Moore-Penrose Inverses and Normal Elements in Rings

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บทคัดย่อ

กำหนดให้ R เป็นริงภายใต้อินโวลูชัน $*$ แล้วเรากล่าวว่า สมาชิก $a \in R$ สามารถหาตัวผกผันมัวร์-เพนโรสได้ ถ้ามีสมาชิก $b \in R$ ซึ่งสอดคล้องกับสมการต่อไปนี้ $aba = a$, $bab = b$, $(ab)^* = ab$ และ $(ba)^* = ba$ และเราเรียก b ว่า ตัวผกผันมัวร์-เพนโรส ของ a เขียนแทนด้วย $a^\dagger$ (ถ้ามีอยู่จริง)

ในวิทยานิพนธ์ฉบับนี้ เราได้หาเงื่อนไขที่จำเป็นและเพียงพอสำหรับการมีอยู่จริงของตัวผกผันมัวร์-เพนโรสของสมาชิกในริงภายใต้อินโวลูชัน (involution) นอกจากนี้เรายังได้ค้นพบตัวผกผันมัวร์-เพนโรสสำหรับผลคูณของ $x_1x_2x_3\cdots x_n$ ได้ โดยที่ $x_1, x_2, x_3, \ldots, x_n$ เป็นสมาชิกที่สามารถหาตัวผกผันมัวร์-เพนโรสได้

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ABSTRACT

Let R be a ring with involution$*$, then $a \in R$ is a Moore-Penrose invertible element if there is $b \in R$ such that $aba = a$, $bab = b$, $(ab)^* = ab$ and $(ba)^* = ba$. b is called Moore-Penrose inverse of a, denoted by $a^\dagger$ (if it exists).

In this thesis, we study Moore-Penrose inverses and normal elements in ring with involution and give the neccessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in ring with involution. Furthermore, we also investigate the existence of the Moore-Penrose inverse for the product $x_1x_2x_3 \cdots x_n$ where $x_1, x_2, \ldots, x_n$ are Moore-Penrose invertible.

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Wannisa Apairat

CONTENTS

CHAPTER 1

Introduction

The original concept of Moore-Penrose inverse started with the work of E. H. Moore between 1910 and 1920. Moore studied the general reciprocal of any matrix and applied it to solve systems of linear equations [10]. R.Penrose rediscovered it later in 1955 [13]. It is called nowadays the Moore-Penrose inverse. There have been many activities in the study of Moore-Penrose inverse since that day. The study has been extended to complex matrix [1], [2], [3], linear operator on Banach or Hilbert spaces [4], [5], C^*-algebra and also in any rings with involution [8].

This research is motivated by the work of [8], which related the concept of well-suppored element in ring with involution to regularity of the element and the existence of the Moore-Penrose inverse and by the work of [11], which gave the characteriztions of normal and Hermitian elements in rings with involution in purely algebraic terms.

In this thesis, we study the Moore-Penrose invertible elements in rings with involution. We generalize the result of Koliha et al. [8] by giving the necessary and sufficient conditions for an element in a ring with involution to be Moore-Penrose invertible. We also investigate the existence of the Moore-Penrose invertible elements in any ring with involution. For a ring R with involution $*$ and $a \in R$, let $a^\dagger$ denote the Moore-Penrose inverse of a (if it exists) and $a^\#$ denote the group inverse of a (if it exists). We prove that if a is Moore-Penrose invertible and a normal element, i.e. $aa^* = a^*a$, then the product $x_1x_2\cdots x_n$ is always Moore-Penrose invertible for $x_1, x_2, \ldots, x_n \in \{a, a^*, a^\dagger, (a^\dagger)^*\}$. We also prove that if a is an EP element, i.e. a is Moore-Penrose invertible, group invertible and $a^\dagger = a^\#$ or a is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$, then x^n is Moore-Penrose invertible for any $x \in \{a, a^*, a^\dagger, (a^\dagger)^*\}$ and for all $n \in \mathbb{N}$. Finally, we show that if $a \in R^\dagger$, then $(aa^*)^n$, $(a^*a)^n$, $(a^*a^\dagger aa)^n$, $a(a^*a)^n$, and $a^*(aa^*)^n$ are Moore-Penrose invertible for all $n \in \mathbb{N}$.

CHAPTER 2

Preliminaries

We will use the notation and glossary of [8], [9] and [11] in order to introduce the notion and the basic properties of Moore-Penrose inverses and normal elements in rings.

Definition 2.1. [11] Let R be a ring, and let $a \in R$. Then a is group invertible if there is an element $b \in R$ such that

$$aba = a, bab = b, ab = ba;$$

b is a group inverse of a and it is unique, denoted by $a^{\#}$. We use $R^{\#}$ to denote the set of all group invertible elements of R.

Proposition 2.1. *[11] $a^{\#}$ is unique.*

Proof. Assume that b and c are group inverses of a. Then

$$\begin{aligned} b &= bab \\ &= bba \\ &= bbaca \\ &= bbaac \\ &= babac \\ &= bac \\ &= bacac \\ &= abacc \\ &= acc \\ &= cac \\ &= c. \end{aligned}$$

$\square$

Definition 2.2. [9] An involution $a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, (a+b)^* = a^* + b^*, (ab)^* = b^*a^*.$$

Definition 2.3. [8] Let R be a ring with involution $*$, and let $a \in R$. Then a is Moore-Penrose invertible (or MP-invertible) if there is an element $b \in R$ such that

$$aba = a, bab = b, (ba)^* = ba \text{ and } (ab)^* = ab;$$

b is a Moore-Penrose inverse of a and it is unique, denoted by $a^\dagger$. We use $R^\dagger$ to denote the set of all Moore-Penrose invertible elements of R.

Proposition 2.2. *[8] $a^\dagger$ is unique.*

Proof. Assume that b and c are MP-inverses of a. Then

$$\begin{aligned} b &= bab \\ &= bacab \\ &= (ba)^*(ca)^*b \\ &= (caba)^*b \\ &= (ca)^*b \\ &= cab \\ &= cacab \\ &= c(ac)^*(ab)^* \\ &= c(abac)^* \\ &= c(ac)^* \\ &= cac \\ &= c. \end{aligned}$$

□

Definition 2.4. [11] An element $a \in R$ satisfying $aa^* = a^*a$ is called normal.

Definition 2.5. [11] An element $a \in R$ satisfying $a = a^*$ is called Hermitian (or symmetric).

Definition 2.6. [9] An element a of a ring R with involution is said to be EP if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$.

Definition 2.7. [8] An element $a \in R$ is left $*$-cancellable if

$$a^*ax = a^*ay \text{ implies } ax = ay;$$

it is right $*$-cancellable if

$$xaa^* = yaa^* \text{ implies } xa = ya;$$

and it is $*$-cancellable if it is both left and right $*$-cancellable.

Definition 2.8. [8] An element $p \in R$ is a projection if p is both a Hermitian element and an idempotent, that is, $p = p^* = p^2$.

Example 2.1. Let $R = \mathbb{M}_2(\mathbb{Z}_2)$ and let the involution $*$ be the matrix transposition. Then R is a ring with involution $*$. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$. Then

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A.$$

So $AAA = A$ and $AA = AA$. Thus A is group invertible, and $A^{\#} = A$. However, A is not MP-invertible.

Indeed, if A was MP-invertible. Then there is a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$ such that $ABA = A, BAB = B, (AB)^* = AB$, and $(BA)^* = BA$.

We consider $ABA = A$, so

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} a+c & a+c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus $a + c = 1$.

Next, we consider $(BA)^* = BA$, so

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right]^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\left[\begin{pmatrix} a & a \\ c & c \end{pmatrix}\right]^T = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$$
$$\begin{pmatrix} a & c \\ a & c \end{pmatrix} = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$$

Thus $a = c$, and hence $1 = a + c = a + a = 2a = 0 \in \mathbb{Z}_2$, which is a contradiction.

Example 2.2. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_2)$. Then there is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ such that

i) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

ii) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

iii) $\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]^* = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

iv) $\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^* = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Thus A is MP-invertible and $A^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The next table shows the inverses, the group inverses and the Moore-Penrose inverses for all elements in the ring $M_2(\mathbb{Z}_2)$ under involution $*$ as the transposition of a matrix.

Elements in $M_2(\mathbb{Z}_2)$	Inverse	Group Inverse	Moore-Penrose Inverse
$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	-	-	-
$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	A_5	A_5	A_5
$A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	A_3	A_3	A_3
$A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	A_4	A_4	A_4
$A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	A_2	A_2	A_2
$A_6 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	-	A_6	-
$A_7 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	-	A_7	-
$A_8 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	-	A_8	-
$A_9 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	-	A_9	-
$A_{10} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	A_{10}	A_{10}	A_{10}
$A_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	A_{11}	A_{11}	A_{11}
$A_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	-	A_{12}	A_{12}
$A_{13} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	-	-	A_{14}

Elements in $M_2(\mathbb{Z}_2)$	Inverse	Group Inverse	Moore-Penrose Inverse
$A_{14} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	-	-	A_{13}
$A_{15} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	-	A_{15}	A_{15}
$A_{16} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	-	A_{16}	A_{16}

Theorem 2.3. *[11] For any $a \in R^{\dagger}$, the following is satisfied:*

(1) $(a^{\dagger})^{\dagger} = a$;

(2) $(a^*)^{\dagger} = (a^{\dagger})^*$;

(3) $(a^*a)^{\dagger} = a^{\dagger}(a^{\dagger})^*$;

(4) $(aa^*)^{\dagger} = (a^{\dagger})^*a^{\dagger}$;

(5) $a^* = a^{\dagger}aa^* = a^*aa^{\dagger}$;

(6) $a^{\dagger} = (a^*a)^{\dagger}a^* = a^*(aa^*)^{\dagger} = (a^*a)^{\#}a^* = a^*(aa^*)^{\#}$;

(7) $(a^*)^{\dagger} = a(a^*a)^{\dagger} = (aa^*)^{\dagger}a$.

Proof. **(1)** We know that a is the Moore-Penrose inverse of $a^{\dagger}$ and also $(a^{\dagger})^{\dagger}$ is the Moore-Penrose inverse of $a^{\dagger}$. Since the Moore-Penrose inverse of $a^{\dagger}$is unique, $(a^{\dagger})^{\dagger} = a$.

(2) We will show that $(a^{\dagger})^*$ is the Moore-Penrose inverse of a^* by direct computation.

i) $a^*(a^{\dagger})^*a^* = (aa^{\dagger}a)^* = a^*$;

ii) $(a^{\dagger})^*a^*(a^{\dagger})^* = (a^{\dagger}aa^{\dagger})^* = (a^{\dagger})^*$;

iii) $(a^*(a^{\dagger})^*)^* = a^{\dagger}a = (a^{\dagger}a)^*$;

iv) $((a^{\dagger})^*a^*)^* = aa^{\dagger} = (aa^{\dagger})^*$.

Thus $(a^{\dagger})^*$ is the Moore-Penrose inverse of a^* and also $(a^*)^{\dagger}$ is the Moore-Penrose inverse of a^*. By the uniqueness of Moore-Penrose inverse, we get $(a*)^{\dagger} = (a^{\dagger})^*$.

(3) we will show that $a^{\dagger}(a^{\dagger})^*$ is the Moore-Penrose inverse of a^*a by direct computation.

i)

$$\begin{aligned} a^*aa^\dagger(a^\dagger)^*a^*a &= a^*aa^\dagger(aa^\dagger)^*a \\ &= a^*aa^\dagger aa^\dagger a \\ &= a^*a; \end{aligned}$$

ii)

$$\begin{aligned} a^\dagger(a^\dagger)^*a^*aa^\dagger(a^\dagger)^* &= a^\dagger(aa^\dagger)^*aa^\dagger(a^\dagger)^* \\ &= a^\dagger aa^\dagger aa^\dagger(a^\dagger)^* \\ &= a^\dagger(a^\dagger)^*; \end{aligned}$$

iii)

$$\begin{aligned} (a^*aa^\dagger(a^\dagger)^*)^* &= ((a^\dagger)^*)^*(aa^\dagger)^*(a^*)^* \\ &= a^\dagger aa^\dagger a \\ &= a^\dagger a = (a^\dagger a)^* \\ &= a^*(a^\dagger)^*(aa^\dagger a)^*(a^\dagger)^* \\ &= a^*aa^\dagger(a^\dagger)^*; \end{aligned}$$

iv)

$$\begin{aligned} (a^\dagger(a^\dagger)^*a^*a)^* &= (a^\dagger aa^\dagger a)^* \\ &= (a^\dagger a)^* \\ &= a^\dagger a \\ &= a^\dagger aa^\dagger a \\ &= a^\dagger(a^\dagger)^*a^*a. \end{aligned}$$

we conclude that $a^\dagger(a^\dagger)^*$ is the Moore-Penrose inverse of a^*a, so $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$.

(4) The proof is similarly to **(3)**

(5) Since $a \in R^\dagger$, $a = aa^\dagger a$, $(aa^\dagger)^* = aa^\dagger$ and $(a^\dagger a)^* = a^\dagger a$. Then $a^* = (aa^\dagger a)^* = (a^\dagger a)^* a^* = a^\dagger a a^*$.
Similarly, $a^* = (aa^\dagger a)^* = a^*(aa^\dagger)^* = a^* a a^\dagger$.

(6) From part **(3)**, $a^\dagger (a^\dagger)^* = (a^* a)^\dagger$. Then $a^\dagger = a^\dagger a a^\dagger = a^\dagger (aa^\dagger)^* = a^\dagger (a^\dagger)^* a^* = (a^* a)^\dagger a^*$. Similarly, by the part **(4)** we get $a^\dagger = a^*(aa^*)^\dagger$.
For the equalities $a^\dagger = (a^* a)^\# a^* = a^*(aa^*)^\#$, we will prove that $(a^* a)^\# = (a^* a)^\dagger$ and $(aa^*)^\# = (aa^*)^\dagger$.
Since a is Moore-Penrose invertible, $a^* a$ and aa^* are also Moore-Penrose invertible.
From the part **(3)**, $(a^* a)^\dagger = a^\dagger (a^*)^\dagger$. Then
i)

$$\begin{aligned} a^* a a^\dagger (a^\dagger)^* a^* a &= a^* a a^\dagger (aa^\dagger)^* a \\ &= a^* a a^\dagger a a^\dagger a \\ &= a^* a; \end{aligned}$$

ii)

$$\begin{aligned} a^\dagger (a^\dagger)^* a^* a a^\dagger (a^\dagger)^* &= a^\dagger (aa^\dagger)^* a a^\dagger (a^\dagger)^* \\ &= a^\dagger a a^\dagger a a^\dagger (a^\dagger)^* \\ &= a^\dagger (a^\dagger)^*; \end{aligned}$$

iii)

$$\begin{aligned} a^\dagger (a^*)^\dagger a^* a &= a^\dagger (aa^\dagger)^* a \\ &= a^\dagger a a^\dagger a \\ &= a^\dagger a \\ &= (a^\dagger a)^* \\ &= a^* (a^\dagger)^* \\ &= a^* (a^\dagger a a^\dagger)^* \\ &= a^* (aa^\dagger)^* (a^\dagger)^* \\ &= a^* a a^\dagger (a^\dagger)^*. \end{aligned}$$

Then $a^\dagger(a^*)^\dagger = (a^*a)^\dagger$ is the group inverse of (a^*a). By the uniqueness od group inverse, $(a^*a)^\# = (a^*a)^\dagger$.

The proof of $(aa^*)^\# = (aa^*)^\dagger$ can be proved in the similar way as $(a^*a)^\# = (a^*a)^\dagger$.

Hence $a^\dagger = (a^*a)^\dagger a^* = (a^*a)^\# a^* = a^*(aa^*)^\dagger = a^*(aa^*)^\#$.

(7) Taking $*$ to the equation $a^\dagger = (a^*a)^\dagger a^*$, we get $(a^\dagger)^* = ((a^*a)^\dagger a^*)^* = a(a^\dagger(a^\dagger)^*)^* = aa^\dagger(a^\dagger)^* = a(a^*a)^\dagger$. Similarly, we applied $*$ to $a^\dagger = a^*(aa^*)^\dagger$, we get $(a^*)^\dagger = (a^*a)^\dagger a$. □

Theorem 2.4. *[9] Let $a \in R$. Then $a \in R^\dagger$ then a is $*$-cancellable and a^*a is group invertible.*

Proof. Let $a \in R^\dagger$ and suppose that $a^*ax = a^*ay$.Then

$$\begin{aligned} ax = aa^\dagger ax = (aa^\dagger)^* ax \\ &= (a^\dagger)^* a^* ax \\ &= (a^\dagger)^* a^* ay \\ &= (aa^\dagger)^* ay \\ &= aa^\dagger ay \\ &= ay. \end{aligned}$$

Similarly, $xaa^* = yaa^*$ implies $xa = ya$. Hence a is $*$-cancellable.

The MP-inverse of a^*a is obtained by verifying that $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$.

Since $(a^*a)^*(a^*a) = a^*a(a^*a)^*$, a^*a is normal.

Since a^*a is normal, a^*a is EP i.e. $(a^*a)^\# = (a^*a)^\dagger$.

□

Lemma 2.5. *[11] If a is group invertible and $ax = xa$ then $a^\# x = xa^\#$.*

Proof. Let $a \in R^{\#}$ and $x \in R$ be such that $ax = xa$. Then

$$\begin{aligned}
a^{\#}x &= a^{\#}aa^{\#}x \\
&= a^{\#}a^{\#}ax \\
&= a^{\#}a^{\#}xa \\
&= a^{\#}a^{\#}xaa^{\#}a \\
&= a^{\#}a^{\#}axa^{\#}a \\
&= a^{\#}xa^{\#}a \\
&= a^{\#}xaa^{\#} \\
&= a^{\#}axa^{\#} \\
&= a^{\#}axa^{\#}aa^{\#} \\
&= a^{\#}axaa^{\#}a^{\#} \\
&= a^{\#}aaxa^{\#}a^{\#} \\
&= axa^{\#}a^{\#} \\
&= xaa^{\#}a^{\#} \\
&= xa^{\#}.
\end{aligned}$$

□

Lemma 2.6. *Let R be a ring with in volution. If* $a \in R^{\#}$ *then* $a^* \in R^{\#}$ *and* $(a^*)^{\#} = (a^{\#})^*$

Proof. We will show that $(a^{\#})^*$ is the group inverse of a^* by di rect computation.
i) $a^*(a^{\#})^*a^* = (aa^{\#}a)^* = a^*$;
ii) $(a^{\#})^*a^*(a^{\#})^* = (a^{\#}aa^{\#})^* = (a^{\#})^*$;
iii) $a^*(a^{\#})^* = (a^{\#}a)^* = (aa^{\#})^* = (a^{\#})^*a^*$.
Thus the group inverse of a^* is $(a^{\#})^*$ i.e $(a^*)^{\#} = (a^{\#})^*$. □

Theorem 2.7. *[11] An element* $a \in R$ *is EP if and only if* a *is group invertible and* $a^{\#}a$ *is Hermitian.*

Proof. ($\Rightarrow$) Let $a \in R$ such that a is EP. Then $a \in R^{\#} \cap R^{\dagger}$ and $a^{\dagger} = a^{\#}$.Thus

$$(a^{\#}a)^* = (a^{\dagger}a)^* = a^{\dagger}a = a^{\#}a.$$

Hence $a^{\#}a$ is Hermitian.

($\Leftarrow$) Let $a \in R$ such that a is group invertible and $(a^{\#}a)^* = a^{\#}a$. Then $aa^{\#}a = a, a^{\#}aa^{\#} = a^{\#}, a^{\#}a = aa^{\#}$. Consider

$$(aa^{\#})^* = (a^{\#}a)^* = a^{\#}a = aa^{\#}.$$

Thus a is MP-invertible and $a^{\#} = a^{\dagger}$. Hence a is EP. $\square$

Lemma 2.8. *Let $a \in R^{\dagger}$ and $b \in R$. If $ab = ba$ and $a^*b = ba^*$ then $a^{\dagger}b = ba^{\dagger}$.*

Proof. Let $a \in R^{\dagger}$ and $binR$.

Since $a \in R^{\dagger}$, aa^* and a^*a are group invertible elements and by Theorem 2.3, $a^{\dagger} = a^*(aa^*)^{\#}$.

According to Lemma 2.5, aa^* is a group invertible element and $(aa^*)b = b(aa^*)$, we get $(aa^*)^{\#}b = b(aa^*)^{\#}$. Then

$$a^{\dagger}b = a^*(aa^*)^{\#}b = a^*b(aa^*)^{\#} = ba^*(aa^*)^{\#} = ba^{\dagger}.$$

Hence $a^{\dagger}b = ba^{\dagger}$. $\square$

Lemma 2.9. *[11] Let $a, b \in R^{\dagger}$. If $ab = ba$ and $a^*b = ba^*$, then $ab \in R^{\dagger}$.*

Proof. Since $ab = ba$ and $a^*b = ba^*$, so $b^*a^* = a^*b^*$ and $b^*a = ab^*$. According to Lemma 2.8, $ab = ba$ and $a^*b = ba^*$ implied $a^{\dagger}b = ba^{\dagger}$.

$ba = ab$ and $b^*a = ab^*$ implied $b^{\dagger}a = ab^{\dagger}$.

$a^*b^* = b^*a^*$ and $a^*b = ba^*$implied $a^*b^{\dagger} = b^{\dagger}a^*$.

Since $ab^{\dagger} = b^{\dagger}a$ and $a^*b^{\dagger} = b^{\dagger}a^*$, we can apply Lemma 2.8 then we obtain $a^{\dagger}b^{\dagger} = b^{\dagger}a^{\dagger}$. We consider

i)

$$\begin{aligned} abb^{\dagger}a^{\dagger}ab &= bab^{\dagger}a^{\dagger}ab \\ &= bb^{\dagger}aa^{\dagger}ab \\ &= bb^{\dagger}ab \\ &= bb^{\dagger}ba \\ &= ba \\ &= a; \end{aligned}$$

ii)

$$\begin{aligned}
b^{\dagger}a^{\dagger}abb^{\dagger}a^{\dagger} &= a^{\dagger}b^{\dagger}abb^{\dagger}a^{\dagger}\\
&= a^{\dagger}ab^{\dagger}bb^{\dagger}a^{\dagger}\\
&= a^{\dagger}ab^{\dagger}a^{\dagger}\\
&= a^{\dagger}aa^{\dagger}b^{\dagger}\\
&= a^{\dagger}b^{\dagger} = b^{\dagger}a^{\dagger};
\end{aligned}$$

iii)

$$\begin{aligned}
(abb^{\dagger}a^{\dagger})^* &= (a^{\dagger})^*(bb^{\dagger})^*a^*\\
&= (a^{\dagger})^*bb^{\dagger}a^*\\
&= (a^{\dagger})^*ba^*b^{\dagger}\\
&= (a^{\dagger})^*a^*bb^{\dagger}\\
&= (aa^{\dagger})^*bb^{\dagger}\\
&= aa^{\dagger}bb^{\dagger}\\
&= aba^{\dagger}b^{\dagger}\\
&= abb^{\dagger}a^{\dagger};
\end{aligned}$$

iv)

$$\begin{aligned}
(a^{\dagger}b^{\dagger}ab)^* &= b^*a^*(b^{\dagger})^*(a^{\dagger})^*\\
&= a^*b^*(b^{\dagger})^*(a^{\dagger})^*\\
&= a^*(b^{\dagger}b)^*(a^{\dagger})^*\\
&= a^*b^{\dagger}b(a^{\dagger})^*\\
&= b^{\dagger}a^*b(a^{\dagger})^*\\
&= b^{\dagger}ba^*(a^{\dagger})^*\\
&= b^{\dagger}b(a^{\dagger}a)^*\\
&= b^{\dagger}ba^{\dagger}a\\
&= b^{\dagger}a^{\dagger}ba\\
&= a^{\dagger}b^{\dagger}ba.
\end{aligned}$$

Therefore $ab \in R^\dagger$, and $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. $\square$

Lemma 2.10. *[11] Let $a \in R^\dagger$. Then a is normal if and only if $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$.*

Proof. ($\Rightarrow$) Let $a \in R^\dagger$ be a normal element. Then $aa^* = a^*a$. By Lemma 2.8, we get $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$.
($\Leftarrow$) Suppose that $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. Now, we obtain $aa^* = a(a^*aa^\dagger) = a(a^*a^\dagger)a = a(a^\dagger a^*)a = a^\dagger aa^*a = a^*a$. Hence a is normal. $\square$

Lemma 2.11. *[9],[11] If $a \in R^\dagger$ is normal, then a is EP.*

Proof. Assume that $a \in R^\dagger$ and a is normal. We will show that a is group invertible. Since $aa^\dagger a = a$, $a^\dagger aa^\dagger = a^\dagger$ and $aa^\dagger = a^\dagger a$, a is group invertible and $a^\# = a^\dagger$. $\square$

Lemma 2.12. *Let $a \in R^\dagger$. Then a is EP if and only if $aa^\dagger = a^\dagger a$.*

Proof. ($\Rightarrow$) Let $a \in R^\dagger$ be an EP element. Then $a^\dagger = a^\#$. Hence

$$aa^\dagger = aa^\# = a^\# a = aa^\dagger.$$

($\Leftarrow$) Let $a \in R^\dagger$ be such that $aa^\dagger = a^\dagger a$. Since $a \in R^\dagger$, $aa^\dagger a = a$ and $a^\dagger aa^\dagger = a^\dagger$. From the assumption, $aa^\dagger = a^\dagger a$. We obtain $a^\dagger$ is the group inverse of a. Since the group inverse of a is unique, $a^\# = a^\dagger$. Hence a is EP. $\square$

Lemma 2.13. *[11] If $a \in R^\dagger$, then $aa^*a \in R^\dagger$ and $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$.*

Proof. Let $a \in R^\dagger$. Consider
(i)

$$\begin{aligned} aa^*aa^\dagger(a^*)^\dagger a^\dagger aa^*a &= aa^*(a^*)^\dagger a^*a \\ &= a(aa^\dagger a)^*a \\ &= aa^*a; \end{aligned}$$

(ii)

$$\begin{aligned} a^\dagger(a^*)^\dagger a^\dagger aa^*aa^\dagger(a^*)^\dagger a^\dagger &= a^\dagger(a^*)^\dagger a^\dagger aa^*(a^*)^\dagger a^\dagger \\ &= a^\dagger(a^*)^\dagger a^*(a^*)^\dagger a^\dagger \\ &= a^\dagger(a^\dagger aa^\dagger)^* a^\dagger \\ &= a^\dagger(a^\dagger)^* a^\dagger \\ &= a^\dagger(a^*)^\dagger a^\dagger; \end{aligned}$$

(iii)

$$\begin{aligned} (aa^*aa^\dagger(a^*)^\dagger a^\dagger)^* &= (aa^*(a^*)^\dagger a^\dagger)^* \\ &= (a(a^\dagger a^*)a^\dagger)^* \\ &= (aa^\dagger aa^\dagger)^* \\ &= (aa^\dagger)^* \\ &= aa^\dagger \\ &= aa^*(aa^*)^\dagger \\ &= aa^*aa^\dagger(a^*)^\dagger a^\dagger; \end{aligned}$$

(iv)

$$\begin{aligned} (a^\dagger(a^*)^\dagger a^\dagger aa^*a)^* &= (a^\dagger(a^*)^\dagger a^*a)^* \\ &= (a^\dagger(aa^\dagger)^*a)^* \\ &= (a^\dagger aa^\dagger a)^* \\ &= (a^\dagger a)^* = a^\dagger a \\ &= (a^*a)^\dagger a^*a \\ &= a^\dagger(a^*)^\dagger a^\dagger aa^*a. \end{aligned}$$

Hence $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$. □

Lemma 2.14. *[11] a is $*$-cancellable if and only if a^* is $*$-cancellable.*

Proof. ($\Rightarrow$) Let a be $*$-cancellable. Suppose that $aa^*x = aa^*y$. Taking $*$ on both sides, we get $x^*aa^* = y^*aa^*$. Since a is $*$-cancellable, $x^*a = y^*a$. Thus $a^*x = a^*y$.

Similarly, $xa^*a = ya^*a$ implies $xa^* = ya^*$. Hence a^* is $*$-cancellable.

($\Leftarrow$) Let a^* be $*$-cancellable. Suppose that $a^*ax = a^*ay$. Taking $*$ on both sides, we get $x^*a^*a = y^*a^*a$. Since a^* is $*$-cancellable, $x^*a^* = y^*a^*$. Thus $ax = ay$. Similarly, $xaa^* = yaa^*$ implies $xa = ya$. Hence a is $*$-cancellable. $\square$

CHAPTER 3

Moore-Penrose Inverses

In this chapter, ring R means an associate ring with involution. We give necessary and sufficient conditions for an element of a ring with involution to be Moore-Penrose invertible. We also investigate the existence of the Moore-Penrose inverse of the product of Moore-Penrose invertible elements.

Proposition 3.1. *Let R be a ring with involution. Suppose 1 is the multiplicative identity of R, the following are satisfied;*
(1) $0^* = 0$*;*
(2) $0^\dagger = 0$*;*
(3) *If $a \in R^\dagger$ then $-a \in R^\dagger$ and $(-a)^\dagger = -a^\dagger$;*
(4) *If a is a projection then $a^\dagger = a$*
(5) $1^\dagger = 1$*;*
(6) *If u is a unit in R then $u^\dagger = u^{-1}$.*

Proof. **(1)**

$$\begin{aligned} 0^* &= (0+0)^* \\ 0^* &= 0^* + 0^* \\ 0^* - 0^* &= 0^* \\ 0 &= 0^*. \end{aligned}$$

(2)

$$\begin{aligned} 0^\dagger &= 0^\dagger \cdot 0 \cdot 0^\dagger \\ 0^\dagger &= 0 \cdot 0^\dagger \\ 0^\dagger &= 0. \end{aligned}$$

(3) Let $a \in R^\dagger$. We will show that $-a \in R^\dagger$ and $(-a)^\dagger = -a^\dagger$.
i) $(-a)(-a^\dagger)(-a) = -aa^\dagger a = -a$;

ii) $(-a^{\dagger})(a)(-a^{\dagger}) = -a^{\dagger}aa^{\dagger} = -a^{\dagger}$;

iii) $((-a^{\dagger})(-a))^{*} = (a^{\dagger}a)^{*} = a^{\dagger}a = (-a^{\dagger})(-a)$;

iv) $((-a)(-a^{\dagger}))^{*} = (aa^{\dagger})^{*} = aa^{\dagger} = (-a)(-a^{\dagger})$.

Since Moore-Penrose inverse of $-a$ is unique, $(-a)^{\dagger} = -a^{\dagger}$. Hence $-a \in R^{\dagger}$ and $(-a)^{\dagger} = -a^{\dagger}$.

(4) Assume that $a^2 = a = a^*$. We will show that $a^{\dagger} = a$. Since $a^2 = a$ and $a^* = a$, $aaa = aa = a$. and $(aa)^* = a^* = a$. Hence $a^{\dagger} = a$.

(5) Assume that 1 is the multiplicative identity of R. We know that $1 = (1^*)^* = (1 \cdot 1^*)^* = 1 \cdot 1^* = 1^*$ and $1^2 = 1$. By **(4)** we get $1^{\dagger} = 1$.

(6) Let u be a unit in R. Then there is $u^{-1} \in R$ such that $uu^{-1} = 1 = u^{-1}u$.

i) $uu^{-1}u = 1u = u$;

ii) $u^{-1}uu^{-1} = 1u^{-1} = u^{-1}$;

iii) $(uu^{-1})^* = 1^* = 1 = uu^{-1}$;

iv) $(u^{-1}u)^* = 1^* = 1 = u^{-1}u$. Hence $u \in R^{\dagger}$ and $u^{\dagger} = u^{-1}$. □

Lemma 3.2. *Let* $a \in R^{\dagger}$. *If* $a^{\dagger}a = aa^{\dagger}$, *then* $a^n \in R^{\dagger}$ *for any* $n \in \mathbb{N}$.

Proof. Let $a \in R^{\dagger}$ be such that $a^{\dagger}a = aa^{\dagger}$. Then

i)

$$\begin{aligned} a^n(a^{\dagger})^n a^n &= (aa^{\dagger}a)^n \\ &= a^n; \end{aligned}$$

ii)

$$\begin{aligned} (a^{\dagger})^n a^n (a^{\dagger})^n &= (a^{\dagger}aa^{\dagger})^n \\ &= (a^{\dagger})^n; \end{aligned}$$

iii)

$$\begin{aligned} [a^n(a^{\dagger})^n]^* &= [(aa^{\dagger})^n]^* \\ &= [(aa^{\dagger})^*]^n \\ &= (aa^{\dagger})^n \\ &= a^n(a^{\dagger})^n; \end{aligned}$$

iv)

$$\begin{aligned}[(a^\dagger)^n a^n]^* &= [(a^\dagger a)^n]^* \\ &= [(a^\dagger a)^*]^n \\ &= (a^\dagger a)^n \\ &= (a^\dagger)^n a^n.\end{aligned}$$

Therefore $a^n \in R^\dagger$ for all $n \in \mathbb{N}$. □

Definition 3.1. An element $a \in R$ is left supported by a projection if $a = pa$ for some projection $p \in R$; it is right supported by a projection if $a = aq$ for some $q \in R$; and it is supported by a projection if it is both left and right supported by a projection.

Proposition 3.3. *Let $a \in R$. Then $aa^\dagger$ and $a^\dagger a$ are projections.*

Proof. It is clear that $(aa^\dagger)^2 = (aa^\dagger a)a^\dagger = aa^\dagger$ and $(aa^\dagger)^* = aa^\dagger$. Thus $aa^\dagger$ is a projection. Similarly, $a^\dagger a$ is also a projection. □

Theorem 3.4. *Let R be a ring with involution and let $a \in R$. Then the following are equivalent:*

(1) *a is MP-invertible.*

(2) *a is left $*$-cancellable, right supported by a projection and a^*a is group invertible.*

(3) *a is right $*$-cancellable, left supported by a projection and aa^* is group invertible.*

(4) *a is $*$-cancellable, supported by a projection and both a^*a and aa^* are group invertible.*

Proof. $(1) \Rightarrow (2), (3), (4)$ Suppose that a is MP-invertible. Let $x, y \in R$ be such that $a^*ax = a^*ay$. Then

$$
\begin{aligned}
ax &= aa^\dagger ax \\
&= (aa^\dagger)^* ax \\
&= (a^\dagger)^* a^* ax \\
&= (a^\dagger)^* a^* ay \\
&= (aa^\dagger)^* ay \\
&= aa^\dagger ay \\
&= ay.
\end{aligned}
$$

Thus a is left $*$-cancellable. Similarly, a is right $*$-cancellable. Let $p = aa^\dagger$ and $q = a^\dagger a$. Then p, q are projections and

$$a = aa^\dagger a = pa = aq.$$

Thus a is left and right supported by a projection. Since a^*a and aa^* are Hermitian, a^*a and aa^* are EP elements. Thus a^*a and aa^* are group invertible. This proves (2) and (3). It is obvious that (2) and (3) implies (4). Hence (4) holds.

$(2) \Rightarrow (1)$ Suppose that a is left $*$-cancellable, right supported by a projection and a^*a is group invertible. Then $a = aq$ for some projection q. Let $b = (a^*a)^\# a^*$. Then

$$a^*aqba = a^*a(a^*a)^\# a^*a = a^*a = a^*aq.$$

Since a is left $*$-cancellable, it is clear that

i)

$$
\begin{aligned}
aba &= aqba \\
&= aq \\
&= a;
\end{aligned}
$$

ii)

$$
\begin{aligned}
bab &= (a^*a)^\# a^*a(a^*a)^\# a^* \\
&= (a^*a)^\# a^* \\
&= b;
\end{aligned}
$$

iii)

$$\begin{aligned}(ab)^* &= [a(a^*a)^{\#}a^*]^*\\ &= a(a^*a)^{\#}a^*\\ &= ab;\end{aligned}$$

and iv)

$$\begin{aligned}(ba)^* &= [(a^*a)^{\#}a^*a]^*\\ &= a^*a(a^*a)^{\#}\\ &= (a^*a)^{\#}a^*a\\ &= ba.\end{aligned}$$

Thus a is MP-invertible and $a^{\dagger} = b$.

$(3) \Rightarrow (1)$ Suppose that a is right $*$-cancellable, left supported by a projection and aa^* is group invertible. Then $a = pa$ for some projection p. Let $b = a^*(a^*a)^{\#}$. Then

$$abpaa^* = aa^*(aa^*)^{\#}aa^* = aa^* = paa^*.$$

Since a is right $*$-cancellable, it is clear that

i)

$$aba = abpa = pa = a;$$

ii)

$$\begin{aligned}bab &= a^*(aa^*)^{\#}aa^*(aa^*)^{\#}\\ &= a^*(aa^*)^{\#}\\ &= b;\end{aligned}$$

iii)

$$\begin{aligned}(ab)^* &= [aa^*(aa^*)^{\#}]^*\\ &= (aa^*)^{\#}aa^*\\ &= aa^*(aa^*)^{\#}\\ &= ab;\end{aligned}$$

and iv)

$$\begin{aligned}(ba)^* &= [a^*(a^*a)^{\#}a]^* \\ &= a^*(aa^*)^{\#}a \\ &= ba.\end{aligned}$$

Thus a is MP-invertible and $a^\dagger = b$.

$(4) \Rightarrow (1)$ It is trivial that (4) implies (3) and (3) implies (1). Thus (4) implies (1). This completes the proof. □

Remark 3.1. If R is a ring with identity, then every element in R is clearly supported by the identity element. Thus Theorem 3.4 is a generalization of Proposition 1.1 in [8].

The following example shows that we cannot omit the condition that a is left(right) supported by a projection for an element a to be MP-invertible.

Example 3.1. Let $R = \{A \in M_3(\mathbb{R}) | a_{ij} = 0 \text{ for all } i \geq j\}$ with the usual matrix addition and multiplication. For any $a = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \in R$, we define $a^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$. Then $*$ is an involution. A computation shows that $abc = \mathbf{0}$ for all $a, b, c \in R$. This implies that $a \in R^{\#}$ if and only if $a = \mathbf{0}$ and $a \in R^\dagger$ if and only if $a = \mathbf{0}$.

Let $a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $ax = \mathbf{0}$ for all $x \in R$. Thus $a^*ax = \mathbf{0}$ implies $ax = \mathbf{0}$ for all $x \in R$. Likewise, $xaa^* = \mathbf{0}$ implies $xa = \mathbf{0}$. Hence a is $*$-cancellable. It is clear that $aa^* = a^*a = \mathbf{0}$ which are group invertible. However, a is not MP-invertible.

Next, we investigate the existence of the Moore-Penrose inverse of the product $x_1x_2\cdots x_n$ given that $x_1, x_2, \ldots, x_n \in R^\dagger$. It is worth noting that $R^\dagger$ is not closed under multiplication.

Example 3.2. Let $R = M_2(\mathbb{Z}_2)$ with the matrix transposition as an involution. Let $a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a, b \in R^\dagger$ but $ab = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin R^\dagger$.

Definition 3.2. A subset Γ of $R^\dagger$ is called star-dagger closed if $x^* \in \Gamma$ and $x^\dagger \in \Gamma$ for all $x \in \Gamma$. For any $x \in R^\dagger$, we define $\Gamma(x) = \{x, x^\dagger, x^*, (x^*)^\dagger\}$.

It is obvious that $R^\dagger$ is star-dagger closed. The following theorem shows that nontrivial star-dagger closed sets exist.

Theorem 3.5. *If $a \in R^\dagger$, then $\Gamma(a)$ is star-dagger closed.*

Proof. Clearly, $a, a^*, (a^\dagger)^* \in R^\dagger$. Since $(a^*)^* = a \in \Gamma(a)$ and $[(a^\dagger)^*]^* = a^\dagger \in \Gamma(a)$, we have $x^* \in \Gamma(a)$ for all $x \in \Gamma(a)$. Obviously, $a^\dagger \in \Gamma(a)$. Since $(a^\dagger)^\dagger = a \in \Gamma(a)$, $(a^*)^\dagger = (a^\dagger)^* \in \Gamma(a)$ and $[(a^\dagger)^*]^\dagger = [(a^*)^\dagger]^\dagger = a^* \in \Gamma(a)$, we conclude that $x^\dagger \in \Gamma(a)$ for all $in\Gamma(a)$. Therefore, $\Gamma(a)$ is star-dagger closed. $\square$

Definition 3.3. A subset Γ of R is called a commuting set if $xy = yx$ for all $x, y \in \Gamma$.

Theorem 3.6. *If $a \in R^\dagger$ is normal, then $\Gamma(a)$ is a commuting set.*

Proof. Suppose that $a \in R$ is normal. Then $aa^* = a^*a$, $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. Thus $a(a^\dagger)^* = (a^\dagger a^*)^* = (a^*a^\dagger)^* = (a^\dagger)^*a$. Similarly, $a^*(a^\dagger)^* = (a^\dagger a)^* = (aa^\dagger)^* = (a^\dagger)^*a^*$ and $a^\dagger(a^\dagger)^* = (a^*a)^\dagger = (aa^*)^\dagger = (a^\dagger)^*a^\dagger$. Hence $\Gamma(a)$ is a commuting set. $\square$

Theorem 3.7. *If $\Gamma \subseteq R^\dagger$ is a commuting and star-dagger closed set, then $xy \in R^\dagger$ for all $x, y \in \Gamma$. Moreover, $(xy)^\dagger = y^\dagger x^\dagger$ for all $x, y \in \Gamma$.*

Proof. Let $x, y \in \Gamma$. Then $x^*, y^*, x^\dagger, y^\dagger \in \Gamma$. Thus

$$\begin{aligned}(xy)(y^\dagger x^\dagger)(xy) &= (xx^\dagger x)(yy^\dagger y) \\ &= xy,\end{aligned}$$

$$\begin{aligned}(y^\dagger x^\dagger)(xy)(y^\dagger x^\dagger) &= (x^\dagger x x^\dagger)(y^\dagger y y^\dagger) \\ &= y^\dagger x^\dagger,\end{aligned}$$

$$
\begin{aligned}
(xyy^\dagger x^\dagger)^* &= (xx^\dagger)^*(yy^\dagger)^* \\
&= xx^\dagger yy^\dagger \\
&= xyy^\dagger x^\dagger,
\end{aligned}
$$

and

$$
\begin{aligned}
(y^\dagger x^\dagger xy)^* &= (y^\dagger y)^*(x^\dagger x)^* \\
&= y^\dagger y x^\dagger x \\
&= y^\dagger x^\dagger xy.
\end{aligned}
$$

This shows that $xy \in R^\dagger$ and $(xy)^\dagger = y^\dagger x^\dagger$. $\square$

Theorem 3.8. *If $\Gamma \subseteq R^\dagger$ is a commuting and star-dagger closed set, then $x_1x_2\cdots x_n \in R^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma$. Moreover, $(x_1x_2\cdots x_n)^\dagger = x_n^\dagger x_{n-1}^\dagger \cdots x_1^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma$.*

Proof. Let $\Gamma \subseteq R^\dagger$ be a commuting and star-dagger closed set. Then $xy = yx$ for all $x, y \in \Gamma$. Let $P(n)$: $\forall x_1, x_2, \ldots, x_n \in \Gamma$, $x_1x_2\cdots x_n \in R^\dagger$ and $(x_1x_2\cdots x_n)^\dagger = x_n^\dagger x_{n-1}^\dagger \cdots x_1^\dagger$ for all $n \in \mathbb{N}$. We will prove it by mathematical induction.

Basis Step: Suppose that $x_1, x_2 \in \Gamma$. According to Theorem 3.7, Γ is a commuting and star-dagger closed set, then $x_1x_2 \in R^\dagger$ and $(x_1x_2)^\dagger = x_2^\dagger x_1^\dagger = x_1^\dagger x_2^\dagger$.

Inductive Step: Assume that the statement $P(k)$: holds.

We consider $P(k+1)$. Suppose that $x_1, x_2, \ldots, x_{k+1} \in \Gamma$. According to Theorem 3.7 $x_1x_2\cdots x_k, x_{k+1} \in R^\dagger$, and $x_1x_2\cdots x_{k+1} = (x_1x_2\cdots x_k)(x_{k+1}) \in R^\dagger$ and

$$
\begin{aligned}
(x_1x_2\cdots x_{k+1})^\dagger &= ((x_1x_2\cdots x_k)(x_{k+1}))^\dagger \\
&= (x_{k+1})^\dagger(x_1x_2\cdots x_k)^\dagger \\
&= (x_{k+1}^\dagger)(x_k^\dagger x_{k-1}^\dagger \cdots x_1^\dagger) \\
&= x_{k+1}^\dagger x_k^\dagger x_{k-1}^\dagger \cdots x_1^\dagger.
\end{aligned}
$$

Thus $x_1x_2\cdots x_n \in R^\dagger$ for all $n \in \mathbb{N}$ and $(x_1x_2\cdots x_n)^\dagger = x_n^\dagger x_{n-1}^\dagger \cdots x_1^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma$. $\square$

Corollary 3.9. *If $a \in R^\dagger$ is normal, then $x_1x_2\cdots x_n \in R^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma(a)$.*

Proof. Since $\Gamma(a)$ is commuting and star-dagger closed set, the result follows from Theorem 3.8. □

Corollary 3.10. *If R is commutative, then $(R^\dagger, \cdot)$ is a subsemigroup of $(R, \cdot)$.*

Proof. Since R is commutative, $R^\dagger$ is a commuting and star-dagger closed set. The result follows Theorem 3.7. □

Theorem 3.11. *If a is an EP element, then $a^\dagger, a^*$ and $(a^\dagger)^*$ are also EP elements.*

Proof. Suppose that a is an EP element. Then $aa^\dagger = a^\dagger a$. Thus

$$a^\dagger(a^\dagger)^\dagger = a^\dagger a = aa^\dagger = (a^\dagger)^\dagger a^\dagger.$$

This means $a^\dagger$ is an EP element. We also have that

$$a^*(a^*)^\dagger = a^*(a^\dagger)^* = (a^\dagger a)^* = (aa^\dagger)^* = (a^*)^\dagger a^*.$$

Thus a^* is an EP element. This implies $(a^\dagger)^*$ is an EP element. □

Theorem 3.12. *If a is an EP element, then $x^n \in R^\dagger$ for all $x \in \Gamma(a)$ and all $n \in \mathbb{N}$.*

Proof. Suppose that a is an EP element. By using Theorem 3.11, we know that $a, a^\dagger, a^\dagger$ and $(a^\dagger)^*$ are EP elements. Thus it suffices to prove that $a^n \in R^\dagger$ for all $n \in \mathbb{N}$. Since a is an EP element, $aa^\dagger = a^\dagger a$. Then

i)

$$\begin{aligned} a^n(a^\dagger)^n a^n &= (aa^\dagger a)^n \\ &= a^n, \end{aligned}$$

ii)

$$\begin{aligned} (a^\dagger)^n a^n (a^\dagger)^n &= (a^\dagger aa^\dagger)^n \\ &= (a^\dagger)^n, \end{aligned}$$

iii)

$$\begin{aligned}
[a^n(a^\dagger)^n]^* &= [(aa^\dagger)^n]^* \\
&= [(aa^\dagger)^*]^n \\
&= (aa^\dagger)^n \\
&= a^n(a^\dagger)^n,
\end{aligned}$$

iv)

$$\begin{aligned}
[(a^\dagger)^n a^n]^* &= [(a^\dagger a)^n]^* \\
&= [(a^\dagger a)^*]^n \\
&= (a^\dagger a)^n \\
&= (a^\dagger)^n a^n.
\end{aligned}$$

This proves that $a^n \in R^\dagger$ and $(a^n)^\dagger = (a^\dagger)^n$ for all $n \in \mathbb{N}$. $\square$

Theorem 3.13. *Let $a \in R^\dagger$. Then the following elements are Moore-Penrose invertible for all $n \in \mathbb{N}$.*

(1) $(aa^*)^n$,

(2) $(a^*a)^n$,

(3) $(a^*a^\dagger aa)^n$,

(4) $a(a^*a)^n$ and

(5) $a^*(aa^*)^n$.

Proof. **(1)** We know that $aa^* \in R^\dagger$ is Hermitian. Thus aa^* is an EP element. By Theorem 3.12, $(aa^*)^n \in R^\dagger$ for all $n \in \mathbb{N}$.

(2) We know that $a^*a \in R^\dagger$ is Hermitian. Thus a^*a is an EP element. By Theorem 3.12, $(a^*a)^n \in R^\dagger$ for all $n \in \mathbb{N}$.

(3) Let $x = a^*a^\dagger aa$. Then $x^* = a^*a^*(a^\dagger)^*a$ and

$$\begin{aligned} xx^* &= a^*a^\dagger aaa^*a^*(a^\dagger)^*a \\ &= a^*(a^\dagger a)^*aa^*(a^\dagger a)^*a \\ &= a^*a^*(a^\dagger)^*aa^*a^\dagger aa \\ &= x^*x. \end{aligned}$$

Thus x is a normal element and hence an EP element. By Theorem 3.12, $(a^*a^\dagger aa)^n \in R^\dagger$ for all $n \in \mathbb{N}$.

(4) Let $x = a(a^*a)^n = (aa^*)^n a$ and $y = [(a^*a)^\dagger]^n a^\dagger = a^\dagger[(aa^*)^\dagger]^n$. Since aa^* and a^*a are EP elements, we have $[(aa^*)^n]^\dagger = [(aa^*)^\dagger]^n$ and $[(a^*a)^n]^\dagger = [(a^*a)^\dagger]^n$. Then

$$\begin{aligned} xy &= (aa^*)^n aa^\dagger[(aa^*)^n]^\dagger \\ &= (aa^*)^{n-1}(aa^*aa^\dagger)[(aa^*)^n]^\dagger \\ &= (aa^*)^{n-1}(aa^*)[(aa^*)^n]^\dagger \\ &= (aa^*)^n[(aa^*)^n]^\dagger \end{aligned}$$

and

$$\begin{aligned} yx &= a^\dagger[(aa^*)^\dagger]^n a(a^*a)^n \\ &= a^\dagger[(aa^*)^\dagger]^{n-1}[(aa^*)^\dagger a](a^*a)^n \\ &= a^\dagger[(aa^*)^\dagger]^{n-1}(a^*)^\dagger(a^*a)^n \\ &= [a^\dagger(a^*)^\dagger]^n(a^*a)^n \\ &= [(a^*a)^n]^\dagger(a^*a)^n. \end{aligned}$$

Thus,

$$\begin{aligned} xyx &= a(a^*a)^n[(a^*a)^n]^\dagger(a^*a)^n \\ &= a(a^*a)^n \\ &= x \end{aligned}$$

and

$$\begin{aligned} yxy &= a^\dagger[(aa^*)^n]^\dagger(aa^*)^n[(aa^*)^n]^\dagger \\ &= a^\dagger[(aa^*)^n]^\dagger \\ &= y. \end{aligned}$$

Since xy and yx are projections, we also have $(xy)^* = xy$ and $(yx)^* = yx$. Therefore $a(a^*a)^n \in R^\dagger$ and $[a(a^*a)^n]^\dagger = [(a^*a)^\dagger]^n a^\dagger$.

5. Since $[a(a^*a)^n]^* = a^*(aa^*)^n$, we conclude that $a^*(aa^*)^n \in R^\dagger$.

$\square$

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