|y_{(k-1)}^{(n)}-p\|+(1-\alpha_k^{(n)})(1-\beta_k^{(n)}h_k^{(n)}\|y_{(k-1)}^{(n)}-p\| \\
& +\alpha_k^{(n)}\|f(p)-p\| \\
\le\ & (1-(1-\alpha)\alpha_k^{(n)}+h_n)\|y_{(k-1)^{(n)}}-p\|+\alpha_n\|f(p)-p\| \\
\le\ & (1+h_n)(1+h_n)^{k-1}\|x_n-p\|+(1+h_n)\sum_{i=0}^{k-2}(1+h_n)^i\alpha_n\|f(p)-p\| \\
& +\alpha_n\|f(p)-p\| \\
=\ & (1+h_n)^k\|x_n-p\|+\sum_{i=0}^{k-1}(1+h_n)^i\alpha_n\|f(p)-p\|. \qquad (4.17)
\end{aligned}
$$

Let $s_n=(1+h_n)^{k-1}+(1+h_n)^{k-2}+\ldots+(1+h_n)+1$. Since $\sum_{n=1}^{\infty}h_n<\infty$, the sequence $\{h_n\}$ converges to 0 and hence there exists a constant $n_0>0$ such that $0\le h_n<1$ for all $n\ge n_0$. Then for any $n\ge n_0$,

$$
\begin{aligned}
s_n &= (1+h_n)^{k-1}+(1+h_n)^{k-2}+\cdots+(1+h_n)+1 \\
&= \frac{(1+h_n)^k-1}{h_n} \\
&= \frac{1+\binom{k}{1}h_n+\binom{k}{2}h_n^2+\cdots+\binom{k}{k}h_k-1}{h_n} \\
&= \binom{k}{1}+\binom{k}{2}h_n+\binom{k}{3}h_n^2+\cdots+\binom{k}{k}h_n^{k-1} \\
&\le \binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k};\quad \text{since}\quad 0\le h_n<1 \\
&= 2^k-1.
\end{aligned}
$$

Then there exists a positive constant C such that $s_n\le C$ for all $n\ge 1$. Now, we can rewrite (4.17) as

$$\|x_{n+1}-p\|\le(1+t_n)\|x_n-p\|+M\alpha_n \qquad (4.18)$$

where $t_n=(1+h_n)^k-1$ and $M=C\|f(p)-p\|$. Since $\sum_{n=1}^{\infty}h_n<\infty$, then $\sum_{n=1}^{\infty}t_n<\infty$. Lemma 2.24 implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. Thus $\|x_n-p\|$ is bounded. Let $L=\sup_{n\ge1}\|x_n-p\|$. We can rewrite (4.18) as

$$\|x_{n+1}-p\|\le\|x_n-p\|+Lt_n+M\alpha_n \quad \text{for} \quad n\ge 1. \qquad (4.19)$$

Now, for any positive integers $m, n \geq 1$, $p \in F$ and induction, we have

$$\|x_{n+m} - p\| \leq \|x_n - p\| + L \sum_{i=n}^{n+m-1} t_i + M \sum_{i=n}^{n+m-1} \alpha_i. \tag{4.20}$$

By (4.19) and taking infimum over $p \in F$, we obtain

$$d(x_{n+1}, F) \leq d(x_n, F) + Lt_n + M\alpha_n.$$

The assumption $\liminf_{n\to\infty} d(x_n, F) = 0$ implies that there exists a subsequence of $\{d(x_n, F)\}$ converging to zero. This result together with the fact $\sum_{n=1}^{\infty}(Lt_n + u_n) < \infty$, and Lemma 2.24, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{4.21}$$

We claim that $\{x_n\}$ is Cauchy in X. Let $\epsilon > 0$ be given. From (4.21), $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, there exists n_0 such that for $n \geq n_0$, we get

$$d(x_n, F) < \frac{\epsilon}{6}, \quad \sum_{i=n_0}^{\infty} t_i < \frac{\epsilon}{3(L+1)} \quad \text{and} \quad \sum_{i=n}^{\infty} \alpha_i < \frac{\epsilon}{3}. \tag{4.22}$$

The first inequality of (4.22) and the definition of infimum, there exists $z_1 \in F$ such that

$$\|x_{n_0} - z_1\| < \frac{\epsilon}{6}. \tag{4.23}$$

Combining (4.20), (4.22) and (4.23), we have

$$\begin{aligned}
\|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - z_1\| + \|x_{n_0} - z_1\| \\
&\leq 2\|x_{n_0} - z_1\| + L \sum_{i=n}^{n_0+m-1} t_i + M \sum_{i=n}^{n_0+m-1} \alpha_i \\
&\leq 2\|x_{n_0} - z_1\| + L \sum_{i=n_0}^{\infty} t_i + M \sum_{i=n_0}^{\infty} \alpha_i \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence in X. But X is a Banach space, so there must be some $q \in X$ such that $x_n \to q$. Since C is closed and $\{x_n\}$ is a sequence in C, we have that $q \in C$. Since $\emptyset \neq F \subseteq C$ and $x_n \to q$ by Lemma 2.28, we have

$$0 = \lim_{n\to\infty} d(x_n, F) = d(q, F).$$

From this and since F is closed, so $q \in F$ by Lemma 2.26. Therefore $\{x_n\}$ converges to a common fixed point of $\{T_i, i = 1, 2, 3, \ldots, k\}$, as desired. □

Corollary 4.7. *Let $X, C, T_i(i = 1, 2, 3, \ldots, k)$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.6. Suppose that conditions in Theorem 4.6 hold and*

(i) the mapping $T_i(i = 1, 2, 3, \ldots, k)$ is asymptotically regular in x_n, that is

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \ \ i = 1, 2, 3, \ldots, k;$$

(ii) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ implies that $\liminf_{n\to\infty} d(x_n, \ F) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i, i = 1, 2, 3, ..., k\}$.

Theorem 4.8. *Let $X, C, \{T_i, i = 1, 2, 3, \ldots, k\}$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.6. Suppose that conditions in Theorem 4.6 hold. Assume further that the mapping $\{T_i, i = 1, 2, 3, \ldots, k\}$ is an asymptotically regular and satisfies condition $(\overline{A})$, then $\{x_n\}$ converges strongly to common fixed point of the family of mappings.*

Proof. Since $\{T_i, i = 1, 2, 3, ..., k\}$ satisfies condition $(\overline{A})$, there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{k} \sum_{i=1}^{k} \|x_n - T_i x_n\| \geq f(d(x_n, F)),$$

for all $n \geq 1$. Since each T_i is asymptotically regular in x_n for $i = 1, 2, 3, \ldots, k$,

$$\liminf_{n\to\infty} f(d(x_n, F)) \leq 0.$$

Since $f : [0, \infty) \to [0, \infty)$, we have that

$$\liminf_{n\to\infty} f(d(x_n, F)) = 0 \tag{4.24}$$

We claim that $\liminf_{n\to\infty} d(x_n, F) = 0$. We prove this by contradiction, assume that

$$\liminf_{n\to\infty} d(x_n, F) \neq 0.$$

From this and $f : [0, \infty) \to [0, \infty)$, we have

$$\liminf_{n\to\infty} d(x_n, F) = L > 0.$$

Since $\liminf_{n\to\infty} d(x_n, F) = L > 0$, for all $\epsilon = L > 0$, there exists $N_1 \in \mathbb{N}$ such that $N > N_1$ implies

$$|\inf_{n\geq N} d(x_n,\ F) - L| < \frac{L}{k}$$

From this we get

$$\frac{(k-1)L}{k} < \inf_{n\geq N} d(x_n, F) < \frac{(k+1)L}{k}, \quad \text{for all} \quad N > N_1,$$

That is

$$\frac{(k-1)L}{k} < d(x_n,\ F), \quad \text{for all} \quad n \geq N > N_1.$$

Since f is nondecreasing,

$$f(\frac{(k-1)L}{k}) \leq f(d(x_n,\ F)), \quad \text{for all} \quad n \geq N > N_1.$$

We get

$$\begin{aligned} f(\frac{(k-1)L}{k}) &\leq \inf f(d(x_n,\ F)), \quad \text{for all} \quad N > N_1 \\ &\leq \lim_{N\to\infty} \inf\{f(d(x_n,\ F)) \ \ ; n \geq N\} \\ &= \liminf_{n\to\infty} f(d(x_n,\ F)). \end{aligned}$$

Since $f(t) > 0$ if $t > 0$, we have

$$0 < f(\frac{(k-1)L}{k}) \leq \liminf_{n\to\infty}\{f(d(x_n, F)),$$

which contradicts (4.24). Hence $\liminf_{n\to\infty} d(x_n, F) = 0$. We see that $\{x_n\}$ converges strongly to a common fixed point of $\{T_i, i = 1, 2, 3, \ldots, k\}$, by Theorem 4.6, as desired. □

4.2 Convergence Theorems in Uniformly Convex Banach Spaces

At the beginning of the section, we restate some results in section 4.1 by using Theorem 3.8 in Chapter 3 and then establish some weak and strong convergence theorems for the iterative scheme (4.15) for a finite family of asymptotically quasi-nonexpasive nonself mapping from C to X by removing the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ from theorems obtained in section 4.1.

Now we restate some results in section 4.1 by using Theorem 3.8 in Chapter 3 in the uniformly convex Banach space.

Corollary 4.9. *Let X be a uniformly convex real Banach space, and let C be a nonempty closed convex bounded subset of X and suppose that a retraction map $P : X \to C$ is nonexpansive. For $i = 1, 2, 3$, let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself-mapping with respect to $\{h_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = \max\{h_1^{(n)}, h_2^{(n)}, h_3^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$ and $\{g_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} e_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined in (4.1) converges strongly to a common fixed point of T_1, T_2 and T_3 if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T_1) \cap F(T_2) \cap F(T_3)) = 0.$$

Corollary 4.10. *Let $X, C, T_i (i = 1, 2, 3)$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.9. Suppose that conditions in Theorem 4.9 hold and*

(i) the mapping $T_i (i = 1, 2, 3)$ is asymptotically regular in x_n, i.e.,

$$\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0, \ i = 1, 2, 3;$$

(ii) $\liminf_{n\to\infty} \|x_n - T_i x_n\| = 0$ implies that $\liminf_{n\to\infty} d(x_n,\ F(T_1) \cap F(T_2) \cap F(T_3)) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Corollary 4.11. *Let $X,\ C,\ T_i(i=1,\ 2,\ 3)$ and the iterative sequence $\{x_n\}$ be as in Theorem 4.9. Suppose that conditions in Theorem 4.9 hold. Assume further that the mapping $T_i(i=1,2,3)$ is asymptotically regular in x_n and satisfies condition $(\overline{A})$ Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Corollary 4.12. *Let X be a uniformly convex real Banach space, and let C be a nonempty closed convex bounded subset of X and suppose that a retraction map $P : X \to C$ is nonexpansive. For $i = 1, 2, 3, ..., k$, let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ such that $F = \cap_{i=1}^k F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = \max_{1\le i\le k}\{h_i^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_i^{(n)}\}$ and $\{\beta_i^n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_i^{(n)} < \infty$ for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, 3, ..., k$ Then the iterative sequence $\{x_n\}$ defined in (4.15) converges strongly to a common fixed point of $\{T_i : i = 1, 2, 3, ..., k\}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Corollary 4.13. *Let X be a uniformly convex real Banach space, and let C be a nonempty closed convex bounded subset of X and suppose that a retraction map $P : X \to C$ is nonexpansive. For $i = 1, 2, 3, ..., k$, let $T_i : C \to X$ be an asymptotically quasi-nonexpansive nonself mapping with respect to $\{h_i^{(n)}\}$ such that $F = \cap_{i=1}^k F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = \max_{1\le i\le k}\{h_i^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_i^{(n)}\}$ and $\{\beta_i^{(n)}\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_i^{(n)} < \infty$ for all $n = 1, 2, 3, ...$ and $i = 1, 2, 3, ..., k$ and the iterative sequence $\{x_n\}$ defined in (4.15). If $\{T_i : i = 1, 2, 3, ..., k\}$ is asymptotically regular and satisfies condition $(\overline{A})$, then $\{x_n\}$ converges strongly to common fixed point of the family of mappings.*

Now, we let X be a real Banach space, and C be a nonempty closed convex bounded subset of X. For each $i = 1, 2, 3, ..., k$, we let T_i be an asymptotically quasi-nonexpasive nonself mapping from C to X with respect to

$\{h_i^{(n)}\}$ such that $\sum_{n=1}^{\infty} h_i^{(n)} < \infty$. Let F denotes the set of common fixed points of $\{T_i : i = 1, 2, 3, ..., k\}$ and assumes that $F \neq \emptyset$. Let $\{\alpha_i^{(n)}\}$ and $\{\beta_i^{(n)}\}$ be sequences in $[0, 1]$ and $\sum_{n=1}^{\infty} \alpha_i^{(n)} < \infty$ and let x_1 be arbitrary element in C and $\{x_n\}$ be the sequence defined in (4.15) . In order to prove our theorems, we need the following lemma:

Lemma 4.14. *Let C be a nonempty closed and convex subset of uniformly convex Banach space X and $\{T_i, i = 1, 2, 3, ..., k\}$ a finite family of asymptotically quasi-nonexpansive nonself mapping from C to X with respect to $\{h_i^{(n)}\}$ such that $\sum_{n=1}^{\infty} h_i^{(n)} < \infty$ for all $i = 1, 2, 3, ..., k$. Let $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and assumes that $F \neq \emptyset$ and let x_1 be arbitrary element in C and $\{x_n\}$ be the sequence defined in* (4.15)*, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F, h_n = \max_{1\leq i\leq k}\{h_i^{(n)}\}$ and $\alpha_n = \max_{1\leq i\leq k}\{\alpha_i^{(n)}\}$ for all n. By proof of Theorem 4.6 and Lemma 2.24, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.

□

Theorem 4.15. *Let C be a nonempty closed and convex subset of uniformly convex Banach space X and $\{T_i, i = 1, 2, 3, ..., k\}$ a finite family of asymptotically quasi-nonexpansive nonself mapping from C to X with respect to $\{h_i^{(n)}\}$ such that $\sum_{n=1}^{\infty} h_i^{(n)} < \infty$ for all $i = 1, 2, 3, ..., k$. Assumes that $F \neq \emptyset$ and let x_1 be arbitrary element in C and $\{x_n\}$ be the sequence defined in* (4.15)*,and $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If T_j is completely continuous for some $j = 1, 2, 3, \ldots, k$, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, 3, ..., k$ and $I - T_i$ is demiclosed at zero for all $i = 1, 2, 3, \ldots, k$, then $\{x_n\}$ converges strongly to a common point of $\{T_i; i = 1, 2, 3, ..., k\}$.*

Proof. Let $p \in F$, then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 4.14 and hence $\{x_n\}$ is bounded. By assumption, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for each $i = 1, 2, 3, \ldots, k$, we have that $\{T_i x_n\}$ is bounded for each $i = 1, 2, 3, \ldots, k$. Assume without loss of generality that T_1 is completely continuous. Then there exists an element $q \in C$

and a subsequence $\{T_1 x_{n_j}\}$ such that $\|T_1 x_{n_j} - q\| \to 0$ as $j \to \infty$. Since

$$\|x_{n_j} - q\| \leq \|x_{n_j} - T_1 x_{n_j}\| + \|T_1 x_{n_j} - q\|,$$

we have $\lim_{j\to\infty} \|x_{n_j} - q\| = 0$. Since each $I - T_i$ is demiclosed at zero for each $i = 1, 2, 3, \ldots, k$, so we have that $(I - T_i)q = 0$, that is $T_i q = q$. Thus $q \in F$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists and hence equal to zero. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i; i = 1, 2, 3, ..., k\}$. □

Theorem 4.16. *Let C be a nonempty closed and convex subset of uniformly convex Banach space X and $\{T_i, i = 1, 2, 3, ..., k\}$ a finite family of asymptotically quasi-nonexpansive nonself mapping from C to X with respect to $\{h_i^{(n)}\}$ such that $\sum_{n=1}^{\infty} h_i^{(n)} < \infty$ for all $i = 1, 2, 3, ..., k$. Assumes that $F \neq \emptyset$ and let x_1 be arbitrary element in C and $\{x_n\}$ be the sequence defined in* (4.15) *and $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If T_j is demicompact for some $j = 1, 2, 3, \ldots, k$, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, 3, ..., k$ and $I - T_i$ is demiclosed at zero for all $i = 1, 2, 3, \ldots, k$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i; i = 1, 2, 3, ..., k\}$.*

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 4.14 and hence $\{x_n\}$ is bounded. Assume without loss of generality that T_1 is demicompact. Then there exists an element $q \in C$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$. By assumption, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, and $I - T_i$ is demiclosed at zero for all $i = 1, 2, 3, \ldots, k$, we have $(I - T_i)q = 0$, that is, $T_i q = q$. Thus $q \in F$. By Lemma 4.14, $\{x_n\}$ converges strongly to q, a common fixed point of $\{T_i; i = 1, 2, 3, ..., k\}$. □

Theorem 4.17. *Let C be a nonempty closed and convex subset of uniformly convex Banach space X and $\{T_i, i = 1, 2, 3, ..., k\}$ a finite family of asymptotically quasi-nonexpansive nonself mapping from C to X with respect to $\{h_i^{(n)}\}$ such that $\sum_{n=1}^{\infty} h_i^{(n)} < \infty$ for all $i = 1, 2, 3, ..., k$. Assumes that $F \neq \emptyset$ and let x_1 be arbitrary element in C and $\{x_n\}$ be the sequence defined in* (4.15) *and $\{\alpha_n\} \subset [\delta, 1 - \delta]$*

for some $\delta \in (0,1)$*. If* X *satisfies Opial's property,* $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for all* $i = 1,2,3,...,k$ *and* $I - T_i$ *is demiclosed at zero for all* $i = 1,2,3,\ldots,k$*, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i; i = 1,2,3,...,k\}$*.*

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 4.14 and hence $\{x_n\}$ is bounded. By Theorem 3.3 and Theorem 2.19, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $q \in C$. Since $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $I - T_i$ is demiclosed at zero for all $i = 1,2,...,k$, so we have $T_i q = q$. Thus $q \in F$.To complete the proof, let $\{x_{n_k}\}$ be another sequence of $\{x_n\}$ that converges to weakly to some $r \in C$. Similarly proof as above, we can prove that $r \in F$. By Lemma 2.25, $q = r$. Therefore $\{x_n\}$ converges weakly to a common fixed point in F. $\square$

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