where $C_k^{ij}$'s are the Clebsch-Gordan coefficients (CGCs). The CGCs are the purely geometrical factors that describe how the other irreps are made from the weight products.

3. The explicit calculation of the SO(9) tensor products

In order to generate the SO(9) irreps by means of an explicit calculation from tensor products of 9-vector and 16-spinor irreps, the following approach is effective, and can also be used to obtain similar tensor products of more than two irreps.

3.1. Vector-vector product

The tensor product of two 9-vector irreps can be decomposed in terms of dimensions as

$$9 \otimes 9 = 44 \oplus 36 \oplus 1,$$

or in terms of Dynkin labels as

$$(1000) \otimes (1000) = (2000) \oplus (0100) \oplus (0000).$$

Firstly, one needs to consider the (2000) subspace. Acting on the top level (the eighth level) of the highest weight in the vector-vector product, $\zeta_1 = \xi_1 \eta_1$, by a series of four negative simple root generators, $E_r^- \equiv E_{\zeta r}^- = I \otimes E_{\eta r}^- + E_{\xi r}^- \otimes I$, the lower levels of weight vectors are obtained:

At the seventh level:

$$\zeta_7 = \frac{1}{\sqrt{2}} E_1^- \zeta_1 = \frac{1}{\sqrt{2}} \left( \xi_1 (E_{\eta 1}^- \eta_1) + (E_{\xi 1}^- \xi_1) \eta_1 \right)$$

$$E_2^- \zeta_1 = E_3^- \zeta_1 = E_4^- \zeta_1 = 0.$$  \hspace{1cm} (12a)

At the sixth level:

$$\zeta_5 \equiv E_2^- \zeta_1 = \frac{1}{\sqrt{2}} (\xi_1 \eta_3 + \xi_3 \eta_1),$$

$$\zeta_4 \equiv \frac{1}{\sqrt{2}} E_1^- \zeta_2 = \xi_2 \eta_2,$$

$$E_3^- \zeta_2 = E_4^- \zeta_2 = 0.$$  \hspace{1cm} (12c)

At the fifth level:

$$\zeta_6 \equiv E_3^- \zeta_4 = \frac{1}{\sqrt{2}} \left( \xi_1 \eta_4 + \xi_4 \eta_1 \right),$$

$$\zeta_5 \equiv E_1^- \zeta_3 = \frac{1}{\sqrt{2}} E_2^- \zeta_4 = \frac{1}{\sqrt{2}} (\xi_2 \eta_3 + \xi_3 \eta_2),$$

$$E_2^- \zeta_3 = E_4^- \zeta_3 = E_1^- \zeta_4 = E_2^- \zeta_4 = E_3^- \zeta_4 = 0.$$  \hspace{1cm} (12f)

At the fourth level:

$$\zeta_7 \equiv E_4^- \zeta_5 = \frac{1}{\sqrt{2}} (\xi_1 \eta_0 + \xi_0 \eta_1),$$

$$\zeta_8 \equiv E_1^- \zeta_6 = E_3^- \zeta_6 = \frac{1}{\sqrt{2}} (\xi_2 \eta_4 + \xi_4 \eta_2),$$

$$\zeta_9 \equiv \frac{1}{\sqrt{2}} E_2^- \zeta_6 = \xi_3 \eta_3,$$  \hspace{1cm} (12h)
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\[ E_2^- \zeta_5 = E_3^- \zeta_5 = E_4^- \zeta_5 = E_4^- \zeta_6 = 0. \]  
(12d)

At the zero level:

\[ \zeta_{21} \equiv E_1^- E_2^- E_3^- E_4^- \zeta_7 = \frac{1}{2} (\xi_1 \eta_{-1} + \xi_2 \eta_{-2} + \xi_3 \eta_{-3} + \xi_4 \eta_{-4}) , \]  
(12m)

\[ \zeta_{22} \equiv \frac{1}{\sqrt{2}} E_2^- E_1^- E_3^- E_4^- \zeta_7 = \frac{1}{2} (\xi_2 \eta_{-2} + \xi_3 \eta_{-3} + \xi_4 \eta_{-4} + \xi_5 \eta_{-5}) , \]  
(12n)

\[ \zeta_{23} \equiv \frac{1}{\sqrt{2}} E_3^- E_2^- E_1^- E_4^- \zeta_7 = \frac{1}{2} (\xi_3 \eta_{-3} + \xi_4 \eta_{-4} + \xi_5 \eta_{-5} + \xi_6 \eta_{-6}) , \]  
(12o)

\[ \zeta_{24} \equiv \frac{1}{\sqrt{3}} E_4^- E_3^- E_2^- E_1^- \zeta_7 = \frac{1}{\sqrt{6}} (\xi_4 \eta_{-4} + 2 \xi_5 \eta_{-5} + \xi_6 \eta_{-6}) , \]  
(12p)

The weight elements of $\zeta_i$ are found by letting $H \zeta \equiv I \otimes H_\eta + H_\xi \otimes I$ act directly on the right hand side of $\zeta_i$.

Note that $\zeta_7$ at the fourth level is the non-degenerate dominant weight $|1000>\.$ and $\zeta_{21,22,23,24}$ at the zero level are the degenerated dominant weights $|0000>$. An upper half of the action series of the negative simple root generators in the (2000) subspace is summarized and shown as the upper-half weight diagram in Fig. 2.

![Figure 2. The upper-half weight diagram for the 44-dimensional irrep.](image)

In the linear combinations of weight vectors of the vector-vector product, there are 81 $(= 9^2)$ CGCs in the (2000), (0100), and (0000) subspaces. As can be seen from the above computation in the (2000) subspace, many of CGCs are zero and the rest has only a few distinct non zero values. Therefore, it suffices to compute only the dominant weights in the tensor product [3, 4, 5].

For the (0100) subspace, its highest weight $\zeta_{1}' \equiv |0100>$ at the seventh level is orthonormal to $\zeta_2$. Up to an overall phase, one can choose it to be

\[ \zeta_{1}' \equiv \frac{1}{\sqrt{2}} (\xi_1 \eta_2 - \xi_2 \eta_1) . \]  
(13a)

For the remaining dominant weight vectors in the (0100) subspace, one obtains at the fourth level:

\[ \zeta_{6}' \equiv E_4^- E_3^- E_2^- \zeta_1' = \frac{1}{\sqrt{2}} (\xi_1 \eta_0 - \xi_0 \eta_1) . \]  
(13b)
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and at the zero level:

\[ \zeta_{17} \equiv E_7^- E_2^- E_5^- E_4^- \zeta_5' = \frac{1}{2} (\xi_1 \eta_{-1} + \xi_2 \eta_{-2} - \xi_3 \eta_3 - \xi_4 \eta_4) \, , \quad (13c) \]

\[ \zeta_{18} \equiv \frac{1}{\sqrt{2}} E_7^- E_1^- E_3^- E_4^- \zeta_5' = \frac{1}{2} (\xi_2 \eta_{-2} + \xi_3 \eta_{-3} - \xi_4 \eta_3 - \xi_4 \eta_4) \, , \quad (13d) \]

\[ \zeta_{19} \equiv \frac{1}{\sqrt{2}} E_7^- E_2^- E_1^- E_4^- \zeta_5' = \frac{1}{2} (\xi_3 \eta_{-3} + \xi_4 \eta_{-4} - \xi_4 \eta_3 - \xi_4 \eta_4) \, , \quad (13e) \]

\[ \zeta_{20} \equiv E_4^- E_3^- E_7^- E_1^- \zeta_5' = \frac{1}{\sqrt{2}} (\xi_4 \eta_{-4} - \xi_4 \eta_4) \, . \quad (13f) \]

Note that \( \zeta_5' \equiv |1000> \) in the (0100) subspace is orthonormal to \( \zeta_7 \equiv |1000> \) in the (2000) subspace. By removing \( \zeta_1, \zeta_4, \zeta_5, \) and \( \zeta_{16} \) from Fig. 2 and relabeling its weight vectors, one obtains the upper-half weight diagram of (0100) subspace.

For the (0000) subspace, its highest weight \( \zeta'' \equiv |0000> \) at the zero level is orthonormal to \( \zeta_{21,22,23,24} \) and to \( \zeta_{17,18,19,20} \). Up to an overall phase, the weight vector \( \zeta'' \) is

\[ \zeta'' \equiv \frac{1}{\sqrt{2}} (\xi_1 \eta_{-1} - \xi_2 \eta_{-2} + \xi_3 \eta_{-3} - \xi_4 \eta_{-4} + \xi_0 \eta_0 - \xi_4 \eta_4 + \xi_3 \eta_3 - \xi_2 \eta_2 + \xi_1 \eta_1) \, . \quad (14) \]

Note that the weights in \( \zeta \) and \( \zeta'' \) are symmetric under the interchange between \( \xi \) and \( \eta \), whereas the weights in \( \zeta' \) are antisymmetric.

3.2. Vector-spinor product

The \( 9 \otimes 16 \) can be decomposed as

\[ 9 \otimes 16 = 128 \oplus 16, \quad (15a) \]

or in terms of Dynkin label

\[ (\xi) \otimes (\psi) = (\xi') \oplus (\zeta'') \, . \quad (15b) \]

Acting on the top level (the ninth level) of the highest weight \( \zeta_1 = \xi_1 \psi_1(o) \) by a series of those four simple root generators, one gets the four-fold degenerate dominant weights \( |0001> \) at the fifth level,

\[ \zeta_{12} \equiv \frac{1}{\sqrt{2}} E_7^- E_2^- E_3^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}} (\xi_1 \psi_{-4(o)} + \xi_2 \psi_{3(o)}) \, , \quad (16a) \]

\[ \zeta_{13} \equiv \frac{1}{\sqrt{2}} E_7^- E_1^- E_3^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}} (\xi_2 \psi_{3(o)} + \xi_3 \psi_{2(o)}) \, , \quad (16b) \]

\[ \zeta_{14} \equiv \frac{1}{\sqrt{2}} E_7^- E_2^- E_1^- E_4^- \zeta_1 = \frac{1}{\sqrt{2}} (\xi_3 \psi_{2(o)} + \xi_4 \psi_{1(o)}) \, , \quad (16c) \]

\[ \zeta_{15} \equiv \frac{1}{\sqrt{2}} E_7^- E_3^- E_2^- E_1^- \zeta_1 = \frac{1}{\sqrt{2}} (\xi_4 \psi_{1(o)} + \xi_0 \psi_{1(o)}) \, . \quad (16d) \]

These are the only dominant weights in the (1001) subspace.

For the (0001) subspace, its highest weight \( \zeta_1' \) at the fifth level is orthonormal to \( \zeta_{12,13,14,15} \). Up to an overall phase, it is

\[ \zeta_1' = \frac{1}{\sqrt{2}} (\xi_1 \psi_{-4(o)} - \xi_2 \psi_{3(o)} + \xi_3 \psi_{2(o)} - \xi_4 \psi_{1(o)} + \xi_0 \psi_{1(o)}) \, . \quad (17) \]
3.3. Spinor-spinor product

The $16 \otimes 16$ can be decomposed as

$$16 \otimes 16 = 126 \oplus 84 \oplus 36 \oplus 9 \oplus 1,$$

or in terms of Dynkin labels as

$$(0001) \otimes (0001) = (0002) \oplus (0010) \oplus (0100) \oplus (1000) \oplus (0000).$$

The dominant weights in the spinor-spinor product can be computed explicitly as above. Their results for the top level in each subspace are as follow:

For the (0002) subspace:

$$\zeta_1 = \psi_1(e) \chi_1(e).$$

For the (0010) subspace:

$$\zeta'_1 = \frac{1}{\sqrt{2}}(\psi_1(e) \chi_1(o) - \psi_1(o) \chi_1(e)).$$

For the (0100) subspace:

$$\zeta''_1 = \frac{1}{2}(\psi_1(e) \chi_2(e) - \psi_2(e) \chi_1(e) + \psi_2(o) \chi_1(o) - \psi_1(o) \chi_2(o)).$$

For the (1000) subspace:

$$\zeta'''_1 = \frac{1}{2\sqrt{2}}(\psi_1(e) \chi_4(o) + \psi_4(o) \chi_1(e) + \psi_2(o) \chi_3(e) + \psi_3(e) \chi_2(o)$$

$$- \psi_1(o) \chi_4(e) - \psi_4(e) \chi_1(o) - \psi_2(e) \chi_3(o) - \psi_3(o) \chi_2(e)).$$

For the (0000) subspace:

$$\zeta''''_1 = \frac{1}{4}(\psi_1(e) \chi_{-1}(o) + \psi_{-1}(e) \chi_1(e) + \psi_2(e) \chi_{-2}(e) + \psi_{-2}(e) \chi_2(e)$$

$$- \psi_3(e) \chi_{-3}(e) - \psi_{-3}(e) \chi_3(e) - \psi_{-1}(e) \chi_4(e) - \psi_{-4}(e) \chi_4(e)$$

$$- \psi_1(o) \chi_{-1}(o) - \psi_{-1}(o) \chi_1(o) + \psi_2(o) \chi_{-2}(o) + \psi_{-2}(o) \chi_2(o)$$

$$- \psi_3(o) \chi_{-3}(o) - \psi_{-3}(o) \chi_3(o) + \psi_{-1}(o) \chi_4(o) + \psi_{-4}(o) \chi_4(o)).$$

Notice that the (0002), (1000) and (0000) subspaces are symmetric in the interchange between $\psi$ and $\chi$ whereas the (0010) and (0100) subspaces are antisymmetric.

4. The Schwinger’s oscillator realization of a coupled tensor operator

At this stage, the way is cleared for constructing an SO(9) operator for general and special purposes.

In a Lie algebra, a tensor operator $T^\xi$ with a number of components equal to the dimension of $\xi$ is defined through commutation relations of its components with the Lie algebra generators [11],

$$[\hat{H}, T^\xi_\xi] = \xi_\xi T^\xi_\xi,$$

$$[E^\pm_\xi, T^\xi_\xi] = \xi_\xi \pm \alpha_r |E^\pm_\xi, \xi_\xi > T^\xi_\xi \pm \alpha_r.$$

Similarly to the method of generating the other irreps from the tensor product of 9-vector and 16-spinor, a coupled tensor operator can be constructed from the 9-vector and 16-spinor operators. In practice, it is more efficient to realize the tensor operators and their components in terms of Schwinger’s bosonic oscillators [12, 13].